IRREDUCIBILITY OF MODULI SPACES OF VECTOR BUNDLES ON K3 SURFACES

KŌTA YOSHIOKA

0. Introduction

Let X be a projective K3 surface defined over $\mathbb{C}$ and H an ample divisor on X. For a coherent sheaf E on X, $v(E) := \operatorname{ch}(E)\sqrt{\operatorname{td}_X} \in H^*(X,\mathbb{Z})$ is the Mukai vector of E, where td_X is the Todd class of X. We denote the moduli space of stable sheaves E of $v(E) = v$ by $M_H(v)$. If v is primitive and H is general (i.e. H does not lie on walls [Y3]), then $M_H(v)$ is a smooth projective scheme. In [Mu1], Mukai showed that $M_H(v)$ has a symplectic structure. In order to get more precise information, Mukai [Mu2] introduced a quite useful notion called Mukai lattice $(H^*(X,\mathbb{Z}), \langle\ \ ,\ \ \rangle)$, where $\langle\ \ ,\ \ \rangle$ is an integral primitive bilinear form on $H^*(X,\mathbb{Z})$. By the language of this lattice, we can write down Riemann-Roch theorem in a simple form. In particular we get that $\dim M_H(v) = \langle v^2\rangle + 2$.

If v is a primitive isotropic vector, then $M_H(v)$ is a surface with a symplectic structure. Mukai proved that $M_H(v)$ is a K3 surface and described the period in terms of Mukai lattice. If v is a primitive Mukai vector of $\langle v^2\rangle > 0$, then $M_H(v)$ is a higher dimensional symplectic manifold. If $\operatorname{rk} v = 1$, then $M_H(v)$ is the Hilbert scheme of points on X. Beauville [B] proved that it is an example of higher dimensional irreducible symplectic manifold. For an irreducible symplectic manifold, Beauville [B] defined the period and proved local Torelli theorem. As an example, he also computed the period of the Hilbert scheme of points on X. For higher rank cases, Mukai [Mu3] (rank 2 case), O'Grady [O] (any rank but primitive c_1 case) and the author [Y4] (asymptotic case) proved that $M_H(v)$ is an irreducible symplectic manifold and described the period of $M_H(v)$ in terms of Mukai lattice. For classification of $M_H(v)$, it is important to determine the period. Indeed, it is a birational invariant ([Mu3]), and affirmative solution of Torelli conjecture will imply that an irreducible symplectic manifold is determined by its period, up to birational equivalence.

In this paper, by using [Y4] extensively, we prove the following theorem, which is expected by many people (for example, see [D], [Mu3], [O]).

Theorem 0.1. *Let $v = l(r+\xi) + a\omega, \xi \in \operatorname{NS}(X)$ be a primitive Mukai vector such that $r+\xi$ is primitive and $l, r > 0$, where we identify $H^0(X,\mathbb{Z})$ with $\mathbb{Z}$ and ω is the fundamental class of X.*

(1) *$M_H(v)$ is not empty for a general ample divisor H if and only if $\langle v^2\rangle \geq -2$.*

(2) *Assume that $\langle v^2\rangle \geq -2$. Then for a general ample divisor H,*

- (2-1) *$M_H(v)$ is obtained by compositions of deformations and birational transformations from $\operatorname{Hilb}_X^{\langle v^2\rangle/2+1}$. In particular $M_H(v)$ is an irreducible symplectic manifold.*
- (2-2) *Let $B_{M_H(v)}$ be Beauville's bilinear form on $H^2(M_H(v),\mathbb{Z})$. Then*
$$\theta_v : (v^\perp, \langle\ \ ,\ \ \rangle) \to (H^2(M_H(v),\mathbb{Z}), B_{M_H(v)})$$
is an isometry which preserves Hodge structures for $\langle v^2\rangle \geq 2$, where $\theta_v : v^\perp \to H^2(M_H(v),\mathbb{Z})$ is the canonical homomorphism defined by using a quasi-universal family.

Here we only use deformations of $M_H(v)$ induced by deformation of complex structures of X.

We shall explain how to prove (2-1) of this theorem. In [Y3], we discussed chamber structure of polarizations. Let $v = l(r+\xi) + a\omega$, $\xi \in \operatorname{NS}(X)$ be a Mukai vector of $l, r > 0$. We choose an ample divisor H on X which does not lie on walls with respect to v. Then for every μ-semi-stable sheaf E of $v(E) = v$, if $F \subset E$ satisfies $(c_1(F), H)/\operatorname{rk} F = (c_1(E), H)/\operatorname{rk} E$, then $c_1(F)/\operatorname{rk} F = c_1(E)/\operatorname{rk} E$. Thus $v(F) = l'(r+\xi) + a'\omega$ for some l', a'. In particular, if v is primitive, then $M_H(v)$ is compact. Let $\mathcal{M}(v)$ be the stack of coherent sheaves E of $v(E) = v$. We shall fix a general ample divisor H with respect to v. $\mathcal{M}(v)^{\mu ss}$ (resp. $\mathcal{M}(v)^{\mu s}$) denotes the open substack of $\mathcal{M}(v)$ consisting of μ-semi-stable sheaves (resp. μ-stable sheaves).

In [Y4], we proved Theorem 0.1 under the assumption $\langle v^2\rangle > 2l^2$ or $l = 1$. Hence we may assume that $\langle v^2\rangle \leq 2l^2$ and $l > 1$. Our proof in [Y4] is systematic applications of birational maps induced by reflections of Mukai lattices. Then Theorem 0.1 (2-2) automatically follows from these birational correspondences in many cases. In [Y4, Lem. 4.5], we proved the following key lemma under the assumption $lr < r_1 < (l+1)r$.

Lemma 0.2. *Let (X,H) be a polarized smooth projective surface of $\operatorname{NS}(X) = \mathbb{Z}H$. Let (r_1, d_1) and (r, d) be pairs of integers such that $r_1, r > 0$ and $dr_1 - rd_1 = 1$. We assume that $lr < r_1$. Let E_1 be a μ-stable vector bundle of $\operatorname{rk}(E_1) = r_1$ and $\deg(E_1) = d_1$, where $\deg(E_1) = (c_1(E_1), H)/(H^2)$.*

1991 *Mathematics Subject Classification.* 14D20.

(1) *Let E be a μ-stable sheaf of* $\operatorname{rk}(E) = lr$ *and* $\deg(E) = ld$. *Then every non-zero homomorphism* $\varphi : E_1 \to E$ *is surjective in codimension 1 and* $\ker\varphi$ *is a μ-stable sheaf.*

(2) *Let E' be a μ-stable vector bundle of* $\operatorname{rk}(E') = r_1 - lr$ *and* $\deg(E') = d_1 - ld$. *Let $\phi : E' \to E_1$ be a non-zero homomorphism. Then ϕ is injective and* $E := \operatorname{coker}\phi$ *is a μ-semi-stable sheaf.*

Recently we realized that the most important condition in Lemma 0.2 and [Y4, Lem. 2.1, Lem. 4.1] is the following fact:

- We consider the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(r_1 - lr, d_1 - ld)$ and (r_1, d_1). Then there is no integral point in the interior of this triangle.

This condition gives a strong restriction on homomorphisms φ, ϕ and the Harder-Narasimhan polygons of $\ker\varphi$ and $\operatorname{coker}\phi$. Hence we can expect to get rid of the additional condition $r_1 < (l+1)r$. This will be done in Preliminaries.

In order to use this lemma, we need to compare $\mathcal{M}(v)^{\mu ss}$ and $\mathcal{M}(v)^{\mu s}$. More precisely, we need some propositions (cf. [Y4, Lem.4.4]) on the dimension counting of various constructible substack of $\mathcal{M}(v)^{\mu ss}$. This is the most important part in this paper. In [D-L], Drezet and Le Potier computed the dimension of the substack of non-semi-stable sheaves. In their computation, the existence of exceptional vector bundle is very important. In our case, we concentrate our consideration on $\mathcal{M}(v)^{\mu ss}$. By our assumption on H, exceptional vector bundle E of $v(E) = r + \xi + b\omega$, $b \in \mathbb{Z}$ is important. Hence we divide our proof into two cases:

A. There is no (-2) vector of the form $r + \xi + b\omega$, i.e. $((\xi^2)+2)/2r \notin \mathbb{Z}$.
B. There is a (-2) vector of the form $r + \xi + b\omega$, i.e. $((\xi^2)+2)/2r \in \mathbb{Z}$.

In section 2, we treat case A. In particular, we prove the following

$$\dim(\mathcal{M}(v)^{\mu ss} \setminus \mathcal{M}(v)^{\mu s}) \leq \langle v^2\rangle. \tag{0.1}$$

Then we can apply Lemma 0.2. For suitable choice of

1. a primitive Mukai vector $v := l(r + dH) + a\omega$ on (X, H) and
2. an exceptional vector bundle E_1 of $v(E_1) = r_1 + d_1 H + a_1\omega$,

we can construct a birational map $M_H(v) \cdots \to M_H(v')$ sending a general μ-stable vector bundle $E \in M_H(v)$ to $F := \operatorname{coker}(E^\vee \to \operatorname{Hom}(E_1, E)^\vee \otimes E_1^\vee) \in M_H(v')$ where H and v' satisfy that (1) $(H^2)/2 = (r_1\langle v^2\rangle/2l + r)r_1/l - r^2 > 0$ and (2) Theorem 0.1 holds for $M_H(v')$. We remark that we need to choose *a sufficiently large* r_1 for the condition (1). In the same way as in [Y4, 4.3], we get Theorem 0.1 for case A.

In section 3, we treat case B. If $\langle v^2\rangle < 2l^2$, then it is known that there is no μ-stable sheaf. Hence we can not apply Lemma 0.2 in this form. When the (-2) vector is $v(\mathcal{O}_X) = 1 + \omega$, T. Nakashima found the following fact:

We set $v = l - a\omega$. Then the inequality $0 \leq \langle v^2\rangle < 2l^2$ implies that $0 \leq a < l$. We assume that $a \geq 2$. Let E be a μ-stable vector bundle of $v(E) = a - l\omega$. Then $H^0(X, E) = H^2(X, E) = 0$ and $\dim H^1(X, E) = l - a$. We consider the universal extension

$$0 \to E \to E' \to H^1(X, E) \otimes \mathcal{O}_X \to 0. \tag{0.2}$$

It is easy to see that E' is a stable vector bundle and we get an immersion $M_H(a - l\omega)^{\mu s, loc} \hookrightarrow M_H(v)$, where $M_H(a - l\omega)^{\mu s, loc}$ is the open subscheme of $M_H(a - l\omega)$ consisting of μ-stable vector bundles.

This result can be easily extended to general cases. Hence what we should do is to prove the irreducibility of $M_H(v)$ and the classification of $M_H(v)$ consisting of non-locally free sheaves. By similar dimension counting as in case A, we shall classify non-locally free components and prove the irreducibility of $M_H(v)$.

1. Preliminaries

1.1. **Notation.** Except 1.4.1 and 1.4.2, we assume that X is a K3 surface. $\mathcal{M}(v)^{ss}$ and $\mathcal{M}(v)^{s}$ denote the open substack of $\mathcal{M}(v)$ consisting of semi-stable sheaves and stable sheaves respectively. Since $\mathcal{M}(v)^{\mu ss}$ is bounded, it is a quotient stack of an open subscheme of some quot-scheme by some general linear group (see Appendix). Hence our dimension counting of substack of $\mathcal{M}(v)^{\mu ss}$ can be regarded as that of subscheme of some quot scheme. $M_H(v)^{\mu s}$ (resp. $M_H(v)^{loc}$) be the open subscheme of $M_H(v)$ consisting of μ-stable sheaves (resp. stable vector bundles).

For a (-2) vector v_0, $R_{v_0} : H^*(X,\mathbb{Z}) \to H^*(X,\mathbb{Z})$ is the reflection defined by v_0, that is, $R_{v_0}(x) = x + \langle v_0, x\rangle v_0, x \in H^*(X,\mathbb{Z})$. Under suitable conditions (cf. [Y4]), R_{v_0} induces a birational map $M_H(v) \cdots \to M_H(\pm R_{v_0}(v))$.

1.2. Estimate on properly semi-stable sheaves.

Lemma 1.1. (1) *Let E be a stable sheaf and F a semi-stable sheaf such that* $v(F)/\operatorname{rk} F = v(E)/\operatorname{rk} E$. *Then* $\operatorname{Hom}(E, F) \otimes E \to F$ *is injective. In particular* $\dim \operatorname{Hom}(E, F) \leq \operatorname{rk} F/\operatorname{rk} E$.

(2) *Let E be a μ-stable sheaf and F a μ-semi-stable sheaf such that* $c_1(F)/\operatorname{rk} F = c_1(E)/\operatorname{rk} E$. *Then* $\operatorname{Hom}(E,F) \otimes E \to F$ *is injective. In particular* $\dim \operatorname{Hom}(E,F) \le \operatorname{rk} F/\operatorname{rk} E$.

Lemma 1.2. *Let v be a Mukai vector of $\langle v^2 \rangle > 0$ (we don't assume the primitivity of v). We set*

$$\mathcal{M}(v)^{pss} := \{E \in \mathcal{M}(v)^{ss} | E \text{ is properly semi-stable } \}. \tag{1.1}$$

Then $\dim \mathcal{M}(v)^{pss} \le \langle v^2 \rangle$. *In particular, if $\mathcal{M}(v)^{ss} \neq \emptyset$, then $\mathcal{M}(v)^s \neq \emptyset$ and* $\dim \mathcal{M}(v)^{ss} = \langle v^2 \rangle + 1$.

Proof. We set $v = lv'$, where v' is a primitive Mukai vector. We shall prove this lemma by induction on l. Let E_1 be a stable sheaf of $v(E_1) = l_1 v'$ and E_2 a semi-stable sheaf of $v(E_2) = l_2 v'$, where $l_1 + l_2 = l$. By induction hypothesis, $\dim \mathcal{M}^{ss}(v_i) = \langle v_i^2 \rangle + 1$, $i = 1, 2$. We shall estimate the dimension of the substack $J(v_1, v_2)$ whose element E fits in an extension

$$0 \to E_1 \to E \to E_2 \to 0. \tag{1.2}$$

By Lemma 1.1, $\dim \operatorname{Ext}^2(E_2, E_1) = \dim \operatorname{Hom}(E_1, E_2) \le l_2/l_1$. Moreover if E_1 is general, then $\operatorname{Hom}(E_1, E_2) = 0$. Hence by Lemma 4.1 in Appendix, we get

$$\begin{aligned} \dim J(v_1, v_2) &\le \dim \mathcal{M}(v_1)^{ss} + \dim \mathcal{M}(v_2)^{ss} + \langle v_1, v_2 \rangle + \max\{l_2/l_1 - 1, 0\} \\ &= \langle v_1^2 \rangle + \langle v_2^2 \rangle + 2 + \langle v_1, v_2 \rangle + \max\{l_2/l_1 - 1, 0\} \\ &= (\langle v^2 \rangle + 1) - (\langle v_1, v_2 \rangle - \max\{l_2/l_1, 1\}). \end{aligned} \tag{1.3}$$

Since

$$\langle v_1, v_2 \rangle = l_1 \frac{\langle v_2^2 \rangle}{2l_2} + l_2 \frac{\langle v_1^2 \rangle}{2l_1} \ge l, \tag{1.4}$$

we get $\dim J(v_1, v_2) \le \langle v^2 \rangle$. Therefore we get our lemma. $\square$

1.3. Semi-stable sheaves of isotropic Mukai vector.

Lemma 1.3. *Let w be a primitive isotropic Mukai vector of* $\operatorname{rk} w > 0$. *Then* $\dim \mathcal{M}(lw)^{ss} = l$.

Proof. Let E be a semi-stable sheaf of $v(E) = lw$. We shall first prove that there are stable sheaves $E_1, E_2, \dots, E_k$ of $v(E_i) = w$ such that

$$E \cong \oplus_{i=1}^k F_i, \tag{1.5}$$

where F_i are S-equivalent to $E_i^{\oplus n_i}$.

Proof of the claim: By the proof of Mukai [Mu2, Prop. 4.4] (Fourier-Mukai transform for $H^*(X, \mathbb{Q})$), there is an element E_1 of $M_H(w)$ such that $\operatorname{Hom}(E_1, E) \neq 0$. By induction hypothesis, $E/E_1 \cong F_1 \oplus F_2 \oplus \cdots \oplus F_k$, where F_1 is S-equivalent to $E_1^{\oplus n_1}$, $n_1 \ge 0$ and F_i, $i > 1$ are S-equivalent to $E_i^{\oplus n_i}$. Since $\operatorname{Ext}^1(E_i, E_1) = 0$ for $i > 1$, $\operatorname{Ext}^1(\oplus_{i>1} F_i, E_1) = 0$. Therefore $\oplus_{i>1} F_i$ is a direct summand of E, which implies our claim.

Let E_1 be a stable sheaf of $v(E_1) = w$. We set

$$\mathcal{J}(l, E_1) := \{E \in \mathcal{M}(lw)^{ss} | E \text{ is } S\text{-equivalent to } E_1^{\oplus l}\}. \tag{1.6}$$

This is a closed substack of $\mathcal{M}(lw)^{ss}$ (see Appendix 4.3). We shall next prove that

$$\dim \mathcal{J}(l, E_1) \le -1. \tag{1.7}$$

Since E_1 is parametrized by the surface $M_H(w)$, (1.7) implies that

$$\dim \mathcal{M}(lw)^{ss} = l. \tag{1.8}$$

For more details, see Appendix 4.3. Proof of (1.7): We set

$$\mathcal{J}(l, E_1, n) := \{E \in \mathcal{J}(l, E_1) | \dim \operatorname{Hom}(E_1, E) = n\}. \tag{1.9}$$

By upper semi-continuity of cohomologies, this is a locally closed substack of $\mathcal{J}(l, E_1)$. If $n = l$, then $\mathcal{J}(l, E_1, l) = \{E_1^{\oplus l}\}$ and it is a closed substack of $\mathcal{J}(l, E_1)$. For an element E of $\mathcal{J}(l, E_1, n)$, there is an exact sequence

$$0 \to \operatorname{Hom}(E_1, E) \otimes E_1 \to E \to E' \to 0 \tag{1.10}$$

where $E' \in \mathcal{J}(l-n, E_1, n')$. The moduli number of E which fits in this type of extension is equal to $\dim \mathcal{J}(l-n, E_1, n') + nn' - n^2$. Indeed, $\operatorname{Hom}(E_1, E) \otimes E_1 (\cong E_1^{\oplus n})$ belongs to $\mathcal{J}(n, E_1, n)$ and $\dim \mathcal{J}(n, E_1, n) = -n^2$. Hence by the proof of Lemma 4.1, we get the equality. Therefore we see that

$$\dim \mathcal{J}(l, E_1) = - \min_{\substack{l = n_1 + n_2 + \cdots + n_s \\ n_1, n_2, \dots, n_s \ge 1}} \left(\sum_{i=1}^s n_i^2 - \sum_{i=1}^{s-1} n_i n_{i+1} \right) \le -1. \tag{1.11}$$

$\square$

Remark 1.1. If $M_H(w)$ has a universal family, then Fourier-Mukai transform is defined [Br]. Then $\mathcal{M}(lw)^{ss}$ is transformed to the stack of 0-dimensional sheaves on $M_H(w)$. In this case, by using [Y1, Thm. 0.4], we can get our lemma.

1.4. **Lemma 0.2 and its extensions.**

1.4.1. *Proof of Lemma 0.2.* In the proof of [Y4, Lem. 4.5], we shall use the inequality $d/r > d_1/r_1$. Then, in the proof of (1), we see that

$$\frac{1}{r(r_1-lr)} > \frac{d}{r} - \frac{\deg(I)}{\operatorname{rk}(I)} \geq \frac{1}{r\operatorname{rk}(I)}. \tag{1.12}$$

Hence $\operatorname{rk}(I) > r_1 - lr$, which is a contradiction. The other arguments are the same. For the proof of (2), we see that

$$\frac{1}{r(r_1-lr)} > \frac{d}{r} - \frac{\deg(\phi(E'))}{\operatorname{rk}(\phi(E'))} \geq \frac{1}{r\operatorname{rk}(\phi(E'))}. \tag{1.13}$$

Hence $\operatorname{rk}(\phi(E')) > r_1 - lr$, which is a contradiction. The other arguments are the same. □

Remark 1.2. In order to define $\deg(E)$ in Lemma 0.2, we assumed that $\operatorname{NS}(X) = \mathbb{Z}H$. However $\deg(E)$ is still defined if $(H^2)|(H,D)$ for all $D \in \operatorname{NS}(X)$. Hence Lemma 0.2 also holds under this assumption.

1.4.2. *Extensions of Lemma 0.2 and [Y4, Lem. 4.1].* We shall extend Lemma 0.2 and [Y4, Lem. 4.1]. Let (X,H) be a polarized smooth projective surface of $\operatorname{NS}(X) = \mathbb{Z}H$.

Lemma 1.4. *Let E be a μ-semi-stable vector bundle of $\operatorname{rk}(E) = lr$ and $\deg(E) = ld$ which is defined by a non-trivial extension*

$$0 \to F_1 \to E \to F_2 \to 0, \tag{1.14}$$

where F_1 and F_2 are μ-stable vector bundles of $\deg(F_1)/\operatorname{rk}(F_1) = \deg(F_2)/\operatorname{rk}(F_2) = d/r$. Let E_1 be a μ-stable vector bundle in Lemma 0.2. Let $\varphi : E_1 \to E$ be a non-trivial homomorphism. Then φ is surjective in codimension 1, or $\varphi(E_1)$ is a subsheaf of E_1. In particular, if $\operatorname{Hom}(E_1, F_1) = 0$, then $\ker\varphi$ is μ-stable.

Proof. We assume that φ is not surjective in codimension 1. By [Y4, Lem. 5.1], $\deg(\varphi(E_1))/\operatorname{rk}(\varphi(E_1)) = d/r$. Assume that $\varphi(E_1) \to F_2$ is not 0. Then the μ-stability of F_2 implies that it is surjective in codimension 1. By the μ-stability of F_1, $F_1 \cap \varphi(E_1) = 0$. Since F_1 is locally free, the proof of [Y4, Lem. 2.1] implies that $E \cong F_1 \oplus F_2$, which is a contradiction. Hence $\varphi(E_1) \to F_2$ is a 0-map, which means that $\varphi(E_1) \subset F_1$. The last assertion follows from the proof of Lemma 0.2. □

Lemma 1.5. *Keep the notations in Lemma 1.4. Assume that $\operatorname{Ext}^1(E_1, F_2) = 0$. Then a non-trivial extension of E by E_1 is μ-stable.*

Proof. Let E' be a non-trivial extension of E by E_1.

$$0 \to E_1 \to E' \to E \to 0. \tag{1.15}$$

We shall prove that E' is μ-stable. We consider the following diagram which is induced by the extension (1.15).

$$\begin{tikzcd}
 & & 0 & 0 & \\
 & & F_2 \arrow[u] \arrow[r, equal] & F_2 \arrow[u] & \\
0 \arrow[r] & E_1 \arrow[r] & E' \arrow[u] \arrow[r] & E \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & E_1 \arrow[u, equal] \arrow[r] & E'' \arrow[u] \arrow[r] & F_1 \arrow[u] \arrow[r] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{1.16}$$

By Serre duality, $\operatorname{Ext}^1(F_2, E_1) = 0$. Hence $\operatorname{Ext}^1(E, E_1) \to \operatorname{Ext}^1(F_1, E_1)$ is injective. Thus the last horizontal sequence of (1.16) does not split. By [Y4, Lem. 4.1], E'' is μ-stable. If the middle vertical sequence splits, then $E \cong F_1 \oplus F_2$, which is a contradiction. Hence E' is a non-trivial extension of F_2 by a locally free sheaf E''. By the construction of E'', the conditions in [Y4, Lem. 4.1] hold. Hence applying [Y4, Lem. 4.1] again, we see that E' is μ-stable. □

1.4.3. *Extension of [Y4, Lem. 4.2].* Let (X,H) be a polarized K3 surface of $\operatorname{Pic}(X)=\mathbb{Z}H$. Let E_1 be an exceptional vector bundle of $v(E_1):=r_1+d_1H+a_1\omega$ and let $v=l(r+dH)+a\omega$ be a primitive Mukai vector of $dr_1-d_1r=1$. We set

$$M_H(v)_i^{\mu s}:=\{E\in M_H(v)^{\mu s}\mid \dim\operatorname{Hom}(E_1,E)=i-\langle v,v(E_1)\rangle\}. \tag{1.17}$$

Assume that $r_1>lr$. In the same way as in [Y4, Lem. 4.2], we see that

Lemma 1.6. (1) *If $\langle v,v(E_1)\rangle<0$, then*

$$\operatorname{codim}_{M_H(v)^{\mu s}} M_H(v)_i^{\mu s}\geq -\langle v,v(E_1)\rangle+1\geq 2 \tag{1.18}$$

for $i\geq 1$.

(2) *If $\langle v,v(E_1)\rangle\geq 0$, then*

$$\operatorname{codim}_{M_H(v)^{\mu s,loc}}(M_H(v)_i^{\mu s}\cap M_H(v)^{\mu s,loc})\geq \langle v,v(E_1)\rangle+1\geq 1 \tag{1.19}$$

for $i>\langle v,v(E_1)\rangle$, where $M_H(v)^{\mu s,loc}=M_H(v)^{\mu s}\cap M_H(v)^{loc}$.

Proof. We only prove the second claim. Hence we assume that $i>\langle v,v(E_1)\rangle$. For $E\in M_H(v)_i^{\mu s}\cap M_H(v)^{\mu s,loc}$, we choose a homomorphism $\phi:E_1\to E$. By Lemma 0.2, ϕ is surjective in codimension 1. We set $G:=\operatorname{coker}(E^\vee\to E_1^\vee)$. By Lemma 0.2, F is a stable sheaf of $v(G)=w:=v(E_1)^\vee-v^\vee$. It is easy to see that $\dim\operatorname{Hom}(E_1^\vee,G)=\dim\operatorname{Ext}^1(E_1,E)+1=\langle v,v(E_1)\rangle+i+1$. Hence $\phi:E_1\to E$ is parametrized by an open subscheme of a projective bundle of dimension $(\langle v,v(E_1)\rangle+i)$ over the subscheme $M_H(w)_i$, where

$$\begin{aligned}M_H(w)_i:&=\{G\in M_H(w)\mid \dim\operatorname{Hom}(E_1^\vee,G)=\langle v,v(E_1)\rangle+i+1\}\\&=\{G\in M_H(w)\mid \dim\operatorname{Hom}(E_1^\vee,G)=\langle w,v(E_1)^\vee\rangle+(i-1)\}.\end{aligned} \tag{1.20}$$

Thus we see that

$$\begin{aligned}\dim M_H(v)_i^{\mu s}\cap M_H(v)^{\mu s,loc}&\leq\langle w^2\rangle+2+(\langle v,v(E_1)\rangle+i)-(i-1)\\&=\langle v^2\rangle+2-(\langle v,v(E_1)\rangle+1).\end{aligned} \tag{1.21}$$

□

2. Case A

2.1. **Estimate.** In this section, we fix a primitive Mukai vector $r+\xi,\xi\in\operatorname{NS}(X)$. We assume that

$$((\xi^2)+2)/2r\notin\mathbb{Z}. \tag{2.1}$$

We shall prove Theorem 0.1 for a primitive Mukai vector $v:=l(r+\xi)+a\omega\in H^*(X,\mathbb{Z})$. We shall first estimate the dimensions of various locally closed substacks of $\mathcal{M}(v)$.

Lemma 2.1. *If $\mathcal{M}(v)^{\mu ss}\neq\emptyset$, then $\langle v^2\rangle\geq 0$. If the equality holds, then $\mathcal{M}(v)^{\mu ss}=\mathcal{M}(v)^{ss}$.*

Proof. Let E be a μ-semi-stable sheaf of $v(E)=v$ and E is S-equivalent to $\oplus_{i=1}^s E_i$ with respect to μ-stability, where $E_i, 1\leq i\leq s$ are μ-stable sheaves. We set

$$v(E_i):=l_i(r+\xi)+a_i\omega, 1\leq i\leq s. \tag{2.2}$$

By our assumption (2.1), $\langle v(E_i)^2\rangle=l_i(l_i(\xi^2)-2ra_i)\neq -2$. Thus $\langle v(E_i)^2\rangle\geq 0$ for all i. Since

$$\frac{\langle v^2\rangle}{l}=\sum_{i=1}^s\frac{\langle v(E_i)^2\rangle}{l_i}, \tag{2.3}$$

we get $\langle v^2\rangle\geq 0$. If $\langle v^2\rangle=0$, then $\langle v(E_i)^2\rangle=0$ for all i. Since $\langle v(E_i)^2\rangle/\operatorname{rk}(E_i)^2=(\xi^2)-2a_i/rl_i$ and $\chi(E_i)/\operatorname{rk}(E_i)=1+a_i/rl_i$, we see that $\chi(E_i)/\operatorname{rk}(E_i)=\chi(E)/\operatorname{rk}(E)$ for all i. Thus E is semi-stable. □

Corollary 2.2. *If $\langle v^2\rangle=0$, then $\mathcal{M}(v)^{\mu ss}$ consists of locally free sheaves.*

Definition 2.1. Let $w=l_0(r+\xi)+a_0\omega$ be the primitive Mukai vector such that $\langle w^2\rangle=0$.

By Lemma 2.1 and Corollary 2.2, $M_H(w)$ consists of μ-stable locally free sheaves.

Lemma 2.3. (1)

$$\dim(\mathcal{M}(v)^{\mu ss}\setminus\mathcal{M}(v)^{ss})\leq\langle v^2\rangle. \tag{2.4}$$

(2) *Assume that $\langle v^2\rangle>0$. Then*

$$\dim(\mathcal{M}(v)^{\mu ss}\setminus\mathcal{M}(v)^{s})\leq\langle v^2\rangle. \tag{2.5}$$

In particular, if $\mathcal{M}(v)^{\mu ss}\neq\emptyset$, then $\mathcal{M}(v)^s\neq\emptyset$ and $\dim\mathcal{M}(v)^{\mu ss}=\langle v^2\rangle+1$.

Proof. By Lemma 1.2, it is sufficient to prove (1). Let F be a μ-semi-stable sheaf of $v(F)=v$. We assume that F is not stable. Let

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = F \tag{2.6}$$

be the Harder-Narasimhan filtration of F. We set

$$v_i := v(F_i/F_{i-1}) = l_i(r+\xi) + a_i\omega, 1 \le i \le s. \tag{2.7}$$

Since $\chi(F_i/F_{i-1})/\operatorname{rk}(F_i/F_{i-1}) > \chi(F_{i+1}/F_i)/\operatorname{rk}(F_{i+1}/F_i)$, we get that

$$\frac{a_0}{l_0} \ge \frac{a_1}{l_1} > \frac{a_2}{l_2} > \cdots > \frac{a_s}{l_s}. \tag{2.8}$$

Let $\mathcal{F}^{HN}(v_1, v_2, \dots, v_s)$ be the substack of $\mathcal{M}(v)^{\mu ss}$ whose element E has the Harder-Narasimhan filtration of the above type. We shall prove that $\dim \mathcal{F}^{HN}(v_1, v_2, \dots, v_s) \le \langle v^2 \rangle$. Since $\operatorname{Hom}(F_i/F_{i-1}, F_j/F_{j-1}) = 0$ for $i<j$, Lemma 4.2 in Appendix implies that

$$\dim \mathcal{F}^{HN}(v_1, v_2, \dots, v_s) = \sum_{i=1}^{s} \dim \mathcal{M}(v_i)^{ss} + \sum_{i<j} \langle v_j, v_i \rangle. \tag{2.9}$$

For $i<j$, by using Lemma 2.1 and (2.8), we see that

$$\begin{aligned} \langle v_i, v_j \rangle &= l_i l_j (\xi^2) - (l_i a_j + l_j a_i) r \\ &= l_i l_j (\xi^2) - 2 l_j a_i r + (a_i l_j - a_j l_i) r \\ &= l_j (l_i (\xi^2) - 2 a_i r) + (a_i l_j - a_j l_i) r \\ &\ge (a_i l_j - a_j l_i) r \ge r \ge 2, \end{aligned} \tag{2.10}$$

where the inequality $r \ge 2$ comes from our assumption (2.1). Hence if $\langle v_i^2 \rangle > 0$ for all i, then, by using Lemma 1.2, we see that

$$\dim \mathcal{F}^{HN}(v_1, v_2, \dots, v_s) \le (\langle v^2 \rangle + 1) - \left(\sum_{i<j} \langle v_i, v_j \rangle - s + 1 \right) \le \langle v^2 \rangle. \tag{2.11}$$

Assume that $\langle v_i^2 \rangle = 0$, i.e. $v_i = l_i' w$, $l_i \in \mathbb{Z}$. Then $i=1$ and $(a_1 l_j - a_j l_1)$ is divisible by l_i'. Hence

$$\begin{aligned} \langle v_1, v_j \rangle - l_1' &= l_1'(\langle w, v_j \rangle - 1) \\ &\ge l_1'(r-1) > 0. \end{aligned} \tag{2.12}$$

In this case, by using Lemma 1.3, we see that

$$\dim \mathcal{F}^{HN}(v_1, v_2, \dots, v_s) \le (\langle v^2 \rangle + 1) - \left(\sum_{i<j} \langle v_i, v_j \rangle - (l_1' + s - 1) + 1 \right) \le \langle v^2 \rangle. \tag{2.13}$$

Hence we get our lemma. □

Proposition 2.4. *Assume that* $\langle v^2 \rangle > 0$. *Then*

$$\dim(\mathcal{M}(v)^s \setminus \mathcal{M}(v)^{\mu s}) \le \langle v^2 \rangle. \tag{2.14}$$

In particular, $\mathcal{M}(v)^{\mu s} \ne \emptyset$, *if* $\mathcal{M}(v)^s \ne \emptyset$.

Proof. Let E be a stable sheaf and E_1 be a μ-stable subsheaf of E such that E/E_1 is torsion free. We set

$$\begin{aligned} v_1 &:= v(E_1) = l_1(r+\xi) + a_1\omega, \\ v_2 &:= v(E/E_1) = l_2(r+\xi) + a_2\omega. \end{aligned} \tag{2.15}$$

Since $\chi(E_1)/\operatorname{rk} E_1 < \chi(E)/\operatorname{rk} E$, we get $\langle v(E_1)^2 \rangle > 0$ and

$$\frac{a_1}{l_1} < \frac{a_2}{l_2}. \tag{2.16}$$

Let $J(v_1, v_2)$ be the substack of $\mathcal{M}(v)^s$ consisting of E which has a subsheaf $F_1 \subset E$. By using Lemma 4.1 in Appendix, we shall estimate $\dim J(v_1, v_2)$. By Lemma 1.1, $\dim \operatorname{Hom}(E_1, E/E_1) \le l_2/l_1$, and if E_1 is general, then $\operatorname{Hom}(E_1, E/E_1) = 0$. We shall bound the dimension of the substack

$$\mathcal{N}(v_1, v_2) := \{(E_1, E_2) \in \mathcal{M}(v_1)^{\mu ss} \times \mathcal{M}(v_2)^{\mu ss} \mid \dim \operatorname{Hom}(E_1, E_2) \ne 0\}. \tag{2.17}$$

For a fixed $E_2 \in \mathcal{M}(v_2)^{ss}$,

$$\#\{E_1^{\vee\vee} \mid E_1 \in \mathcal{M}(v_1)^{\mu s}, \operatorname{Hom}(E_1, E_2) \ne 0\} < \infty. \tag{2.18}$$

Hence, by using [Y1, Thm. 0.4], we see that

$$\dim\{E_1 \in \mathcal{M}(v_1)^{\mu s} \mid \operatorname{Hom}(E_1, E_2) \ne 0\} \le \dim \mathcal{M}(v_1)^{\mu s} - 2 - (\operatorname{rk} v_1 - 1). \tag{2.19}$$

Thus $\dim \mathcal{N}(v_1, v_2) \leq \dim \mathcal{M}(v_1)^{\mu ss} + \dim \mathcal{M}(v_2)^{\mu ss} - 3$. Moreover, taking (1.5) into account, if $l_1 \neq l_0$ and $v_2 = l_2' w, l_2' \in \mathbb{Z}$, then we get $\mathcal{N}(v_1, v_2) = \emptyset$.

If $\langle v_2^2 \rangle > 0$, then Lemma 2.3 implies that $\dim \mathcal{M}(v_2)^{\mu ss} = \langle v_2^2 \rangle + 1$. Hence Lemma 4.1 implies that

$$\begin{aligned} \dim \mathcal{M}(v)^s - \dim J(v_1, v_2) &= \min \left(\langle v_1, v_2 \rangle - \frac{l_2}{l_1} + 2, \langle v_1, v_2 \rangle - 1 \right) \\ &= l_1 \frac{\langle v_2^2 \rangle}{2 l_2} + l_2 \frac{\langle v_1^2 \rangle}{2 l_1} - \max \left(\frac{l_2}{l_1} - 2, 1 \right) > 0. \end{aligned} \tag{2.20}$$

We next treat the case where $\langle v_2^2 \rangle = 0$. Then $v_2 = l_2' w, l_2' \in \mathbb{Z}$. By Lemma 2.3 (1) and Lemma 1.3, $\dim \mathcal{M}(v_2)^{\mu ss} = \langle v_2^2 \rangle + l_2'$. If $l_1 = l_0$, then $l_2/l_1 = l_2'$. So we see that

$$\begin{aligned} \dim \mathcal{M}(v)^s - \dim J(v_1, v_2) &= \min \left(l_2'(\langle v_1, w \rangle - 1 - 1) + 3, l_2'(\langle v_1, w \rangle - 1) \right) \\ &= \min \{ l_2'((a_0 l_1 - a_1 l_0) r - 2) + 3, l_2'((a_0 l_1 - a_1 l_0) r - 1) \} > 0. \end{aligned} \tag{2.21}$$

□

Remark 2.1. By the proof of Lemma 2.3 and Proposition 2.4, we see that

$$\operatorname{codim}_{\mathcal{M}(v)^{\mu ss}} (\mathcal{M}(v)^{\mu ss} \setminus \mathcal{M}(v)^{\mu s}) \geq 2 \text{ for } r \geq 3. \tag{2.22}$$

Moreover if $r = 2$, then the general member E of $\mathcal{M}(v)^s \setminus \mathcal{M}(v)^{\mu s}$ fits in the following exact sequence

$$0 \to E_1 \to E \to E_2 \to 0 \tag{2.23}$$

where E_1 is a μ-stable vector bundle and E_2 is a μ-stable vector bundle of $v(E_2) = w$.

Indeed, if $\operatorname{codim}_{\mathcal{M}(v)^s} (\mathcal{M}(v)^s \setminus \mathcal{M}(v)^{\mu s}) = 1$, then (1) $\langle v_2^2 \rangle > 0$, $l_1 = l_2 = \langle v_1^2 \rangle / 2 = \langle v_2^2 \rangle / 2 = 1$, or (2) $r = 2$, $l_2' = 1$ and $a_0 l_1 - a_1 l_0 = 1$. By (2.16), case (1) does not occur.

2.2. **Proof of Theorem 0.1 for case A.**

2.2.1. *The case of* $r > 2$. In the same way as in the proof of [Y4, Thm. 0.1], we shall prove Theorem 0.1 if $r > 2$. For more details, see [Y4, sect. 4]. Let $v = l(r + \xi) + a\omega$ be a primitive Mukai vector on a K3 surface X such that $r + \xi, \xi \in \operatorname{NS}(X)$ is primitive. Then we can find a primitive Mukai vector $v' = l(r + d'H') + a'\omega$ on a polarized K3 surface (X', H') of $\operatorname{Pic}(X') = \mathbb{Z}H'$ and an exceptional vector bundle G of $v(G) := v_1 = r_1 + d_1 H' + a_1 \omega'$ such that (1) $r + d'H'$ is primitive, (2) $\langle v^2 \rangle = \langle (v')^2 \rangle$, (3) $a \equiv a' \mod l$, (4) $d' r_1 - d_1 r = 1$ and (5) $\langle v(G), v' \rangle = -1$, where ω' is the fundamental class of X'. Then we get

$$-R_{v_1}(v')^\vee = (r_1 - lr) - (d_1 - ld')H' + (a_1 - a')\omega'. \tag{2.24}$$

We may assume that $r_1 - lr > 0$. Since $r_1 - lr$ and $d_1 - ld'$ are relatively prime, [Y4, Thm. 0.1] implies that $M_{H'}(-R_{v_1}(v')^\vee) \neq \emptyset$ and Theorem 0.1 holds for this space. By using Lemma 0.2 (2), we see that $\mathcal{M}(v')^{\mu ss} \neq \emptyset$. Then Lemma 2.3 and Proposition 2.4 imply that $M_{H'}(v')^{\mu s} \neq \emptyset$. Applying Lemma 0.2 again, we get a birational map

$$M_{H'}(v') \cdots \to M_{H'}(-R_{v_1}(v')^\vee), \tag{2.25}$$

which means that Theorem 0.1 (1) and (2-1) hold for $M_{H'}(v')$. Then [Y4, Prop. 1.1] implies that Theorem 0.1 (1) and (2-1) also hold for $M_H(v)$, where H is a general ample divisor on X. Moreover by Remark 2.1, (2-2) also holds if $r > 2$. □

2.2.2. *The case of* $r = 2$. We next treat the case where $r = 2$. It is sufficient to extend the birational map (2.25) to a general member $E \in M_{H'}(v')$ which fits in (2.23). Let G be the exceptional vector bundle in 2.2.1. By using Lemma 1.4 and 1.5, we can prove the following: Assume that $\langle v', v(G) \rangle = -1$. Then, for a general member E which fits in the exact sequence (2.23),

(1) $\operatorname{Ext}^1(G, E) = 0$,
(2) $\phi : \operatorname{Hom}(G, E) \otimes G \to E$ is surjective in codimension 1 and $\ker \phi$ is stable.

Proof. Since $\langle (v')^2 \rangle > 0$, we can write $v' = xw - y\omega$, where $x, y \in \mathbb{Q}$ and $x, y > 0$. Since $\langle v', v(G) \rangle = x\langle w, v(G) \rangle + y \operatorname{rk}(G) = -1$, $\langle w, v(G) \rangle < 0$. Hence $\langle v(E_1), v(G) \rangle \geq 0$ and $\langle v(E_2), v(G) \rangle < 0$. Applying Lemma 1.6, we may assume that $\operatorname{Hom}(G, E_1) = \operatorname{Ext}^1(G, E_2) = 0$. By Lemma 1.5, we can use the same argument as in the proof of [Y4, Lem. 4.2]. Hence (1) holds. Applying Lemma 1.4, we get (2). □

Therefore the dual of $\phi : E^\vee \to G^\vee$ is injective and the cokernel is stable. Thus the reflection induces a birational map $M_H(v) \setminus Z \to M_H(v(G)^\vee - v^\vee)$ such that $\operatorname{codim}_{M_H(v)} Z \geq 2$. Therefore Theorem 0.1 also holds for this case. □

2.2.3. *Some remarks.* Under the assumptions for case A, the following holds.

(1) $M_H(v) \neq \emptyset$ if and only if $\langle v^2 \rangle \geq 0$.
(2) $M_H(v)^{\mu s} \neq \emptyset$ if and only if $\langle v^2 \rangle \geq 0$.

In particular, $M_H(v)$ contains μ-stable vector bundles if $M_H(v) \neq \emptyset$.

3. Case B

3.1. **The case of $\langle v^2 \rangle < 2l^2$.** In this section, we assume that there is a (-2) vector v_0 of the form $v_0 = r + \xi + b\omega, b \in \mathbb{Z}$. Let E_0 be the element of $M_H(v_0)$. We shall prove Theorem 0.1 for a primitive Mukai vector $v := lv_0 - a\omega$. By [Y4, Thm. 0.1], we may assume that $\langle v^2 \rangle \leq 2l^2$. We first treat the case where $\langle v^2 \rangle < 2l^2$. Clearly $\langle v^2 \rangle \geq -2$, if $M_H(v) \neq \emptyset$. If $\langle v^2 \rangle = -2$, then $2l(a \operatorname{rk} v_0 - l) = -2$. Hence we get $l = 1$, which was treated in [Y4]. So we assume that $\langle v^2 \rangle \geq 0$, that is

$$l \leq a \operatorname{rk} v_0 < 2l. \tag{3.1}$$

Based on the next key lemma, we shall prove Theorem 0.1 in 3.1.2 and 3.1.3.

Lemma 3.1. $\operatorname{codim}_{M_H(v)}(M_H(v) \setminus M_H(v)^{loc}) \geq 2$ *unless*

$$v = \begin{cases} (\operatorname{rk} v_0)v_0 - \omega, \\ lv_0 - (l+1)\omega, \ \operatorname{rk} v_0 = 1. \end{cases} \tag{3.2}$$

3.1.1. *Proof of Lemma 3.1.* We set

$$\mathcal{M}(v,n)^s := \{E \in \mathcal{M}(v)^s | \dim(E^{\vee\vee}/E) = n\}. \tag{3.3}$$

For $E \in \mathcal{M}(v,n)^s$, let

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E^{\vee\vee} \tag{3.4}$$

be the Harder-Narasimhan filtration of $E^{\vee\vee}$. We set

$$v_i := v(F_i/F_{i-1}) = l_i v_0 - a_i \omega, 1 \leq i \leq s. \tag{3.5}$$

We shall estimate the codimension of the substack

$$\mathcal{M}(v,n;v_1,v_2,\dots,v_s) := \{E \in \mathcal{M}(v,n)^s | E^{\vee\vee} \in \mathcal{F}^{HN}(v_1,v_2,\dots,v_s)\}, \tag{3.6}$$

where $\mathcal{F}^{HN}(v_1,v_2,\dots,v_s)$ is defined as in the proof of Lemma 2.3. We divide our consideration into two cases

(I) $s \geq 2$, that is, $E^{\vee\vee}$ is not semi-stable.
(II) $s = 1$, that is, $E^{\vee\vee}$ is semi-stable.

Case (I). (I-a) If $\langle v_1^2 \rangle \geq 0$, then we can use almost the same arguments as in the proof of Lemma 2.3. The difference comes from the inequality $r \geq 2$ which was used in (2.10) and (2.12). Thus

$$\dim \mathcal{F}^{HN}(v_1,v_2,\dots,v_s) \leq \langle v(E^{\vee\vee})^2 \rangle + 1. \tag{3.7}$$

The equality holds only if $r = 1$ and $\langle v_1^2 \rangle = 0$. Since F_s is locally free, F_1 is also locally free. On the other hand, if $r = 1$ and $\langle v_1^2 \rangle = 0$, then $\mathcal{M}(v_1)^{ss}$ consists of non-locally free sheaves (cf. Lemma 1.3). Therefore the equality does not hold. Hence, by using [Y1, Thm. 0.4], we see that

$$\begin{aligned} \dim \mathcal{M}(v,n;v_1,v_2,\dots,v_s) &< \langle (v-n\omega)^2 \rangle + 1 + n(\operatorname{rk} v + 1) \\ &\leq \langle v^2 \rangle + 1 - n(\operatorname{rk} v - 1). \end{aligned} \tag{3.8}$$

Thus we get a desired estimate

$$\operatorname{codim}_{\mathcal{M}(v)^s} \mathcal{M}(v,n;v_1,v_2,\dots,v_s) \geq 2. \tag{3.9}$$

(I-b) We assume that $\langle v_1^2 \rangle < 0$. Then $F_1 = E_0^{\oplus l_1}$, which implies that $\dim \mathcal{M}(v_1)^{ss} = -l_1^2 = \langle v_1^2 \rangle + l_1^2$. For convenience sake, we set

$$v_2' := \sum_{i=2}^{s} v_i = l_2' v_0 - a_2' \omega. \tag{3.10}$$

(I-b-1) We first assume that

$$\langle (v_2')^2 \rangle = 2l_2'(a_2' \operatorname{rk} v_0 - l_2') > 0. \tag{3.11}$$

Then, since $\langle v_i^2 \rangle \geq 0$ for all $i \geq 2$, $\dim \mathcal{F}^{HN}(v_2,\dots,v_s) \leq \langle (v_2')^2 \rangle + 1$. Thus we get

$$\langle v(E^{\vee\vee})^2 \rangle + 1 - \dim \mathcal{F}^{HN}(v_1,v_2,\dots,v_s) \geq \langle v_1, v_2' \rangle - l_1^2. \tag{3.12}$$

We shall prove that

$$\langle v_1, v_2' \rangle - l_1^2 + n(l \operatorname{rk} v_0 - 1) \geq 2. \tag{3.13}$$

Proof. Since E is stable, $F_1 \cap E$ satisfies that

$$\frac{\chi(F_1 \cap E)}{\operatorname{rk} F_1} < \frac{\chi(E)}{\operatorname{rk} E}. \tag{3.14}$$

Since $\chi(F_1 \cap E) \geq \chi(F_1) - n$ and $v_1 = l_1 v_0$ (i.e. $a_1 = 0$), we see that

$$nl_2' - a_2' l_1 > 0. \tag{3.15}$$

By our assumption (3.1), $(a_2' + n) \operatorname{rk} v_0 < 2(l_1 + l_2')$. By using (3.11), we see that

$$n \leq \frac{2l_1 + l_2' - 1}{\operatorname{rk} v_0}. \tag{3.16}$$

By using (3.11) and (3.15), we see that

$$nl_1 \operatorname{rk} v_0 \geq l_1^2 + 1. \tag{3.17}$$

Assume that $\operatorname{rk} v_0 \geq 2$. Then (3.16) implies that $n < l_1 + l_2'/2$. By using (3.11) and (3.17), we see that

$$\begin{aligned} \langle v_1, v_2' \rangle - l_1^2 + n(l \operatorname{rk} v_0 - 1) =& l_1(-2l_2' + a_2' \operatorname{rk} v_0) + n((l_1 + l_2') \operatorname{rk} v_0 - 1) - l_1^2 \\ =& -l_1 l_2' + l_1(a_2' \operatorname{rk} v_0 - l_2') + nl_1 \operatorname{rk} v_0 + l_1 l_2' + l_2'(n \operatorname{rk} v_0 - l_1) - n - l_1^2 \\ \geq& l_1(a_2' \operatorname{rk} v_0 - l_2') + l_2'(n \operatorname{rk} v_0 - l_1) + 1 - n \\ \geq& l_1 + l_2' + 1 - n > 1. \end{aligned} \tag{3.18}$$

Assume that $\operatorname{rk} v_0 = 1$. Then similar computations work if $a_2' \operatorname{rk} v_0 - l_2' > 1$. So we assume that $\operatorname{rk} v_0 = a_2' \operatorname{rk} v_0 - l_2' = 1$. Then (3.17) implies that $n - l_1 > 0$. Hence we see that

$$l_1(-2l_2' + a_2' \operatorname{rk} v_0) + n((l_1 + l_2') \operatorname{rk} v_0 - 1) - l_1^2 = (l_1 + l_2' - 1)(n - l_1) \geq (l - 1) \geq 1. \tag{3.19}$$

If the equality holds, then $l = 2$ and $n - l_1 = 1$. In this case, we get that $l_1 = l_2' = 1$ and $n = 2$. By (3.15), we get a contradiction. Thus the left hand side of (3.19) is greater than or equal to 2. Therefore (3.13) holds. □

By (3.13) and [Y1, Thm. 0.4], we get a desired estimate

$$\operatorname{codim}_{\mathcal{M}(v)^s} \mathcal{M}(v, n; v_1, v_2, \dots, v_s) \geq 2. \tag{3.20}$$

(I-b-2) We next assume that

$$\langle (v_2')^2 \rangle = 2l_2'(a_2' \operatorname{rk} v_0 - l_2') = 0. \tag{3.21}$$

Then $s = 2$ and $\langle v_0, v_2' \rangle = -l_2'$. Since $\operatorname{Hom}(E_0, F_2/F_1) = 0$, $\dim \operatorname{Hom}(F_2/F_1, E_0) \geq l_2'$. By Lemma 1.1, coevaluation map

$$\psi : F_2/F_1 \to E_0 \otimes \operatorname{Hom}(F_2/F_1, E_0)^\vee \tag{3.22}$$

is surjective in codimension 1 (cf. Lemma 1.1). Therefore we get

(i) $\dim \operatorname{Hom}(F_2/F_1, E_0) = l_2'$,
(ii) $\dim \operatorname{Ext}^1(F_2/F_1, E_0) = 0$ and
(iii) ψ is isomorphic in codimension 1.

By the definition of Harder-Narasimhan filtration, ψ is not isomorphic. Thus F_2/F_1 is not locally free. By (ii), $E^{\vee\vee} \cong F_1 \oplus F_2/F_1$, which contradicts the locally freeness of $E^{\vee\vee}$. Thus this case does not occur. By (I-a), (I-b-1) and (I-b-2), we get a desired bound for the case (I).

Case (II). We divide our consideration into three cases (II-a) $\langle v^2 \rangle > 0$, (II-b) $\langle v^2 \rangle = 0$ and (II-c) $\langle v^2 \rangle < 0$.

(II-a) If $\langle v_1^2 \rangle > 0$, then

$$\begin{aligned} \dim \mathcal{M}(v, n; v_1) &= \dim \mathcal{M}(v_1) + n(\operatorname{rk} v + 1) \\ &= \langle v_1^2 \rangle + 1 + n(\operatorname{rk} v + 1) \\ &= \langle v^2 \rangle + 1 - n(\operatorname{rk} v - 1). \end{aligned} \tag{3.23}$$

Hence if $\operatorname{rk} v \geq 3$, then $\operatorname{codim}_{\mathcal{M}(v)^s} \mathcal{M}(v, n; v_1) \geq 2$. If $\operatorname{rk} v = 2$, then the condition (3.1) implies that $a = 3$. Hence we get $\langle v_1^2 \rangle = \langle v^2 \rangle - 2n \operatorname{rk} v \leq 0$. Therefore this case does not occur.

(II-b) If $\langle v_1^2 \rangle = 0$, then the argument in (I-b-2) implies that $\mathcal{M}(v_1)^{ss}$ consists of non-locally free sheaves, which is a contradiction.

(II-c) We assume that $\langle v_1^2 \rangle < 0$, that is, $E^{\vee\vee} = E_0^{\oplus l}$. Then (3.1) implies that $n \operatorname{rk} v_0 - l \geq 0$.

(II-c-1) We first assume that $n \operatorname{rk} v_0 - l \geq 1$. Then we get

$$\begin{aligned} \operatorname{codim}_{\mathcal{M}(v)^s} \mathcal{M}(v,n) &= (2nl \operatorname{rk} v_0 - 2l^2 + 1) - (n(l \operatorname{rk} v_0 + 1) - l^2) \\ &= n(l \operatorname{rk} v_0 - 1) - (l^2 - 1) \\ &\geq \frac{l+1}{\operatorname{rk} v_0}(l \operatorname{rk} v_0 - 1) - (l^2 - 1) \\ &= \frac{(l+1)(\operatorname{rk} v_0 - 1)}{\operatorname{rk} v_0} \geq 0, \end{aligned} \tag{3.24}$$

and the equality holds if and only if $\operatorname{rk} v_0 = 1$ and $n \operatorname{rk} v_0 - l = 1$. By the computation of (3.24), it is easy to show that $\operatorname{codim}_{\mathcal{M}(v)^s} \mathcal{M}(v,n) \geq 2$, if $\operatorname{rk} v_0 \geq 2$, or $n \operatorname{rk} v_0 - l \geq 2$.

(II-c-2) If $n \operatorname{rk} v_0 - l = 0$, then the primitivity of v implies that $n = 1$. Therefore we get a desired bound for the case (II).

By (I) and (II), we complete the proof of Lemma 3.1. □

3.1.2. *Components containing locally free sheaves.* We shall prove Theorem 0.1 unless $v = (\operatorname{rk} v_0)v_0 - \omega$, or $\operatorname{rk} v_0 = 1$ and $v = lv_0 - (l+1)\omega$. For a locally free sheaf $E \in \mathcal{M}(v)^s$, we consider the dual of E. Since $\langle v_0, v\rangle = a \operatorname{rk} v_0 - 2l < 0$ and E is stable, $l' := \dim \operatorname{Ext}^2(E_0, E) \geq 2l - a \operatorname{rk} v_0 > 0$. By Serre duality, we get an exact sequence

$$0 \to (E_0^\vee)^{\oplus l'} \to E^\vee \to F \to 0, \tag{3.25}$$

where $F \in \mathcal{M}(v^\vee - l'v_0^\vee)^{\mu ss}$. Then we see that

$$\operatorname{Hom}(E_0^\vee, F) = \operatorname{Ext}^2(E_0^\vee, F) = 0. \tag{3.26}$$

Since $l' < l$, we see that

$$\begin{aligned} \langle v(F)^2\rangle &= 2l(a \operatorname{rk} v_0 - l) - 2l'(l' - 2l + a \operatorname{rk} v_0) \\ &= 2(l - l')(a \operatorname{rk} v_0 - l + l') \\ &\geq 2(l - l')l > 2(l - l')^2. \end{aligned} \tag{3.27}$$

Hence, by [Y4, Lem. 4.4], we get $\dim \mathcal{M}(v^\vee - l'v_0^\vee)^{\mu ss} = \langle (v - l'v_0)^2\rangle + 1$. Taking into account (3.26), we see that the moduli number of $E^\vee$ which fits in the exact sequence (3.25) is given by

$$\dim \mathcal{M}(v^\vee - l'v_0^\vee)^{\mu ss} + \dim Gr(\langle v^\vee - l'v_0^\vee, v_0^\vee\rangle, l') = \langle v^2\rangle + 1 - l'(l' + \langle v_0, v\rangle), \tag{3.28}$$

(cf. [Y4, Lem. 2.6]). We set

$$M_H(v)^* := \left\{ E \in M_H(v)^{loc} \;\middle|\; \begin{array}{l} \operatorname{Ext}^1(E_0, E) = 0, \\ \operatorname{coker}(\operatorname{Hom}(E, E_0) \otimes E_0^\vee \to E^\vee) \in M_H(R_{v_0}(v^\vee)) \end{array} \right\}. \tag{3.29}$$

Then, by using [Y4, Lem. 4.4], Lemma 3.1 and (3.28), we see that $\operatorname{codim}_{M_H(v)}(M_H(v) \setminus M_H(v)^*) \geq 2$. Since Theorem 0.1 holds for $M_H(R_{v_0}(v^\vee))$, $M_H(R_{v_0}(v^\vee))$ is irreducible. Therefore the morphism $M_H(v)^* \to M_H(R_{v_0}(v^\vee))$ is birational, if $M_H(v) \neq \emptyset$. Conversely, for a μ-stable vector bundle $F \in M_H(R_{v_0}(v^\vee))$, we consider the universal extension

$$0 \to \operatorname{Ext}^1(F, E_0^\vee)^\vee \otimes E_0^\vee \to E' \to F \to 0. \tag{3.30}$$

We claim that $(E')^\vee$ is a stable sheaf of $v((E')^\vee) = v$. This means that $M_H(v) \neq \emptyset$ and Theorem 0.1 holds for this case.

Proof of the claim: Assume that $(E')^\vee$ is not stable. Since $(E')^\vee$ is a μ-semi-stable vector bundle, there is a subbundle G such that

(1) $(E')^\vee/G$ is torsion free,
(2) $v(G) = l_1 v_0 - a_1\omega$ and
(3) $a_1/l_1 < a/l$.

Since $F^\vee$ is μ-stable and $(E')^\vee/G$ is torsion free, $F^\vee \cap G = 0$, or $F^\vee$. If $F^\vee \cap G = 0$, then $G \to \operatorname{Ext}^1(F, E_0) \otimes E_0$ is injective. Since the slopes of G and E_0 are the same and G is locally free, we see that $G \cong E_0^{\oplus l_1}$, which means that (3.30) is not the universal extension. Therefore $F^\vee \cap G$ must be equal to $F^\vee$. This means that G contains $F^\vee$ and $v(G/F^\vee) = (l_1 + l - a \operatorname{rk} v_0)v_0 - (a_1 - a)\omega$. Since $G/F^\vee$ is a subsheaf of $\operatorname{Ext}^1(F, E_0) \otimes E_0$, $a_1 - a \geq 0$. On the other hand, (3) and $l_1 \leq l$ implies that $a_1 < a$, which is a contradiction. Hence $(E')^\vee$ is stable. □

3.1.3. *Non-locally free components.* We shall prove Theorem 0.1 for

$$v = \begin{cases} (\operatorname{rk} v_0)v_0 - \omega, \\ lv_0 - (l+1)\omega, \ \operatorname{rk} v_0 = 1. \end{cases} \tag{3.31}$$

Proposition 3.2. *Assume that $v = (\operatorname{rk} v_0)v_0 - \omega$. Then $M_H(v) \cong X$.*

Proof. By the argument in (I-b-2), $E \in M_H(v)$ satisfies that $E^{\vee\vee} = E_0^{\oplus \operatorname{rk} v_0}$. Since $v = v(E_0^{\oplus \operatorname{rk} v_0}) - \omega$, E is the kernel of a quotient $E_0^{\oplus \operatorname{rk} v_0} \to \mathbb{C}_x, x \in X$. We shall construct a family of stable sheaves $\{\mathcal{E}_x\}_{x \in X}$ of $v(\mathcal{E}_x) = v$ and prove our proposition. For convenience sake, we set $X_i := X$, $i = 1,2$. Let $\Delta \subset X_2 \times X_1$ be the diagonal. We denote the projections $X_2 \times X_1 \to X_i$, $i = 1,2$ by p_i. We shall consider the evaluation map

$$\phi : E_0^\vee \boxtimes E_0 \to (E_0^\vee \boxtimes E_0)_{|\Delta} \to \mathcal{O}_\Delta. \tag{3.32}$$

Then it is easy to see that $\phi_x := \phi_{|\{x\} \times X_1}, x \in X_2$ is surjective and the induced homomorphism

$$\operatorname{Hom}(E_0, (E_0^\vee \boxtimes E_0)_{|\{x\} \times X}) \to \operatorname{Hom}(E_0, \mathbb{C}_x) \tag{3.33}$$

is an isomorphism. Hence $\ker \phi_x$ is stable. Since $\mathcal{O}_\Delta$ is flat over X_2, $\mathcal{E} := \ker \phi$ is flat over X_2 and $\mathcal{E}_{|\{x\} \times X} = \ker \phi_x, x \in X_2$ is stable. Thus we get a morphism $X_2 \to M_H(v)$, which is an isomorphism. $\square$

Corollary 3.3. *R_{v_0} is the Fourier-Mukai transform defined by $\mathcal{E}$.*

Proof. Let $\mathcal{F} : \mathbf{D}(X_1) \to \mathbf{D}(X_2)$ be the functor defined by

$$\mathcal{F}(x) := \mathbf{R} \operatorname{Hom}_{p_2}(\mathcal{E}, p_1^*(x)), x \in \mathbf{D}(X_1) \tag{3.34}$$

and $\widehat{\mathcal{F}} : \mathbf{D}(X_2) \to \mathbf{D}(X_1)$ the functor defined by

$$\widehat{\mathcal{F}}(y) := \mathbf{R} p_{1*}(\mathcal{E} \otimes p_2^*(y)), y \in \mathbf{D}(X_2). \tag{3.35}$$

Then $\widehat{\mathcal{F}}[2]$ gives the inverse of $\mathcal{F}$. Let E be a coherent sheaf on X_1 such that

$$\begin{aligned} &\operatorname{Hom}(E_0, E) = 0, \\ &\operatorname{Ext}^2(E_0, E) = 0. \end{aligned} \tag{3.36}$$

We shall prove that E satisfies WIT_1 for $\mathcal{F}$, i.e.

$$\operatorname{Hom}_{p_2}(\mathcal{E}, p_1^*(E)) = \operatorname{Ext}^2_{p_2}(\mathcal{E}, p_1^*(E)) = 0 \tag{3.37}$$

and $\mathcal{F}^1 := \operatorname{Ext}^1_{p_2}(\mathcal{E}, p_1^*(E))$ fits in the universal extension of E by E_0: Since $\mathcal{O}_\Delta$ is flat over X_1 and K_X is trivial, we get

$$\mathcal{H}om^i(\mathcal{O}_\Delta, p_1^*(E)) = \begin{cases} 0, \ i = 0,1, \\ \mathcal{O}_\Delta \otimes p_1^*(E), \ i = 2. \end{cases} \tag{3.38}$$

By using local-global spectral sequence, (3.36) and (3.38), we see that (3.37) holds and we get an exact sequence

$$0 \to \operatorname{Ext}^1(E_0, E) \otimes E_0 \to \operatorname{Ext}^1_{p_2}(\mathcal{E}, p_1^*(E)) \to E \to 0. \tag{3.39}$$

We claim that this sequence gives the universal extension of E by E_0.
Proof of the claim: If it is not the universal extension, since $\operatorname{Ext}^1(E_0, E) \cong \operatorname{Ext}^1(E, E_0)^\vee$, E_0 must be a direct summand of $\mathcal{F}^1(E)$. Since E satisfies WIT_1 for $\mathcal{F}$, $\mathcal{F}^1(E)$ satisfies WIT_1 for $\widehat{\mathcal{F}}$, which implies that E_0 also satisfies WIT_1 for $\widehat{\mathcal{F}}$. In particular, $R^2 p_{1*}(\mathcal{E} \otimes p_2^*(E_0)) = 0$. On the other hand, a direct computation shows that $R^2 p_{1*}(\mathcal{E} \otimes p_2^*(E_0)) \cong E_0$, which is a contradiction. Therefore (3.39) is the universal extension. Thus $\mathcal{F}^1(E)$ is the reflection of E by v_0. As in [Y5], we define $\mathcal{F}^H : H^*(X_1, \mathbb{Z}) \to H^*(X_2, \mathbb{Z})$ by

$$\mathcal{F}^H(x) := p_{2*}\left(\operatorname{ch}(\mathcal{E})^\vee p_1^* \sqrt{\operatorname{td}_{X_1}} p_2^* \sqrt{\operatorname{td}_{X_2}} p_1^*(x)\right), x \in H^*(X_1, \mathbb{Z}). \tag{3.40}$$

Since $\operatorname{ch}(\mathcal{E})^\vee = p_1^*(\operatorname{ch}(E_0))^\vee p_2^*(\operatorname{ch}(E_0)) - \operatorname{ch}(\mathcal{O}_\Delta)^\vee$, we also see that $\mathcal{F}^H(x) = -(x + \langle x, v(E_0) \rangle v(E_0))$. $\square$

Finally we shall treat $M_H(lv_0 - (l+1)\omega)$, $\operatorname{rk} v_0 = 1$.

Proposition 3.4. *If $\operatorname{rk} v_0 = 1$, then $M_H(lv_0 - (l+1)\omega) \cong \operatorname{Hilb}_X^{l+1}$ and θ_v is an isometry of Hodge structures.*

Proof. We may assume that $v_0 = 1 + \omega$. Let E be an element of $M_H(l-\omega)$. We shall first prove that $E^{\vee\vee} \cong \mathcal{O}_X^{\oplus l}$. To see this, it is sufficient to prove that

$$n := \dim(E^{\vee\vee}/E) = l + 1. \tag{3.41}$$

Proof of (3.41): Since $\chi(E) = l - 1$ and E is stable, Serre duality implies that $\dim H^0(X, E^\vee) \geq l - 1$. Hence we get an exact sequence

$$0 \to \mathcal{O}_X^{\oplus(l-1)} \to E^\vee \to I_Z \to 0 \tag{3.42}$$

where $I_Z \in \mathrm{Hilb}_X^{l+1-n}$. If $n = 0$, then E is locally free. By taking the dual of (3.42), we get a section of E, which contradicts the stability of E. We assume that $0 < n < l + 1$. Then $\dim \mathrm{Ext}^1(I_Z, \mathcal{O}_X) = \dim H^1(X, I_Z) = l - n$. Hence we get a decomposition $E^\vee = \mathcal{O}_X^{\oplus(n-1)} \oplus F$, where F fits in an exact sequence

$$0 \to \mathcal{O}_X^{\oplus(l-n)} \to F \to I_Z \to 0. \tag{3.43}$$

Then $E^{\vee\vee}$ has a subsheaf $\mathcal{O}_X^{\oplus n}$. The stability of E implies that

$$\frac{\chi(\mathcal{O}_X^{\oplus n} \cap E)}{n} < \frac{\chi(E)}{l}. \tag{3.44}$$

Since $\chi(\mathcal{O}_X^{\oplus n} \cap E) \geq \chi(\mathcal{O}_X^{\oplus n}) - n = n$, this is impossible. Therefore (3.42) holds, which implies that $E^{\vee\vee} = \mathcal{O}_X^{\oplus l}$.

Conversely for a general quotient $\phi : \mathcal{O}_X^{\oplus l} \to \oplus_{i=1}^{l+1}\mathbb{C}_{x_i}, x_1, x_2, \dots, x_{l+1} \in X$, it is easy to see that $\ker\phi$ is stable. Thus $M_H(l-\omega) \neq \emptyset$.

We shall prove that it is isomorphic to Hilb_X^{l+1}. For this purpose, we shall consider a functor $\mathcal{G} : \mathbf{D}(X_1) \to \mathbf{D}(X_2)_{op}$ which is the composition of reflection by $v(\mathcal{O}_X)$ with the taking dual functor:

$$\mathcal{G}(x) := \mathbf{R}\,\mathrm{Hom}_{p_2}(p_1^*(x), I_\Delta), x \in \mathbf{D}(X_1), \tag{3.45}$$

where we use the same notation as in Proposition 3.2 and $\mathbf{D}(X_2)_{op}$ is the opposite category of $\mathbf{D}(X_2)$. Then $\mathcal{G}$ gives an equivalence of categories. For $E \in M_H(l-\omega)$, we shall prove that

(a) $\mathrm{Ext}_{p_2}^i(p_1^*(E), I_\Delta) = 0$, $i = 0, 2$ and
(b) $\mathcal{G}^1(E) := \mathrm{Ext}_{p_2}^1(p_1^*(E), I_\Delta)$ is an ideal sheaf of colength $l+1$.

Then the map $M_H(l-\omega) \to \mathrm{Hilb}_X^{l+1}$ sending E to $\mathcal{G}^1(E)$ gives an isomorphism of moduli spaces. The second assertion follow from [Y5, Prop. 2.5] or a direct computation by using the equality

$$\mathcal{G}(\mathcal{E}) = \mathbf{R}\,\mathrm{Hom}_{p_{M_H(l-\omega)}}(\mathcal{E}, \mathcal{O}_{M_H(l-\omega)\times X_1}) \boxtimes \mathcal{O}_{X_2} - \mathbf{R}\mathcal{H}om(\mathcal{E}, \mathcal{O}_{M_H(l-\omega)\times X_2}) \tag{3.46}$$

as an element of Grothendieck group of $M_H(l-\omega)\times X_2$, where $\mathcal{E}$ is a quasi-universal family on $M_H(l-\omega)\times X$.

Proof of (a), (b): By Serre duality and the stability of E, $\mathrm{Ext}^2(E, I_x) = 0$ for all $x \in X$. Also we see that $\mathrm{Hom}(E, I_x) = 0$ for $x \notin \mathrm{Supp}(E^{\vee\vee}/E)$. Hence by the base change theorem and its proof, we see that (a) holds and $\mathcal{G}^1(E)$ is torsion free. It is easy to see that $v(\mathcal{G}(E)) = -R_{v(\mathcal{O}_X)}(v(E)^\vee) = -1 + l\omega$. Hence $v(\mathcal{G}^1(E)) = 1 - l\omega$. Therefore $\mathcal{G}^1(E)$ is an ideal sheaf of colength $l+1$.

□

3.2. **The case of $\langle v^2\rangle = 2l^2$.** We next treat the case where $\langle v^2\rangle = 2l^2$. Let E be a general member of $M_H(v) \setminus M_H(v)^{\mu s}$. Then by the proof of [Y4, Lem. 4.4], E fits in an exact sequence

$$0 \to E_1 \to E \to E_0 \to 0 \tag{3.47}$$

where E_1 is a μ-stable vector bundle.

Indeed, if $\langle v^2\rangle = 2l^2$, then the primitivity of v implies that (i) $v = lv_0 - 2\omega$, $l = \mathrm{rk}\, v_0$ or (ii) $v = lv_0 - \omega$, $2l = \mathrm{rk}\, v_0$. In particular $\mathrm{rk}\, v_0 > 1$. In the notation of the proof of [Y4, Lem. 4.4], we see that $k = 1$. Hence E fits in the above exact sequence. Then in the same way as in 2.2.2, we get Theorem 0.1 in this case. □

3.3. **Some remarks.** By the proof of Theorem 0.1 for case B and [Y4, Lem. 4.4], we also get the following.

(1) $M_H(v)^{\mu s} \neq \emptyset$ if and only if $\langle v^2\rangle \geq 2l^2$.
(2) $M_H(v)^{loc} = \emptyset$ if and only if $v = (\mathrm{rk}\, v_0)v_0 - \omega$, or $\mathrm{rk}\, v_0 = 1$ and $v = lv_0 - (l+1)\omega$.

4. Appendix

In this appendix, we shall explain our method for dimension counting of substacks of $\mathcal{M}(v)^{\mu ss}$. Since most results are appeared in another forms (cf. [D-R], [H-N]), we only give an outline.

4.1. **Notation.** For an ample divisor H' on X, let $Q(mH',v)$ be the open subscheme of the quot-scheme $\mathrm{Quot}_{\mathcal{O}_X(-mH')^{\oplus N}/X/\mathbb{C}}$ consisting of points

$$\lambda : \mathcal{O}_X(-mH')^{\oplus N} \to E \tag{4.1}$$

such that

1. $v(E) = v$,
2. λ induces an isomorphism $H^0(X, \mathcal{O}_X^{\oplus N}) \cong H^0(X, E(mH'))$,
3. $H^i(X, E(mH')) = 0$, $i > 0$.

Let $\mathcal{O}_{Q(mH',v)\times X}(-mH')^{\oplus N} \to \mathcal{Q}_v$ be the universal quotient. We set $V_v := \mathcal{O}_X(-mH')^{\oplus N}$. For our purpose, the choice of mH is not so important. Hence we simply denote $Q(mH',v)$ by $Q(v)$.

Let $q_v : Q(v) \to \mathcal{M}(v)$ be the natural map. We denote the pull-backs $q_v^{-1}(\mathcal{M}(v)^{\mu ss}), q_v^{-1}(\mathcal{M}(v)^{ss}), \dots$ by $Q(v)^{\mu ss}, Q(v)^{ss}, \dots$ respectively. If we choose a suitable $Q(v)$, then $q_v : Q(v)^{\mu ss} \to \mathcal{M}(v)^{\mu ss}$ is surjective and $\mathcal{M}(v)^{\mu ss}$ is a quotient stack of $Q(v)^{\mu ss}$ by a natural action of $G_v := GL(N)$:

$$\mathcal{M}(v)^{\mu ss} \cong [Q(v)^{\mu ss}/G_v]\,. \tag{4.2}$$

¿From now on, we assume that $q_v : Q(v)^{\mu ss} \to \mathcal{M}(v)^{\mu ss}$ is surjective.

4.2. **Stack of filtrations.**

Definition 4.1. $\mathcal{F}(v_1, v_2)$ is the stack of filtrations $F_1 \subset E$, $E \in \mathcal{M}(v)$ such that

1. F_1 is a μ-semi-stable sheaf of $v(F_1) = v_1$.
2. E/F_1 is a μ-semi-stable sheaf of $v(E/F_1) = v_2$.

Let $p_v : \mathcal{F}(v_1, v_2) \to \mathcal{M}(v)^{\mu ss}$ be the projection sending $(F_1 \subset E)$ to E and $p_{v_1,v_2} : \mathcal{F}(v_1, v_2) \to \mathcal{M}(v_1)^{\mu ss} \times \mathcal{M}(v_2)^{\mu ss}$ the morphism sending $(F_1 \subset E)$ to $(F_1, E/F_1)$.

We consider an open subscheme $F(v_1, v_2)$ of $\mathrm{Quot}_{\mathcal{Q}_v/Q(v)^{\mu ss}\times X/Q(v)^{\mu ss}}$ consisting of quotients $(\mathcal{Q}_v)_x \to E_2, x \in Q(v)^{\mu ss}$ such that E_2 is a μ-semi-stable sheaf of $v(E_2) = v_2$. Then

$$\mathcal{F}(v_1, v_2) = [F(v_1, v_2)/G_v]. \tag{4.3}$$

We shall give another expression of $\mathcal{F}(v_1, v_2)$ which is useful to compute the dimensions of substacks of $\mathcal{F}(v_1, v_2)$ and its projections to $\mathcal{M}(v)$.

We shall choose $Q(mH',v)^{\mu ss}$, $Q(mH',v_1)^{\mu ss}$ and $Q(mH',v_2)^{\mu ss}$ for the same mH'. Then $V_v = V_{v_1} \oplus V_{v_2}$. For simplicity, we set $\mathcal{Q}_i := \mathcal{Q}_{v_i}, V_i := V_{v_i}, \dots$ and $\mathcal{K}_i$ are the universal subsheaves of $\mathcal{O}_{Q(v_i)^{\mu ss}} \otimes V_i$, $i = 1, 2$. We define a scheme $\varpi : Y \to Q(v_1)^{\mu ss} \times Q(v_2)^{\mu ss}$ by

$$Y := \{\psi : (\mathcal{K}_2)_{x_2} \to (\mathcal{Q}_1)_{x_1} | (x_1, x_2) \in Q(v_1)^{\mu ss} \times Q(v_2)^{\mu ss}\}. \tag{4.4}$$

Then Y parameterizes subsheaves $K \subset V$ such that $K \cap V_1 = (\mathcal{K}_1)_{x_1}$ and $K/K \cap V_1 = (\mathcal{K}_2)_{x_2}$: For a quotient $\psi : (\mathcal{K}_2)_{x_2} \to (\mathcal{Q}_1)_{x_1}$, the subsheaf K of V_v is defined by

$$K := \{(a_1, a_2) \in V_1 \oplus V_2 | a_2 \in (\mathcal{K}_2)_{x_2}, \psi(a_2) = a_1 \mathrm{mod}(\mathcal{K}_1)_{x_1}\}. \tag{4.5}$$

Considering the quotient $V_v \to V_v/K$, Y also parameterizes the following exact and commutative diagram:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & (\mathcal{Q}_1)_{x_1} \arrow[u] \arrow[r] & E \arrow[u] \arrow[r] & (\mathcal{Q}_2)_{x_2} \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & V_1 \arrow[u] \arrow[r] & V_v \arrow[u] \arrow[r] & V_2 \arrow[u] \arrow[r] & 0
\end{tikzcd} \tag{4.6}$$

Let $\xi : Y \times G_v \to F(v_1, v_2)$ be the morphism sending $(y : V_v \to E, g) \in Y \times G_2$ to

$$(y \circ g : V_v \to E, E \to \mathcal{Q}_{x_2}) \in F(v_1, v_2), \tag{4.7}$$

where $\varpi(y) = (x_1, x_2)$. Let P be the parabolic subgroup of G_v fixing V_1. Then Y has a natural action of P and ξ induces a morphism $Y \times_P G_v \to F(v_1, v_2)$, which is G_v-equivariant. It is easy to see that this morphism is an isomorphism (cf. [Y2, appendix]). Therefore

$$\begin{aligned} \mathcal{F}(v_1, v_2) &\cong [Y \times_P G_v/G_v] \\ &\cong [Y/P]\,. \end{aligned} \tag{4.8}$$

By using (4.8), we shall prove the following.

Lemma 4.1. *We set*

$$\begin{aligned}\mathcal{N}^n(v_1,v_2):&=\{(E_1,E_2)\in\mathcal{M}(v_1)^{\mu ss}\times\mathcal{M}(v_2)^{\mu ss}|\dim\operatorname{Hom}(E_1,E_2)=n\},\\ \mathcal{F}^n(v_1,v_2):&=p_{v_1,v_2}^{-1}(\mathcal{N}^n(v_1,v_2))\\ &=\{(F_1\subset E)\in\mathcal{F}(v_1,v_2)|\dim\operatorname{Hom}(F_1,E/F_1)=n\}.\end{aligned}\tag{4.9}$$

Then,

$$\dim\mathcal{F}^n(v_1,v_2)=\dim\mathcal{N}^n(v_1,v_2)+\langle v_1,v_2\rangle+n.\tag{4.10}$$

Proof. We set

$$Q^n(v_1,v_2):=\{(x_1,x_2)\in Q(v_1)^{\mu ss}\times Q(v_2)^{\mu ss}|\dim\operatorname{Hom}((\mathcal{Q}_1)_{x_1},(\mathcal{Q}_2)_{x_2})=n\}.\tag{4.11}$$

For $(x_1,x_2)\in Q^n(v_1,v_2)$, there is an exact sequence

$$0\to\operatorname{Hom}((\mathcal{Q}_2)_{x_2},(\mathcal{Q}_1)_{x_1})\to\operatorname{Hom}(V_2,(\mathcal{Q}_1)_{x_1})\to\operatorname{Hom}((\mathcal{K}_2)_{x_2},(\mathcal{Q}_1)_{x_1})\to\operatorname{Ext}^1((\mathcal{Q}_2)_{x_2},(\mathcal{Q}_1)_{x_1})\to 0.\tag{4.12}$$

Since $\dim\operatorname{Hom}(V_2,(\mathcal{Q}_1)_{x_1})=\operatorname{rk}V_1\operatorname{rk}V_2$ and $\dim\operatorname{Ext}^2((\mathcal{Q}_2)_{x_2},(\mathcal{Q}_1)_{x_1})=n$, $\operatorname{Hom}(\mathcal{K}_2',\mathcal{Q}_1')$ is a locally free sheaf of rank $\langle v_1,v_2\rangle+\operatorname{rk}V_1\operatorname{rk}V_2+n$ on $Q^n(v_1,v_2)$, where $\mathcal{K}_2'$ and $\mathcal{Q}_1'$ are pull-backs of $\mathcal{K}_2$ and $\mathcal{Q}_1$ to $Q^n(v_1,v_2)$ respectively. We set $Y^n:=\mathbb{V}(\operatorname{Hom}(\mathcal{K}_2',\mathcal{Q}_1')^\vee)\to Q^n(v_1,v_2)$. Then $\mathcal{F}^n(v_1,v_2)\cong[Y^n/P]$. Hence we get that

$$\begin{aligned}\dim\mathcal{F}^n(v_1,v_2)&=\dim Y^n-\dim P\\ &=\dim Q^n(v_1,v_2)+\langle v_1,v_2\rangle+n-(\dim G_1+\dim G_2)\\ &=\dim\mathcal{N}^n(v_1,v_2)+\langle v_1,v_2\rangle+n.\end{aligned}\tag{4.13}$$

□

By similar method as in the proof of Lemma 4.1, we can also prove the following.

Lemma 4.2. *Let $\mathcal{F}^0(v_1,v_2,\dots,v_s)$ be the stack of filtrations*

$$0\subset F_1\subset F_2\subset\cdots\subset F_s=E, E\in\mathcal{M}(v)\tag{4.14}$$

such that

1. *F_i/F_{i-1}, $1\le i\le s$ are semi-stable of $v(F_i/F_{i-1})=v_i$.*
2. *$\operatorname{Hom}(F_i/F_{i-1},F_j/F_{j-1})=0$, $i<j$.*

Then

$$\dim\mathcal{F}^0(v_1,v_2,\dots,v_s)=\sum_{i=1}^s\mathcal{M}(v_i)^{ss}+\sum_{i<j}\langle v_i,v_j\rangle.\tag{4.15}$$

4.3. **Supplement for the proof of Lemma 1.3.** We shall explain how to derive (1.8) from (1.7). Let $q_l:Q(lw)^{ss}\to\overline{M}_H(lw):=Q(lw)^{ss}/G_{lw}$ be the quotient map. For a sequence of positive integers $l_1\le l_2\le\cdots\le l_s$ of $\sum_{i=1}^s l_i=l$, we set

$$\begin{aligned}\overline{M}_H(lw;l_1,l_2,\dots,l_s):&=\{\oplus_{i=1}^s E_i^{\oplus l_i}\in\overline{M}_H(lw)|E_1,E_2,\dots,E_s\in M_H(w),E_i\ne E_j\text{ for }i\ne j\},\\ Q(lw;l_1,l_2,\dots,l_s)^{ss}:&=q_l^{-1}(\overline{M}_H(lw;l_1,l_2,\dots,l_s)).\end{aligned}\tag{4.16}$$

For simplicity, we set $G:=G_{lw}$ and $G_i:=G_{l_iw}$, $i=1,2,\dots,s$. For quotients $\phi_i:V_i\to\mathcal{Q}_{x_i}\in Q(l_iw;l_i)^{ss}$, $i=1,2,\dots,s$ and an element $g\in G$, we define a quotient

$$(\oplus_{i=1}^s\phi_i)\circ g:V\to\oplus_{i=1}^s\mathcal{Q}_{x_i}.\tag{4.17}$$

It will define a morphism $\prod_{i=1}^s Q(l_iw;l_i)^{ss}\times G\to Q(lw)$. Let $\prod_{i=1}^s Q(l_iw;l_i)^{ss}\times_{\prod_i G_i}G$ be the quotient of $\prod_{i=1}^s Q(l_iw;l_i)^{ss}\times G$ by a natural action of $\prod_i G_i$. Then the above morphism induces a morphism

$$\pi_{l_1,l_2,\dots,l_s}:\prod_{i=1}^s Q(l_iw;l_i)^{ss}\times_{\prod_i G_i}G\to Q(lw).\tag{4.18}$$

By the construction of $\pi_{l_1,l_2,\dots,l_s}$, $\operatorname{im}\pi_{l_1,l_2,\dots,l_s}$ contains $Q(lw;l_1,l_2,\dots,l_s)^{ss}$. Hence

$$\dim\left[Q(lw;l_1,l_2,\dots,l_s)^{ss}/G\right]\le\sum_{i=1}^s\dim\left[Q(l_iw;l_i)^{ss}/G_i\right].\tag{4.19}$$

We shall prove that

$$\dim\left[Q(l_iw;l_i)^{ss}/G_i\right]\le 1.\tag{4.20}$$

Then $\dim\left[Q(lw;l_1,l_2,\dots,l_s)^{ss}/G\right]\le s\le l$. Clearly $\dim\left[Q(lw;1,1,\dots,1)^{ss}/G\right]=l$. Hence $\dim[Q(lw)^{ss}/G]=l$.

Proof of (4.20): Since $q_{l_i} : Q(l_i w; l_i)^{ss} \to \overline{M}_H(l_i w; l_i)$ is surjective and $\dim \overline{M}_H(l_i w; l_i) = 2$, it is sufficient to prove that

$$\dim \left[q_{l_i}^{-1}(E^{\oplus l_i})/G_i \right] \le -1, \ E \in M_H(w). \tag{4.21}$$

By definition, $\mathcal{J}(l_i, E) = [q_{l_i}^{-1}(E^{\oplus l_i})/G_i]$. Hence we obtain this claim from (1.7).

Acknowledgement. This paper heavily depends on Mukai's wonderful works [Mu1,2,3]. I would like to thank his papers, and also him for valuable discussion on his works. I would also like to thank Max Planck Institut für Mathematik for support and hospitality.

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Max Planck Institut für Mathematik, Vitatsgasse 7, D-53111 Bonn, Germany & Department of mathematics, Faculty of Science, Kobe University, Kobe, 657, Japan

E-mail address: `yoshioka@mpim-bonn.mpg.de, yoshioka@math.kobe-u.ac.jp`