Systoles of arithmetic surfaces and the Markoff spectrum

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1 Introduction

In [10] (1879) Markoff proved a celebrated theorem about Diophantine approximation. For a real number θ and (rational) integers p, q let

$$\nu(\theta) = \inf\{c : |\theta - p/q| < c/q^2 \text{ for infinitely many } q\}.$$

Then there exists a discrete set of values ν_i, the so-called *Markoff spectrum*, decreasing to $1/3$, with $1/3$ as unique cluster point, such that if $\nu(\theta) > 1/3$ then $\nu(\theta)$ equals one of the ν_i.

The Markoff spectrum is related to the positive infimum of indefinite quadra-tic forms. Namely, let

$$f(X,Y) = aX^2 + bXY + cY^2, \quad D(f) = b^2 - 4ac > 0,$$

and let

$$\mu(f) = \inf\{|f(m,n)| : (m,n) \in \mathsf{Z} \times \mathsf{Z} \setminus (0,0)\}.$$

If $\mu(f)/\sqrt{D(f)} > 1/3$, then $\mu(f)/\sqrt{D(f)}$ is in the Markoff spectrum. In this case f is equivalent to a form with integer coefficients and such a form is called a *Markoff form.*

In [11] (1880) Markoff also showed that the numbers ν_i can be calculated by the integer solutions of the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz.$$

A positive integer z appearing in a solution is called a *Markoff number* and $z/\sqrt{9z^2-4}$ is in the Markoff spectrum.

Markoff proved these results using the theory of continued fractions, see also Dickson [6] (1930). Frobenius [7] (1913) opened the way for a proof in the context of indefinite quadratic forms which was completed later, see in particular Cassels [2] (1959).

The theory of the Markoff spectrum got a new impetus with its interpretation in the context of hyperbolic geometry, see Cohn [3] (1955), [4] (1971), Lehner/Sheingorn [9] (1984), Haas [8] (1986) and in particular the surveys [13] (1993) and Cusick/Flahive [5] (1989). See also Schmidt/Sheingorn [14] for a recent contribution.

In this context, Riemann surfaces of constant negative curvature -1 and their closed geodesics are considered. To be more precise I introduce some notation. Let H be the hyperbolic plane (the upper half plane). Let $\Gamma(N)$ be the principal congruence subgroup of the modular group of level N. Then we have the following theorem of Lehner/Sheingorn [9]. Let u be a simple (without self intersection) closed geodesic of the Riemann surface $\mathsf{H}/\Gamma(3)$. Then there exists a Markoff number z such that

$$3z = 2\cosh(L(u)/4)$$

where $L(u)$ denotes the length of u. Conversely, if z is a Markoff number then there exists a simple closed geodesic u on $\mathsf{H}/\Gamma(3)$ such that

$$3z = 2\cosh(L(u)/4).$$

One proof of this result uses the fact that simple closed geodesics in $\mathsf{H}/\Gamma(3)$ are in a large distance from the cusps where the distance has to be defined in an appropriate way.

In this paper I give some new geometric interpretations in the theory of the Markoff spectrum. A measure for the distance of a closed geodesic of a Riemann surface M to the nearest cusp of M is introduced which corresponds exactly to the Markoff forms giving thus a geometric interpretation of these forms. This measure can be used for all closed geodesics. But of particular interest is its application to the systoles (the shortest closed geodesic) of the surfaces $\mathsf{H}/\Gamma(N)$. In [15] I gave a method for calculating the number of systoles of $\mathsf{H}/\Gamma(N)$. Thereby, systoles are classified with respect to the measure introduced above. In [15] this measure was called the *degree* of a systole. It turns out that the systoles in $\mathsf{H}/\Gamma(N)$ with a high degree with respect to N (which means that they are far away from all cusps) correspond exactly to (integer) multiples of the simple closed geodesics of $\mathsf{H}/\Gamma(3)$. So, instead of simple closed geodesics of $\mathsf{H}/\Gamma(3)$ in the theorem above, one can consider systoles of $\mathsf{H}/\Gamma(N), N = 3,4,5,\ldots$. This shows that the Markoff spectrum is not only related to $\Gamma(3)$, but to all principal congruence subgroups $\Gamma(N)$. Actually, there is a relation to even more subgroups of the modular group since the definition of the degree of a systole can be extended to them.

Conversely, the theory of the Markoff spectrum gives new results concerning the number of systoles of $\mathsf{H}/\Gamma(N)$. I shall classify all systoles of degree Δ in $\mathsf{H}/\Gamma(N), N = 3,4,\ldots$, with $\Delta/N \geq 1/3$.

The paper is organized as follows. In section 2 the measure for the systoles in $\mathsf{H}/\Gamma(N)$ is explained. In section 3 the results in the context of the theory of the Markoff spectrum are given. I also notice the relation to the so-called uniqueness conjecture and give some new results in this context. In section 4 some generalizations are treated.

2 Systoles on $\mathsf{H}/\Gamma(N)$

Definition (i) A *surface M* is a Riemann surface of constant negative curvature -1. If M has non-empty boundary, then this should consist of a finite number of disjoint components which are simple closed geodesics. They are called *boundary geodesics*.
(ii) In order to simplify to notation I shall sometimes say that the cusps of a surface M are closed geodesics of length zero. Consequently, they are also treated as components of the boundary of M and they are also called boundary geodesics.
(iii) H denotes the upper half plane.
(iv) A *Fuchsian group* is a discrete subgroup of

$$SL(2,\mathsf{R}) \doteq \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a,b,c,d \in \mathsf{R}, ad-bc=1 \right\}.$$

(v) Let $\gamma \in SL(2,\mathsf{R})$. Then $tr(\gamma)$ is the *trace* of γ.

The following result is well known.

Theorem 1 *(i) A surface M can be written as $M = \mathsf{H}/\Gamma$ for a Fuchsian group Γ.*
(ii) Let $\gamma \in \Gamma$, $|tr(\gamma)| > 2$. Then M contains a corresponding closed geodesic u such that for an integer n

$$|tr(\gamma)| = 2\cosh(nL(u)/2)$$

where $L(u)$ is the length of u. □

Definition (i) A closed geodesic u of a surface $M = \mathsf{H}/\Gamma$ is in this paper considered as having no orientation and as primitive (we only go around it once; equivalently, the corresponding $\gamma \in \Gamma$ is not a power of another element in Γ).
(ii) I denote by $tr(u)$ the quantity $2\cosh(L(u)/2)$.
(iii) To u corresponds the union of the conjugacy classes of an element $\gamma \in \Gamma$ and of its inverse (since our closed geodesics have no orientation). I shall say that an element of this extended conjugacy class *corresponds* to u.
(iv) A *systole* of a surface M is a shortest closed inner geodesic of M where "inner" means that it is not a boundary geodesic.

Remark 1 Let u be a closed inner geodesic in a surface M. Assume that u is not simple. In this case, if we take u as a point set, then u has a subset v' which is homotopic to a closed geodesic $v \neq u$. v is shorter than v' and hence shorter than u. If M is not a pair of pants (see the following definition), then v' can be chosen such that v is an inner geodesic of M. It follows that a systole of M is always simple if M is not a pair of pants.

Definition A *pair of pants* is a surface M of genus 0 with three boundary components which are simple closed geodesics or cusps (a sphere with three holes, or two holes and one puncture, or one hole and two puntures, or three puncures; hereby the holes are simple closed geodesics in the hyperbolic metric).

For a proof of the following two results see for example Buser [1].

Proposition 1 *Let M be a pair of pants with boundary geodesics a, b, c. Then the fundamental group of M is a free group of two generators. The generators A and B can be chosen such that they correspond to a and b, respectively, and such that AB correspond to c.* □

Lemma 1 *Let M be a pair of pants with boundary geodesics a, b, c. Then there exists a unique common orthogonal t between a and b. If the length of a and b are fixed and the length of c varies (and is finite), then $L(c)$ grows if and only if $L(t)$ grows.* □

Remark 2 Let M be a pair of pants with boundary geodesics a, b, c and let t be the common orthogonal between a and b. If $L(a)$ and $L(b)$ are fixed and $L(c)$ varies (and is finite), then Lemma 1 indicates that the distance between a and b can qualitatively be measured by the length of c. This is of particular interest if a is a cusp since then the length of t is infinite and gives no information while the length of c remains a free parameter.

Definition Let $N > 2$ be an integer. Let

$$\Gamma(N) = \left\{ \begin{bmatrix} 1+aN & bN \\ cN & 1+dN \end{bmatrix} \in SL(2,\mathsf{Z}) \middle| a, b, c, d \in \mathsf{Z} \right\}$$

where

$$SL(2,\mathsf{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathsf{Z}, ad - bc = 1 \right\}.$$

I remark that $\Gamma(N)$ has no elliptic elements and that $\mathsf{H}/\Gamma(N)$ is a surface with cusps. If $\gamma \in \Gamma(N)$, then $tr(\gamma) \equiv \pm 2 \bmod (N^2)$ as it is easy to see. The following result is an immediate consequence.

Lemma 2 *The systoles of $\mathsf{H}/\Gamma(N)$ have trace $N^2 - 2$. Every non-trivial element $\gamma \in \Gamma(N)$ with $tr(\gamma) = 2$ corresponds to a cusp and every $\gamma \in \Gamma(N)$, with $|tr(\gamma)| = N^2 - 2$ to a systole.* □

Lemma 3 *Let u be a systole in $\mathsf{H}/\Gamma(N)$ and let $U \in \Gamma(N)$ correspond to u. Let $V \in \Gamma(N)$ correspond to a cusp v in $\mathsf{H}/\Gamma(N)$. Let*

$$U = \begin{bmatrix} 1+aN & bN \\ cN & 1+dN \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 1+a'N & b'N \\ c'N & 1-a'N \end{bmatrix}.$$

Then

$$tr(UV) = 2 + N^2[(2a+N)a' + cb' + bc' - 1].$$

Proof. By Lemma 2 we have $d = -a-N$. The lemma follows by a calculation. □

Definition (i) Let U correspond to a systole and V to a cusp in $\mathsf{H}/\Gamma(N)$. Let

$$U = \begin{bmatrix} 1+aN & bN \\ cN & 1-aN-N^2 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 1+a'N & b'N \\ c'N & 1-a'N \end{bmatrix}.$$

Then I shall write $U = U(a,b,c)$ and $V = V(a',b',c')$.
(ii) For U and V as above define

$$\Delta(U,V) = |(2a+N)a' + cb' + bc'|.$$

(iii) For a systole u in $\mathsf{H}/\Gamma(N)$ define

$$\Delta(u) = \min\{\Delta(U,V) : U \text{ corresponds to } u, V \text{ corresponds to a cusp}\}.$$

$\Delta(u)$ is called the *degree* of u.

Lemma 4 *For the calculation of the degree of a systole u in $\mathsf{H}/\Gamma(N)$ it is sufficient to consider elements $V = V(a',b',c')$ corresponding to a cusp such that a',b',c' have no common factor bigger than 1.*

Proof. If ν is a common factor of a',b',c', then $W = W(a'/\nu, b'/\nu', c'/\nu)$ also corresponds to a cusp in $\mathsf{H}/\Gamma(N)$. The lemma now follows by the definition of $\Delta(u)$. □

Proposition 2 *Let u be a systole in $\mathsf{H}/\Gamma(N)$ and let $U = U(a,b,c)$ correspond to u. Then*

$$\Delta(u) = \min\{|cm^2 - (2a+N)mn - bn^2| : (m,n) \in \mathsf{Z}\times\mathsf{Z}\setminus(0,0)\}.$$

Proof. Let S correspond to u and T correspond to a cusp in $\mathsf{H}/\Gamma(N)$ such that $\Delta(S,T) = \Delta(u)$. Then there exists an element $W \in SL(2,\mathsf{Z})$ such that $U = WSW^{-1}$ or $U = WS^{-1}W^{-1}$. Since by definition $\Delta(S,T) = \Delta(S^{-1},T)$ we can assume that $U = WSW^{-1}$. Let $V = WTW^{-1}$. Then V corresponds to a cusp in $\mathsf{H}/\Gamma(N)$ (by Lemma 2) and $\Delta(U,V) = \Delta(S,T) = \Delta(u)$ which shows that for the calculation of the degree of u it is sufficient to consider U. By Lemma 4 it is sufficient to consider $V = V(a',b',c')$ corresponding to a cusp such that a', b', c' have no common factor. Since the determinant of V is 1, we have

$$a'^2 = -b'c'$$

which implies that $|b'|$ and $|c'|$ are squares. Therefore, we can set

$$b' = \beta^2, c' = -\gamma^2, a' = -\beta\gamma$$

where β and γ are integers. I notice that the signs can be choosen in this way since $\Delta(S,T) = \Delta(S,T^{-1})$.

Conversely, take integers s and t, not both zero. $V' = V'(-st, s^2, -t^2)$ then corresponds to a cusp in $\mathsf{H}/\Gamma(N)$ by Lemma 2. The proposition follows. □

Remark 3 The degree of a systole corresponds to the distance to the nearest cusp if this distance is measured as indicated in Remark 2. This is so since by Proposition 1 the element U corresponding to a systole u and the element V corresponding to a cusp in $\mathsf{H}/\Gamma(N)$ generate the fundamental group of a pair of pants where the third boundary geodesic has trace $2 + N^2(\Delta(U,V) - 1)$.

3 Markoff forms and systoles

Definition (i) A *Markoff number* is a positive integer z which appears in an integer solution of the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz. \tag{1}$$

A solution of (1) in integers (x, y, z) with $0 < x \leq y \leq z$ is called a *Markoff triple.*

(iii) Let $f(X,Y) = aX^2 + bXY + cY^2$ be a quadratic form with discrimant $D(f) = b^2 - 4ac > 0$. Let

$$\mu(f) = \inf\{|f(m,n)| : (m,n) \in \mathsf{Z} \times \mathsf{Z} \setminus (0,0)\}.$$

If $\mu(f)/\sqrt{D(f)} > 1/3$ and if the coefficients a, b, c of f are integers with no non-trivial common factor, then f is called a *Markoff form.*

(iv) Two quatratic forms $f(X,Y) = aX^2 + bXY + cY^2$ and $g(X,Y) = a'X^2 + b'XY + c'Y^2$ are *equivalent* if there exists a matrix

$$Z = \begin{bmatrix} s & t \\ u & v \end{bmatrix}, sv - tu = \pm 1$$

such that $g(sX + tY, uX + vY) = f(X, Y)$.

Remark 4 Let (x, y, z) be a Markoff triple. Then x, y, z are mutually co-prime, therefore there exists an integer q, $0 \leq q \leq z/2$ such that $qx \equiv y \bmod (z)$ or $-qx \equiv y \bmod (z)$. Moreover, by (1) there exists an integer r such that $q^2 + 1 = rz$.

Theorem 2 *Let $f(X, Y) = aX^2 + bXY + cY^2$ be a quadratic form with discrimant $D(f) > 0$ and $\mu(f)/\sqrt{D(f)} > 1/3$. Then f is equivalent to a multiple of a Markoff form. Moreover, there exists a Markoff triple (x, y, z) with q, r defined as in Remark 4 such that $f(X, Y)$ is a multiple of the form*

$$g(X, Y) = zX^2 - (3z - 2q)XY - (3q - r)Y^2$$

and $\mu(g) = z$.

Proof. See for example Cassels [2]. □

Theorem 3 *Let u be a simple closed geodesic of $\mathsf{H}/\Gamma(3)$. Then there exists a Markoff number z such that*

$$3z = 2\cosh(L(u)/4)$$

where $L(u)$ denotes the length of u. Conversely, if z is a Markoff number then there exists a simple closed geodesic u in $\mathsf{H}/\Gamma(3)$ such that

$$3z = 2\cosh(L(u)/4).$$

Moreover, if

$$W = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

is a matrix corresponding to a closed geodesic u in $\mathsf{H}/\Gamma(3)$, then u is simple if and only if

$$\gamma X^2 - (\alpha - \delta)XY - \beta Y^2$$

is a multiple of a Markoff form.

Proof. See Lehner/Sheingorn [9]. □

Now we can formulate the corresponding results in the context of systoles.

Theorem 4 *Let (x, y, z) be a Markoff triple. Let q and r be defined as in Remark 4. Let $N = 3z$, $a = -q$, $b = 3q - r$, $c = z$. Then*

$$U = U(a,b,c) = \begin{bmatrix} 1+aN & bN \\ cN & 1-aN-N^2 \end{bmatrix} = \begin{bmatrix} 1-3qz & 3z(3q-r) \\ 3z^2 & 1+3qz-9z^2 \end{bmatrix}$$

corresponds to a systole in $\mathsf{H}/\Gamma(3z)$ of degree z.

Proof. The determinant of $U(a, b, c)$ is

$$1 - N^2(a^2 + aN + bc + 1) = 1 - N^2(q^2 - 3qz + (3k - r)z + 1) = 1$$

since $q^2 + 1 = rz$. It follows by Lemma 2 that U corresponds to a systole u in $\mathsf{H}/\Gamma(3z)$. By Proposition 2 the degree of u is

$$\Delta(u) = \min\{|cm^2 - (2a + N)mn - bn^2| : (m, n) \in \mathsf{Z} \times \mathsf{Z} \setminus (0,0)\}.$$

This is the same as

$$\Delta(u) = \min\{|zm^2 - (3z - 2q)mn - (3q - r)n^2| : (m, n) \in \mathsf{Z} \times \mathsf{Z} \setminus (0,0)\}.$$

By Theorem 2 this is a multiple of a Markoff form and $\Delta(u) = z$. □

Corollary 1 *Let u be one of the systoles described in Theorem 4. Then there exists a simple closed geodesic u' in $\mathsf{H}/\Gamma(3)$ of the same length as u. Moreover, u' is the image of u under a covering map and the same matrices correspond to both u and u'.*

Proof. By Theorem 4, u lies in $\mathsf{H}/\Gamma(N)$ with $N = 3z$ for a Markoff number z. Since $\Gamma(3z)$ is a normal subgroup of $\Gamma(3)$, it follows that $\mathsf{H}/\Gamma(N)$ is a cover of $\mathsf{H}/\Gamma(3)$. By Theorem 3 the image u' of u under this covering map is simple. □

Remark 5 Let u' be a simple closed geodesic in $\mathsf{H}/\Gamma(3)$. As remarked in the introduction, u' is characterized by the fact that it is far away from the cusps, a property which is of course shared by its cover, the systole u of the same length in some $\mathsf{H}/\Gamma(3z)$. We said that in this paper closed geodesics are always primitive. But let as drop this restriction for a moment and consider multiples of the closed geodesic u'. They share of course the same property as u' since they are as far away from the cusps as u'. They also have local covers of the same length which are systoles in some $\mathsf{H}/\Gamma(N)$ as we shall see in the next theorem. In fact, all systoles in $\mathsf{H}/\Gamma(N)$ of a high degree with respect to N are obtained in this way.

Theorem 5 *Let $Syst(\Gamma(N), N \geq 3)$ be the set of all lengths of systoles u in a $\mathsf{H}/\Gamma(N)$ with $\Delta(u)/N \geq 1/3$. Let $\mathcal{S}(\Gamma(3)) \times \mathsf{Z}_+$ be the set of all lengths of the simple closed geodesics of $\mathsf{H}/\Gamma(3)$ multiplied by a positive integer. Then $Syst(\Gamma(N), N \geq 3)$ and $\mathcal{S}(\Gamma(3)) \times \mathsf{Z}_+$ are isomorphic and the isomorphism is given by identification of corresponding matrices.*

Proof. (i) Remark that

$$\frac{\Delta(u)}{\sqrt{N^2-4}} > \frac{\Delta(u)}{N}$$

and that

$$\frac{n}{\sqrt{(3n+1)^2-4}} < \frac{1}{3}, n \in \mathsf{Z}_+,$$

so it is no restriction if we replace the condition $\Delta(u)/\sqrt{N^2-4} > 1/3$ by the condition $\Delta(u)/N \geq 1/3$, and vice versa.

(ii) For every Markoff triple (x, y, z) define q and r as in Remark 4 and let u_z be the simple closed geodesic in $\mathsf{H}/\Gamma(3)$ with $3z = 2\cosh(L(u_z)/4)$ as it is possible by Theorem 3. Let

$$U_z = \begin{bmatrix} 1-3qz & 3z(3q-r) \\ 3z^2 & 1+3qz-9z^2 \end{bmatrix} = \begin{bmatrix} a_z & b_z \\ c_z & d_z \end{bmatrix}$$

be a corresponding matrix as in Theorem 4. Then an n-fold multiple of the closed geodesic u_z (compare Remark 5) corresponds to U_z^n. We have to show that U_z^n corresponds to a systole $u(n)$ in an appropriate $\mathsf{H}/\Gamma(N)$ and that $\Delta(u(n))/N \geq 1/3$.

U_z is conjugate in $SL(2, \mathsf{R})$ to a matrix

$$\begin{bmatrix} exp(s_z) & 0 \\ 0 & exp(-s_z) \end{bmatrix}$$

and $2\cosh s_z = tr(U_z) = 9z^2 - 2$. Therefore $2\cosh(s_z/2) = 3z$. Then

$$tr(U_z^n) = 2\cosh(ns_z) = (2\cosh(ns_z/2))^2 - 2.$$

It follows that $N_z(n) = 2\cosh(ns_z/2)$ is an integer and U_z^n corresponds to a systole $u(n)$ in $\mathsf{H}/\Gamma(N_z(n))$.

Let

$$U_z^n = \begin{bmatrix} a_z(n) & b_z(n) \\ c_z(n) & d_z(n) \end{bmatrix}.$$

Then it is easy to see that there exists an integer $t(n)$ such that $t(n)b_z = b_z(n)$, $t(n)c_z = c_z(n)$, and $t(n)(a_z - d_z) = a_z(n) - d_z(n)$ which implies that the quadratic form

$$f(X,Y) = c_z(n)X^2 - (a_z(n) - d_z(n))XY - b_z(n)Y^2$$

is a multiple of the form

$$c_zX^2 - (a_z - d_z)XY - b_zY^2$$

which implies by Theorem 4 that it is a multiple of a Markoff form and hence $\Delta(u(n))/N_z(n) \geq 1/3$.

(iii) Conversely, let u be a systole in $\mathsf{H}/\Gamma(N)$ with $\Delta(u)/N \geq 1/3$. Let $U = U(a,b,c)$ correspond to u. Then

$$f(X,Y) = cX^2 - (2a+N)XY - bY^2$$

is a multiple of a form $g(X,Y)$ by Theorem 4 where

$$g(X,Y) = zX^2 - (3z-2q)XY - (3q-r)Y^2.$$

Here z is a Markoff number and q and r are defined as in Remark 4. We therefore have

$$D(f) = N^2 - 4 = \frac{c^2}{z^2}D(g) = \frac{c^2}{z^2}(9z^2-4).$$

Hence (N,c) is a solution of Pell's equation

$$p^2 - \frac{9z^2-4}{z^2}q^2 = 4. \tag{2}$$

The smallest solution in positive integers of (2) is $(3z, z)$ as it is easy to see. Instead of $3z$ we can write $2\cosh(s_z/2)$ as we have seen in (ii). All other positive integer solutions of (2) have the form (P_n, Q_n), n a positive integer, with

$$P_n = 2\cosh(ns_z/2)$$

and

$$T_n = \frac{2z\sinh(ns_z/2)}{\sqrt{9z^2-4}},$$

see for example [12], pg. 93, for this fact. This implies that N equals one of the $N_z(n)$ in (ii) and u is the corresponding $u(n)$ in (ii). This proves the theorem. $\square$

Remark 6 Let (x, y, z) be a Markoff triple. The so-called *uniqueness conjecture* says that a Markoff triple is already determined by z. I shall give an equivalent formulation in the context of systoles.

Definition Let u be a closed geodesic in $\mathsf{H}/\Gamma(N)$. The *extended isometry class* of u contains a closed geodesic v of $\mathsf{H}/\Gamma(N)$ if this surface has a (possibly orientation reversing) automorphism ϕ with $\phi(u) = v$.

Remark that systoles in the same extended isometry class have the same degree since the degree is a geometric measure, compare Remark 3.

Theorem 6 *The uniqueness conjecture is equivalent to the conjecture that for no N, the surface $\mathsf{H}/\Gamma(N)$ has more than one extended isometry class of Markoff systoles where a Markoff systole u is characterized by $\Delta(u)/N \geq 1/3$.*

Proof. By Theorem 5, we already have a complete list of N such that $\mathsf{H}/\Gamma(N)$ contains a Markoff systole. Moreover, we can restrict to the Markoff systoles of "minimal" length, appearing in Theorem 4, which have the same length as the simple closed geodesics in $\mathsf{H}/\Gamma(3)$. I further remark that it follows by an appropriate proof of Theorem 3 that the uniqueness conjecture is equivalent to the conjecture that if two simple closed geodesics in $\mathsf{H}/\Gamma(3)$ have the same length, then they are in the same extended isometry class. Since the surfaces $\mathsf{H}/\Gamma(3z)$, z a Markoff number, are covers of $\mathsf{H}/\Gamma(3)$ (compare Corollary 1), the theorem follows. $\square$

I also notice the following results which I did not find in the literature.

Lemma 5 *Let (x, y, z) and (x', y', z') be two Markoff triples and let q and q' be defined as in Remark 4. If $z = z'$ and $q = q'$, then $x = x'$ and $y = y'$.*

Proof. By Theorem 4, (x, y, z) and q define a matrix $U = U(a, b, c)$ corresponding to a systole in $\mathsf{H}/\Gamma(3z)$ with $a = -q$ and $c = z$. So, for both triples, this matrix is the same which implies that the corresponding systoles are (trivialy) in the same extended isometry class. Hence Theorem 6 implies the lemma. $\square$

Theorem 7 *Let (x, y, z) be a Markoff triple. Let*

$$\zeta = \frac{3z - \sqrt{5z^2 - 4}}{2}$$

and let

$$Q(z) = \{n \in \mathbf{Z} : \zeta \leq n \leq z/2, n^2 \equiv -1 \bmod (z)\}.$$

If $Q(z)$ has only one element, then (x, y, z) is determined by z.

Proof. Let

$$U = \begin{bmatrix} 1 - 3qz & 3z(3q - r) \\ 3z^2 & 1 + 3qz - 9z^2 \end{bmatrix}$$

be defined as in Theorem 4. Then $det(U) = 1$ implies that

$$q^2 - 3qz + 1 + (3q - r)z = 0 \tag{3}$$

Let

$$f(X, Y) = zX^2 - (-2q + 3z)XY - (3q - r)Y^2.$$

Then $3q - r \geq z$ since $|f(0, 1)| \geq z$. It follows by (3) that $q^2 - 3qz + 1 + z^2 \leq 0$ which implies $q \geq \zeta$ and therefore $q \in Q(z)$. The theorem now follows by Lemma 5. □

Remark 7 However there exist integers m such that $Q(m)$ has more than one element, for example $Q(1130) = \{437, 467\}$, $(\zeta = 431.6)$, or $Q(2005) = \{782, 822\}$, $(\zeta = 765.8)$.

Corollary 2 *A Markoff triple (x, y, z) is determined by z if z is a power of a prime.*

Proof. For $p = 2$ the claim is obvious since no Markoff number is a multiple of 4, so assume that $z = p^n$, $p \neq 2$ a prime, such that $Q(z)$ has two different elements a and b. Then $a^2 - b^2 \equiv 0 \bmod (z)$ by definition and hence $a + b \equiv 0 \bmod (p)$ and $a - b \equiv 0 \bmod (p)$ which gives $a \equiv 0 \bmod (p)$ which is impossible. □

4 Generalizations

The definition of the degree of a systole can be easily generalized to other closed geodesics of $\mathsf{H}/\Gamma(N)$ as follows. Let u be a closed geodesic and U a corresponding matrix. Then

$$U = \begin{bmatrix} 1+aN & bN \\ cN & 1-aN-kN^2 \end{bmatrix}$$

for a non-zero integer k. We can thus write $U = U_k(a,b,c)$. Then the degree of u is defined as (compare Proposition 2)

$$\Delta(u) = \min\{cm^2 - (2a+kN)mn - bn^2 : (m,n) \in \mathsf{Z} \times \mathsf{Z} \setminus (0,0)\}.$$

So, instead of systoles we could work with closed geodesics with a fixed k (which determines the length of the geodesic). In the case of systoles however, the situation is particular nice.

The definition of the degree of a systole (and of other closed geodesics) can be applied for other subgroups of $SL(2,\mathsf{Z})$ than $\Gamma(N)$.

Definition For N a positive integer let $\Gamma_1(N)$ and $\Gamma_2(N)$ be the subgroups of $SL(2,\mathsf{Z})$ containing the matrices of the form

$$\begin{bmatrix} 1+aN & bN \\ c & 1+dN \end{bmatrix}$$

and

$$\begin{bmatrix} 1+2aN & 2bN \\ 2c & 1+2dN \end{bmatrix},$$

respectively, where a, b, c, d are integers.

Remark that this notation is slightly different from the notation in [15].

Remark 8 We have seen in Theorem 3 that the Markoff numbers are closely related to the simple closed geodesics of $\mathsf{H}/\Gamma(3)$. There is another well known surface with the same relation, namely the modular one punctured torus. This surface M has genus 1 and one puncture and it has three different systoles (this determines the surface). M can be written as H/Γ' where Γ' is a subgroup of $SL(2,\mathsf{Z})$. The lengths of the simple closed geodesics of M are just half of the lengths of the simple closed geodesics of $\mathsf{H}/\Gamma(3)$.

Concerning the systoles we have a similar situation. Instead of a systole u of degree z in $\mathsf{H}/\Gamma(3z)$, z a Markoff number, we can use a systole v which has half of the length of u in another congruence subgroup of $SL(2,\mathsf{Z})$, but which has the same degree. Moreover, u has a corresponding matrix which is the square of a (certain) corresponding matrix of v.

Theorem 8 *Let z be a Markoff number. Let $N = 3z+2$. Then there exists a systole of degree z in $\mathsf{H}/\Gamma_1(N)$ if z is odd and in $\mathsf{H}/\Gamma_2(N/4)$ if z is even. Moreover $tr(u) = z$.*

Proof. Let (x, y, z) be a Markoff triple and let q and r be defined as in Remark 4. Assume that z is odd. Then z and $3z+2$ are coprime, so that there exists an integer t with

$$q + tz \equiv 0 \bmod (3z+2).$$

Let $b = -r - 2qt - 3q - t^2 z - 3tz$. Let

$$V = \begin{bmatrix} -q - tz - 3z & b \\ z & q + tz \end{bmatrix}.$$

Then $V \in \Gamma_1(3z+2)$ and $|tr(V)| = 3z$. It is easy to see that the systoles in $\mathsf{H}/\Gamma_1(N)$ have trace $N-2$, so U corresponds to a systole v.

Let

$$X^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}.$$

Then

$$X^n W X^{-n} = \begin{bmatrix} \alpha + n\gamma & * \\ \gamma & \delta - n\gamma \end{bmatrix}.$$

Let

$$U = \begin{bmatrix} 1 - 3qz & 3z(3q - r) \\ 3z^2 & 1 + 3qz - 9z^2 \end{bmatrix} \in \Gamma(3z)$$

as in Theorem 4. Then $X^n U X^{-n}$ also corresponds to a systole u of degree z in $\mathsf{H}/\Gamma(3z)$. Set $n = -t - 3$. Then a calculation gives

$$-V^2 = X^n U X^{-n}$$

which proves that the degree of the systole v equals the degree of u and $2L(v) = L(u)$.

If z is even, then the proof is similar and is left to the reader (one has to use the fact that $z \equiv 2 \bmod 8$ which is easy to see.) □

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