Reidemeister numbers

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Abstract

In [5] we have conjectured that the Reidemeister number is infinite as long as an endomorphism of a discrete group is injective and the group has exponential growth. In the paper we prove this conjecture for any automorphism of a non-elementary, Gromov hyperbolic group. We also prove some generalisations of this result. The main results of the paper have topological counterparts. Finally, we propose an analog of Selberg theory for twisted conjugacy classes.

0 Introduction

Let G be a finitely generated group and $\phi : G \to G$ an endomorphism. Two elements $\alpha, \alpha' \in G$ are said to be $\phi - conjugate$ iff there exists $\gamma \in G$ with

$$\alpha' = \gamma\alpha\phi(\gamma)^{-1}.$$

We shall write $\{x\}_\phi$ for the ϕ-conjugacy class of the element $x \in G$. The number of ϕ-conjugacy classes is called the *Reidemeister number* of an endomorphism ϕ, denoted by $R(\phi)$. If ϕ is the identity map then the ϕ-conjugacy classes are the usual conjugacy classes in the group G.

We note that $R(\phi)$ is infinite if group G is free Abelian and the action of ϕ on G has 1 as eigenvalue [6] .

In [5] we have conjectured that the Reidemeister number is infinite as long as an endomorphism ϕ is injective and the group G has exponential growth.

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In this paper we prove our conjecture for any automorphism of any non-elementary (i.e. not virtually cyclic), Gromov hyperbolic group. We also prove some generalisations of this result. The work of G. Levitt and M. Lustig [8] plays the key role in this new development of the subject.

Main results of this paper have their topological counterparts. Let X to be a connected, compact polyhedron and $f : X \to X$ to be a continuous map. Let $p : \tilde{X} \to X$ be the universal cover of X and $\tilde{f} : \tilde{X} \to \tilde{X}$ a lifting of f, i.e. $p \circ \tilde{f} = f \circ p$. Two liftings $\tilde{f}$ and $\tilde{f}'$ are called *conjugate* if there is a elementγ in the deck transformation group $\Gamma \cong \pi_1(X)$ such that $\tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1}$. The subset $p(Fix(\tilde{f})) \subset Fix(f)$ is called *the fixed point class of f determined by the lifting class* $[\tilde{f}]$. Two fixed points x_0 and x_1 of f belong to the same fixed point class iff there is a path c from x_0 to x_1 such that $c \cong f \circ c$ (homotopy relative endpoints). This fact can be considered as an equivalent definition of a non-empty fixed point class. Every map f has only finitely many non-empty fixed point classes, each a compact subset of X. A fixed point class is called *essential* if its index is nonzero. The number of lifting classes of f (and hence the number of fixed point classes, empty or not) is called the *Reidemeister number* of f, denoted $R(f)$. This is a positive integer or infinity. The number of essential fixed point classes is called the *Nielsen number* of f, denoted by $N(f)$.The Nielsen number is always finite.

We prove that the topological Reidemeister number $R(f)$ is infinite for any homeomorphism f of a compact polyhedron X with a non-elementary, Gromov hyperbolic fundamental group.

In the case of pseudo-Anosov homeomorphisms of surfaces we propose an analog of the Selberg theory for twisted conjugacy classes and we obtain an asymptotic expansion for the number of twisted conjugacy classes or for the number of Nielsen fixed point classes whose norm is at most x.

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1 Twisted conjugacy classes and Reidemeister number of group endomorphism

Lemma 1 *If G is a group and ϕ is an endomorphism of G then an element $x \in G$ is always ϕ-conjugate to its image $\phi(x)$.*

PROOF. If $\gamma = x^{-1}$, then one has immediately $\gamma x = \phi(x)\phi(\gamma)$. The existence of a γ satisfying this equation implies that x and $\phi(x)$ are ϕ-conjugate.

The mapping torus $M(\phi)$ of the group endomorphism $\phi : G \to G$ is obtained from group G by adding a new generator z and adding the relations $z^{-1}gz = \phi(g)$ for all $g \in G$. This means that $M(\phi)$ is a semi-direct product of G with Z.

Lemma 2 *Two elements x, y of G are ϕ-conjugate if and only if xz and yz are conjugate in the usual sense in $M(\phi)$. Therefore $R(\phi)$ is the number of usual conjugacy classes in the coset $G \cdot z$ of G in $M(\phi)$.*

PROOF. If x and y are ϕ-conjugate, then there is a $\gamma \in G$ such that $\gamma x = y\phi(\gamma)$. This implies $\gamma x = yz\gamma z^{-1}$ and therefore $\gamma(xz) = (yz)\gamma$. So xz and yz are conjugate in the usual sense in $M(\phi)$. Conversely suppose xz and yz are conjugate in $M(\phi)$. Then there is a $\gamma z^n \in M(\phi)$ with $\gamma z^n xz = yz\gamma z^n$. From the relation $zxz^{-1} = \phi(x)$, we obtain $\gamma\phi^n(x)z^{n+1} = y\phi(\gamma)z^{n+1}$ and therefore $\gamma\phi^n(x) = y\phi(\gamma)$. This shows that $\phi^n(x)$ and y are ϕ-conjugate. However, by lemma 1 , x and $\phi^n(x)$ are ϕ-conjugate, so x and y must be ϕ-conjugate.

Lemma 3 *(T.Delzant) Let J be a non-elementary, Gromov hyperbolic group. Let K be a normal subgroup with abelian quotient. Then every coset C of J mod K contains infinitely many conjugacy classes.*

PROOF. (See [8]). Fix u in the coset C under consideration. Suppose for a moment that we can find $c, d \in K$, generating a free group of rank 2, such that $uc^{\infty} \neq c^{-\infty}$ and $ud^{\infty} \neq d^{-\infty}$ (recall that we denote $g^{-\infty} = \lim_{n\to+\infty} g^{-n}$ for g of infinite order). Consider $x_k = c^k uc^k$ and $y_k = d^k ud^k$. For k large, the above inequalities imply that these two elements have infinite order, and do not generate a virtually cyclic group because ${x_k}^{+\infty}$ and ${x_k}^{-\infty}$(respectively ${y_k}^{+\infty}$ and ${y_k}^{-\infty}$) is close to $c^{+\infty}$ and $c^{-\infty}$ (respectively $d^{+\infty}$ and $d^{-\infty}$). Fix k, and consider the elements $z_n = {x_k}^{n+1}{y_k}^{-n}$. They belong to the coset

C, because J/K is abelian, and their stable norm goes to infinity with n. Therefore C contains infinitely many conjugacy classes. Let us now construct c, d as above. Choose $a, b \in K$ generating a free group of rank 2. We first explain how to get c. There is a problem only if $ua^\infty = a^{-\infty}$ and $ub^\infty = b^{-\infty}$. In that case there exists integers p, q with $ua^p u^{-1} = a^{-p}$ and $ub^q u^{-1} = b^{-q}$. We take $c = a^p b^q$, noting that $ucu^{-1} = a^{-p} b^{-q}$ is different from $c^{-1} = b^{-q} a^{-p}$.

Once we have c, we choose $c^* \in K$ with $< c, c^* >$ free of rank 2, and we obtain d by applying the preceding argument using c^* and cc^* instead of a and b. The group $< c, d >$ is free of rank 2 because d is a positive word in c^* and cc^*.

1.1 Automorphisms of Gromov hyperbolic groups

Let now ϕ be an automorphism of the Gromov hyperbolic group G and let $\|\|$ denote the word metric with respect to some finite generating set for G. The automorphism ϕ is called hyperbolic if there is an integer m and a number $\lambda > 1$ such that, for all $g \in G$ we have $max(\|\phi^m(g)\|, \|\phi^{-m}(g)\|) \geq \lambda\|g\|$. For example a pseudo-Anosov homeomorphism of a closed surface of genus larger then one induces a hyperbolic automorphism on the level of fundamental group. Also, an automorphism of finitely generated free group with no nontrivial periodic conjugacy classes is hyperbolic.

Lemma 4 *[2] The mapping torus $M(\phi)$ of a hyperbolic automorphism ϕ is Gromov hyperbolic group.*

Theorem 1 *The Reidemeister number $R(\phi)$ is infinite if group G is Gromov hyperbolic, non-elementary, and ϕ is hyperbolic automorphism.*

PROOF. The proof immediately folows from lemmas 2, 3 and 4 .

The following theorem, actually , was proved (implicitly) in the paper of Cohen and Lustig [3] in Proposition 5.4

Theorem 2 *The Reidemeister number $R(\phi)$ is infinite if group G is a free group F_n and ϕ fixes a nontrivial conjugacy class in F_n.*

PROOF. Let $D(\phi)$ is the graph with a vertex $V(v)$ for each $v \in F_n$ and an oriented edge x from $V(v)$ to $V(w)$ whenever $w = x^{-1} v \phi(x)$. The component of $D(\phi)$ containing vertex $V(v)$ is denoted $D_v(\phi)$. The graph $D(\phi)$ was introduced by Goldstein and Turner[9]. Since ϕ fixes non-trivial conjugacy class in F_n we can choose a non-trivial word X such that $\phi(X) = v^{-1}Xv$ for some $v \in F_n$. Notice that $X^{-1}(X^n v)\phi(X) = X^n v$. Thus there is, for each $n \in Z$, a loop based at $V(X^n v)$ which reads off the word X. Such a loop is carried by a graph containing a simple closed curve. If infinitely many of the vertices $V(X^n v)$ were contained in the same component $D_w(\phi)$ then, since the lenghts of these loops are bounded(in fact equal to $\|X\|$), there would be an infinite family of pairwise disjoint simple closed curve in $D_w(\phi)$. This is impossible since $rank(\pi_1(D_w(\phi)))$ is finite [3]. Hence there exist infinitely many components which contain non-trivial loops labelled X based at vertices of the form $V(X^n v)$. This means that the number of twisted conjugacy classes is also infinite.

Let us now consider an outer automorphism $\Phi \in OutG$ corresponding to automorphism of $\phi \in AutG$ and viewed as a collection of ordinary automorphisms $\alpha \in AutG$. We define $\alpha, \beta \in \Phi$ to be isogredient if $\beta = i_h \cdot \alpha \cdot {i_h}^{-1}$ for some $h \in G$, with $i_h(g) = hgh^{-1}$.

Lemma 5 *[8] The set $S(\Phi)$ of isogredience classes is infinite if group G is Gromov hyperbolic, non-elementary, and Φ has finite order in the group $OutG$.*

Proof. Let J be the subgroup of $AutG$ consisting of all automorphisms whose image in $OutG$ is a power of Φ. The exact sequence $1 \to K \to J \to <\Phi> \to 1$, with $K = G/Center$ and $<\Phi>$ finite, shows that J is hyperbolic, non-elementary. The set of automorphisms $\alpha \in \Phi$ is a coset of J mod K. If $\alpha, \beta \in \Phi$ are isogredient they are conjugate in J. The proof is therefore concluded by applying lemma 3 .

Theorem 3 *[8] For any $\Phi \in OutG$, with G Gromov hyperbolic, non-elementary, the set $S(\Phi)$ of isogredience classes is infinite.*

Proof. We describe here main steps of the proof in [8]. By lemma 5, we may assume that Φ has infinite order. By Paulin's theorem [12] Φ preserves some R-tree T with nontrivial minimal small action of G(recall that an action of G is small if all ars stabilisers are virtually cyclic; the action of G on T is always irreducible(no global fixed point, no invariant line, no invariant end)). This means that there is an R-tree T equipped with an isometric

action of G whose length function satisfies $l \cdot \Phi = \lambda l$ for some $\lambda \geq 1$.
Step 1. Suppose $\lambda = 1$. Then $S(\Phi)$ is infinite.
Step 2. Suppose $\lambda > 1$. Assume that arc stabilisers are finite, and there exists $N_0 \in N$ such that, for every $Q \in T$, the action of $StabQ$ on $\pi_o(T - Q)$ has at most N_0 orbits. Then $S(\Phi)$ is infinite.
Step 3. If $\lambda > 1$, then T has finite arc stabilisers. If $\lambda > 1$ then from work of Bestwina-Feighn [2] it follows that there exists $N_0 \in N$ such that, for every $Q \in T$, the action of $StabQ$ on $\pi_o(T - (Q))$ has at most N_0 orbits.

Theorem 4 *The Reidemeister number $R(\phi)$ is infinite if group G is Gromov hyperbolic, non-elementary, and ϕ is any automorphism of G.*

PROOF. By definition, the automorphisms $\beta = i_m \cdot \alpha$ and $\gamma = i_n \cdot \alpha$ are isogredient if and only if there exists $g \in G$ with $\gamma = i_g \cdot \beta \cdot {i_g}^{-1}$,or equivalently $n = gm\alpha(g^{-1})c$ with c in center of G. So, the set $S(\Phi)$ of isogredience classes of automorphisms representing Φ may be identified to the set of twisted conjugacy classes of $G \mod$ its center. If ϕ is automorphism of finite order in $AutG$, then the theorem immediately follows from lemma 5 . If an automorphism ϕ has infinite order in $AutG$ then theorem follows from theorem 3 .

1.2 Reduction to Injective Endomorphisms and the Co-Hopf property

A group G is called co-Hopf if every monomorphism of G into itself is an isomorphism. It is fairly immediate to see that a freely decomposable group is not co-Hopf.

Lemma 6 *([15]) Let G be a non-elementary, torsion-free, Gromov hyperbolic group. Then G is co-Hopf if and only if G is freely indecomposable.*

Theorem 5 *The Reidemeister number $R(\phi)$ is infinite if group G is Gromov hyperbolic, non-elementary,torsion free, freely indecomposable and ϕ is any monomorphism of G into itself.*

PROOF. The proof follows from lemma 6 and theorem 4 .

1.2.1 Reduction to Injective Endomorphisms

Let G be a group and $\phi : G \to G$ an endomorphism. We shall call an element $x \in G$ nilpotent if there is an $n \in I\!N$ such that $\phi^n(x) = id$. Let N be the set of all nilpotent elements of G.

Lemma 7 *The set N is a normal subgroup of G. We have $\phi(N) \subset N$ and $\phi^{-1}(N) = N$. Thus ϕ induces an endomorphism $[\phi/N](xN) := \phi(x)N$. The endomorphism $[\phi/N] : G/N \to G/N$ is injective, and we have $R(\phi) = R([\phi/N])$.*

PROOF.

(i) Let $x \in N, g \in G$. Then for some $n \in I\!N$ we have $\phi^n(x) = id$. Therefore $\phi^n(gxg^{-1}) = \phi^n(gg^{-1}) = id$. This shows that $gxg^{-1} \in N$ so N is a normal subgroup of G.

(ii) Let $x \in N$ and choose n such that $\phi^n(x) = id$. Then $\phi^{n-1}(\phi(x)) = id$ so $\phi(x) \in N$. Therefore $\phi(N) \subset N$.

(iii) If $\phi(x) \in N$ then there is an n such that $\phi^n(\phi(x)) = id$. Therefore $\phi^{n+1}(x) = id$ so $x \in N$. This show that $\phi^{-1}(N) \subset N$. The converse inclusion follows from (ii).

(iv) We shal write $\mathcal{R}(\phi)$ for the set of ϕ-conjugacy classes of elements of G. We shall now show that the map $x \to xN$ induces a bijection $\mathcal{R}(\phi) \to \mathcal{R}([\phi/N])$. Suppose $x, y \in G$ are ϕ-conjugate. Then there is a $g \in G$ with $gx = y\phi(g)$. Projecting to the quotient group G/N we have $gnxN = yN\phi(g)N$, so $gNxN = yN[\phi/N](gN)$. This means that xN and yN are $[\phi/N]$-conjugate in G/N. Conversely suppose that xN and yN are $[\phi/N]$-conjugate in G/N. Then there is a $gN \in G/N$ such that $gNxN = yN[\phi/N](gN)$. In other words $gx\phi(g)^{-1}y^{-1}) = id$. Therefore $\phi^n(g)\phi^n(x) = \phi^n(y)\phi^n(\phi(g))$.

This shows that $\phi^n(x)$ and $\phi^n(y)$ are ϕ-conjugate. Howeever by lemma 1 x and $\phi^n(x)$ are ϕ-conjugate as are y and $\phi^n(y)$. Therefore x and y are ϕ-conjugate .

(v) We have shown that x and y are ϕ-conjugate iff xN and yN are ϕ/N-conjugate. From this it follows that $x \to xN$ induces a bijection from $\mathcal{R}(\phi)$ to $\mathcal{R}([\phi/N])$. Therefore $R(\phi) = R([\phi/N])$.

Theorem 6 *The Reidemeister number $R(\phi)$ is infinite if group G/N is Gromov hyperbolic, non-elementary,torsion free, freely indecomposable and ϕ is any endomorphism of G into itself.*

PROOF. The proof follows from lemma 7 and theorems 4 and 5 .

1.3 Reidemeister coincidence number

Let G be a finitely generated group and $\phi, \psi : G \to G$ two endomorphisms. Two elements $\alpha, \alpha' \in G$ are said to be $(\phi, \psi) -$ *conjugate* iff there exists $\gamma \in G$ with

$$\alpha' = \psi(\gamma)\alpha\phi(\gamma)^{-1}.$$

The number of (ϕ, ψ)-conjugacy classes is called the Reidemeister coincidence number of an endomorphisms ϕ and ψ, denoted by $R(\phi, \psi)$. If ψ is the identity map then the (ϕ, id)-conjugacy classes are the ϕ- conjugacy classes in the group G. The Reidemeister coincidence number $R(\phi, \psi)$ has useful applications in Nielsen coincidence theory.

Lemma 8 *Let $\phi, \psi : G \to G$ are two automorphisms. Two elements x, y of G are $\psi^{-1}\phi$-conjugate if and only if elements $\psi(x)$ and $\psi(y)$ are (ψ, ϕ)-conjugate. Therefore the Reidemeister number $R(\psi^{-1}\phi)$ is equal to $R(\phi, \psi)$.*

PROOF. If x and y are $\psi^{-1}\phi$-conjugate, then there is a $\gamma \in G$ such that $x = \gamma y \psi^{-1}\phi(\gamma^{-1})$. This implies $\psi(x) = \psi(\gamma)\psi(y)\phi(\gamma^{-1})$. So $\psi(x)$ and $\psi(y)$ are (ϕ, ψ)-conjugate. The converse statement follows if we move in opposite direction in previous implications.

Theorem 7 *The Reidemeister number $R(\phi, \psi)$ is infinite if group G is Gromov hyperbolic, non-elementary, and ϕ, ψ are any automorphisms of G.*

PROOF. From theorem 4 it follows that the Reidemeister number $R(\psi^{-1}\phi)$ of an automorphism $\psi^{-1}\phi$ is infinite.The proof is therefore concluded by applying lemma 8 .

Theorem 8 *The Reidemeister number $R(\phi, \psi)$ is infinite if group G is Gromov hyperbolic, non-elementary,torsion free, freely indecomposable and ϕ, ψ are any monomorphisms of G into itself.*

PROOF. The proof follows from lemma 6 and theorems 5 and 7 .

2 Reidemeister number of continuous map

Let now X to be a connected, compact polyhedron and $f : X \to X$ to be a continuous map. Let a specific lifting $\tilde{f} : \tilde{X} \to \tilde{X}$ be chosen as reference. Then every lifting of f can be written uniquely as $\gamma \circ \tilde{f}$, with $\gamma \in \Gamma$. So the elements of Γ serve as coordinates of liftings with respect to the reference $\tilde{f}$. Now, for every $\gamma \in \Gamma$, the composition $\tilde{f} \circ \gamma$ is a lifting of f; so there is a unique $\gamma' \in \Gamma$ such that $\gamma' \circ \tilde{f} = \tilde{f} \circ \gamma$. This correspondence $\gamma \to \gamma'$ is determined by the reference $\tilde{f}$, and is obviously a homomorphism.

Definition 1 *The endomorphism $\tilde{f}_* : \Gamma \to \Gamma$ determined by the lifting $\tilde{f}$ of f is defined by*

$$\tilde{f}_*(\gamma) \circ \tilde{f} = \tilde{f} \circ \gamma.$$

It is well known that $\Gamma \cong \pi_1(X)$. We shall identify $\pi = \pi_1(X, x_0)$ and Γ in the following way. Pick base points $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0) \subset \tilde{X}$ once and for all. Now points of $\tilde{X}$ are in 1-1 correspondence with homotopy classes of paths in X which start at x_0: for $\tilde{x} \in \tilde{X}$ take any path in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}$ and project it onto X; conversely, for a path c starting at x_0, lift it to a path in $\tilde{X}$ which starts at $\tilde{x}_0$, and then take its endpoint. In this way, we identify a point of $\tilde{X}$ with a path class $\langle c \rangle$ in X starting from x_0. Under this identification, $\tilde{x}_0 = \langle e \rangle$ is the unit element in $\pi_1(X, x_0)$. The action of the loop class $\alpha = \langle a \rangle \in \pi_1(X, x_0)$ on $\tilde{X}$ is then given by

$$\alpha = \langle a \rangle : \langle c \rangle \to \alpha \cdot c = \langle a \cdot c \rangle.$$

Now we have the following relationship between $\tilde{f}_* : \pi \to \pi$ and

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(X, f(x_0)).$$

Lemma 9 *Suppose $\tilde{f}(\tilde{x}_0) = \langle w \rangle$. Then the following diagram commutes:*

$$\begin{array}{ccc}
\pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(X, f(x_0)) \\
 & \tilde{f}_* \searrow & \downarrow w_* \\
 & & \pi_1(X, x_0)
\end{array}$$

where w_* is the isomorphism induced by the path w.

In other words, for every $\alpha = \langle a \rangle \in \pi_1(X, x_0)$, we have

$$\tilde{f}_*(\langle a \rangle) = \langle w(f \circ a)w^{-1} \rangle.$$

Remark 1 *In particular, if $x_0 \in p(Fix(\tilde{f}))$ and $\tilde{x}_0 \in Fix(\tilde{f})$, then $\tilde{f}_* = f_*$.*

Lemma 10 *Lifting classes of f (and hence fixed point classes, empty or not) are in 1-1 correspondence with $\tilde{f}_*$-conjugacy classes in π, the lifting class $[\gamma \circ \tilde{f}]$ corresponding to the $\tilde{f}_*$-conjugacy class of γ. We therefore have $R(f) = R(\tilde{f}_*)$.*

We shall say that the fixed point class $p(Fix(\gamma \circ \tilde{f}))$, which is labeled with the lifting class $[\gamma \circ \tilde{f}]$, *corresponds* to the $\tilde{f}_*$-conjugacy class of γ. Thus $\tilde{f}_*$-conjugacy classes in π serve as coordinates for fixed point classes of f, once a reference lifting $\tilde{f}$ is chosen.

Theorem 9 *Reidemeister number $R(f)$ is infinite for any homeomorphism f of a compact polyhedron X with a non-elementary, Gromov hyperbolic fundamental group $\pi_1(X, x_0)$.*

Proof. Proof immediately follows from theorem 4 and lemma 10 .

Theorem 10 *The Reidemeister number $R(f)$ is infinite if the fundamental group $\pi_1(X, x_0)$ is a Gromov hyperbolic, non-elementary,torsion free, freely indecomposable and the map f induces a monomorphism $\tilde{f}_*$ of $\pi_1(X, x_0)$ into itself.*

Proof. The proof follows from lemma 10 and theorem 5 .

Let now f is any continious map of X . Let N be the set of all nilpotent elements of $\pi_1(X, x_0)$ under $\tilde{f}_*$.

Theorem 11 *For any map f the Reidemeister number $R(f)$ is infinite if the group $\pi_1(X, x_0)/N$ is a Gromov hyperbolic, non-elementary,torsion free, freely indecomposable.*

Proof. The proof follows from lemma 10 and theorem 6 .

2.0.1 Reidemeister coincidence number of maps

Let now $f, g : X \to X$ two homeomorphisms of X. The Reidemeister coincidence number of f and g, denoted by $R(f,g)$ is simply defined to be the Reidemeister number $R(\phi,\psi)$, where ϕ and ψ are induced fundamental groups automorphisms of f and g(see [17]).

Theorem 12 *Reidemeister number $R(f,g)$ is infinite for any homeomorphisms f and g of a compact polyhedron X with a non-elementary, Gromov hyperbolic fundamental group.*

Proof. Proof immediately follows from theorem 7 .

3 Pseudo-Anosov homeomorphism and asymptotic expansion

Now we assume X to be a compact surface of negative Euler characteristic and $f : X \to X$ is a pseudo-Anosov homeomorphism, i.e. there is a number $\lambda > 1$ and a pair of transverse measured foliations (F^s, μ^s) and (F^u, μ^u) such that $f(F^s, \mu^s) = (F^s, \frac{1}{\lambda}\mu^s)$ and $f(F^u, \mu^u) = (F^u, \lambda\mu^u)$. The mapping torus T_f of $f : X \to X$ is the space obtained from $X \times [0,1]$ by identifying $(x,1)$ with $(f(x),0)$ for all $x \in X$. Take the base point x_0 of X as the base point of T_f. According to van Kampen's Theorem, the fundamental group $\pi_1(T_f, x_0)$ is obtained from $\pi_1(X, x_0)$ by adding a new generator z and adding the relations $z^{-1}gz = \tilde{f}_*(g)$ for all $g \in \pi_1(X, x_0)$, where z is the generator of $\pi_1(S^1, x_0)$. This means that $\pi_1(T_f, x_0)$ is a semi-direct product of $\pi_1(X, x_0)$ with Z.

From Lemma 2 it follows that two elements x, y of $\pi_1(X, x_0)$ are $\tilde{f}_*$-conjugate if and only if xz and yz are conjugate in the usual sense in $\pi_1(T_f, x_0)$. Therefore $R(f) = R(\tilde{f}_*)$ is the number of usual conjugacy classes in the coset $\pi_1(X, x_0) \cdot z$ of $\pi_1(X, x_0)$ in $\pi_1(T_f, x_0)$. There is a canonical projection $\tau : T_f \to R/Z$ given by $(x,s) \mapsto s$. This induces a map $\pi_1(\tau) : G = \pi_1(T_f, x_0) \to Z$.

We see that the Reidemeister number $R(f)$ is equal to the number of homotopy classes of closed paths γ in T_f whose projections onto R/Z are homotopic to the path

$$\begin{array}{rccc} \sigma : & [0,1] & \to & R/Z \\ & s & \mapsto & s. \end{array}$$

Corresponding to this, there is a group-theoretical interpretation of $R(f)$ as the number of usual conjugacy classes of elements $\gamma \in \pi_1(T_f)$ satisfying $\pi_1(\tau)(\gamma) = z$.

Lemma 11 *[16, 11] The interior of the mapping torus $Int(T_f)$ admits a hyperbolic structure of finite volume if and only if f is isotopic to a pseudo-Anosov homeomorphism.*

So, if the surface X is closed and f is isotopic to a pseudo-Anosov homeomorphism, the mapping torus T_f can be realised as a hyperbolic 3-manifold , $H^3/\pi_1(T_f, x_0)$, where H^3 is the Poincare upper half space $\{(x, y, z) : z > 0, (x, y) \in R^2\}$ with the metric $ds^2 = (dx^2 + dy^2 + dz^2)/z^2$.

The closed geodesics on a hyperbolic manifold are in one-to-one correspondence with the free homotopy classes of loops. These classes of loops are in one-to-one correspondence with the conjugacy classes of loxodromic elements in the fundamental group of the hyperbolic manifold.

This correspondence allows Ch. Epstein (see [4], p.127) to study the asymptotics of such functions as $p_n(x) = \#\{$ primitive closed geodesics of length less than x represented by an element of the form $gz^n\}$ using the Selberg trace formula. A primitive closed geodesic is one which is not an iterate of another closed geodesic.

Phillips and Sarnak [13] have generalised results of Epstein in the following way. Let M be a compact Riemannian manifold of negative curvature. Let $\psi : G \to \Lambda$ be a surjective homomorphism of the fundamental group G of M onto an abelian group Λ. Let r be the rank of Λ and let m be the order of the torsion of Λ; i.e., Λ is isomorphic to $Z^r \cdot F$, F being finite of order m. For $\beta \in \Lambda$ let $p_\beta(x)$ be the number of primitive closed geodesics γ on M of length at most x for which $\psi(\gamma) = \beta$.

Theorem 13 *[13] We have the asymptotic expansion*

$$p_\beta(x) \sim \frac{e^{(n-1)x}}{mx^{r/2+1}}(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + ...)$$

as $x \to \infty$. Here C_0 is independent of β; C_0 is the determinant of a certain period matrix of harmonic 1-forms on M.

The proof of this result makes routine use of the Selberg trace formula. If $\Lambda = H_1(M, Z)$ then theorem 12 gives the asymptotic of the number of primitive closed geodesics of length at most x lying in fixed homology class. In the more general case of variable negative curvature similar asymptotic was obtained by Pollicott and Sharp [14]. They used dynamical approach based on the geodesic flow. We will only need an asymptotic for $p_1(x)$. Note, that closed geodesics which are represented by an element of the form gz are automatically primitive, because they wrap exactly once around the mapping torus (once around generator z). Taking $\Lambda = Z$,a surjective homomorphism $\pi_1(\tau) : G = \pi_1(T_f, x_0) \to Z$ and $\beta = z$-generator of group Z in theorem 10 we have the following asymptotic expansion [4, 13, 14, 1]

$$p_1(x) = \frac{e^{hx}}{x^{3/2}}\left(\sum_{n=0}^{N} \frac{C_n}{x^{n/2}} + o\left(\frac{1}{x^{N/2}}\right)\right), \tag{1}$$

for any $N > 0$, where $h = \dim T_f - 1 = 2$ is the topological entropy of the geodesic flow on the unit-tangent bundle ST_f, the constant $C_0 > 0$, constants C_n vanish if n is odd. So in particular, using $N = 1$ we have

$$p_1(x) \sim C_0 \frac{e^{hx}}{x^{3/2}}, x \to \infty \tag{2}$$

Since $C_0 > 0$ we see that $p_1(x)$ goes to infinity as x does.

Now, using one-to-one correspondences in lemma 2 and lemma 10 we define the norm of a fixed point class, or of the corresponding lifting class, or of the corresponding twisted conjugacy class $\{g\}_{\tilde{f}_*}$ in the fundamental group of the surface $\pi_1(X, x_0)$, as the length of the primitive closed geodesic γ on T_f, which is represented by an element of the form gz. So, for example, the norm function l^* on the set of twisted conjugacy classes equals $l^* = l \circ B$, where l is length function on geodesics ($l(\gamma)$ is the length of the primitive closed geodesic γ) and B is a bijection between the set of twisted conjugacy classes $\{g\}_{\tilde{f}_*}$ in the fundamental group of the surface $\pi = \pi_1(X, x_0)$ and the set of closed geodesics represented by an elements of the form gz in the fundamental group $G := \pi_1(T_f, x_0)$. We introduce the following counting functions

FPC(x)= # {fixed point classes of f of norm less than x},
L(x)= # {lifting classes of f of norm less than x},
Tw(x)= #{twisted conjugacy classes for $\tilde{f}_*$ in the fundamental group of surface of norm less than x}. We have following asymptotic expansion

Theorem 14 *Let X be a closed surface of negative Euler characteristic and let $f : X \to X$ be a pseudo-Anosov homeomorphism. Then*

$$FPC(x) = L(x) = Tw(x) = \frac{e^{2x}}{x^{3/2}}(\sum_{n=0}^{N} \frac{C_n}{x^{n/2}} + o(\frac{1}{x^{N/2}})),$$

where the constant $C_0 > 0$ depends on the volume of the hyperbolic 3-manifold T_f, and the constants C_n vanish if n is odd.

Proof. The proof follows from lemmas 2 and 10 and asymptotic expansion (1).

Corollary 1 *For pseudo-Anosov homeomorphisms of closed surfaces the Reidemeister number is infinite.*

As we see, Theorem 14 gives analytical proof of Theorem 1 in the case of pseudo-Anosov automorphism of surface group.

Theorem 14 indicates also, that we can develop an analog of Selberg theory for twisted conjugacy classes of group automorphism. We have the following question in this direction

Question 1 *Is there an asymptotic of counting function for twisted conjugacy classes of hyperbolic automorphism of Gromov hyperbolic group?*

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