Shintani Function and Its application to Automorphic L-Functions for Classical Groups

II. Rankin-Selberg Convolution of Two Variables for O(m,2)

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II. Rankin-Selberg Convolution of Two Variables for O(m,2)

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Introduction

This paper is concerned with a certain Rankin-Selberg convolution of *two variables* for a cusp form F on an orthogonal group of signature $(m-1, 2)$. The main object is to show that the convolution splits into a product of two standard L-functions attached to F.

To be more precise, let $H = O(m-1, 2)$ be an orthogonal group suitably embedded in $G_1 = O(m, 2)$ (cf. §2.1). Let B_1 be a $\mathbf{Q}$-minimal parabolic subgroup of G_1. Then its Levi component is isomorphic to $GL_1 \times GL_1 \times G_0$, where $G_0 = O(m-2)$ is a definite orthogonal group. For an automorphic form φ on G_0, we can attach an Eisenstein series $E(*, \varphi; s, s_0)$ of two variables s and s_0 on G_1 with respect to B_1. Let F be a cusp form on H and suppose that F and φ are Hecke eigenforms. The convolution we study in this paper is defined by

$$\Xi_{F,\varphi}(s, s_0) = \int_{H_{\mathbf{Q}}\backslash H_{\mathbf{A}}} F(h)\, E(h, \varphi; s, s_0)\, dh.$$

The main result (Theorem 3.1) asserts that, under certain assumptions on orthogonal groups, the integral $\Xi_{F,\varphi}(s, s_0)$ decomposes into a product of an Euler product $\dfrac{L(F; s)\, L(F; s_0)}{L(\varphi; s+\frac{1}{2})\, L(\varphi; s_0+\frac{1}{2})}$ and a local factor at infinity. Here $L(F; s)$ and $L(\varphi; s)$

denote the standard L-functions associated with F and φ. The local factor at infinity can be explicitly calculated if F is holomorphic (Theorem 3.2).

To prove the results, we employ two main ingredients; Shintani functions (§4.3) and generalized Whittaker functions (§5.1). They have been introduced and studied in [MS1] and [Su] respectively, where we proved certain formulas relating certain integrals of these functions to some Euler factors (see Proposition 4.1 and Proposition 5.1). These formulas are essential in the calculation of the local integrals.

We now explain a brief account of the paper. In §1, we review the definition of the standard L-functions for orthogonal groups. We introduce various orthogonal groups and their embeddings in §2. The main results of this paper is stated in §3 (Theorem 3.1 and Theorem 3.2). The next two sections are devoted to the proof of Theorem 3.1. In §4, we first recall the definition of Shintani functions. Using the basic identity proved in [MS1], we show that $\Xi_{F,\varphi}(s, s_0)$ is a product of the quotient $L(F; s)/L(\varphi; s+\frac{1}{2})$ of L-functions and a certain integral $\Lambda_\infty(F, \varphi; s, s_0)$ at the infinite place. In §5, we prove that the integral $\Lambda_\infty(F, \varphi; s, s_0)$ is expressed as a certain integral of generalized Whittaker function. By virtue of the result of [Su], we see that the integral is a product of $L(F; s_0)/L(\varphi; s_0+\frac{1}{2})$ and a local factor $d_\infty(F, \varphi; s, s_0)$ that depends only on the data at the infinite place. Theorem 3.1 is proved by combining these results. In the final section, we calculate $d_\infty(F, \varphi; s, s_0)$ in an explicit form in the case where F is holomorphic.

In the forthcoming paper ([MS2]), applying the above results to the case $m = 3$, we study the pullback of Eisenstein series on $PGSp_2$ (isogeneous to $O(3, 2)$) to a Hilbert modular group (isogeneous to $O(2, 2)$).

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Notation

Let $\mathbf{Q}_v$ be the completion of $\mathbf{Q}$ at a prime v of $\mathbf{Q}$ and $\mathbf{A}$ be the adele ring of

$\mathbf{Q}$. For a linear algebraic group X defined over $\mathbf{Q}$, we denote by X_v (resp. $X_{\mathbf{A}}$) the group of $\mathbf{Q}_v$ (resp. $\mathbf{A}$) -rational points of X. For each prime v of $\mathbf{Q}$, $|\ |_v$ stands for the normalized valuation of $\mathbf{Q}_v$ given by $d(ax) = |a|_v\, dx$ $(a \in \mathbf{Q}_v^\times)$, where dx is a Haar measure on $\mathbf{Q}_v$. Let $|x|_{\mathbf{A}} = \prod_v |x_v|_v$ be the norm of an idele $x = (x_v) \in \mathbf{A}^\times$. The finite part of $\mathbf{A}$ is denoted by $\mathbf{A}_f$. We fix the additive character ψ of $\mathbf{A}$ trivial on $\mathbf{Q}$ so that $\psi(x_\infty) = \mathbf{e}[x_\infty] := \exp(2\pi i x_\infty)$ for $x_\infty \in \mathbf{R}$.

For a symmetric matrix $A \in M_r$, we put $A(x, y) = {}^t xAy$ and $A[x] = {}^t xAx$ for r-column vectors x and y. We write 0_r for the zero column vector of size r.

§1. Review of the standard L-functions for orthogonal groups

Let $S \in M_m(\mathbf{Z})$ be a non-degenerate even integral symmetric matrix. Assume that S is maximal; namely, $L = \mathbf{Z}^m$ is a maximal $\mathbf{Z}$-lattice with respect to S. Let $G = O(S)$ be the orthogonal group of S. By maximality of S, $K_p = G(\mathbf{Z}_p)$ is a maximal open compact subgroup of $G_p = G(\mathbf{Q}_p)$ for every p. Let $H(G_p, K_p)$ be the algebra of compactly supported bi-K_p invariant functions on G_p. Then the Hecke algebra $H(G_p, K_p)$ is isomorphic to the affine algebra $\mathbf{C}[X_1^{\pm 1}, \cdots, X_{\nu_p}^{\pm 1}]^{W_{\nu_p}}$ via the Satake homomorphism, where ν_p is the Witt index of S at p and W_{ν_p} is the subgroup of the automorphism group of $\mathbf{C}[X_1^{\pm 1}, \cdots, X_{\nu_p}^{\pm 1}]$ generated by the involutions $X_i \to X_i^{-1}$ and the permutations of the indeterminates $X_1, \cdots, X_{\nu_p}$ (see [Sa]). It follows that each $\mathbf{C}$-algebra homomorphism λ_p of $H(G_p, K_p)$ to $\mathbf{C}$ determines the *Satake parameter* $(\alpha_1, \cdots, \alpha_{\nu_p}) \in (\mathbf{C}^\times)^{\nu_p}/W_{\nu_p}$.

To define an L-factor attached to λ_p, we introduce certain invariants of S at p. Let $L_p^* = S^{-1}L_p$ be the dual lattice of $L_p = \mathbf{Z}_p^m$ with respect to S. Put $L_p' = \{x \in L_p^* \mid S[x] \in p^{-1}\mathbf{Z}_p\}$. By maximality of S, L_p' is a $\mathbf{Z}_p$-lattice containing L_p. We set $\partial_p(S) = \dim_{\mathbf{Z}_p/p\mathbf{Z}_p} L_p'/L_p$. It is known that $0 \le \partial_p(S) \le 2$ and that $\partial_p(S) = 0$ for almost all p(cf. [Su]). Let $n_{o,p} = m - 2\nu_p(S)$ be the dimension of the maximal anisotropic subspace of $\mathbf{Q}_p^m$ $(0 \le n_{o,p} \le 4)$. We define the standard L-factor $L_p(\lambda_p; s)$ as follows:

$$L_p(\lambda_p; s) = A_{S,p}(s) \cdot \prod_{i=1}^{\nu_p} (1-\alpha_i p^{-s})^{-1} (1-\alpha_i^{-1} p^{-s})^{-1}, \tag{1.1}$$

$$A_{S,p}(s) = \begin{cases} 1 & \text{if } (n_{o,p}(S), \partial_p(S)) = (0,0) \text{ or } (1,0) \\ (1+p^{1/2-s}) & (1,1) \\ (1-p^{-2s})^{-1} & (2,0) \\ (1-p^{-s})^{-1} & (2,1) \\ (1-p^{-s})^{-1}(1+p^{1-s}) & (2,2) \\ (1-p^{-1/2-s})^{-1} & (3,1) \\ (1-p^{-1/2-s})^{-1}(1+p^{1/2-s}) & (3,2) \\ (1-p^{-s})^{-1}(1-p^{-1-s})^{-1} & (4,2). \end{cases}$$

We denote by $M(K_f)$ the space of automorphic forms on $G_{\mathbf{A}}$ that are invariant under right $K_f = \prod_{p<\infty} K_p$ -translations (see [BJ]). By definition, $f \in M(K_f)$ is a smooth function on $G_{\mathbf{Q}} \backslash G_{\mathbf{A}} / K_f$ which satisfies the following conditions:

(1.2) The function f is $Z(\mathrm{Lie}(G_\infty)_{\mathbf{C}})$-finite, where $Z(\mathrm{Lie}(G_\infty)_{\mathbf{C}})$ is the center of the universal enveloping algebra of the complexified Lie algebra $\mathrm{Lie}(G_\infty)_{\mathbf{C}}$ of G_∞.

(1.3) For any $g_f \in G_{\mathbf{A}_f}$, the function $g_\infty \to f(g_\infty g_f)$ is of moderate growth on G_∞.

The subspace of cusp forms in $M(K_f)$ is denoted by $S(K_f)$. The Hecke algebra $\bigotimes'_{p<\infty} H(G_p, K_p)$ acts on $M(K_f)$ by convolution on the right:

$$f * \varphi(h) = \int_{G_{\mathbf{A}_f}} f(gx^{-1})\varphi(x)dx \quad (f \in M(K_f),\ \varphi \in \textstyle\bigotimes'_{p<\infty} H(G_p, K_p),\ g \in G_{\mathbf{A}}).$$

We say that $f \in M(K_f)$ is a *Hecke eigenform* if, for each rational prime p, we have $f * \varphi_p = \lambda_{f,p}(\varphi_p) \cdot f$ for any $\varphi_p \in H(G_p, K_p)$ with $\lambda_{f,p} \in \mathrm{Hom}_{\mathbf{C}}(H(G_p, K_p), \mathbf{C})$. The global standard L-function $L(f; s)$ for a Hecke eigenform f is defined by the Euler product

(1.4) $$L(f; s) = \prod_{p<\infty} L_p(\lambda_{f,p}; s).$$

§2. Embedding of orthogonal groups

2.1 Let Q be a non-degenerate even symmetric matrix of rank $m-2 \geq 1$. Let $R = \begin{bmatrix} Q & -Q\lambda_o \\ -{}^t\lambda_o Q & -2a \end{bmatrix}$ with $\lambda_o \in Q^{-1}\mathbf{Z}^{m-2}$ and $a \in \mathbf{Z}$, and put $\Delta = Q[\lambda_o] + 2a$. Let

$$S = \begin{bmatrix} & & 1 \\ & Q & \\ 1 & & \end{bmatrix}, T = \begin{bmatrix} & & 1 \\ & R & \\ 1 & & \end{bmatrix}, S_1 = \begin{bmatrix} & & 1 \\ & S & \\ 1 & & \end{bmatrix}.$$

In what follows, we assume $Q > 0$ and $\Delta > 0$. Then the signatures of Q, R, S, T and S_1 are given by $(m-2, 0), (m-2, 1), (m-1, 1), (m-1, 2)$ and $(m, 2)$ respectively. We consider the orthogonal groups

$$G_o = O(Q, V_o), V_o = \mathbf{Q}^{m-2}; H_o = O(R, W_o), W_o = \mathbf{Q}^{m-1};$$

$$G = O(S, V), V = \mathbf{Q}^m; \ H = O(T, W), W = \mathbf{Q}^{m+1};$$

$$G_1 = O(S_1, V_1), V_1 = \mathbf{Q}^{m+2}.$$

Define embeddings $V \xrightarrow{j_o} W \xrightarrow{j} V_1$ of vector spaces by

$$j_o(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}) = \begin{bmatrix} v_1 \\ v_2 \\ 0 \\ v_3 \end{bmatrix} \quad (v_1, v_3 \in \mathbf{Q}, v_2 \in \mathbf{Q}^{m-2}),$$

$$j(\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}) = \begin{bmatrix} -aw_3 - Q(\lambda_o, w_2) \\ w_1 \\ w_2 \\ w_4 \\ w_3 \end{bmatrix} \quad (w_1, w_3, w_4 \in \mathbf{Q}, w_2 \in \mathbf{Q}^{m-2}).$$

Then we have $T[j_o(v)] = S[v]$, $S_1[j(w)] = T[w]$ for $v \in V, w \in W$. Moreover we see $j_o(V) \perp \mathbf{Q}\xi$ and $j(W) \perp \mathbf{Q}\eta$ with respect to T and S_1 respectively, where we put

$$\xi = \begin{bmatrix} 0 \\ \lambda_o \\ 1 \\ 0 \end{bmatrix} \Delta^{-1} \in W, \quad \eta = \begin{bmatrix} a \\ 0 \\ \lambda_o \\ 0 \\ 1 \end{bmatrix} \in V_1.$$

We define the embeddings $G \xrightarrow{\iota_o} H \xrightarrow{\iota} G_1$ of orthogonal groups to be

$$\iota_o(g)(t\xi + j_o(v)) = t\xi + j_o(gv),$$

$$\iota(h)(t\eta + j(w)) = t\eta + j(hw)$$

for $t \in \mathbf{Q}$, $v \in V$, $w \in W$, $g \in G$ and $h \in H$. Then $\iota_o(G)$ (resp. $\iota(H)$) is the isotropy subgroup of ξ (resp. η) in H (resp. G_1). For $x \in \mathbf{Q}^{m-2}$ and $y \in \mathbf{Q}^{m-1}$, put $n_G(x) = \begin{bmatrix} 1 & -{}^t xQ & -2^{-1}Q[x] \\ 0 & 1_{m-2} & x \\ 0 & 0 & 1 \end{bmatrix} \in G_\mathbf{Q}$ and $n_H(y) = \begin{bmatrix} 1 & -{}^t yR & -2^{-1}R[y] \\ 0 & 1_{m-1} & y \\ 0 & 0 & 1 \end{bmatrix} \in H_\mathbf{Q}$. Then $P_\mathbf{Q} = \{n_G(x) \begin{bmatrix} t & 0 & 0 \\ 0 & g_o & 0 \\ 0 & 0 & t^{-1} \end{bmatrix} \mid x \in \mathbf{Q}^{m-2}, t \in \mathbf{Q}^\times, g_o \in G_{0,\mathbf{Q}}\}$ and $P_{H,\mathbf{Q}} = \{n_H(y) \begin{bmatrix} t & 0 & 0 \\ 0 & h_o & 0 \\ 0 & 0 & t^{-1} \end{bmatrix} \mid y \in \mathbf{Q}^{m-1}, t \in \mathbf{Q}^\times, h_o \in H_{0,\mathbf{Q}}\}$ are maximal $\mathbf{Q}$-parabolic subgroups of G and H respectively. The following is easily verified (cf. [MS1, §2]).

Lemma 2.1

(i) *For* $p = n_G(x) \begin{bmatrix} t & 0 & 0 \\ 0 & g_o & 0 \\ 0 & 0 & t^{-1} \end{bmatrix} \in P$, *we have*

$$\iota_o(p) = n_H(\begin{bmatrix} x \\ 0 \end{bmatrix}) \times \begin{bmatrix} t & 0 & 0 & 0 \\ 0 & g_o & (1-g_o)\lambda_o & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t^{-1} \end{bmatrix}.$$

(ii) $\iota(\iota_o(g)) = \begin{bmatrix} 1 & {}^t\lambda S(1-g) & {}^t\lambda S(g-1)\lambda \\ 0 & g & (1-g)\lambda \\ 0 & 0 & 1 \end{bmatrix}$, *where* $g \in G$ *and* $\lambda = \begin{bmatrix} 0 \\ \lambda_o \\ 0 \end{bmatrix} \in V$.

In what follows, we often regard G (resp. H) as a subgroup of H (resp. G_1) via ι_o (resp. ι) and simply write g (resp. h) for $\iota_o(g)$ (resp. $\iota(h)$) if there is no fear of confusion.

2.2 We define symmetric domains D and D_1 of type (IV) as follows:

$$D = \{z \in \mathbf{C}^{m-1} \mid R[\mathrm{Im}\, z] < 0\}, \quad D_1 = \{Z \in \mathbf{C}^m \mid S[\mathrm{Im}\, Z] < 0\}.$$

Note that both D and D_1 have two connected components. For $z \in D$ and $Z \in D_1$, put

$$z^\sim = \begin{bmatrix} -2^{-1}R[z] \\ z \\ 1 \end{bmatrix} \in W_{\mathbf{C}}, \quad Z^\sim = \begin{bmatrix} -2^{-1}S[Z] \\ Z \\ 1 \end{bmatrix} \in V_{1,\mathbf{C}}.$$

The action of H_∞ (resp. $G_{1,\infty}$) on D (resp. D_1) and the automorphic factor J_H : $H_\infty \times D \to \mathbf{C}^\times$ (resp. $J_{G_1} : G_{1,\infty} \times D_1 \to \mathbf{C}^\times$) are defined to be

$$h \cdot z^\sim = (h\langle z\rangle)^\sim \cdot J_H(h, z), \quad g_1 \cdot Z^\sim = (g_1\langle Z\rangle)^\sim \cdot J_{G_1}(g_1, Z)$$

($h \in H_\infty$, $g_1 \in G_{1,\infty}$, $z \in D$, $Z \in D_1$). These actions are transitive and holomorphic. Put $U_\infty = \{h \in H_\infty \mid h\langle z_o\rangle = z_o\}$ and $K_{1,\infty} = \{g_1 \in G_{1,\infty} \mid g_1\langle Z_o\rangle = Z_o\}$ where $z_o = \begin{bmatrix} \lambda_o i \\ i \end{bmatrix} \in D$ and $Z_o = \begin{bmatrix} 2^{-1}\Delta i \\ \lambda_o \\ -i \end{bmatrix} \in D_1$. We note that U_∞ and $K_{1,\infty}$ are compact and isomorphic to $O(m-1) \times SO(2)$ and $O(m) \times SO(2)$ respectively. Put $v_o = \begin{bmatrix} -2^{-1}\Delta \\ 0_{m-2} \\ 1 \end{bmatrix} \in V_{\mathbf{R}}$. Since $S[v_o] = -\Delta < 0$, the subgroup $K_\infty = \{g \in G_\infty \mid gv_o = v_o\}$ of G_∞ is compact and isomorphic to $O(m-1)$.

Lemma 2.2

(i) *We have* $\iota_o(K_\infty) \subset U_\infty^1 = \{u \in U_\infty \mid J_H(u, z_o) = 1\}$.

(ii) *For* $h \in H_\infty$, *we have* $J_{G_1}(\iota(h), Z_0) = (-iz'')\cdot J_H(h, z_0)$ *where* $h\langle z_0\rangle = \begin{bmatrix} z' \\ z'' \end{bmatrix} \in D$ $(z' \in \mathbf{C}^{m-2}, z'' \in \mathbf{C})$.

(iii) *We have* $\iota(U_\infty) \subset K_{1,\infty}$ *and* $J_{G_1}(\iota(u), Z_0) = J_H(u, z_0)$ *for* $u \in U_\infty$.

Proof. Observe $z_0^\sim = \begin{bmatrix} -2^{-1}\Delta \\ \lambda_0 i \\ i \\ 1 \end{bmatrix} = j_0(v_0) + \xi\cdot\Delta i$. Then, for $k \in K_\infty$, we have $\iota_0(k)z_0^\sim = j_0(kv_0) + \xi\cdot\Delta i = j_0(v_0) + \xi\cdot\Delta i = z_0^\sim$, which implies (i). The remaining parts follow from the observation that $Z_0^\sim = \begin{bmatrix} -a-Q[\lambda_0] \\ 2^{-1}\Delta i \\ \lambda_0 \\ -i \\ 1 \end{bmatrix} = j(z_0^\sim)\cdot(-i)$. *q.e.d.*

2.3 We define a holomorphic embedding $\rho : D \to D_1$ by

$$\rho(z) = \begin{bmatrix} -\frac{1}{2}z''^{-1}Q[z'] + Q(\lambda_0, z') + az'' \\ z''^{-1}z' \\ z''^{-1} \end{bmatrix} = z''^{-1}\begin{bmatrix} -2^{-1}R[z] \\ z' \\ 1 \end{bmatrix}$$

for $z = \begin{bmatrix} z' \\ z'' \end{bmatrix} \in D$ $(z' \in \mathbf{C}^{m-2}, z'' \in \mathbf{C})$. Note that $S[\mathrm{Im}\,\rho(z)] = \frac{1}{|z''|^2} R[\mathrm{Im}\,z]$ for $z \in D$ and that $\rho(z_0) = Z_0$. The following is easily verified.

Lemma 2.3 *Let* $z = \begin{bmatrix} z' \\ z'' \end{bmatrix} \in D$ $(z' \in \mathbf{C}^{m-2}, z'' \in \mathbf{C})$.

(i) *We have* $j(z^\sim) = \rho(z)^\sim \cdot z''$.

(ii) *For* $h \in H_\infty$, *let* $h\langle z\rangle = \begin{bmatrix} z_1' \\ z_1'' \end{bmatrix}$. *Then we have*

$$\iota(h)\langle\rho(z)\rangle = \rho(h\langle z\rangle), \quad J_{G_1}(\iota(h), \rho(z)) = z''^{-1} z_1'' \cdot J_H(h, z).$$

§3. Main results

3.1 From now on we assume that Q and R are maximal. Let $K_{o,p}$, K_p, U_p and $K_{1,p}$ be the groups of $\mathbf{Z}_p$-rational points of G_o, G, H and G_1, which are maximal open compact subgroups of $G_{o,p}$, G_p, H_p and $G_{1,p}$ respectively. Put $K_{o,f} = \prod_{p<\infty} K_{o,p}$. Define K_f, U_f and $G_{1,f}$ similarly. Throughout this paper, we assume that the condition

$$\partial_p(Q) = \partial_p(R) \tag{3.1}$$

holds for every p. Under this assumption, we have

$$\iota_o(G_p) \cap U_p = \iota_o(K_p), \quad \iota(H_p) \cap K_{1,p} = \iota(U_p) \tag{3.2}$$

for every p (cf. [MS1, Proposition 3.7]).

3.2 Let $S(U_f)$ be the space of cusp forms on $H_{\mathbf{A}}$ invariant under right U_f-translations (see §1). We fix an even positive integer l and denote by $S_l(U_f)$ the space of $F \in S(U_f)$ satisfying $F(hu_\infty) = F(h)\, J_H(u_\infty, z_o)^{-l}$ for $h \in H_{\mathbf{A}}$ and $u_\infty \in U_\infty$. Put $K_o = G_{o,\infty} K_{o,f}$. Note that $G_{o,\infty}$ is compact and that $G_{o,\mathbf{Q}} \backslash G_{o,\mathbf{A}} / K_o$ is a finite set since Q is positive definite. Let $M(K_o)$ be the space of automorphic forms on $G_{o,\mathbf{A}}$ invariant under right K_o-translations.

3.3 Let

$$B_{1,\mathbf{Q}} = \left\{ \begin{bmatrix} t & * & * & * & * \\ 0 & t_o & * & * & * \\ 0 & 0 & g_o & * & * \\ 0 & 0 & 0 & t_o^{-1} & * \\ 0 & 0 & 0 & 0 & t^{-1} \end{bmatrix} \in G_{1,\mathbf{Q}} \;\middle|\; t, t_o \in \mathbf{Q}^\times, g_o \in G_{o,\mathbf{Q}} \right\}$$

be a $\mathbf{Q}$-minimal parabolic subgroup of G_1. Then each $g_1 \in G_{1,\mathbf{A}}$ is decomposed into $b_1(g_1)\, k_1(g_1)$ with

$$b_1(g_1) = \begin{bmatrix} \alpha(g_1) & * & * & * & * \\ 0 & \alpha_0(g_1) & * & * & * \\ 0 & 0 & \beta_0(g_1) & * & * \\ 0 & 0 & 0 & \alpha_0(g_1)^{-1} & * \\ 0 & 0 & 0 & 0 & \alpha(g_1)^{-1} \end{bmatrix} \in B_{1,\mathbf{A}},$$

$\alpha(g_1), \alpha_0(g_1) \in \mathbf{A}^\times$, $\beta_0(g_1) \in G_{0,\mathbf{A}}$ and $k_1(g_1) = \prod_{v \leq \infty} k_1(g_1)_v \in K_1 = K_{1,\infty}K_{1,f}$. We can choose $\alpha(g_1)$ and $\alpha_0(g_1)$ so that $\alpha(g_1)_\infty$, $\alpha_0(g_1)_\infty > 0$. The Eisenstein series $\mathbb{E}(g_1, \varphi, l; s, s_0)$ attached to $\varphi \in M(K_0)$ with respect to B_1 is defined as follows:

$$(3.3)\quad \mathbb{E}(g_1, \varphi, l; s, s_0) = \sum_{\gamma_1 \in B_{1,\mathbf{Q}} \backslash G_{1,\mathbf{Q}}} \varphi(\beta_0(\gamma_1 g_1))\, |\alpha(\gamma_1 g_1)|_{\mathbf{A}}^{s+m/2}\, |\alpha_0(\gamma_1 g_1)|_{\mathbf{A}}^{s_0+(m-2)/2} \times J_{G_1}(k_1(\gamma_1 g_1)_\infty, Z_0)^l$$

($g_1 \in G_{1,\mathbf{A}}$, $(s, s_0) \in \mathbf{C}^2$). Thanks to Langlands ([La]), the series (3.3) converges absolutely in the region $\{(s, s_0) \in \mathbf{C}^2 \mid \mathrm{Re}\, s - \mathrm{Re}\, s_0 > 1, \mathrm{Re}\, s > \frac{m-2}{2}\}$ and can be continued to a meromorphic function of (s, s_0) on $\mathbf{C}^2$.

3.4 Let $F \in S_l(U_f)$ and $\varphi \in M(K_0)$ be Hecke eigenforms and $L(F; s)$ and $L(\varphi; s)$ be the corresponding standard L-functions (see §1). The object of this paper is to study the following Rankin-Selberg convolution of two variables:

$$(3.4)\qquad \Xi_{F,\varphi}(s, s_0) = \int_{H_{\mathbf{Q}} \backslash H_{\mathbf{A}}} F(h)\, \mathbb{E}(\iota(h), \varphi; s - \tfrac{1}{2}, s_0 - \tfrac{1}{2})\, dh.$$

The integral (3.4) can be continued to a meromorphic function of (s, s_0) on $\mathbf{C}^2$. The main result of this paper is stated as follows.

Theorem 3.1 *Let* $F \in S_l(U_f)$ *and* $\varphi \in M(K_0)$ *be Hecke eigenforms. Assume that* Q *and* R *are maximal and that the condition (3.1) is satisfied. Then we have*

$$\Xi_{F,\varphi}(s, s_0) = \frac{L(F; s)\, L(F; s_0)}{L(\varphi; s + \frac{1}{2})\, L(\varphi; s_0 + \frac{1}{2})}$$

$$\times \zeta(s + s_0)^{-1}\, \zeta(s - s_0 + 1)^{-1} \times \begin{cases} 1 & \textit{if } m \textit{ is even} \\ \zeta(2s)^{-1}\, \zeta(2s_0)^{-1} & \textit{if } m \textit{ is odd} \end{cases}$$

$$\times\, d_\infty(F, \varphi; s, s_0).$$

Here the local factor at infinity $d_\infty(F, \varphi; s, s_0)$ *is given by the integral*

$$\int_{\iota_0(P_\infty)\backslash H_\infty} dh \int_{\mathbf{R}^\times} dt^\times\, W_{F,\varphi}(\begin{bmatrix} t & & \\ & 1_{m-1} & \\ & & t^{-1} \end{bmatrix} h)\, |t|^{s_0 - \frac{m-1}{2}}$$

$$|\alpha_0(h)|^{s_0 + \frac{m-3}{2}}\, |\alpha(h)|^{s + \frac{m-1}{2}}\, J_{G_1}(k_1(h)_\infty, Z_0)^l,$$

where $W_{F,\varphi}$ *is a generalized Whittaker function associated with* F, φ *(for the precise definition, see §5.1).*

Remark. The local factor $d_\infty(F, \varphi; s, s_0)$ depends only on l and $W_{F,\varphi}|_{H_\infty}$.

3.5 Let $S_l^{hol}(U_f)$ be the space of holomorphic cusp forms on H of weight l. By definition, $F \in S_l^{hol}(U_f)$ is an element of $S_l(U_f)$ such that, for every $h_f \in H(\mathbf{A}_f)$, the function $F(z; h_f) := F(h_\infty h_f)\, J_H(h_\infty, z_0)^l$ is holomorphic in $z = h_\infty\langle z_0\rangle \in D$. We can calculate $d_\infty(F, \varphi; s, s_0)$ explicitly in the case where $F \in S_l^{hol}(U_f)$.

Theorem 3.2 *Assume* $F \in S_l^{hol}(U_f)$. *Then we have*

$$d_\infty(F, \varphi; s, s_0) = c \cdot W_{F,\varphi}(1) \cdot 2^{-(s+s_0)} \pi^{-s_0} \times \frac{\Gamma(s+l-\frac{m-1}{2})\, \Gamma(s_0+l-\frac{m-1}{2})}{\Gamma(\frac{s+s_0+l}{2})\, \Gamma(\frac{s-s_0+l+1}{2})},$$

where $c = e^{2\pi} \Delta^{\frac{m-2}{2}} (\det Q)^{-\frac{1}{2}}\, 2^{-(2l-m)}\, \pi^{-(l-m+\frac{1}{2})}$.

§4. Shintani functions and the basic identity

4.1 Let $M(K)$ be the space of automorphic forms on G_A invariant under right $K = K_\infty K_f$-translations (see §1). The *Shintani function* associated with $F \in S_l(U_f)$ and $f \in M(K)$ is given by

$$(4.1) \qquad \omega_{F,f}(h) = \int_{G_Q \backslash G_A} F(gh)\, f(g)\, dg \qquad (h \in H_A).$$

(cf. [MS1, §1]). Observe that $\omega_{F,f}$ is an eigen function under the action of $\bigotimes'_{p<\infty} \mathcal{H}(H_p, U_p)$ on the right and that of $\bigotimes'_{p<\infty} \mathcal{H}(G_p, K_p)$ on the left, if F and f are Hecke eigenforms. In [MS1, Theorem 1.6], we have proved

Proposition 4.1 *Assume (3.1). Let* $F \in S_l(U_f)$ *and* $f \in M(K)$ *be Hecke eigenforms. Then*

$$\int_{G_{A_f} \backslash H_{A_f}} \omega_{F,f}(\beta(h)^{-1}h)\, |\alpha(h)|_{A_f}^{s+\frac{m-1}{2}}\, dh = \frac{L(F;\, s)}{L(f;\, s+\frac{1}{2})} \times \begin{cases} 1 & \textit{if } m \textit{ is even} \\ \zeta(2s)^{-1} & \textit{if } m \textit{ is odd.} \end{cases}$$

4.2 Let $P_1 = \left\{ \begin{bmatrix} t & * & * \\ 0 & g & * \\ 0 & 0 & t^{-1} \end{bmatrix} \in G_1 \;\middle|\; t \neq 0,\, g \in G \right\}$ be a maximal parabolic subgroup of G_1. Then each $g_1 \in G_{1,A}$ is decomposed into

$$\begin{bmatrix} \alpha(g_1) & * & * \\ 0 & \beta(g_1) & * \\ 0 & 0 & \alpha(g_1)^{-1} \end{bmatrix} k_1(g_1),$$ where $\alpha(g_1) \in A^\times$, $\beta(g_1) \in G_A$ and $k_1(g_1) \in K_1 = K_{1,\infty}K_{1,f}$. For $f \in M(K)$ and $l \in \mathbf{Z}_{\geq 0}$, we define the Eisenstein series on $G_{1,A}$ with respect to P_1 by

$$(4.2) \quad E(g_1, f, l;\, s) = \sum_{\gamma_1 \in P_{1,Q} \backslash G_{1,Q}} f(\beta(\gamma_1 g_1))\, |\alpha(\gamma_1 g_1)|_A^{s+m/2}\, J_{G_1}(k_1(\gamma_1 g_1)_\infty, Z_0)^l.$$

The series (4.2) can be continued to a meromorphic function of s on $\mathbf{C}$.

4.3 Let us consider the convolution

$$(4.3) \qquad Z_{F,f}(s) = \int_{H_Q \backslash H_A} F(h)\, E(\iota(h), f, l; s - \tfrac{1}{2})\, dh.$$

Proposition 4.2 (The basic identity) *For* $F \in S_f(U_f)$ *and* $f \in M(K)$, *we have*

$$(4.4) \qquad Z_{F,f}(s) = \int_{G_A \backslash H_A} \omega_{F,f}(\beta(h)^{-1}h)\, |\alpha(h)|_A^{s+\frac{m-1}{2}}\, J_{G_1}(k_1(h)_\infty, Z_o)^l\, dh.$$

Proof. While this result has been already mentioned in [MS1, §1.8, Remark], we give a sketch of proof for completeness. First observe the following facts (cf. [MS1, §2]):

(a) $G_1 = P_1 \cdot \iota(H) \cup P_1 \cdot \Upsilon_o \cdot \iota(H)$ (disjoint union),

where $\Upsilon_o = \begin{bmatrix} & & 1 \\ & 1_m & \\ 1 & & \end{bmatrix} \begin{bmatrix} 1 & -{}^tX_oS & -2^{-1}S[X_o] \\ 0 & 1_m & X_o \\ 0 & 0 & 1 \end{bmatrix} \in G_{1,Q}$, $X_o = \begin{bmatrix} a \\ 0_{m-2} \\ 1 \end{bmatrix} \in Q^m$.

(b) $P_1 \cap \iota(H) = \iota(\iota_o(G))$.

(c) $\Upsilon_o^{-1} P_1 \Upsilon_o \cap \iota(H) = \iota(P')$, where $P'_Q = \{h \in H_Q \mid h \cdot \begin{bmatrix} -a \\ 0_{m-2} \\ 1 \\ -1 \end{bmatrix} = t \cdot \begin{bmatrix} -a \\ 0_{m-2} \\ 1 \\ -1 \end{bmatrix}, t \in$

$Q^\times\}$ is a maximal Q-parabolic subgroup of H.

(d) $\Upsilon_o \cdot \iota(N') \cdot \Upsilon_o^{-1} \subset N_1$, where N' (resp. N_1) is the unipotent radical of P' (resp. P_1).

By (a), (b) and (c), we have

$$E(\iota(h), f, l; s - \tfrac{1}{2}) = \sum_{\gamma \in G_Q \backslash H_Q} f(\beta(\gamma h))\, |\alpha(\gamma h)|_A^{s+\frac{m-1}{2}}\, J_{G_1}(k_1(\gamma h)_\infty, Z_o)^l$$

$$+ \sum_{\gamma \in P'_Q \backslash H_Q} f(\beta(\Upsilon_o \gamma h))\, |\alpha(\Upsilon_o \gamma h)|_A^{s+\frac{m-1}{2}}\, J_{G_1}(k_1(\Upsilon_o \gamma h)_\infty, Z_o)^l.$$

Then $Z_{F,f}(s)$ equals

$$\int_{G_Q\backslash H_A} F(h)\, f(\beta(h))\, |\alpha(h)|_A^{s+\frac{m-1}{2}}\, J_{G_1}(k_1(h)_\infty, Z_0)^l\, dh$$

$$+ \int_{P'_Q\backslash H_A} F(h)\, f(\beta(\Upsilon_0 h))\, |\alpha(\Upsilon_0 h)|_A^{s+\frac{m-1}{2}}\, J_{G_1}(k_1(\Upsilon_0 h)_\infty, Z_0)^l\, dh.$$

The first term of the above sum is equal to the right hand side of (4.4), since $\beta(gh) = g\cdot\beta(h)$ and $|\alpha(gh)|_A = |\alpha(h)|_A$ for $g \in G_A$ and $h \in H_A$. By Lemma 2.2 and the decomposition $H_A = P'_A U$, the second term is equal to

$$\int_{P'_Q\backslash P'_A} F(p')\, f(\beta(\Upsilon_0 p'))\, |\alpha(\Upsilon_0 p')|_A^{s+\frac{m-1}{2}}\, J_{G_1}(k_1(\Upsilon_0 p')_\infty, Z_0)^l\, d_l p'. \tag{4.5}$$

Let $P' = M'N'$ be a Levi decomposition of P' and dm' (resp. dn') be a Haar measure on M'_A (resp. N'_A). Then $d_l p' = \mu(m')dm'dn'$ with the module $\mu(m') = \dfrac{d(m'n'm'^{-1})}{dn'}$ of M'. In view of (d), (4.5) is equal to

$$\int_{M'_Q\backslash M'_A} f(\beta(\Upsilon_0 m'))\, |\alpha(\Upsilon_0 m')|_A^{s+\frac{m-1}{2}}\, J_{G_1}(k_1(\Upsilon_0 m')_\infty, Z_0)^l\, \Big\{\int_{N'_Q\backslash N'_A} F(n'm')dn'\Big\}\, \mu(m')dm'.$$

Since F is cuspidal, the integral over $N'_Q\backslash N'_A$ vanishes and hence the proposition is proved. *q.e.d.*

4.4 Combining Proposition 4.1 and Proposition 4.2, we get the following result.

Corollary 4.3 *Under the same assumptions and notation of Proposition 3.1, we have*

$$Z_{F,f}(s) = \frac{L(F;\, s)}{L(f;\, s+\frac{1}{2})} \times \begin{Bmatrix} 1 & \textit{if } m \textit{ is even} \\ \zeta(2s)^{-1} & \textit{if } m \textit{ is odd} \end{Bmatrix} \times Z^{(\infty)}_{F,f}(s),$$

where

$$Z^{(\infty)}_{F,f}(s) = \int_{G_\infty\backslash H_\infty} \omega_{F,f}(\beta(h)^{-1}h)\, |\alpha(h))|^{s+\frac{m-1}{2}}\, J_{G_1}(k_1(h), Z_0)^l\, dh. \tag{4.6}$$

4.5 To proceed further, we consider the Eisenstein series $E_G(g, \varphi; s_o)$ on G_A attached to $\varphi \in M(K_o)$ with respect to P defined as follows. Decompose $g \in G_A$ into

$$\begin{bmatrix} \alpha_o(g) & * & * \\ 0 & \beta_o(g) & * \\ 0 & 0 & \alpha_o(g)^{-1} \end{bmatrix} k(g)$$

where $\alpha_o(g) \in A^\times$, $\beta_o(g) \in G_{0,A}$, $k(g) \in K = K_\infty K_f$. We set

$$E_G(g, \varphi; s_o) = \sum_{\gamma \in P_Q \backslash G_Q} \varphi(\beta_o(\gamma g))\, |\alpha_o(\gamma g)|_A^{s_o + \frac{m-2}{2}}. \tag{4.7}$$

It is known that the series (4.7) can be continued to a meromorphic function of s_o on $\mathbf{C}$ and that $E_G(*, \varphi; s_o) \in M(K)$. The following facts are easily verified.

Lemma 4.4 *In the region* $\{(s, s_o) \in \mathbf{C}^2 \mid \mathrm{Re}\, s - \mathrm{Re}\, s_o > 1, \mathrm{Re}\, s_o > \frac{m-2}{2}\}$, *we have*

$$E(g_1, E_G(*, \varphi; s_o), l; s) = \mathbb{E}(g_1, \varphi, l; s, s_o).$$

Lemma 4.5 *If* $\varphi \in M(K_o)$ *is a Hecke eigenform, then* $E_G(*, \varphi; s_o)$ *is also a Hecke eigenform and* $L(E_G(*, \varphi; s_o); s) = L(\varphi; s) \cdot \zeta(s + s_o)\, \zeta(s - s_o)$.

4.6 Applying Corollary 4.3 to $f = E_G(*, \varphi; s_o)$ and using Lemma 4.4 and Lemma 4.5, we obtain the following result.

Proposition 4.6 *Under the same assumptions of Theorem 3.1, we have*

$$\Xi_{F,\varphi}(s, s_o) = \frac{L(F; s)}{L(\varphi; s+\frac{1}{2})} \cdot \zeta(s + s_o)^{-1}\, \zeta(s - s_o + 1)^{-1} \cdot \begin{Bmatrix} 1 & \text{if m is even} \\ \zeta(2s)^{-1} & \text{if m is odd} \end{Bmatrix}$$
$$\times\ \Lambda_\infty(F, \varphi; s, s_o),$$

where we put $\Lambda_\infty(F, \varphi; s, s_o) = Z^{(\infty)}_{F, E_G(*,\varphi;s_o)}(s)$.

§5. Generalized Whittaker functions and the calculation of $\Lambda_\infty(F, \varphi; s, s_0)$

5.1 We first recall the definition of the generalized Whittaker function $W_{F,\varphi}$ attached to $F \in S_l(U_f)$ and $\varphi \in M(K_0)$ (for detail, see [Su, §1]). The Fourier coefficient F_μ of F at $\mu \in \mathbf{Q}^{m-1}$ is given by

$$F_\mu(h) = \int_{\mathbf{Q}^{m-1}\backslash \mathbf{A}^{m-1}} F(n_H(y)h)\, \psi(-R(\mu, y))\, dy \qquad (h \in H_{\mathbf{A}})$$

(for the definition of ψ, see Notation). Then we have

$$(5.1) \qquad F(n_H(y)h) = \sum_{\mu \in \mathbf{Q}^{m-1}} F_\mu(h)\, \psi(R(\mu, y)) \qquad (h \in H_{\mathbf{A}},\ y \in \mathbf{A}^{m-1}).$$

Note that, for $F \in S_l^{\mathrm{hol}}(U_f)$, we have $F_\mu = 0$ unless $R[\mu] < 0$. The *generalized Whittaker function* $W_{F,\varphi}$ is defined by

$$(5.2) \qquad W_{F,\varphi}(h) = \int_{G_{0,\mathbf{Q}}\backslash G_{0,\mathbf{A}}} F_{-\xi_0}\left(\begin{bmatrix} 1 & & \\ & g_0 & \\ & & 1 \end{bmatrix} h\right) \varphi(g_0)\, dg_0,$$

where $\xi_0 = \begin{bmatrix} \lambda_0 \\ 1 \end{bmatrix} \Delta^{-1} \in \mathbf{Q}^{m-1}$ and G_0 is embedded into H_0 via $g_0 \to \begin{bmatrix} g_0 & (1-g_0)\lambda_0 \\ 0 & 1 \end{bmatrix}$. Under this embedding, G_0 is the isotropy subgroup of ξ_0 in H_0.

The function $W_{F,\varphi}$ has the following properties:

(a) $W_{F,\varphi}(n_H(y)\, h) = \psi(-R(\xi_0, y))\, W_{F,\varphi}(h) \qquad (h \in H_{\mathbf{A}},\ y \in \mathbf{A}^{m-1})$

(b) If both F and φ are Hecke eigenforms, $W_{F,\varphi}$ is an eigen function under the action of $\bigotimes'_{p<\infty} \mathcal{H}(H_p, U_p)$ on the right and that of $\bigotimes'_{p<\infty} \mathcal{H}(G_{0,p}, K_{0,p})$ on the left.

We need the following formula later.

Proposition 5.1 ([Su], Theorem 1) *Let the assumptions be the same as in Theorem 3.1. For* $h \in H_\infty$, *we have*

$$\int_{A_f^\times} W_{F,\varphi}\left(\begin{bmatrix} t & & \\ & 1_{m-1} & \\ & & t^{-1}\end{bmatrix} h\right) |t|_{A_f}^{s_0-\frac{m-1}{2}} d^\times t$$

$$= \frac{L(F;\, s_0)}{L(\varphi;\, s_0+\frac{1}{2})} \times \begin{cases} 1 & \textit{if } m \textit{ is even} \\ \zeta(2s_0)^{-1} & \textit{if } m \textit{ is odd.} \end{cases} \times W_{F,\varphi}(h)$$

5.2 We now go back to the calculation of the integral

$$\Lambda_\infty(F, \varphi;\, s, s_0) = \int_{G_\infty\backslash H_\infty} \omega_{F,E_G(*,\varphi;s_0-\frac{1}{2})}(\beta(h)^{-1}h)\, |\alpha(h)|^{s+\frac{m-1}{2}}\, J_{G_1}(k_1(h), Z_0)^l\, dh.$$

First note that, for $g \in G_A$, we can choose $\alpha(\iota_0(g)) = 1$, $\alpha_0(\iota_0(g)) = \alpha_0(g)$ and $\beta_0(\iota_0(g)) = \beta_0(g)$ in view of Lemma 2.1 (ii). Unwinding the Eisenstein series $E_G(*, \varphi;\, s_0 - \frac{1}{2})$ in the integral $\Lambda_\infty(F, \varphi;\, s, s_0)$ and using the decomposition $G_A = P_A K_\infty K_f$, we get

$$(5.3)\quad \Lambda_\infty(F, \varphi;\, s, s_0) = \int_{G_\infty\backslash H_\infty} dh \int_{P_Q\backslash P_A} d_l p \int_{K_\infty} dk\, F(pk\beta(h)^{-1}h)\, \varphi(\beta_0(p))$$

$$|\alpha_0(p)|_A^{s_0+\frac{m-3}{2}}\, |\alpha(h)|_\infty^{s+\frac{m-1}{2}}\, J_{G_1}(k_1(h)_\infty, Z_0)^l.$$

Here $d_l p$ is a left invariant measure on P_A given by $d_l p = |t|_A^{-(m-2)}\, dx\, d^\times t\, dg_0$, where $p = n(x)\begin{bmatrix} t & & \\ & g_0 & \\ & & t^{-1}\end{bmatrix}$ $(x \in A^{m-2},\, t \in A^\times,\, g_0 \in G_{0,A})$. We may suppose that $\beta(h) \in P_\infty$. For $h \in H_\infty$, we see $|\alpha_0(p\beta(h))|_A = |\alpha_0(p)|_A\, |\alpha_0(\beta(h))|_\infty$ and $\varphi(\beta_0(p{\cdot}\beta(h))) = \varphi(\beta_0(p)\beta_0(h)) = \varphi(\beta_0(p))$, since φ is right $G_{0,\infty}$-invariant. Thus, changing the variable p into $p{\cdot}\beta(h)$ in (5.3), we obtain

$$\Lambda_\infty(F, \varphi;\, s, s_0) = \int_{G_\infty\backslash H_\infty} dh \int_{K'_\infty} dk' \int_{P_Q\backslash P_A} d_l p\, F(pk'h)\, \varphi(\beta_0(p))$$

$$|\alpha_0(p)|_A^{s_0+\frac{m-3}{2}}\, |\alpha_0(\beta(h))|_\infty^{s_0+\frac{m-3}{2}}\, |\alpha(h)|_\infty^{s+\frac{m-1}{2}}\, J_{G_1}(k_1(h)_\infty, Z_0)^l,$$

where dk' is the normalized Haar measure on $K'_\infty = \beta(h)K_\infty\beta(h)^{-1}$. For a while, we fix $h \in H_\infty$ and let $k' \in K'_\infty$. Since $k' \in G_\infty$, we have $|\alpha(k'h)|_\infty = |\alpha(h)|_\infty$ and $k_1(k'h)_\infty = k_1(h)_\infty$. Next observe that $\beta(k'h) \in k'\beta(h)K_\infty = \beta(h)K_\infty$, which implies $|\alpha_0(\beta(k'h)|_\infty = |\alpha_0(\beta(h))|_\infty = |\alpha_0(h)|_\infty$. Since $G_\infty = P_\infty K'_\infty$, we have proved the following:

Lemma 5.2 *We have*

$$\Lambda_\infty(F, \varphi; s, s_0) = \int_{\iota_0(P_\infty)\backslash H_\infty} \left\{ \int_{P_Q\backslash P_A} F(ph)\, \varphi(\beta_0(p))\, |\alpha_0(p)|_A^{s_0+\frac{m-3}{2}}\, d_l p \right\}$$

$$|\alpha_0(h)|_\infty^{s_0+\frac{m-3}{2}}\, |\alpha(h)|_\infty^{s+\frac{m-1}{2}}\, J_{G_1}(k_1(h)_\infty, Z_0)^l\, dh.$$

5.3 In view of Proposition 4.6 and Lemma 5.2, the proof of Theorem 3.1 is now reduced to the following.

Proposition 5.3 *Let the assumptions be the same as in Theorem 3.1. For* $h \in H_\infty$, *we have*

$$(5.4)\quad \int_{P_Q\backslash P_A} F(ph)\, \varphi(\beta_0(p))\, |\alpha_0(p)|_A^{s_0+\frac{m-3}{2}}\, d_l p = \int_{A^\times} W_{F,\varphi}\left(\begin{bmatrix} t & & \\ & 1_{m-1} & \\ & & t^{-1} \end{bmatrix} h\right) |t|_A^{s_0-\frac{m-1}{2}}\, d^\times t$$

$$= \frac{L(F; s_0)}{L(f; s_0+\frac{1}{2})} \times \begin{cases} 1 & \textit{if } m \textit{ is even} \\ \zeta(2s_0)^{-1} & \textit{if } m \textit{ is odd} \end{cases} \times \int_{R^\times} W_{F,\varphi}\left(\begin{bmatrix} t & & \\ & 1_{m-1} & \\ & & t^{-1} \end{bmatrix} h\right) |t|_\infty^{s_0-\frac{m-1}{2}}\, d^\times t.$$

Proof. The left hand side of (5.4) is equal to

$$\int_{G_{0,Q}\backslash G_{0,A}} dg_0 \int_{Q^\times\backslash A^\times} d^\times t \int_{Q^{m-2}\backslash A^{m-2}} dx\, F(\iota_0(n_G(x) \begin{bmatrix} t & & \\ & g_0 & \\ & & t^{-1} \end{bmatrix})\cdot h)\, \varphi(g_0)\, |t|_A^{s_0-\frac{m-1}{2}}.$$

By Lemma 2.1 (i) and (5.1), the integral $\displaystyle\int_{Q^{m-2}\backslash A^{m-2}} F(\iota_0(n_G(x) \begin{bmatrix} t & & \\ & g_0 & \\ & & t^{-1} \end{bmatrix})\cdot h)\, dx$

equals

$$\sum_{\mu\in Q^{m-1}} F_\mu(\begin{bmatrix} t & & \\ & \begin{matrix} g_0 & (1-g_0)\lambda_0 \\ 0 & 1 \end{matrix} & \\ & & t^{-1} \end{bmatrix} h) \int_{Q^{m-2}\backslash A^{m-2}} \psi(R(\mu, \begin{bmatrix} x \\ 0 \end{bmatrix})) \, dx.$$

Since the integral in the above formula is equal to one if $\mu = u\cdot(-\xi_0)$ for some $u \in \mathbf{Q}^\times$ and equal to zero otherwise, and since $F_{u\mu}(h) = F_\mu(\begin{bmatrix} u & & \\ & 1_{m-1} & \\ & & u^{-1} \end{bmatrix} h)$ for $u \in \mathbf{Q}^\times$, $\mu \in \mathbf{Q}^{m-1}$ and $h \in H_A$, the left hand side of (5.4) equals

$$\int_{A^\times} \{ \int_{G_{0,Q}\backslash G_{0,A}} F_{-\xi_0}(\begin{bmatrix} t & & \\ & \begin{matrix} g_0 & (1-g_0)\lambda_0 \\ 0 & 1 \end{matrix} & \\ & & t^{-1} \end{bmatrix} h) \, \varphi(g_0) \, dg_0 \} \, |t|_A^{s_0 - \frac{m-1}{2}} \, d^\times t.$$

This proves the first equality of the proposition. The second one follows from Proposition 5.1. *q.e.d.*

§6. Proof of Theorem 3.2

6.1 In this section we always assume that $F \in S_l^{hol}$. For $h_f \in H_{A_f}$, the function $F(z; h_f) = F(h_\infty h_f) \, J_H(h_\infty, z_0)^l$ $(h_\infty \in H_\infty, z = h_\infty\langle z_0\rangle \in D)$ admits a Fourier expansion: $F(z; h_f) = \sum_\mu a_F(\mu; h_f) \, \mathbf{e}[R(\mu, z)]$ where μ runs over the set $\{\mu \in \mathbf{Q}^{m-1} \mid R[\mu] < 0, -\mu i$ and z are in the same connected component of $D\}$. Then $F_\mu(h_\infty h_f)$ is equal to $a_F(\mu; h_f) \, \mathbf{e}[R(\mu, h_\infty\langle z_0\rangle)]) \, J_H(h_\infty, z_0)^{-l}$ if $-\mu i$ and $h_\infty\langle z_0\rangle$ are in the same connected component of D and equal to zero otherwise. Let $D_\pm = \{z = \begin{bmatrix} z' \\ z'' \end{bmatrix} \in D \mid \pm \operatorname{Im}(z'') > 0\}$. We note that $z_0 \in D_+$.

Lemma 6.1 *Let* $h \in H_\infty$ *and assume that* $h\langle z_0\rangle = \begin{bmatrix} z' \\ z'' \end{bmatrix} \in D_+$. *Then we have*

$$\int_{\mathbf{R}^\times} W_{F,\varphi}(\begin{bmatrix} t & & \\ & 1_{m-1} & \\ & & t^{-1} \end{bmatrix} h)\, |t|_\infty^{s_o - \frac{m-1}{2}}\, d^\times t$$

$$= e^{2\pi}\, W_{F,\varphi}(1)\cdot\Gamma(s_o + l - \frac{m-1}{2})\,(-2\pi i z'')^{-(s_o + l - \frac{m-1}{2})} \cdot J_H(h, z_o)^{-l}.$$

Proof. Let $t \in \mathbf{R}^\times$. By definition (5.2) of $W_{F,\varphi}$ and the above remark, we have

$$(6.1)\quad W_{F,\varphi}(\begin{bmatrix} t & & \\ & 1_{m-1} & \\ & & t^{-1} \end{bmatrix} h)$$

$$= \delta(t > 0) \int_{G_{o,Q}\backslash G_{o,A}} a_F(-\xi_o; \begin{bmatrix} 1 & & \\ & g_{o,f} & \\ & & 1 \end{bmatrix})\, \mathbf{e}[R(-\xi_o, \begin{bmatrix} t & & \\ & g_{o,\infty} & \\ & & t^{-1} \end{bmatrix} h\langle z_o\rangle)]$$

$$\times J_H(\begin{bmatrix} t & & \\ & g_{o,\infty} & \\ & & t^{-1} \end{bmatrix} h, z_o)^{-l}\, \varphi(g_o)\, dg_o$$

$$= \delta(t > 0)\, t^l\, J_H(h, z_o)^{-l}\, \mathbf{e}[t\, z''] \int_{G_{o,Q}\backslash G_{o,A}} a_F(-\xi_o; \begin{bmatrix} 1 & & \\ & g_{o,f} & \\ & & 1 \end{bmatrix})\, \varphi(g_o)\, dg_o,$$

where $\delta(t > 0)$ is equal to one if $t > 0$ and equal to zero otherwise. Note that $\begin{bmatrix} t & & \\ & g_{o,\infty} & \\ & & t^{-1} \end{bmatrix} h\langle z_o\rangle \in D_+$ if and only if $t > 0$. Setting $t = 1$ and $h = 1$ in (6.1), we get

$$\int_{G_{o,Q}\backslash G_{o,A}} a_F(-\xi_o; \begin{bmatrix} 1 & & \\ & g_{o,f} & \\ & & 1 \end{bmatrix})\, \varphi(g_o)\, dg_o = e^{2\pi}\, W_{F,\varphi}(1)$$

and hence

$$W_{F,\varphi}(\begin{bmatrix} t & & \\ & 1_{m-1} & \\ & & t^{-1} \end{bmatrix} h) = \delta(t > 0)\cdot e^{2\pi}\, W_{F,\varphi}(1)\cdot t^l\, J_H(h, z_o)^{-l}\, \mathbf{e}[t\, z''].$$

The lemma is an immediate consequence of the above equality. *q.e.d.*

6.2 Let $H_\infty^+ = \{h \in H_\infty \mid h\langle z_0\rangle \in D_+\}$ and $P_\infty^+ = \{n_G(x)\begin{bmatrix} t & & \\ & g_0 & \\ & & t^{-1}\end{bmatrix} \in P_\infty \mid t > 0\}$.

Then $\iota_0(P_\infty^+) \subset H_\infty^+$. Lemma 6.1 implies that $d_\infty(F, \varphi; s, s_0)$ is equal to

$$e^{2\pi} W_{F,\varphi}(1)\,\Gamma(s_0 + l - \tfrac{m-1}{2})\cdot \int_{\iota_0(P_\infty^+)\backslash H_\infty^+} (-2\pi i z'')^{-(s_0+l-\frac{m-1}{2})}$$
$$\times J_H(h, z_0)^{-l}\, J_{G_1}(k_1(h)_\infty, Z_0)^l\, |\alpha_0(h)|^{s_0+\frac{m-3}{2}}\, |\alpha(h)|^{s+\frac{m-1}{2}}\, dh,$$

where $h\langle z_0\rangle = \begin{bmatrix} z' \\ z'' \end{bmatrix}$. By Lemma 2.2 (ii), we see that $J_{G_1}(k_1(h)_\infty, Z_0) =$ $\alpha(h)\cdot J_{G_1}(\iota(h), Z_0) = (-iz'')\,\alpha(h)\, J_H(h, z_0)$. Thus we get

$$(6.2)\quad d_\infty(F, \varphi; s, s_0) = e^{2\pi} W_{F,\varphi}(1)\,\Gamma(s_0 + l - \tfrac{m-1}{2})\,(2\pi)^{-(s_0+l-\frac{m-1}{2})}\cdot I(s, s_0),$$

where

$$(6.3)\quad I(s, s_0) = \int_{\iota_0(P_\infty^+)\backslash H_\infty^+} (-iz'')^{-(s_0-\frac{m-1}{2})}\, |\alpha_0(h)|^{s_0+\frac{m-3}{2}}\, |\alpha(h)|^{s+l+\frac{m-1}{2}}\, dh.$$

Lemma 6.2 *For* $h \in H_\infty^+$, *we have*

$$\alpha(h) = |z''|^{-1}\, |J_H(h, z_0)|^{-1},\quad \alpha_0(h) = |z''|\cdot|J_H(h, z_0)|^{-1}\cdot(\mathrm{Im}\, z'')^{-1},$$

where $z = \begin{bmatrix} z' \\ z'' \end{bmatrix} = h\langle z_0\rangle \in D_+$.

Proof. The first formula follows from $|J_{G_1}(\iota(h), Z_0)| = \alpha(h)^{-1}$ and Lemma 2.2 (ii).

By Lemma 2.3, we have $(\iota(h)\langle Z_0\rangle)^\sim = (\rho(z))^\sim = \begin{bmatrix} * \\ z''^{-1} \\ 1 \end{bmatrix}$. On the other hand, we have

$$(\iota(h)\langle Z_0\rangle)^\sim = J_{G_1}(\iota(h), Z_0)^{-1}\,\iota(h)\, Z_0^\sim$$

$$= J_{G_1}(\iota(h), Z_o)^{-1} J_{G_1}(k_1(h), Z_o)\begin{bmatrix} \alpha(h) & * & * & * & * \\ 0 & \alpha_o(h) & * & * & * \\ 0 & 0 & \beta_o(h) & * & * \\ 0 & 0 & 0 & \alpha_o(h)^{-1} & \gamma(h) \\ 0 & 0 & 0 & 0 & \alpha(h)^{-1} \end{bmatrix}\begin{bmatrix} -a-Q[\lambda_o] \\ 2^{-1}\Delta i \\ \lambda_o \\ -i \\ 1 \end{bmatrix}$$

$$= \alpha(h)\cdot\begin{bmatrix} * \\ -i\alpha_o(h)^{-1}+\gamma(h) \\ \alpha(h)^{-1} \end{bmatrix}$$

with some $\gamma(h) \in \mathbf{R}$. This implies that $z''^{-1} = -\,i\,\alpha_o(h)^{-1}\,\alpha(h) + \alpha(h)\,\gamma(h)$ and hence that $\dfrac{\mathrm{Im}(z'')}{|z''|^2} = \alpha_o(h)^{-1}\,\alpha(h)$. We are done. *q.e.d.*

Let $G^+_{0,\infty}$ and $H^+_{0,\infty}$ be the identity components of $G_{0,\infty}$ and $H_{0,\infty}$. Since $\iota_o(P^+_\infty) \setminus H^+_\infty / U_\infty \approx \{n_H(\begin{bmatrix} 0_{m-2} \\ v \end{bmatrix})\cdot\begin{bmatrix} 1 & & \\ & h_o & \\ & & 1 \end{bmatrix} \mid v \in \mathbf{R},\, h_o \in G^+_{0,\infty} \setminus H^+_{0,\infty}\}$ and $J_H(n_H(\begin{bmatrix} 0_{m-2} \\ v \end{bmatrix})\cdot\begin{bmatrix} 1 & & \\ & h_o & \\ & & 1 \end{bmatrix}, z_o) = 1$, we get

$$I(s, s_o) = \int_{\mathbf{R}} dv \int_{H^+_{0,\infty}} dh_o\,(-\,iz'')^{-(s_o-\frac{m-1}{2})}\,|z''|^{-(s-s_o+l+1)}\,(\mathrm{Im}\,z'')^{-(s_o+\frac{m-3}{2})}.$$

Here $z = \begin{bmatrix} z' \\ z'' \end{bmatrix} = n_H(\begin{bmatrix} 0_{m-2} \\ v \end{bmatrix})\begin{bmatrix} 1 & & \\ & h_o & \\ & & 1 \end{bmatrix}\langle z_o\rangle$ and dh_o is the Haar measure on $H^+_{0,\infty}$ (its normalization will be given in §6.3). Note that z'' is, as a function of h_o, left $G^+_{0,\infty}$-invariant. It is easy to see that $z'' = i\cdot A(h_o) + v$, where $A(h_o) > 0$ is the $(m-1)$-th component of $h_o\begin{bmatrix} \lambda_o \\ 1 \end{bmatrix}$. By a straightforward calculation, we obtain

$$(6.4)\quad I(s, s_o) = \pi\cdot 2^{-(s+l-\frac{m+1}{2})}\frac{\Gamma(s+l-\frac{m-1}{2})}{\Gamma(\frac{s+s_o+l-m+2}{2})\,\Gamma(\frac{s-s_o+l+1}{2})}$$

$$\times \int_{G^+_{0,\infty}\backslash H^+_{0,\infty}} A(h_0)^{-(s+s_0+l-1)}\, dh_0.$$

6.3 To normalize the Haar measure dh_0 on $H^+_{0,\infty}$, we consider a symmetric space on which $H^+_{0,\infty}$ acts. For $x \in \mathbf{R}^{m-2}$, put $x^\sim = \begin{bmatrix} x \\ 1 \end{bmatrix} \in \mathbf{R}^{m-1}$. Let $D_0 = \{x \in \mathbf{R}^{m-2} \mid R[x^\sim] < 0\}$. We see $\lambda_0 \in D_0$. For $h_0 \in H^+_{0,\infty}$, we define the action $x \to h_0\langle x\rangle$ on D_0 and the automorphic factor $J_0(h_0, x) \in \mathbf{C}^\times$ by $h_0 \cdot x^\sim = J_0(h_0, x)\, (h_0\langle x\rangle)^\sim$. The action of $H^+_{0,\infty}$ on D_0 is transitive and the isotropy subgroup of λ_0 in $H^+_{0,\infty}$ is $G^+_{0,\infty}$. For $x \in D_0$, put

$$r(x) = -\Delta^{-1}R[x^\sim] = \frac{\Delta - Q[x-\lambda_0]}{\Delta} \tag{6.5}$$

(recall that $\Delta = Q[\lambda_0] + 2a > 0$). We see that $0 < r(x) \le 1$ and $r(\lambda_0) = 1$. Moreover we have $r(h_0\langle x\rangle) = J_0(h_0, x)^{-2}\, r(x)$ for $h_0 \in H^+_{0,\infty}$ and $x \in D_0$. It follows that

$$A(h_0) = J_0(h_0, \lambda_0) = r(h_0\langle\lambda_0\rangle)^{-1/2} \quad (h_0 \in H^+_{0,\infty}). \tag{6.6}$$

Define the invariant measure on D_0 by $d\mu(x) = r(x)^{-\frac{m-1}{2}}\, dx_1 \cdots dx_{m-2}$. We normalize the Haar measure dh_0 on $H^+_{0,\infty}$ by

$$\int_{H^+_{0,\infty}} f(h_0)\, dh_0 = \int_{D_0} \Big(\int_{G^+_{0,\infty}} f(h_0 g_0)\, dg_0 \Big)\, d(h_0\langle\lambda_0\rangle) \qquad (f \in C^\infty_c(H^+_{0,\infty})), \tag{6.7}$$

where dg_0 is the Haar measure on $G^+_{0,\infty}$ with total volume one.

6.4 We now complete the calculation of $I(s, s_0)$, from which Theorem 3.2 follows (see (6.2)).

Lemma 6.3 *We have*

$$\int_{H^+_{0,\infty}} A(h_0)^{-s}\, dh_0 = \Delta^{(m-2)/2}\, (\det Q)^{-1/2}\, \pi^{(m-2)/2}\, \frac{\Gamma(\frac{s-m+3}{2})}{\Gamma(\frac{s+1}{2})}. \tag{6.8}$$

Proof. By (6.6), the left hand side of (6.8) equals

$$\int_{D_o} r(x)^{s/2}\, d\mu(x) = \int_{Q[x-\lambda_o]<\Delta} r(x)^{(s-m+1)/2}\, dx_1 \cdots dx_{m-2}$$

$$= \Delta^{(m-2)/2} \int_{Q[x]<1} (1 - Q[x])^{(s-m+1)/2}\, dx_1 \cdots dx_{m-2}$$

$$= \Delta^{(m-2)/2} (\det Q)^{-1/2} \int_{x_1^2+\cdots+x_{m-2}^2<1} (1 - x_1^2 - \cdots - x_{m-2}^2)^{(s-m+1)/2} dx_1 \cdots dx_{m-2},$$

which is equal to the right hand side of (6.8) by a well-known formula. *q.e.d.*

The following result follows from (6.4) and the previous lemma.

Proposition 6.4 *We have*

$$I(s, s_o) = (\det Q)^{-1/2}\, \Delta^{(m-2)/2}\, 2^{-(s+l-(m+1)/2)}\, \pi^{m/2} \frac{\Gamma(s+l-\frac{m-1}{2})}{\Gamma(\frac{s+s_o+l}{2})\, \Gamma(\frac{s-s_o+l+1}{2})} .$$

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