On the Mumford-Narasimhan problem

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To Prof. M. Narasimhan on his 70th birthday

Thirty years ago, after appearing of the papers [T1] David Mumford asked me about an application of the proposed technique to the "Schottki problem for vector bundles". This problem is a non-abelian analogy of the classical cover $f\colon \mathbb{C}^g \to J(C)$ where the target space is the place where the theta functions of an algebraic curve C live. More precisely, if $S_g^r \subset \mathrm{CLRep}(\pi_1(C), \mathrm{SL}(r,\mathbb{C}))$ is the subset of the representations with trivial "a"-periods and $f\colon S_g^r \to M_r^{ss}(C)$ is the forgetful map from the space of flat bundles to the moduli space of semi-stable vector bundles on C then the question is what kind of image we have. The spectrum of the cases is the following

(1) surjectivity $f(S_g^r) = M_r^{ss}(C)$ is the best answer, predicted by the classical case;

(2) for general curve $f(S_g^r)$ contains a Zariski open set in $M_r^{ss}(C)$;

(3) other possibilities are too bad to discuss them.

Many times after we discussed this problem with M. Narasimhan reducing the question to new approaches like Hitchin's Higgs bundles [H] and so on. The point was the following: by the Narasimhan-Sesadri theorem the restriction of f to the unitary Schottki space

$$uS_g^r = \mathrm{CLRep}(\pi_1(C), \mathrm{SU}(r)) \subset S_g^r$$

is an embedding. Moreover the differential of f can be investigated. C. Florentino proved that around uS_g^r this differential is an isomorphism. Long time after we hadn't any information about its behavior. In this paper we prove at least that the second case is realized. That is for general smooth curve C general vector bundle admits the Schotki representation. It is quite relevant to dedicate this paper to Prof. M.Narasimhan on his 70th birthday in spite of the fact that this is the first step only to an expected perfect solution to the Mumford-Narasimhan problem. The main idea is

(1) to prove the statement for special curves and

(2) to extend this statement using the standard technique of complex analysis to neighborhoods of special curves.

This problem lies on the boundary connecting algebraic geometry and complex analysis. But for our special curves the analysis can be reduce to a simple algebraic geometry. These curves are called *large limit* curves and we would not discuss here why. But it is quite easy to understand from section 3 which isn't necessery for the main result proof. Such curves were used succesfully by Ciro Ciliberto, Angelo Lopez, Rick Miranda and Lucia Caporaso already in algebro-geometrical set up (see, for example [CLM]). Here we use them in the complex gauge theoretical set up. Moreover, the correlation functions of all local quantum fields in a two dimensional conformal field theory can be recovered from the partition function when all channels (tubes of the pair of pants decomposition) of the surface are constricted down to nodes. This procedure produces our reducuble curve P_Γ with uniquely defined complex structure. Details of such approach can be found in [T3].

1 Trinion decompositions and holomorphic flat connections on smooth algebraic curves

For an oriented compact surface Σ of genus g a trinion decomposition is given by a maximal set of disjoint, isotopy inequivalent circles

$$(C_1, ..., C_{3g-3}) \subset \Sigma \tag{1.1}$$

(see [HT]). Removing these circles we get

$$\Sigma - \{C_1, ..., C_{3g-3}\} = \cup_{i=1}^{2g-2} \tilde{v}_i \tag{1.2}$$

the finite set of trinions (or "pairs of pants") that is a trinion decomposition of Σ. Such decomposition defines *dual trivalent graph* Γ such that

(1) the set of its vertices
$$V(\Gamma) = \{v_i\} = \{\tilde{v}_i\} \tag{1.3}$$
is the set of trinions;

(2) and with the set of edges
$$E(\Gamma) = \{e\} = \{C_e\}; \tag{1.4}$$

(3) and two vertices v and v' are joined by edge e iff there exists the corresponding circle
$$C_e = \partial\tilde{v} \cap \partial\tilde{v'}. \tag{1.5}$$

Actualy we can start with any 3-valent graph Γ with the set of edges $E(\Gamma)$, then $|E(\Gamma)| = 3g-3$, and with the set of vertices $V(\Gamma)$, $\quad |V(|\Gamma)| = 2g-2$ where $|S|$ is the cardinality of a finite set S.

Topologically it is equivalent to a 3-dimensional handlebody H_Γ with boundary $\partial H_\Gamma = \Sigma_\Gamma$ where Σ_Γ is the Riemann surface given by the pumping up trick (see [T2]): we pump up every edge of Γ to a tube and every vertex to a trinion (= 2-sphere with 3 holes). By the construction our Riemann surface Σ_Γ has a trinions (or "pair of pants") decomposition given by removing all tubes.

For a visualization we can input our graph
$$i \colon \Gamma \hookrightarrow H_\Gamma \tag{1.6}$$
in the handlebody by the natural way. Then we can see that

(1) an orientation $\vec{e}$ of an edge $e \in E(\Gamma)$ gives the orientation $\vec{C_e}$ of the corresponding circle from the collection (1.1);

(2) sending any 1-cycle on Σ_Γ to the cycle on the handlebody H_Γ we get an epimorphism of the fundamental groups:
$$r \colon \pi_1(\Sigma_\Gamma) \to \pi_1(\Gamma) = \pi_1(H_\Gamma) \to 1 \tag{1.7}$$
the kernel of which is the free group with g generators
$$ker \quad r = F_g. \tag{1.8}$$

(3) According to the corresponding exact sequence

$$1 \to ker \quad r \to \pi_1(\Sigma_\Gamma) \to \pi_1(\Gamma) = \pi_1(H_\Gamma) \to 1 \tag{1.9}$$

we can choose standard generators of $\pi_1(\Sigma_\Gamma)$

$$\pi_1(\Sigma_\Gamma) =< a_1, ..., a_g, b_1, ..., b_g | \prod_{i=1}^{g} [a_i, b_i] = 1 >$$

such that

$$ker \quad r = F_g =< a_1, ..., a_g > \tag{1.10}$$

and

$$\pi_1(\Gamma) =< r(b_1), ..., r(b_g) >;$$

(4) obviously whole the collection of cycles $\{[C_e]\}$ (1.1) lies in $ker \quad r$ and moreover

$$ker \quad r = F_g =< [C_1], ..., [C_{3g-3}] >; \tag{1.11}$$

Recall that a path of lenth 1 on Γ is just an oriented edge $\vec{e}$. Let the set $P_1(\Gamma) = \vec{E}(\Gamma)$ be the set of 1-paths on Γ. Every such path $\vec{e}$ has vertices of two types

$$v_s(\vec{e}), \quad v_t(\vec{e}) \in V(\Gamma) \tag{1.12}$$

- the *source* and the *target* - which are equal for loops.

A path of length d in Γ is an ordered sequence $(\vec{e}_1, ..., \vec{e}_d)$ of oriented edges such that for every i

$$v_t(\vec{e}_i) = v_s(\vec{e}_{i+1}). \tag{1.13}$$

If $\vec{e}_{d+1} = \vec{e}_1$, then our path is a loop. A path $(\vec{e}_1, ..., \vec{e}_d)$ (or a loop) is called irreducible if $e_i \neq e_{i+1}$ for every i (including $i = d+1$ for a loop).

Every path $(\vec{e}_1, ..., \vec{e}_d) \in P_d(\Gamma)$ defines two vertices

$$v_s((\vec{e}_1, ..., \vec{e}_d)), \quad v_t((\vec{e}_1, ..., \vec{e}_d)) \in V(\Gamma) \tag{1.14}$$

- the source and the target and the coincidence of which shows that our parth is a loop.

Let

$$L_d(\Gamma) \subset P_d(\Gamma) \tag{1.15}$$

be the set of oriented irreducible loops of length d. For every vertex $v \in V(\Gamma)$ we have the set of irreducible loops marked by v

$$L_d(\Gamma)_v = \{l \in L_d(\Gamma) | v \subset l\}. \tag{1.16}$$

The union

$$L_\infty(\Gamma)_v = \cup_{d=1}^{\infty} L_d(\Gamma)_v \tag{1.17}$$

admits a group structure

$$\pi_1^C(\Gamma)_v = L_\infty(\Gamma)_v. \tag{1.18}$$

if we consider only irreducible fragments of the compositions.

Obviously, this group depends on the marking point v.

Let $\pi_1(\Gamma)$ be the standard fundamental group of Γ (as a 1-complex). Then the natural epimorphism $r \colon \pi_1^C(\Gamma) \to \pi_1(\Gamma)$ is an isomorphism. Obviously, if Γ is a 3-valent graph of genus g then $\pi_1(\Gamma) = F_g$ is a free group with g generators.

Now let us fix a point $p \in \Sigma_\Gamma$ (suppose it coincides with $i(v)$ (1.6)) and join all circles with p by any system of paths. Then we can consider the oriented circles $\vec{C}_e$ as elements of the group $ker \quad r$ (1.9) that is as elements of the fundamental group $\pi_1(\Sigma_\Gamma)$.

Sending every combinatorial loop

$$l = (\vec{e}_1, ..., \vec{e}_d) \in L_\infty(\Gamma)_v$$

to the ordered product

$$Int(l) = \vec{C}_{e_1} \cdot ... \cdot \vec{C}_{e_d} \in ker \quad r \subset \pi_1(\Sigma_\Gamma) \tag{1.19}$$

we get a homomorphism

$$Int \colon \pi_1(\Gamma) = \pi_1(H_\Gamma) \to \pi_1(\Gamma) \tag{1.20}$$

Recall that both of these groups are free groups with g generators and it is easy to see that Int is an isomorphism.

Our surface Σ_Γ doesn't carry any "natural" complex structure. Moreover after the fixing of the previous basis of the fundamental group (1.9)- (1.10) the space of complex structures on Σ_Γ turns to be the Teichmuller space τ_g. If we fix such a structure $I \in \tau_g$ then we can consider the space of classes of representations

$$A^{na} = CLRep(\pi_1(\Sigma), SL(2, \mathbb{C})) \tag{1.21}$$

as the full space $A^{na}(\Sigma_I)$ of all holomorphic flat connections on topologicaly trivial vector bundle on Σ_I. Any representation ρ with class

$$[\rho] \in CLRep(\pi_1(\Sigma), SL(2,\mathbb{C})) \tag{1.22}$$

defines a holomorphic vector bundle just by the standard construction

$$E = U \times \mathbb{C}^2/(\pi_1, \rho) \tag{1.23}$$

where U is the universal cover of Σ_I with the natural action of the fundamental group of the base.

Of course, in the set of such bundles there are non-stable but indecomposable vector bundles and even semi-stable decomposable ones. Let us remove the representations corresponding to non-stable bundles and get the space

$$CLRep^{I-ss}(\pi_1(\Sigma_I), \mathrm{SL}(2,\mathbb{C})) \subset CLRep(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{C})) \tag{1.24}$$

of classes of representations which give stable, semi-stable and decomposable semi-stable bundles.

Obviously this Zariski dense (=containing a Zariski open set) subspace depends on the complex structure $I \in \tau_g$. Moreover we have the holomorphic forgetful map

$$f\colon A^{na}_{I-ss} = CLRep^{I-ss}(\pi_1(\Sigma), SL(2,\mathbb{C})) \to M^{ss}(\Sigma_I) \tag{1.25}$$

where the target space is the moduli space of semi-stable topologicaly trivial vector bundles. This is the affine bundle over the cotangent bundle of $M^{ss}(\Sigma_I)$. Indeed, any fiber $f^{-1}(E)$ is the affine space of holomorphic flat connections on E and the difference of any two connections is a traceless Higgs field

$$\phi\colon E \to E(K_{\Sigma_I}) \in T^*_{\Sigma_I} M^{ss}(\Sigma_I) \tag{1.26}$$

that is a covector to $M^{ss}(\Sigma_I)$ at E.

Every affine bundle is given by a 1-cocycle of the vector bundle (see [T1]). In our case this is a cocycle

$$\varepsilon_{na} \in H^1(M^{ss}, \Omega) \tag{1.27}$$

and

$$H^1(M^{ss}, \Omega) = H^{1,1}(M^{ss}, \mathbb{C})$$

by the Dolbault theorem. In our case this class is precisely the class of the polarization that is the class of theta divisor:

Proposition 1.1 *The cohomology class (1.27) is given by the formula*

$$\varepsilon_{na} = [\Theta_{na}] = [\omega_{na}] \tag{1.28}$$

where ω is the standard symplectic structure on the moduli space.

Corollary 1.1 *The forgetful projection f (1.25) doesn't admit any holomorphic section.*

That is, the answer is precisely the same as in the abelian case.

Remark By the Narasimhan-Sesadri theorem there exists the non-holomorphic section

$$M^{ss}(\Sigma_\Gamma) = CLRep(\pi_1(\Sigma), \mathrm{SU}(2)) \hookrightarrow CLRep(\pi_1(\Sigma), \mathrm{SL}(2, \mathbb{C})) \tag{1.29}$$

Now let us perform the trinion decomposition corresponding to Σ_Γ (1.1)-(1.2). We can construct the space of holomorphic flat bundles $CLRep(\pi_1(\Sigma), \mathrm{SL}(2, \mathbb{C}))$ gluing the spaces of $\mathrm{SL}(2, \mathbb{C})$-flat connections over the trinions $\{\tilde{v}_i\}$ (1.2).

For one trinion $\tilde{v}$ (which is 2-sphere with 3 holes) the space of $\mathrm{SL}(2, \mathbb{C})$-flat connections is given as the classes representations space

$$A^{na}(\tilde{v}) = CLRep(\pi_1(\tilde{v}), \mathrm{SL}(2, \mathbb{C})) \tag{1.30}$$

where $\pi_1(\tilde{v}) = F_2$ is the free group with 2 generators.

Definition 1.1 For the free group F_g with g generators the classes representations space

$$S_g = CLRep(F_g, \mathrm{SL}(2, \mathbb{C})) \tag{1.31}$$

is called the Schottki space of genus g.

So $A^{na}(\tilde{v}) = S_2$ is the Schottki space of genus 2.

If two vertices v_i and v_j are joined by edge e_l that is if the circle C_l corresponding to e_l is a boundary component of trinions $\tilde{v}_i$ and $\tilde{v}_j$:

$$C_l = \partial\tilde{v}_i \cap \partial\tilde{v}_j \tag{1.32}$$

then we can glue $A^{na}(\tilde{v}_i)$ and $A^{na}(\tilde{v}_j)$ along the boudary component C_l by just the same way as we do this for $\mathrm{SU}(2)$ representations (see for example

[JW]): let $Conj(\mathrm{SL}(2,\mathbb{C}))$ be the set of conjugation classes of elements of $\mathrm{SL}(2,\mathbb{C})$. Then we have the natural map

$$conj\colon\ \mathrm{SL}(2,\mathbb{C}) \to Conj(\mathrm{SL}(2,\mathbb{C})). \tag{1.33}$$

Two representations $\rho_i, \quad [\rho_i] \in A^{na}(\tilde{v}_i)$ and $\rho_j, \quad [\rho_j] \in A^{na}(\tilde{v}_j)$ are glued iff

$$conj(\rho_i([C_l])) = conj(\rho_j([C_l])). \tag{1.34}$$

The ambiguity of such gluing is the stabilizer $Z(\rho_i([C_l]))$ of the monodromy around this loop. For example for a semi-simple element $m \in \mathrm{SL}(2,\mathbb{C})$ the stabilizer $Z(m) = \mathbb{C}^*$. The result of such gluing is

$$A^{na}(\tilde{v}_i) * A^{na}(\tilde{v}_j) = S_3 \tag{1.35}$$

if $C_l = \tilde{\partial(v_i)} \bigcap \tilde{\partial(v_j)}$. Gluing all trinions $\tilde{v}_i$ we get our surface and gluing all spaces of flat bundles $A^{na}(\tilde{v})$ we get the space $CLRep(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{C}))$ of flat bundles on Σ_Γ.

The exact sequence (1.9) defines the canonical embedding of the Schottki space (1.31)

$$i\colon S_g \hookrightarrow CLRep(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{C})) \tag{1.36}$$

such a way that the image is in a sense a complete intersection. Namely, consider the space of representations $Rep(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{C}))$ (without the diagonal adjoint factorization), such that

$$\mathrm{CLRep}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{C})) = Rep(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{C}))/PGL(2,\mathbb{C}) \tag{1.37}$$

where the last action is the natural diagonal adjoint action of $\mathrm{SL}(2,\mathbb{C})$ on the space of representations. So, let

$$p\colon Rep(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{C})) \to \mathrm{CLRep}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{C})) \tag{1.38}$$

be the natural projection to the factor by this action.

Geometrically this means that we fixed a point $p \in \Sigma_I$ and a trivialization $E_p = \mathbb{C}^2$ of the fiber of vector bundle E over this point. The moduli space of vector bundles with such additional structure $\widetilde{M^{ss}(\Sigma_I)}$ admits the structure of principal $PGL(2,\mathbb{C})$-bundle over $M^{ss}(\Sigma_I)$:

$$\phi\colon \widetilde{M^{ss}(\Sigma_I)} \to M^{ss}(\Sigma_I) \tag{1.39}$$

where the group $\mathrm{SL}(2,\mathbb{C})$ modulo ± 1 acts on $E_p = \mathbb{C}^2$ as (2 X 2)-matrices with determinant 1.

The moduli space of flat vector bundles with such additional structure is $Rep(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{C}))$ and it admits the structure p (1.38) of a principal $PGL(2,\mathbb{C})$-bundle over $\mathrm{CLRep}(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{C}))$.

Now every element $\gamma \in \pi_1(\Sigma)$ defines regular functions

$$c_{ij}(\gamma) \colon Rep(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{C})) \to \mathbb{C} \tag{1.40}$$

by the formula

$$c_{ij}(\gamma)(\rho) = (\rho(\gamma))_{ij}$$

where ij is the matrix element of the corresponding matrix.

Then the half basis $(a_1, ..., a_g)$ (1.10) defines $3g$ functions

$$c_{11}(a_1), c_{12}(a_1), c_{21}(a_1), ..., c_{11}(a_g), c_{12}(a_g), c_{21}(a_g) \tag{1.41}$$

and a system of divisors

$$D_{ij}(a_l) = \{c_{ij}(a_l) = 0, \text{ if } i \neq j, \} \tag{1.42}$$

and

$$D_{11}(a_l) = \{c_{11}(a_l) = 1\}.$$

These divisors are not invariant with respect to the diagonal adjoint action of $\mathrm{SL}(2,\mathbb{C})$. But the complete intersection

$$\bigcap_{ij,l} D_{ij}(a_l) = \phi^{-1}(i(S_g)) \tag{1.43}$$

is the preimage of the Schottki space (1.36).

Returning to the affine bundle (1.25) consider a bundle $E \in M^{ss}(\Sigma_I)$ and the fiber

$$f^{-1}(E) = \mathbb{C}_E^{3g-3} \tag{1.44}$$

Such affine subspace of $CLRep(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{C}))$ is called a *packet* of classes of representations. This is an affine space over the space of Higgs fields $H^0(\Sigma_I, adE \otimes T^*\Sigma_I)$.

Now consider the restriction of the principal $\mathrm{SL}(2,\mathbb{C})$-bundle (1.38) to this affine space

$$p \colon p^{-1}(\mathbb{C}_E^{3g-3}) \to \mathbb{C}_E^{3g-3}. \tag{1.45}$$

Obviously the space $p^{-1}(\mathbb{C}_E^{3g-3})$ is an affine algebraic variety.

Restricting the functions (1.40) to this affine variety we get the complete intersection

$$\phi^{-1}(i(S_g)) \cap p^{-1}(\mathbb{C}_E^{3g-3}) \tag{1.46}$$

and E *admits a Schottki representation iff this complete intersection is non empty.*

Non emptyness is an (Zariski) open condition thus to solve the Mumford-Narasimhan problem for general curve and general vector bundle we need to find one curve with this property.

That is our strategy is the same as C. Ciliberto, A. Lopez and R. Miranda in [CLM] for the computation of corank of the Gaussian-Wahl map for general curve. Moreover, the probe curves are the same.

2 3-valent graphs and ll-curves

Sending every vertex $v \in V(\Gamma)$ to the complex Riemann sphere $P_v = \mathbb{CP}^1$ with a triple of points $(p_{e_1}, p_{e_2}, p_{e_3})$ corresponding to edges of the star $S(v)$ of the vertex (= the set of edges incident to v) and gluing two components P_v and $P_{v'}$ such that $\partial e = v, v'$ along the points $p_e \in P_v, P_{v'}$ we obtain a reducible algebraic curve P_Γ with the following properties:

(1) arithmetical genus of the connected reducible curve P_{Γ_g} is equal to g;

(2) for any 3-valent graph Γ_g the curve P_{Γ_g} is Deligne - Mumford stable

(3) hence the curve P_{Γ_g} defines a point of the Deligne-Mumford compactification $\overline{\mathcal{M}_g}$ (with the same notation);

So in the Deligne-Mumford compactification $\overline{\mathcal{M}_g}$ of the moduli space $\mathcal{M}_g$ of smooth curves of genus g we get a finite configuration of points

$$\mathcal{P} \subset \overline{\mathcal{M}_g} \tag{2.1}$$

enumerated by the set of 3-valent graphs TG_g - the set of large limit curves.

In the algebraic geometry framework these curves were described by Artamkin in [A]:

(1) on every curve P_Γ the canonical class K_Γ is a line bundle for which the restriction to every component $P_v, v \in V(\Gamma)$ is the sheaf of meromorphic differentials ω with simple poles at $p_{e_1}, p_{e_2}, p_{e_3}$ where $e_1 \bigcup e_2 \bigcup e_3 = S(v)$:

$$K_\Gamma|_{P_v} = K_{P_v}(p_{e_1} + p_{e_2} + p_{e_3}) = \mathcal{O}_{P_\Gamma}(1); \tag{2.2}$$

(2) thus every holomorphic section s of K_Γ is a collection of meromorphic differentials $\{\omega_v\}$ on the components P_v with poles at $\bigcup_{e\in E(\Gamma)} p_e$ with constraints: for every e such that $\partial e = v, v'$ one has

$$res_{p_e}\omega_v + res_{p_{v'}}\omega_{v'} = 0; \tag{2.3}$$

(3) beside of these equalities we have 2g-2 linear relations: for every $v \in V(\Gamma)$ with $(e_1, e_2, e_3) = S(v)$

$$res_{p_{e_1}}\omega_v + res_{p_{e_2}}\omega_v + res_{p_{e_3}}\omega_v = 0; \tag{2.4}$$

(4) from here we have the right number g for the dimension of holomorphic differential space $H^0(P_\Gamma), \mathcal{O}(K_\Gamma))$;

(5) but the properties of the canonical complete linear system $|K_\Gamma|$ depend on the topology of Γ.

Definition 2.1 The minimal number of edges that may be removed to make the graph disconnected is called the *thickness* $th(\Gamma)$ of the graph.

Obviously for a 3-valent graph Γ the number $th(\Gamma) \leq 3$.

In [A] Artamkin proved the following

Proposition 2.1 *The canonical linear system* $|K_{P_\Gamma}|$ *is*

(1) base points free iff $th(\Gamma) \geq 2$*;*

(2) very ample iff $th(\Gamma) \geq 3$.

The double canonical systems on ll-curves are much more regular:

(1) every holomorphic section s of $\mathcal{O}(2K_\Gamma)$ is a collection of meromorphic quadratic differentials $\{\omega_v\}$ on the components P_v with poles of degree 2 at $\bigcup_{e\in E(\Gamma)} p_e$;

(2) every such quadratic differential defines *biresidue* $bires_{p_e}$ at every pole ($Res^2_{p_e}$ in notations of [T1]) and

(3) the constraints in this case are the following: for every e such that $\partial e = v, v'$

$$bires_{p_e}\omega_v + bires_{p_{v'}}\omega_{v'} = 0; \tag{2.5}$$

(4) thus the system of nodes $\{p_e\}$ of our ll-curve P_Γ defines the decomposition

$$H^0(P_\Gamma, \mathcal{O}(2K_\Gamma))^* = \mathbb{C}^{E(\Gamma)} \tag{2.6}$$

where for every quadratic differential ω the value of the linear form H_e is given by the formula

$$H_e(\omega) = bires_{p_e}\omega; \tag{2.7}$$

(5) thus for a 3-valent graph Γ_g the large limit curve $P_\Gamma \in \overline{\mathcal{M}_g}$ is a smooth point of $\overline{\mathcal{M}_g}$ as orbifold.

Recall that the fiber of the tangent bundle of the moduli space at a point P_Γ is given by the equality

$$T_{P_\Gamma}\overline{\mathcal{M}_g} = H^0(P_\Gamma, \mathcal{O}_{P_\Gamma}(2K_{P_\Gamma}) \tag{2.8}$$

and the decomposition (2.6) is given by the following geometrical way: the double canonical map of P_Γ given by the complete linear system $|2K_\Gamma|$

$$\phi_{2K_\Gamma} \colon P_\Gamma \to \mathbb{P}^{3g-4} \tag{2.9}$$

has as the target space the projectivization of the tangent space of the moduli space $\overline{\mathcal{M}_g}$ at the point P_Γ.

Then the images of nodes

$$\{\phi_{2K_\Gamma}(\bigcup_{e \in E(\Gamma)} p_e)\} \tag{2.10}$$

define the configuration of

$$3g - 3 = rkH^0(P_\Gamma, \mathcal{O}(2K_{P_\Gamma}))$$

linear independent points (since the components are coming to irreducible conics and every conic is defined by any triple of points on it). Thus this configuration of points gives a decomposition of the tangent space

$$T_{P_\Gamma}\overline{\mathcal{M}_g} = \bigoplus_{e \in E(\Gamma)} \mathbb{C}_e \tag{2.11}$$

where $\mathbb{P}\mathbb{C}_e = \phi_{2K_\Gamma}(p_e)$ coincides with the decomposition (2.6).

Let $e \subset \Gamma$ be an edge of a 3-valent graph Γ with two vertices $v, v' = \partial e$ and with two stars $S(v) = e, e_1, e_2$ and $S(v') = e, e'_1, e'_2$. From this we can get a graph Γ'_e of genus $g-1$ by the following construction:

(1) remove e and get two 2-valent vertices v and v' with stars $S(v) = e_1, e_2$ and $S(v') = e'_1, e'_2$;

(2) consider the pair e_1, e_2 as the first new edge e_{new} and e'_1, e'_2 as the second new edge e'_{new};

(3) so, we get a new graph Γ' with fixed pair of (a priori) disjoint edges $(e_{new} \bigcup e'_{new}) \subset \Gamma'$;

(4) obviously genus of Γ' is equal to $g-1$.

Coming to ll-curves we have the correspondence:

$$(p_e \in P_\Gamma) \longleftrightarrow (p_{e_{new}}, p_{e'_{new}} \subset P_{\Gamma'}) \tag{2.12}$$

and vice versa.

Obviously this operation corresponds to the projection of the canonical curve $P_\Gamma \subset \mathbb{P}^{g-1}$ from the node p_e to the canonical curve $P_{\Gamma'} \subset \mathbb{P}^{g-2}$.

Remark This projection is an analog of the Fano double projection for Fano threefolds.

So, we have the correspondence

$$C \subset \mathcal{P}_g \times \mathcal{P}_{g-1} \tag{2.13}$$

with two projections:

$$p_g \colon C \to \mathcal{P}_g \tag{2.14}$$

with the fiber

$$p_g^{-1}(\Gamma) = E(\Gamma)$$

and

$$p_{g-1}\colon C \to \mathcal{P}_{g-1} \tag{2.15}$$

with the fiber

$$p_{g-1}^{-1}(\Gamma') = S^2(E(\Gamma')).$$

Now we blow down this edge in our graph. We get a new graph with 4-valent vertex $v_{new} = v = v'$ with the star $S(v_{new}) = e_1, e_2, e'_1, e'_2$. There are 3 partitions of this set to pairs: the old one $(e_1, e_2)|(e'_1, e'_2)$ and two new ones:

$$(e_1, e'_2|e_2, e'_1) \text{ and } (e_1, e'_1|e_2, e'_2). \tag{2.16}$$

Now we can blow up the vertex v_{new} to the edge e_{new} with vertices $\partial e_{new} = v_{new}, v'_{new}$ and stars

(1) $S(v_{new}) = e_{new}, e_1, e_2$ and $S(v'_{new}) = e_{new}, e'_1, e'_2$. This is our starting graph Γ.

(2) $S(v_{new}) = e'_{new}, e_1, e'_2$ and $S(v'_{new}) = e'_{new}, e'_1, e_2$. This is the first new graph Γ'.

(3) $S(v_{new}) = e''_{new}, e_1, e'_1$ and $S(v'_{new}) = e''_{new}, e'_2, e'_2$. This is the second new graph Γ''.

As a result of this construction every obtained graph has distinguished edge that is we have a triple of flags

$$(e \subset \Gamma), (e'_{new} \subset \Gamma'), (e''_{new} \subset \Gamma''). \tag{2.17}$$

Such triple we call a *nest* of flags.

The correspondence (2.14) gives a nest of graphs of genus $g - 1$

$$(e_1, e'_1 \subset \Gamma_1), (e_2, e'_2 \subset \Gamma_2), ((e_3, e'_3 \subset \Gamma_3)) \tag{2.18}$$

Every nest is defined uniquely by any flag $(e \subset)\Gamma$ or $(e_i, e'_i \subset \Gamma_i)$ from the triple.

Moreover, let $S(e)$ be the star of an edge e that is the union of stars of the boundary $v, v' = \partial e$

$$S(e) = S(v) \bigcup S(v'). \tag{2.19}$$

Then we have the canonical identification of graphs

$$\Gamma - S(e) = \Gamma' - S(e_{new}) = \Gamma'' - S(e_{new}). \tag{2.20}$$

Remark It is easy to see that if e is a loop this construction gives the same graph with the same loop e again.

3 Special 1-parameter deformations of large limit curves

Now we are ready to construct 1-parameter family of reducible curves with $2g-3$ rational components.

For our flag $e \subset \Gamma$ consider two componets $P_v, P_{v'}$ of the curve P_Γ with common point p_e where $v, v' = \partial e$. Remove this point and glue P_v and $P_{v'}$ by a tube that is, consider the connected sum

$$P_v \#_{p(e)} P_{v'} = P_{v,v'} = S^2. \tag{3.1}$$

This is a 2-sphere with two pairs of points (p_{e_1}, p_{e_2}) and $(p_{e'_1}, p_{e'_2})$ where $(e, e_1, e_2) = S(v)$ and $(e, e'_1, e'_2) = S(v')$. If we fix a complex structure on S^2 and consider the double cover

$$\phi \colon E \to \mathbb{P}^1 \tag{3.2}$$

with ramification points

$$W = p_{e_1} \bigcup p_{e_2} \bigcup p_{e'_1} \bigcup p_{e'_1} \tag{3.3}$$

we obtain an elliptic curve E with a point of second order

$$\sigma = p_{e_1} + p_{e_2} - p_{e'_1} - p_{e'_2} \in Pic(E)_2. \tag{3.4}$$

So the moduli space of complex structures on S^2 with quadruple of points divided in two pairs is equal to $\mathcal{M}_1^2$ - the moduli space of smooth elliptic curves with fixed point of order 2.

Every such complex structure $\tau \in \mathcal{M}_1^2$ on S^2 with quadruple of points divided in two pairs and the standard complex structures on all others components glued as before define a stable algebraic reducuble curve $P_{e\subset\Gamma,\tau}$. Thus we obtain an embedding

$$\psi_{e\subset\Gamma} \colon \mathcal{M}_1^2 \to \overline{\mathcal{M}_g}. \tag{3.5}$$

The moduli space

$$\mathcal{M}_1^2 = \mathbb{P}^1 - (\sigma, \sigma', \sigma'') \tag{3.6}$$

is the projective line without 3 points. These 3 points correspond to 3 possibility to divide 4 points $p_{e_1}, p_{e_2}, p_{e'_1}, p_{e'_2}$ in 2 pairs that is a choice of a point of order 2 on an elliptic curve.

It is easy to see that

Proposition 3.1 *(1) The map $\psi_{e\subset\Gamma}$ (3.5) can be extended to a map*

$$\psi_{e\subset\Gamma}\colon \mathbb{P}^1 \to \overline{\mathcal{M}_g}; \tag{3.7}$$

(2)

$$\psi_{e\subset\Gamma}(\sigma) = p_e \in P_\Gamma \tag{3.8}$$

where σ is given by (3.4) and P_Γ is a large limit curve with fixed node corresponding to the edge e,

(3)

$$\psi_{e\subset\Gamma}(\sigma') = P_{\Gamma'}; \tag{3.9}$$

and

(4)

$$\psi_{e\subset\Gamma}(\sigma'') = P_{\Gamma''}; \tag{3.10}$$

where the triple

$$(e \subset \Gamma), (e_{new} \subset \Gamma'), (e_{new} \subset \Gamma'') \tag{3.11}$$

is a nest of flags (2.17).

(5) Now we can identify the tangent direction

$$T\psi_{e\subset\Gamma}(\mathbb{P}^1)_{P_\Gamma} = \mathbb{C}_e \tag{3.12}$$

from the decomposition (2.11) with the corresponding node of the double canonical curve. We obtain the uniquely defined 2-canonical model.

This identification of images of nodes under the double canonical embedding and the directions of special deformations cancels the projective transformation ambiguity.

Remark If edge e is a loop with a vertex v such that the star $S(v) = e, e'$ then P_v is a rational curve with one double point p_e and the smooth point $p_{e'}$. The operation of connected summing around double point p_e gives a smooth 2-torus T^2 with fixed point $p_{e'}$ and the isotopy class $a \in H_1(T^2, \mathbb{Z})$ which is the class of the neck of gluing tube. The class a mod 2 gives a point of order 2 on T^2. The space $\mathcal{M}_1^2$ of complex structures on T^2 with such additional data is $\mathbb{C}^* - 1$ that is $\mathbb{P}^1$ without 3 points again. In this case the rational curve $\psi_{e\subset\Gamma}(\mathcal{M}_1^2)$ admits the compactification by the double point corresponding to P_Γ.

So, the Deligne-Mumford compactification $\overline{\mathcal{M}_g}$ contains the configuration $\mathcal{C}$ of rational curves

$$\{C\} = \mathcal{C}, \text{ where every } C = \psi_{e\subset\Gamma}(\mathbb{P}^1) \tag{3.13}$$

for some flag $e \subset \Gamma$. For three flags from the same nest (2.17) the curve C is the same. We can consider this configuration of rational curves as a reducible curve

$$\mathcal{C} = \bigcup C \tag{3.14}$$

that is the union of all components. It is easy to see that

Proposition 3.2 *(1) for every component C of $\mathcal{C}$*

$$C \bigcap \mathcal{P}_g = P_\Gamma + P_{\Gamma'} + P_{\Gamma''} \tag{3.15}$$

where $(\Gamma, \Gamma', \Gamma'')$ is a nest of graphs (2.17);

(2) for a pair C, C' of componets either the intersection $C \bigcap C'$ is empty or it is transversal and

$$C \bigcap C' \in \mathcal{P} \tag{3.16}$$

(3) the set $S(\Gamma)$ of components through every point $P_\Gamma \in \mathcal{P}$ is enumerated by the set $E(\Gamma)$;

Now let ETG_g be the set of 3-valent flags of genus g with the projection to the set of graphs

$$\gamma \colon ETG_g \to TG_g, \quad \gamma^{-1}(\Gamma) = E(\Gamma) \tag{3.17}$$

and $Com(\mathcal{C})$ be the set of components of the reducible curve $\mathcal{C}$. Then we have 3-cover

$$c \colon ETG_g \to Com(\mathcal{C}) \tag{3.18}$$

with remification along flags $e \subset \Gamma$, where e is a loop in Γ. Let LTG_g be the set of such loop flags. Then one has

$$3 \cdot |Com(\mathcal{C})| - |LTG_g| = 3(g-1) \cdot |TG_g|. \tag{3.19}$$

Thus, roughly speaking, the cardinality of $Com(\mathcal{C})$ is equal to $(g-1)$ times the cardinality of $\mathcal{P}_g$.

We saw that if an edge $e \in E(\Gamma)$ is not a loop then the map $\psi_{e\subset\Gamma}$ (3.7) is an embedding and the corresponding component C of the reducible curve $\mathcal{S} \subset \overline{\mathcal{M}_g}$ is a smooth rational curve with fixed 3 points $(P_\Gamma, P_{\Gamma'}, P_{\Gamma''})$ such that the corresponding 3 graphs can be equipped with flags structures forming the nest (2.17). Over every of such points the tangent space admits decomposition (2.11).

Geometry of the other curves from the family of curves parametrized by C is very near to the geometry of large limit curves: let $e \in E(\Gamma)$, $v, v' = \partial e$ and $P_{E,\sigma}$ is the point of C corresponding to the elliptic curve (3.2) with a point of order 2 (3.4). Then $P_{E,\sigma}$ has 2g-4 old components

$$\bigcup_{v''\neq v,v'} P_{v''} \tag{3.20}$$

with triples of points and 3g-1 nodes $p_{e'}, e \neq e'$ and one new component $P_{v,v'}$ with the quadruple of points (see (3.1). Then

(1) the canonical class $K_{P_{E,\sigma}}$ is a line bundle: a restriction of it to every component $P_{v''}, v'' \neq v, v'$ is the sheaf of meromorphic differentials ω with simple poles at $p_e, p_{e'}, p_{e''}$ where $\{e, e', e''\} = S(v)$;

(2) the restriction of the canonical class to the component $P_{v,v'}$ is the sheaf of meromorphic differentials ω with simple poles at $p_{e_1}, p_{e_2}, p_{e'_1}, p_{e''_2}$ (see (3.1) and (3.3)).

(3) Thus
$$c(K_{P_{E,\sigma}}) = (2, 1, ..., 1) \in NS_{P_{E,\sigma}} \tag{3.21}$$
where the first coordinate corresponds to $P_{v,v'}$.

(4) Again every holomorphic section s of the canonical class is a collection of meromorphic differentials $\{\omega_{v''}\}$ on the components $P_{v''}$ with poles at $p_e, p_{e'}, p_{e''}$ where $\{e, e', e''\} = S(v)$ and a meromorphic differential $\omega_{v,v'}$ on the component $P_{v,v'}$ with poles at the quadruple with the same constraints (2.3) and (2.4).

(5) The canonical map defined by the complete linear system $|K_{P_{E,\sigma}}|$
$$\phi_K \colon P_{E,\sigma} \to \mathbb{P}^{g-1} \tag{3.22}$$
sends $P_{v''}$ to a configuration of lines and $P_{v,v'}$ to a conic in $\mathbb{P}^{g-1}$.

(6) Again the dimension of the space of quadratic differentials on $P_{E,\sigma}$ is equal to 3g-3 and this curve is an orbifold smooth point of $\overline{\mathcal{M}_g}$.

(7) The double canonical map of $P_{E,\sigma}$ given by the complete linear system $|2K_\Gamma|$ is an embedding

$$\phi_{2K_{P_{E,\sigma}}} : P_{E,\sigma} \to \mathbb{P}^{3g-4} = \mathbb{P}T\overline{\mathcal{M}_g}_{P_{E,\sigma}}. \tag{3.23}$$

(see (2.11)).

(8) The images of nodes

$$\{\phi_{2K_{P_{E,\sigma}}}(p_{e'})\}, \quad e \neq e' \in E(\Gamma) \tag{3.24}$$

define the decomposition of the restriction of the tangent bundle

$$T\overline{\mathcal{M}_g}|_C = TC \bigoplus (\bigoplus_{e\neq e' \in E(\Gamma)} L_{e'}) \tag{3.25}$$

where the fiber of the line bundle $L_{e'}$ over a point is the component of the decomposition (2.11).

(9) Thus every line bundle $L_{e'}$ from the previous decomposition is the tautological line bundle

$$L_{e'} = \mathcal{O}_C(-1), \tag{3.26}$$

(10) hence the previous decomposition is

$$T\overline{\mathcal{M}_g}|_C = \mathcal{O}_C(2) \bigoplus (3g-4)\mathcal{O}_C(-1). \tag{3.27}$$

(11) The restriction to C of the canonical class of $\overline{\mathcal{M}_g}$ is

$$K_{\overline{\mathcal{M}_g}}|_C = \mathcal{O}_C(3(g-2)). \tag{3.28}$$

For a singular curve the moduli spaces of vector bundles are not compact. These moduli spaces admit the compactifications by torsion free sheaves. These sheaves are not local free only over nodes and the theory of the compactification is very close to the theory for algebraic surfaces. All details of this theory can be found in forthcoming Artamkin paper [A]. All constructions are very close to the complex gauge theory on smooth compact Riemann surfaces. We will see this in the following section where we apply the main constructions of the complex gauge theory from section 1 to ll-curves.

4 The complex gauge theory on ll-curves

A vector bundle E on P_Γ is called topologicaly trivial if the restrictions $E|_{P_v}$ are trivial for all $v \in V(\Gamma)$. We begin with the description of the moduli spaces of topologicaly trivial

(1) line bundles $Pic_0(P_\Gamma)$,

(2) rk 2 semi-stable bundles M^{ss}_{vb}.

Since each component P_v is projective line the restriction of any topologicaly trivial line bundle equals

$$L|_{P_v} = \mathcal{O}_{P_v}. \tag{4.1}$$

To describe a line bundle on P_Γ consider a collection of any trivializations of L on all components P_v of P_Γ and denote the line bundle with such addition structure by L_0.

Now to get a line bundle L on P_Γ we have to concord every pair of line bundles $\mathcal{O}_{P_v}, \mathcal{O}_{P_{v'}}$ at common point p_e if $v, v' = \partial e$. Under our trivializations such concordance is given by a multiplicative constant $a(\vec{e}) \in \mathbb{C}^*$ of oriented edge such that under the orientation reversing involution i_e

$$a(i_e(\vec{e})) = a(\vec{e})^{-1}. \tag{4.2}$$

Thus L_0 defines a map

$$a \colon \vec{E}(\Gamma) \to \mathbb{C}^*. \tag{4.3}$$

subjecting (4.2).

The changing of trivialization is given by a function

$$\tilde{\lambda} \colon V(\Gamma) \to \mathbb{C}^* \tag{4.4}$$

which acts on the functions a by the formula

$$\tilde{\lambda}(a(\vec{e})) = \tilde{\lambda}(v_s) \cdot a(\vec{e}) \cdot \tilde{\lambda}(v_t)^{-1} \tag{4.5}$$

where $\partial\vec{e} = v_s \bigcup v_t$ and v_s is the source of arrow and v_t is the target.

Geometrically the construction of a concordance $a(\vec{e})$ at a point p_e for ll-curve plays the role of the period (monodromy) $\rho([C_e])$ of a flat connection $[\rho] \in \mathrm{CLRep}(\pi_1(\Sigma_\Gamma), \mathbb{C}^*)$ along the cycle $\vec{C}_e$ for non-singular curve Σ_Γ from the previous section.

Hence if $\mathcal{A}_{\mathbb{C}}$ is the space of trivialized topologicaly trivial line bundles that is the space of functions a (4.2) subjecting (4.3) and $\mathcal{G}_{\mathbb{C}}$ be the group of changing of trivializations (4.4). Then the moduli space of line bundles on P_Γ

$$Pic_0(P_\Gamma) = Hom(ker \quad r, \mathbb{C}^*) = (\mathbb{C}^*)^g \tag{4.6}$$

is the abelian Schottky space.

For rk 2 bundle E on P_Γ we have the same type description: consider a collection of trivializations of restrictions $E|_{P_v} = \mathcal{O} \oplus \mathcal{O}$ on all components of P_Γ and denote the bundle with such addition structure by E_0.

Now we have to glue every pair of bundles $\mathcal{O}_{P_v} \oplus \mathcal{O}_{P_v}, \mathcal{O}_{P_{v'}} \oplus \mathcal{O}_{P_{v'}}$ at common point p_e if $v, v' = \partial e$. Such gluing is given by a function

$$a \colon \vec{E}(\Gamma) \to SL(2, \mathbb{C}) \tag{4.7}$$

subjection to the equation

$$a(i_e(\vec{e})) = a(\vec{e})^{-1}. \tag{4.8}$$

The changing of the trivialization is given by the function

$$\tilde{g} \colon V(\Gamma) \to SL(2, \mathbb{C}) \tag{4.9}$$

which acts on the functions a by the formula

$$\tilde{g}(a(\vec{e})) = \tilde{u}(v_s) \cdot a(\vec{e}) \cdot \tilde{g}(v_t)^{-1} \tag{4.10}$$

where $\partial \vec{e} = v_s \bigcup v_t$.

We would like to emphisize again that geometrically the construction of a concordance $a(\vec{e})$ at a point p_e for ll-curve plays the role of the period (monodromy) $\rho([C_e])$ of a flat connection $[\rho] \in \mathrm{CLRep}(\pi_1(\Sigma_\Gamma), \mathrm{SL}(2, \mathbb{C}))$ along the cycle $\vec{C}_e$ for non-singular curve Σ_Γ from the previous section.

Again if $\mathcal{A}_{\mathbb{C}}$ is the space of trivialized topologicaly trivial rk 2 bundles with the group of trivializations $\mathcal{G}_{\mathbb{C}}$ acting by the formula (4.10) then the moduli space of semi-stable topologicaly trivial vector bundles on P_Γ is the quotient

$$M_{vb}^{ss}(P_\Gamma) = \mathcal{A}_{\mathbb{C}} / \mathcal{G}_{\mathbb{C}}. \tag{4.11}$$

Again it is easy to see that

$$M_{vb}^{ss}(P_\Gamma) = \mathrm{CLRep}(ker \quad r, SL(2, \mathbb{C})) = S_g \tag{4.12}$$

(see (1.31)) is the Schottky space of genus g.

Now we would like to describe the space of holomorphic flat connections on vector bundles over this moduli space.

To do this let us draw the trinion decomposition corresponding to Σ_Γ (1.1)- (1.2) again. We can construct the space of holomorphic flat bundles gluing the spaces of $\mathrm{SL}(2,\mathbb{C})$-flat connections on the every component $P_v - \bigcup_{e\in S(v)} p_e$ that is on $\mathbb{P}^1$ without 3 points.

Such spaces of $\mathrm{SL}(2,\mathbb{C})$-flat connections are given as the classes of representations spaces

$$A^{na}(P_v - \bigcup_{e\in S(v)} p_e) = CLRep(\pi_1(P_v - \bigcup_{e\in S(v)} p_e), \mathrm{SL}(2,\mathbb{C})) \tag{4.13}$$

where $\pi_1(\tilde{v}) = F_2$ is the free group with 2 generators. Thus we can identify the spaces

$$A^{na}(P_v - \bigcup_{e\in S(v)} p_e) = A^{na}(\tilde{v}) = S_2 \tag{4.14}$$

(see (1.30)) where the last one is the Schottki space of genus 2.

If two vertices v and v' are joined by the edge e that is if the components P_v and $P_{v'}$ intersect at p_e then we can glue $A^{na}(P_v - \bigcup_{e\in S(v)} p_e)$ and $A^{na}(P_{v'} - \bigcup_{e\in S(v')} p_e)$ along this intersection point p_e just in the same way as we do this for the smooth case (1.32)- (1.35). Again for a semi-simple element $m \in \mathrm{SL}(2,\mathbb{C})$ the stabilizer $Z(m) = \mathbb{C}^*$. The result of such gluing is

$$A^{na}(P_v - \bigcup_{e\in S(v)} p_e) * A^{na}(P_{v'} - \bigcup_{e\in S(v')} p_e)) = S_3 \tag{4.15}$$

if $P_v \bigcap P_{v'} = p_e$. Gluing all components we get our ll-curve P_Γ and gluing all spaces of flat bundles $A^{na}(P_v - \bigcup_{e\in S(v)} p_e))$ we get the same space $\mathrm{CLRep}(\pi_1(\Sigma_\Gamma), \mathrm{SL}(2,\mathbb{C}))$ of flat bundles on P_Γ as for the smooth case:

$$A^{na}(P_\Gamma) = \mathrm{CLRep}(\pi_1(\Sigma_\Gamma), \mathrm{SL}(2,\mathbb{C})) = A^{na}(\Sigma_\Gamma). \tag{4.16}$$

The direct interpretation gives the following coincidence:

Proposition 4.1 *(1) The forgetful map (1.25) for ll-curve P_Γ*

$$f\colon A^{na}(P_\Gamma) = \mathrm{CLRep}(\pi_1(\Sigma_\Gamma), \mathrm{SL}(2,\mathbb{C})) \to M^{ss}_{vb}(P_\Gamma) = \tag{4.17}$$

$$= \mathrm{CLRep}(ker \quad r, \mathrm{SL}(2,\mathbb{C})) = S_g$$

is the natural map induced by the restriction of representations to the kernel $ker \quad r \subset \pi_1(\Sigma_\Gamma)$ *(see (1.7)- (1.9)).*

(2) This forgetful map admits a holomorphic section

$$s\colon \mathrm{CLRep}(ker \quad r, \mathrm{SL}(2,\mathbb{C})) = S_g \hookrightarrow CLRep(\pi_1(\Sigma_\Gamma), \mathrm{SL}(2,\mathbb{C})) \tag{4.18}$$

induced by epimorphism r *(1.7) and isomorphism* Int *(1.20).*

(3) Thus the affine bundle (4.17) is the Higgs fields vector bundle (or the Hitchin bundle from [H]), which we may consider as the cotangent bundle to the singular variety $M^{ss}_{vb}(P_\Gamma) = S_g$*.*

(4) For every complex structure $I \in \tau_g$ *(see the text between formula (1.20) and (1.21)) we have the rational map*

$$F_{(\Gamma,I)}\colon M^{ss}_{vb}(P_\Gamma) = S_g \to M^{ss}(\Sigma_I) \tag{4.19}$$

which is the composition of section s *(4.18) and forgetful map* f *(1.25).*

Again consider the space of representations $Rep(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{C}))$ (without the diagonal adjoint factorization) (1.37) with the projection p (1.38). The geometrical meaning of this procedure is almost the same as before: we fixed a component P_v and a trivialization $E|_{P_v} = \mathbb{C}^2$ of the vector bundle E over this component. The moduli space of vector bundles with such additional structure

$$\widetilde{M^{ss}_{vb}(P_\Gamma)} = (\mathrm{SL}(2,\mathbb{C}))^g \tag{4.20}$$

is a non singular algebraic variety with the structure of principal $PGL(2,\mathbb{C})$-bundle over S_g:

$$\phi(\mathrm{SL}(2,\mathbb{C}))^g \to S_g \tag{4.21}$$

where the group $\mathrm{SL}(2,\mathbb{C})$ modulo ± 1 acts on $E|_{P_v} = \mathbb{C}^2$ as (2 X 2)-matrices with determinant 1.

The moduli space of flat vector bundles on P_Γ with such additional structure is $Rep(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{C}))$ with the structure p (1.38) of a principal $PGL(2,\mathbb{C})$-bundle. The moduli space of flat bundles $Rep(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{C}))$ can be described as the preimage of 1 for the algebraic map

$$com\colon (\mathrm{SL}(2,\mathbb{C}))^{2g} = \prod \mathrm{SL}(2,\mathbb{C})_{a_i} \times \prod \mathrm{SL}(2,\mathbb{C})_{b_i} \to \mathrm{SL}(2,\mathbb{C}) \tag{4.22}$$

$$com(g(a_1),...,g(a_g),g(b_1),...,g(b_g)) = \prod_{i=1}^{g}[g(a_i),g(b_i)]$$

(see (1.9)):

$$com^{-1}(1) = Rep(\pi_1(\Sigma), \mathrm{SL}(2,\mathbb{C}))$$

which is an algebraic variety obviously.

Again regular functions (1.40)-(1.41) defines the divisors (1.42) with the intersection (1.43).

But now the affine bundle (4.17) over $M^{ss}_{vb}(P_\Gamma)$ is a vector bundle because it admits the holomorphic section (4.18). It has as the fiber over a point $E \in M^{ss}_{vb}(P_\Gamma)$ the space

$$f^{-1}(E) = H^0(P_\Gamma, adE \times K_\Gamma) = \mathbb{C}^{3g-3}_E \tag{4.23}$$

which is a *packet* of classes of representations of $\pi_1(\Sigma_\Gamma)$ (see (1.2), (1.6)).

Again consider the restriction of the principal $\mathrm{SL}(2,\mathbb{C})$-bundle (1.38) to this affine space

$$p\colon p^{-1}(H^0(P_\Gamma, adE \times K_\Gamma)) \to H^0(P_\Gamma, adE \times K_\Gamma). \tag{4.24}$$

Restricting the functions (1.40) to this affine variety we get the complete intersection

$$i(S_g) \cap (H^0(P_\Gamma, adE \times K_\Gamma)) = E \tag{4.25}$$

as a point of the moduli space $M^{ss}_{vb}(P_\Gamma) = S_g$. That is for every vector bundle E on P_Γ

(1) *this complete intersection is non empty*, and

(2) *there exists unique class of the Schottki representation.*

From this we get immideately our final result:

Theorem 4.1 *On general curve general stable vector bundle admits the Schottki representation.*

Remark Of course all our constructions are valid for vector bundle of any rank. For simplicity we worked with rk 2.

In this paper we uses only direct "classical" arguments. But obviously these results can be refined by using the Higgs bundles technique for ll-curves. In particular we know (see [H]) that the Higgs pairs space $T^*M^{ss}(P_\Gamma)$ admits a partial compactification $\overline{T^*M^{ss}(P_\Gamma)}$ completing the fibers of the holomorphic moment map. This partial compactification defines a compactification

$$\overline{S_g} \subset \overline{T^*M^{ss}(P_\Gamma)}. \tag{4.26}$$

of the moduli space of vector bundles on ll-curves by torsion free sheaves. Now we postpone this beautifull subject to the next paper of the serie refining [T3].

References

[A] I. V. Artamkin "Algebraic geometry of ll-curves." appears in Izv. RAN,ser.Math.

[CLM] C. Ciliberto, A. Lopez and R. Miranda "Projective degenerations of K3 surfaces, Gaussian maps and Fano threefolds", Invent. Math. v.114, 1993, p. 641- 667.

[HT] Hatcher A. and Thurston W. "A presentation for the mapping class group of closed oriented surface", Topology, 19 (3),1980,221-237.

[H] Hitchin N. "Stable bundles and integral systems." Duke Math. Jour. ,v.54, n.1, !987, 91-114.

[JW] L. C. Jeffrey and J. Weitsman, Bohr–Sommerfeld orbits in the moduli space of flat connections and the Verlinde dimension formula, Commun. Math. Phys. **150** (1992), 593–630

[T1] A.N. Tyurin, Periods and principal parts on a Riemann surface with a flat structure, Izv. Akad. Nauk SSSR, Ser. Mat., 41:6, (1977), 1425-1443, Math. USSR-Izv., 11, 1979.

A.N. Tyurin, On periods of quadratic differentials. Uspekhi Mat.Nauk, 33:6, (1978), 59-88; Russian Math. Surveys, 33:6 (1979), 169-221.

[T2] A. Tyurin, “Three mathematical facets of SU(2)-spin networks”, math.DG/0011035, 1-20.

[T3] A. Tyurin,"Quantization, Classical and Quantum Field Theory and Theta-Functions", Les publication du CRM, a paraitre 2002.