On the moduli of pairs of stable sheaves related by non-trivial morphisms

B. V. Karpov*

March 26, 2001

Introduction

Let $\mathcal{M}(r, c_1, \delta)$ denote the moduli space of semi-stable sheaves by Gieseker on $\mathbb{P}^2$ over $\mathbb{C}$ with the rank r, the first Chern class c_1 and Euler characteristic (alternated sum of dimensions of cohomology spaces) equal to $-\delta$. It is known [12] that $\mathcal{M}(r, c_1, \delta)$ is a projective scheme, it parametrizes classes of semi-stable sheaves that have isomorphic direct sums of quotients of Jordan-Hölder filtration. In this paper we deal with the case when r and c_1 are coprime. Then $\mathcal{M}(r, c_1, \delta)$ is smooth and carries a universal family [12, Ex.6.8, Th.6.11]. Besides, in this case semi-stability by Gieseker is equivalent to stability by Mumford – Takemoto, so the word "stable" is used in this paper for the latter notion. Since the Jordan-Hölder filtration of a stable sheaf is trivial, we shall often write "$X \in \mathcal{M}(r, c_1, \delta)$" meaning that X is a stable sheaf with $r(X) = r$, $c_1(X) = c_1$ and $-\chi(X) = \delta$. Also, we use short notations $\mathcal{M}_v$ or $\mathcal{M}(v)$ for $\mathcal{M}(r, c_1, \delta)$, where $v = (r, c_1, \delta)$ is a triple of integers with $r > 0$.

For a sheaf X without torsion denote

$$Q_X = X^{**}/X \, ,$$

this is zero-dimensional sheaf i.e. having $r(Q_X) = 0$ and $c_1(Q_X) = 0$. For $r > 1$ the moduli space $\mathcal{M}(r, c_1, \delta)$ has the following stratification:

$$\mathcal{M}(r, c_1, \delta) = \bigcup_{p \geqslant 0} \mathcal{M}^{(p)}(r, c_1, \delta) \, ,$$

where $\mathcal{M}^{(p)}(r, c_1, \delta)$ parametrizes the sheaves X such that $h^0(Q_X) = p$. We will use also notation $\mathcal{M}_v^{(p)}$. Note that the number of strata is finite, it is equal to $\delta - \delta_{\min}$, where $\delta_{\min}$ is the lowest integer such that $\mathcal{M}(r, c_1, \delta_{\min})$ is non-empty (see details in 3). Clearly, distinct $\mathcal{M}^{(p)}(r, c_1, \delta)$ are disjoint and $\mathcal{M}^{(0)}(r, c_1, \delta)$ is an open subset of $\mathcal{M}(r, c_1, \delta)$ parametrizing locally free sheaves. In other words, $\mathcal{M}_v^{(0)}$ is the moduli space of vector bundles.

Let $v = (r, c_1, \delta)$ and $w = (r', c_1', \delta')$ be two triples of integers such that the moduli spaces $\mathcal{M}_v$ and $\mathcal{M}_w$ are non-empty and

$$\begin{vmatrix} r & c_1 \\ r' & c_1' \end{vmatrix} = 1 \, . \qquad (1)$$

*email: karpov@karpov.mccme.ru

In this paper we study the moduli space $\widetilde{\mathcal{M}}_{v\to w}$, which parameterizes the triples

$$\{(X,Y,[\varphi]) \mid X\in\mathcal{M}_v,\ Y\in\mathcal{M}_w,\ [\varphi]\in P(\mathrm{Hom}(X,Y))\}\,.$$

We consider only the case $r(X)<r(Y)$, the case $r(X)>r(Y)$ being dual in some sense.

In particular, we show that the image of natural map

$$\Phi:\widetilde{\mathcal{M}}_{v\to w}\longrightarrow\mathcal{M}_v\,,\qquad (X,Y,[\varphi])\mapsto X$$

is the union of strata $\mathcal{M}_v^{(p)}$, where the lower and the upper bounds of range of values of p depend on the triples v and w (Theorem 3.6).

Besides, Φ is surjective iff $\mathcal{M}_v^{(0)}\subset\mathrm{im}\,\Phi$, in this case the preimage $\Phi^{-1}(\mathcal{M}_v^{(0)})$ of moduli space of bundles is isomorphic to the projectivization of a vector bundle over the product $\mathcal{M}_v^{(0)}\times\mathcal{M}_{w-v}$ (Theorem 4.2). Note that all moduli spaces under consideration are smooth and carry universal families because (1) implies coprimeness of ranks and c_1's for the sheaves.

Preimages of the rest strata are more complicated. In 4.4 we describe $\Phi^{-1}(X)$ as a union of parts numerated by subsheaves of Q_X, where $X\in\mathcal{M}_v^{(p)}$ and $p>0$.

This paper is motivated by the following problem, which can be regarded as mathematical testing of S-duality conjecture ([8],[9],[10]). Consider the sequence $\{r_n\}_{n\in\mathbb{Z}}$ determined by recurrent relations and initial values

$$r_{n+1}+r_{n-1}=3\,r_n\,,\qquad r_0=r_{-1}=1\,. \tag{2}$$

Several members of this sequence near r_0 are the following:

$$\begin{array}{ccccccccccccc} \dots & r_{-5} & r_{-4} & r_{-3} & r_{-2} & r_{-1} & r_0 & r_1 & r_2 & r_3 & r_4 & r_5 & \dots \\ \dots & 34 & 13 & 5 & 2 & 1 & 1 & 2 & 5 & 13 & 34 & 89 & \dots \end{array}$$

It is easily seen that $r_n=r_{-n-1}$ i.e. the sequence is symmetric w.r.to the "middle" between r_{-1} and r_0.

Consider the moduli spaces

$$\mathcal{M}_n(\delta):=\mathcal{M}(r_n,-r_{n-1},\delta)\,.$$

Note that r_n and r_{n-1} are coprime because of the formula

$$\begin{vmatrix} r_{n-1} & r_n \\ r_n & r_{n+1}\end{vmatrix}=1\,, \tag{3}$$

which follows from (2) (see Section 5). Hence, as it was pointed above, each $\mathcal{M}_n(\delta)$ is smooth and carries a universal family $\mathcal{E}_{n,\delta}\to\mathcal{M}_n(\delta)\times\mathbb{P}^2$, which is defined up to tensoring by the pull-back of an invertible sheaf on $\mathcal{M}_n(\delta)$. Suppose that universal families are chosen by a reasonable way, e.g. via construction that describes $\mathcal{E}_{n,\delta}$ simultaneously, see for example [5]. Let $\pi:\mathcal{M}_n(\delta)\times\mathbb{P}^2\to\mathcal{M}_n(\delta)$ be the projection and put $\mathcal{H}_{n,\delta}=R^1\pi_*\mathcal{E}_{n,\delta}$. This is a locally free sheaf of rank δ over $\mathcal{M}_n(\delta)$, herewith the dimension of $\mathcal{M}_n(\delta)$ is equal to $2r_n\delta$. Mathematical testing of S-duality in this case is examining the functions

$$Z_{\mathbf{b},n}(\tau)=\sum_{k\geqslant 0}c_{top}(\mathcal{H}_{n,k})^{2r_n}q^k\,,\qquad\text{where}\quad q=e^{2\pi i\tau}\,,\quad \mathrm{Im}\,\tau>0\,, \tag{4}$$

for modular properties. Note that each of the moduli spaces $\mathcal{M}_n(0)$ is zero-dimensional and parametrizes a unique sheaf, which is a sheaf of sections of an exceptional bundle E_n (see [1], or [4], [6] on exceptional bundles). Herewith, $E_{-1} = \mathcal{O}_{\mathbb{P}^2}(-2)$, $E_0 = \mathcal{O}_{\mathbb{P}^2}(-1)$ and the other bundles E_n are obtained from the canonical exact triples

$$0 \longrightarrow E_{n-1} \xrightarrow{ev^*} E_n \otimes \mathrm{Hom}(E_n, E_{n+1}) \xrightarrow{ev} E_{n+1} \longrightarrow 0\,, \tag{5}$$

which in view of canonical isomorphism $\mathrm{Hom}(E_n, E_{n+1}) = \mathrm{Hom}(E_{n-1}, E_n)^*$ give maps of the form $(E_n, E_{n+1}) \overset{L}{\mapsto} (E_{n-1}, E_n)$ and $(E_{n-1}, E_n) \overset{R}{\mapsto} (E_n, E_{n+1})$ called mutations.

Hypothetically, there should be transformations of moduli spaces $\mathcal{M}_n(\delta)$ concordant with the mutations of exceptional bundles $E_n = \mathcal{M}_n(0)$. These transformations should clarify relations between the functions $Z_{\mathbf{b},n}(\tau)$. It seems quite natural to seek such a transformations starting with looking for possible constructions with the sheaves. This paper provides a first move in this direction.

In Section 5 we reformulate description of $\operatorname{im}\Phi$ for the case when $v = (r_n, -r_{n-1}, \delta)$, $w = (r_{n+1}, -r_n, \delta')$ and $n \geqslant 0$ (Theorem 5.4) and prove that the map

$$\Psi : \widetilde{\mathcal{M}}_{v\to w} \longrightarrow \mathcal{M}_w\,, \quad (X, Y, [\varphi]) \longmapsto Y$$

isn't surjective for almost all values of δ' (Proposition 5.5).

The author is grateful to S. A. Kuleshov and A. S. Tikhomirov for useful discussions and to Max-Planck-Institut für Mathematik in Bonn for hospitality.

1 Preliminaries

We start with some notation. For a coherent sheaf X over $\mathbb{P}^2$ put

$$h^i(X) = \dim H^i(X)\,, \qquad \chi(X) = \sum_i (-1)^i h^i(X)\,, \qquad \mu(X) = \frac{c_1(X)}{r(X)}\,,$$

the last value is called *slope*. We represent the image of X in $K_0(\mathbb{P}^2)$ by the vector

$$v_X = (r(X), c_1(X), -\chi(X))\,, \tag{6}$$

which is additive by X. The last component of v_X is related with $c_2(X)$ by Riemann – Roch theorem

$$-\chi(X) = c_2(X) - c_1(X)\frac{c_1(X)+3}{2} - r(X)\,. \tag{7}$$

Let Y be also a coherent sheaf over $\mathbb{P}^2$. Put

$$h^i(X,Y) = \dim \mathrm{Ext}^i(X,Y)\,, \qquad \chi(X,Y) = \sum_i (-1)^i h^i(X,Y)\,.$$

Euler characteristic $\chi(X,Y)$ is expressed via $\chi(X)$ and $\chi(Y)$ by Riemann – Roch formula

$$\chi(X,Y) = -r(X)r(Y) - 3r(Y)c_1(X) - c_1(X)c_1(Y) + r(X)\chi(Y) + r(Y)\chi(X)\,. \tag{8}$$

This expression defines non-symmetric bilinear form on $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \cong K_0(\mathbb{P}^2)$, where the identification is given by (6).

1.1 Remark 1. If X and Y are semistable sheaves such that $0 < \mu(Y) - \mu(X) < 3$, then $h^0(Y,X) = h^2(Y,X) = 0$ and $h^1(Y,X) = -\chi(Y,X)$. In particular, these equalities take place if $r(X)c_1(Y) - r(Y)c_1(X) = 1$.

2. For any semi-stable sheaf X with $-3 < \mu(X) < 0$ one has: $h^0(X) = h^2(X) = 0$ and $-\chi(X) = h^1(X)$.

Indeed, the first statement follows immediately from stability properties and Serre duality. Note that the relation $r(X)c_1(Y) - r(Y)c_1(X) = 1$ implies the inequality $0 < \mu(Y) - \mu(X) < 1$. The second statement is a particular case of the first one for $Y = \mathcal{O}_{\mathbb{P}^2}$.

Let $v = (r, c_1, \delta)$ be a triple of integers such that $r > 0$, g.c.d.$(r, c_1) = 1$ and the moduli space $\mathcal{M}_v = \mathcal{M}(r, c_1, \delta)$ is non-empty. Then the dimension of the moduli space is given by the formula

$$\dim \mathcal{M}_v = 1 - \chi(v, v)\,. \tag{9}$$

Indeed, let X be stable sheaf with $v_X = v$, then $h^2(X,X) = h^0(X, X(-3)) = 0$ and $h^0(X,X) = 1$, thus $h^1(X,X) = \dim \mathcal{M}_v = 1 - \chi(X,X)$.

Applying formula (9) to $v = (r_n, -r_{n-1}, \delta)$ we have:

$$\dim \mathcal{M}_n(\delta) = 2r_n\delta\,.$$

In particular, $\dim \mathcal{M}_n(0) = 0$ and $\mathcal{M}_n(0)$ consists of one point E_n, which is exceptional vector bundle. Remind that there are two equivalent definitions of exceptional sheaves on $\mathbb{P}^2$: a sheaf E is exceptional if

(i) $h^0(E,E) = 1$ and $h^j(E,E) = 0$ for $j > 0$ ([4], [6]); or

(ii) E is stable and $\chi(E,E) = 1$ ([1]).

Any exceptional sheaf on $\mathbb{P}^2$ is locally free. The simplest argument is the triviality of $\mathrm{Ext}^1(E,E)$ and free group action on $\mathbb{P}^2$. Besides, this follows from 3.3.

The exceptional bundles E_n are known to satisfy the following cohomological conditions: $h^j(E_{n+1}, E_n) = 0$ for all j, $h^0(E_n, E_{n+1}) = 3$ and $h^j(E_n, E_{n+1}) = 0$ for $j > 0$. On the language of theory of exceptional sheaves, each pair (E_n, E_{n+1}) is exceptional and the sequence $\{E_n\}_{n\in\mathbb{Z}}$ is generated by the pair $(E_{-1}, E_0) = (\mathcal{O}_{\mathbb{P}^2}(-2), \mathcal{O}_{\mathbb{P}^2}(-1))$.

1.2 Lemma. ([9]) *Suppose there is an exact triple of sheaves*

$$0 \longrightarrow X \longrightarrow \tilde{X} \longrightarrow Q \longrightarrow 0\,,$$

where $r(Q) = 0$ and $c_1(Q) = 0$. Then the following statements hold:
a) if $\tilde{X}$ is (semi-)stable, then X is (semi-)stable;
b) if $\tilde{X}$ is torsion-free and X is (semi-)stable, then $\tilde{X}$ is (semi-)stable.

Proof. We have: $r(X) = r(\tilde{X})$ and $c_1(X) = c_1(\tilde{X})$. The statement a) is trivial because X has no torsion as a subsheaf of a torsion-free sheaf and any subsheaf of X is also a subsheaf of $\tilde{X}$. Let us prove b).

Suppose F is a subsheaf of $\tilde{X}$ such that $0 < r(F) < r(\tilde{X})$, then there is the commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & F & \xrightarrow{\mathrm{id}} & F & & \\
 & & & & \downarrow & & \downarrow\psi & & \\
 & & 0 & \longrightarrow X \longrightarrow & \tilde{X} & \longrightarrow & Q & \longrightarrow & 0 \\
 & & & \mathrm{id}\downarrow & \downarrow & & \downarrow & & \\
0 \longrightarrow \ker\psi & \longrightarrow & & X \longrightarrow & G & \longrightarrow & \tilde{Q} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Here $\tilde{Q} = \operatorname{coker}\psi$. We have: $r(\operatorname{im}\psi) = 0$ and $c_1(\operatorname{im}\psi) = 0$ because $\operatorname{im}\psi$ is a subsheaf of Q; whence $r(\ker\psi) = r(F)$, $c_1(\ker\psi) = c_1(F)$, and $\mu(\ker\psi) = \mu(F)$. But $\ker\psi$ is a subsheaf of X, so $\mu(\ker\psi) < \mu(X)$ (respectively $\leqslant$) as far as X is stable (resp. semistable). Therefore, $\mu(F) < \mu(\tilde{X})$ (resp. $\leqslant$) that means stability (semi-stability) of $\tilde{X}$. $\square$

Suppose $r > 1$ and $X \in \mathcal{M}^{(p)}(r, c_1, \delta)$. From the lemma it follows that taking X to the reflexive span X^{**} defines a map $f : \mathcal{M}^{(p)}(r, c_1, \delta) \longrightarrow \mathcal{M}^{(0)}(r, c_1, \delta - p)$. The preimage $f^{-1}(E)$ of a point $E \in \mathcal{M}^{(0)}(r, c_1, \delta - p)$ is Grothendieck's quotient scheme $\operatorname{Quot}(E, p)$, which parametrizes epimorphisms $E \twoheadrightarrow Q$ modulo automorphisms of Q, where $r(Q) = 0$, $c_1(Q) = 0$ and $h^0(Q) = p$. It is known that $\operatorname{Quot}(E, p)$ for locally free sheaf E is an irreducible [3] projective [7, Th. 2.2.4] scheme of dimension $p(r+1)$. We have:

$$\dim \mathcal{M}^{(p)}(r, c_1, \delta) = \dim \mathcal{M}^{(0)}(r, c_1, \delta - p) + p(r+1) = 1 - \chi(E, E) + p(r+1)\,.$$

From (8) it follows that $\chi(E, E) - \chi(X, X) = 2rp$. Therefore, we obtain the following statement.

1.3 Proposition. $\operatorname{codim}_{\mathcal{M}_v} \mathcal{M}_v^{(p)} = p(r-1)$ *for* $r > 1$. $\square$

Now let us define the moduli space $\widetilde{\mathcal{M}}_{v\to w}$. We are assuming that the vectors $v = (r, c_1, \delta)$ and $w = (r', c_1', \delta')$ satisfy (1) and the moduli spaces $\mathcal{M}_v$ and $\mathcal{M}_w$ are non-empty. In particular, r and c_1 are coprime as well as r' and c_1', so the moduli spaces $\mathcal{M}_v$ and $\mathcal{M}_w$ are smooth, projective and carry universal families $\mathcal{X}$ and $\mathcal{Y}$, these are coherent sheaves over $\mathcal{M}_v \times \mathbb{P}^2$ and $\mathcal{M}_w \times \mathbb{P}^2$ respectively. Denote by π_v, π_w and π the projections of the product $\mathcal{M}_v \times \mathcal{M}_w \times \mathbb{P}^2$ onto $\mathcal{M}_v \times \mathbb{P}^2$, $\mathcal{M}_w \times \mathbb{P}^2$ and $\mathcal{M}_v \times \mathcal{M}_w$, and put: $\tilde{\mathcal{X}} = \pi_v^* \mathcal{X}$, $\tilde{\mathcal{Y}} = \pi_w^* \mathcal{Y}$. Suppose $z = (X, Y)$ is a closed point in $\mathcal{M}_v \times \mathcal{M}_w$ and consider the base change homomorphism

$$\tau^i(z) : \mathcal{E}xt^i_\pi(\tilde{\mathcal{Y}}, \tilde{\mathcal{X}} \otimes \omega_\pi) \otimes_{\mathcal{M}_v \times \mathcal{M}_w} \mathbb{C}(z) \longrightarrow \operatorname{Ext}^i_{\mathbb{P}^2}(Y, X \otimes \mathcal{O}_{\mathbb{P}^2}(-3))$$

for relative Ext-sheaves. Clearly, for $i = 3$ the last space is trivial, hence, by [13, Th. 1.4], $\tau^3(z)$ is an isomorphism for all z. It follows that the sheaf $\mathcal{E}xt^3_\pi(\tilde{\mathcal{Y}}, \tilde{\mathcal{X}} \otimes \omega_\pi)$ has trivial fibers at all closed points, whence we have: $\mathcal{E}xt^3_\pi(\tilde{\mathcal{Y}}, \tilde{\mathcal{X}} \otimes \omega_\pi) = 0$. Applying the quoted statement again, we conclude that $\tau^2(z)$ is an isomorphism for all z. By Serre duality we have: $\operatorname{Ext}^2_{\mathbb{P}^2}(Y, X \otimes \mathcal{O}_{\mathbb{P}^2}(-3)) = \operatorname{Hom}_{\mathbb{P}^2}(X, Y)^*$, therefore the scheme

$$\widetilde{\mathcal{M}}_{v\to w} := \mathbb{P}(\mathcal{E}xt^2_\pi(\tilde{\mathcal{Y}}, \tilde{\mathcal{X}} \otimes \omega_\pi))$$

admits the interpretation given in the introduction, i.e. parametrizes the triples $(X, Y, [\varphi])$, where $\varphi \in P\operatorname{Hom}(X, Y)$. The morphisms $\Phi : \widetilde{\mathcal{M}}_{v\to w} \to \mathcal{M}_v$ and $\Psi : \widetilde{\mathcal{M}}_{v\to w} \to \mathcal{M}_w$ are defined as compositions of the natural morphism $\mathbb{P}(\mathcal{E}xt^2_\pi(\tilde{\mathcal{Y}}, \tilde{\mathcal{X}} \otimes \omega_\pi)) \longrightarrow \mathcal{M}_v \times \mathcal{M}_w$ with the projections of the last product onto $\mathcal{M}_v$ and $\mathcal{M}_w$ respectively.

2 Properties of pairs of stable sheaves

In this section we give some properties of stable sheaves X, Y on $\mathbb{P}^2$, invariants of which satisfy the conditions

$$\begin{vmatrix} r(X) & c_1(X) \\ r(Y) & c_1(Y) \end{vmatrix} = 1 \quad \text{and} \quad r(X) < r(Y)\,. \tag{10}$$

Note that all statements of this section hold in a more general situation, namely, for stable sheaves w.r.to an ample divisor H on a smooth projective surface, herewith one should consider $c_1 \cdot H$ instead of c_1.

In particular, all the results can be applied for $X \in \mathcal{M}_n(\delta)$ and $Y \in \mathcal{M}_{n+1}(\delta')$, where $n \geqslant 0$ (see Section 5).

So, let X and Y be stable sheaves satisfying (10).

2.1 Proposition. *Any non-trivial morphism $\varphi : X \longrightarrow Y$ is an embedding and the sheaf $F = \operatorname{coker}\varphi$ either is stable or has a torsion.*

Proof. Since $\operatorname{im}\varphi$ is a quotient-sheaf of X and a subsheaf of Y, we have $0 < r(\operatorname{im}\varphi) \leqslant r(X) < r(Y)$, hence $\mu(X) \leqslant \mu(\operatorname{im}\varphi) < \mu(Y)$. Consider the plane with coordinates $(r,\, c_1)$, a sheaf E we will represent by point $(r(E),\, c_1(E))$ denoted also by E. Herewith, the slope $\mu(E) = \frac{c_1(E)}{r(E)}$ is equal to the tangent of the angle between r-axis and the line OE.

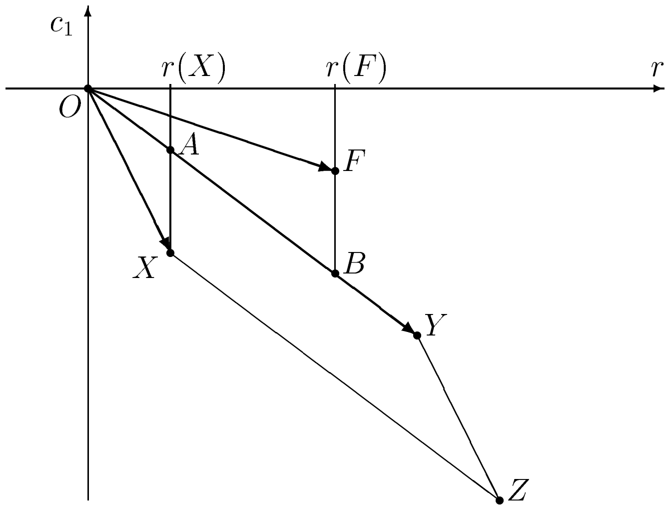

Fig. 1.

From (10) it follows that inside the parallelogram $XOYZ$ there is no point with integer coordinates, but the point $\operatorname{im}\varphi$ must belong to the triangle OXA (see Fig. 1). Therefore, there is only possibility $(r(\operatorname{im}\varphi),\, c_1(\operatorname{im}\varphi)) = (r(X),\, c_1(X))$. But then $r(\ker\varphi) = 0$, this implies that $\ker\varphi = 0$ because X has no torsion.

Let us show that if $F = \operatorname{coker}\varphi$ is torsion free then F is stable. Since $(r(F),\, c_1(F)) = (r(Y),\, c_1(Y)) - (r(X),\, c_1(X))$, the vectors $\overrightarrow{OF}$ and $\overrightarrow{OY}$ form a basis in the integral lattice, hence in the triangle OFB there is no integral points except O and F. Thus, for any quotient sheaf G of F such that $0 < r(G) < r(F)$ and $\mu(G) \leqslant \mu(F)$ one has $\mu(G) < \mu(Y)$ this contradicts stability of Y. Therefore, F is stable. $\square$

It follows that any non-trivial map $\varphi \in \operatorname{Hom}(X, Y)$ defines an exact triple

$$0 \longrightarrow X \xrightarrow{\varphi} Y \longrightarrow F \longrightarrow 0\,, \tag{11}$$

in which the sheaf F is stable whenever it has no torsion. Let $T = T(F)$ be the torsion subsheaf of F.

2.2 Proposition. *The sheaf T has zero-dimensional support and is a subsheaf of Q_X. In particular, if X is locally free then $T = 0$ and F is stable.*

Proof. Consider the following commutative diagram, in which F' is torsion free

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & T & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & F & \longrightarrow & 0 \\
 & & \psi\downarrow & & \|\,\mathrm{id} & & \downarrow & & \\
0 & \longrightarrow & X_T & \xrightarrow{\varphi'} & Y & \longrightarrow & F' & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & T & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array} \tag{12}$$

We have: $r(X) = r(X_T)$ because $r(T) = 0$. Suppose $c_1(T) = c_1(X_T) - c_1(X) > 0$, then $\mu(X_T) > \mu(X)$ and $\mu(X_T) > \mu(Y)$ since in the triangle OXA (see Fig. 1) there are no integral points except O and X. This contradicts stability of Y. Thus, $c_1(T) = 0$, this means that $\dim(\operatorname{supp} T) = 0$.

Now let us show that $T \subset Q_X$. We have: $r(X) = r(X_T)$, $c_1(X_T) = c_1(X)$, and the embedding ψ from diagram (12) induces isomorphism of reflexive spans $X^{**} \cong X_T^{**}$. Therefore, we have the diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & T & & \\
 & & \downarrow & & & & \downarrow i & & \\
0 & \longrightarrow & X & \longrightarrow & X^{**} & \longrightarrow & Q_X & \longrightarrow & 0 \\
 & & \psi\downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & X_T & \longrightarrow & {X_T}^{**} & \longrightarrow & Q_{X_T} & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & T & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array} \tag{13}$$

from which the required statement follows. □

2.3 Remark 1) The extension

$$0 \longrightarrow X \xrightarrow{\psi} X_T \longrightarrow T \longrightarrow 0\,, \tag{14}$$

in the left column of diagram (13) is defined by the image of $i \in \mathrm{Hom}(T, Q_X)$ from the right column under the map

$$\partial_T : \mathrm{Hom}(T, Q_X) \longrightarrow \mathrm{Ext}^1(T, X)$$

obtained by applying the functor $\mathrm{Hom}(T, -)$ to the exact triple

$$0 \longrightarrow X \longrightarrow X^{**} \longrightarrow Q_X \longrightarrow 0\,. \tag{15}$$

2) The sheaf X_T is stable by Lemma 1.2 since it is torsion free as a subsheaf of X^{**}. Besides, Proposition 2.1 applied to the morphism $\varphi' : X_T \longrightarrow Y$ from diagram (12) implies that F' is stable.

3) It is readily seen that ∂_T is an isomorphism, so there is one-to-one correspondence between the pairs (T, i), where $T \overset{i}{\hookrightarrow} Q_X$, modulo automorphisms of T and pairs (X_T, ψ) modulo automorphisms of X_T. Roughly speaking, subsheaves of Q_X are in one-to-one correspondence with stable sheaves X_T such that $X \subset X_T \subset X^{**}$.

2.4 Proposition. *Suppose X and F are stable sheaves, $\begin{vmatrix} r(X) & c_1(X) \\ r(F) & c_1(F) \end{vmatrix} = 1$ and the extension (11) is defined by cocycle $e \in \mathrm{Ext}^1(F, X)$. Then the following statements hold:*

1) if X is locally free then Y is stable;

2) if X is not locally free then Y is stable iff $e \notin \operatorname{im} \partial$, where $\partial : \mathrm{Hom}(F, Q_X) \longrightarrow \mathrm{Ext}^1(F, X)$ obtained by applying the functor $\mathrm{Hom}(F, -)$ to the exact triple (15).

The proof is in [11, Lemma 3.2]. (See similar statement in [9, 4.2].) □

Suppose X and F are as above. Let us describe some necessary and sufficient conditions for existence of stable Y in the middle of (11). Applying the functor $\mathrm{Hom}(F, -)$ to the exact triple (15), we have the exact sequence

$$0 \longrightarrow \mathrm{Hom}(F, Q_X) \xrightarrow{\partial} \mathrm{Ext}^1(F, X) \longrightarrow \mathrm{Ext}^1(F, X^{**}) \longrightarrow \mathrm{Ext}^1(F, Q_X) \longrightarrow 0\,,$$

where the zeroes stand for $\mathrm{Hom}(F, X^{**})$ and $\mathrm{Ext}^2(F, X)$ because of 1.1. We see that ∂ is an embedding, so we will identify $\mathrm{Hom}(F, Q_X)$ with the subspace $\operatorname{im} \partial \subset \mathrm{Ext}^1(F, X)$ in similar cases. We have:

$$h^1(F, X) - h^0(F, Q_X) = h^1(F, X^{**}) - h^1(F, Q_X)\,.$$

Applying the functor $\mathrm{Hom}(-, Q_X)$ to the exact triple

$$0 \longrightarrow F \longrightarrow F^{**} \longrightarrow Q_F \longrightarrow 0$$

and taking into account the triviality of $\mathrm{Ext}^1(F^{**}, Q_X)$, we get the exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(F, Q_X) \longrightarrow \mathrm{Ext}^2(Q_F, Q_X) \longrightarrow \mathrm{Ext}^2(F, Q_X) \longrightarrow \cdots\,.$$

Here $\mathrm{Ext}^2(F, Q_X) \cong \mathrm{Hom}(Q_X, F \otimes \mathcal{O}_{\mathbb{P}^2}(-3))^* = 0$ and $\mathrm{Ext}^2(Q_F, Q_X) \cong \mathrm{Hom}(Q_X, Q_F)^*$, hence $h^1(F, Q_X) = h^0(Q_X, Q_F)$. Therefore,

$$h^1(F, X) - h^0(F, Q_X) = -\chi(F, X^{**}) - h^0(Q_X, Q_F)$$

and we obtain the first assertion of the following

2.5 Proposition. *Suppose X, F are as in the condition of 2.4; then the following statements are true:*

1. There is a stable sheaf Y included in the triple (11) iff $h^0(Q_X, Q_F) < -\chi(F, X^{**})$.

2. There is $F_0 \in \mathcal{M}(v_F)$ and exact tripe $0 \to X \to Y_0 \to F_0 \to 0$ with stable Y_0 if and only if $-\chi(F, X^{**}) > 0$.

Proof. To prove 2 note that one can choose F_0 to be locally free or such that $\operatorname{supp} Q_{F_0} \cap \operatorname{supp} Q_X = \emptyset$, then $h^0(Q_X, Q_{F_0}) = 0$. $\square$

In the rest of this section we examine the question when Y is locally free, still assuming that the sheaves X and Y are stable, satisfy (10) and there is an exact triple of the form

$$0 \longrightarrow X \xrightarrow{\varphi} Y \longrightarrow F \longrightarrow 0. \tag{11}$$

2.6 Proposition. *Suppose that F in (11) has no torsion, then the following statements hold:*

1. The sheaf Q_X is a subsheaf of Q_Y. In particular, if Y is locally free than X is locally free too.

2. Let X be locally free, then Y is locally free iff the cocycle $e \in \mathrm{Ext}^1(F, X)$ corresponding to the extension (11) doesn't belong to the image of the map

$$\mathrm{Ext}^1(F', X) \longrightarrow \mathrm{Ext}^1(F, X) \tag{16}$$

for any sheaf F' such that $F \subsetneqq F' \subset F^{**}$.

Proof. The embeddings of X and Y into their reflexive spans define the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & & T & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 \longrightarrow & X & \longrightarrow & Y & \longrightarrow & F & \longrightarrow 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 \longrightarrow & X^{**} & \longrightarrow & Y^{**} & \longrightarrow & \tilde{F} & \longrightarrow 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow T \longrightarrow & & Q_X & \longrightarrow & Q_Y & \longrightarrow & Q & \longrightarrow 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{17}$$

It follows that T is a torsion sheaf as a subsheaf of Q_X. On the other hand, T is a subsheaf of F, whence $T = 0$, so, we obtain the first statement.

Note that $T = T(F)$ is the torsion subsheaf of F as far as F/T has no torsion being a subsheaf of $\tilde{F}$, which is torsion free by 2.2.

Now let us prove 2. Suppose e is in the image of the map (16), then the triple (11) is included into the following diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & F & \longrightarrow & 0 \\ & & {\scriptstyle \mathrm{id}} \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X & \longrightarrow & Y' & \longrightarrow & F' & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & Q & \xrightarrow{\mathrm{id}} & Q & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} \tag{18}$$

where Q is zero-dimensional sheaf. From the middle row of the diagram it follows that Y' has no torsion as an extension of torsion free sheaves. Hence, the extension in middle column is also non-trivial, thus Y is not locally free.

Now let Y be non-locally-free, than the embeddings of X and Y into reflexive spans induce the diagram of the form (18), in which $Y' = Y^{**}$, $Q = Q_Y$ and $X = X^{**}$. It follows that e belongs to the image of the map (16), herewith, F' has no torsion by 2.2, whence $F \subsetneqq F' \subset F^{**}$. $\square$

Note that if we have two sheaves F' and F'' such that $F \subsetneqq F' \subset F'' \subset F^{**}$, then there are maps

$$\mathrm{Ext}^1(F'', X) \longrightarrow \mathrm{Ext}^1(F', X) \longrightarrow \mathrm{Ext}^1(F, X)\,,$$

hence to determine whether is Y locally free or not, it is sufficient to consider minimal (by inclusion) sheaves F' i.e. non-trivial extensions of the form

$$0 \longrightarrow F \longrightarrow F' \longrightarrow \mathcal{O}_x \longrightarrow 0\,.$$

Applying to this triple the functor $\mathrm{Hom}(-, X)$, one can see that $\mathrm{Ext}^1(F', X)$ is a subspace of $\mathrm{Ext}^1(F, X)$ of codimension $r(X)$. Varying the point $x \in \mathbb{P}^2$, we obtain two-dimensional family of such subspaces.

2.7 Corollary. Suppose F in (11) has no torsion, then for $r(X) > 2$ there is an open subset in $\mathrm{Ext}^1(F, X)$, elements of which correspond to the triples (11) with locally free Y's. $\square$

Now assume that we have arbitrary F i.e. F can have a torsion. According to 2.2, the torsion subsheaf $T = T(F) \subset F$ is also a subsheaf of Q_X.

2.8 Proposition. *The sheaf Y in (11) can be locally free only if $T(F) = Q_X$.*

Proof. If Y is locally free, then in (17) one has: $Y = Y^{**}$ and $Q_Y = 0$. $\square$

Therefore, if F has a torsion, then to study when a locally free Y exists in the middle of (11) one can "get rid of torsion" by considering the diagram (12) with $T = Q_X$, $X_T = X^{**}$, and apply Proposition 2.6 to the triple $0 \to X^{**} \to Y \to F' \to 0$, where $F' = F/T(F)$.

3 The image of Φ

First we recall the construction of bound for invariants of stable sheaves over $\mathbb{P}^2$ [1]. For $\alpha \in \mathbb{Q}$ let r_α be the smallest $r > 0$ such that $r\alpha \in \mathbb{Z}$ and put

$$\Delta_\alpha = \frac{1}{2}\left(1 - \frac{1}{r_\alpha}\right).$$

For a pair (α, β) of rational numbers such that $3 + \alpha - \beta \neq 0$ define a new rational number

$$\alpha \centerdot \beta = \frac{\alpha + \beta}{2} + \frac{\Delta_\beta - \Delta_\alpha}{3 + \alpha - \beta}.$$

Consider the following map $\varepsilon : \{\frac{p}{2^q} | p, q \in \mathbb{Z}, q \geqslant 0\} \to \mathbb{Q}$ defined inductively:

$$\varepsilon(n) = n \ \text{ for } n \in \mathbb{Z}; \qquad \varepsilon\left(\frac{2p+1}{2^{q+1}}\right) = \varepsilon\left(\frac{p}{2^q}\right) \centerdot \varepsilon\left(\frac{p+1}{2^q}\right).$$

It is proved in [1] that $\mathfrak{E} = \operatorname{im} \varepsilon \subset \mathbb{Q}$ is exactly the set of slopes of exceptional bundles and each exceptional bundle is uniquely determined by its slope.

In [1, Theorem B] necessary and sufficient conditions are proved for existence of stable sheaves on $\mathbb{P}^2$ with given topological invariants. Here we reformulate this result in terms of our coordinates $(r, c_1, -\chi)$.

3.1 Theorem. **([1])** *Let $v = (r, c_1, \delta)$ be triple of integers with $r > 1$. Suppose $\alpha \in \mathfrak{E}$ is the largest element such that $r_\alpha \leqslant \frac{r}{2}$ and $\alpha \leqslant \frac{c_1}{r}$, and $\beta \in \mathfrak{E}$ is the smallest element such that $r_\beta \leqslant \frac{r}{2}$ and $\frac{c_1}{r} \leqslant \beta$. Then $\mathcal{M}(r, c_1, \delta) \neq \emptyset$ iff*

$$-\chi(v, E_\alpha) \geqslant 0 \ \text{and} \ -\chi(E_\beta, v) \geqslant 0, \tag{19}$$

where E_α and E_β are the exceptional bundles with the slopes α and β respectively. $\square$

Applying formula (8), we see that

$$-\chi(v, E_\alpha) \geqslant 0 \quad \Longleftrightarrow \quad \delta \geqslant -r - (3 + \alpha)c_1 + \frac{r}{r_\alpha}\chi(E_\alpha)$$

and

$$-\chi(E_\beta, v) \geqslant 0 \quad \Longleftrightarrow \quad \delta \geqslant -r - \beta(3r + c_1) + \frac{r}{r_\beta}\chi(E_\beta).$$

Therefore, there is a unique integer $\delta_{\min}(v)$ such that the system of conditions (19) is equivalent to the inequality $\delta \geqslant \delta_{\min}(v)$. Clearly, $\delta_{\min}(v)$ depends only on r and c_1.

3.2 Definition. We call vector $v = (r, c_1, \delta)$ *minimal* if $\delta = \delta_{\min}(v)$. In other words, v is minimal iff there is a stable sheaf X with $v_X = v$, but there is no stable sheaf X' with the same rank and c_1 and $-\chi(X') < \delta$. Also, we call a sheaf X *minimal* if X is stable and v_X is minimal. Put $\delta_{\min}(X) := \delta_{\min}(v_X)$.

Remark 1.1 implies that if we pose the restriction $-3r < c_1 < 0$, then $\delta_{\min}(v) \geqslant 0$. In fact, in this case for any stable sheaf X with $v_X = v$ we have: $\delta = h^1(X)$. Other simple properties of minimal sheaves are collected in the following statement.

3.3 Proposition. *1. Any minimal sheaf is locally free. 2. Exceptional bundles are minimal. 3. For $r > 1$ the maximal number p of the stratum $\mathcal{M}_v^{(p)} \subset \mathcal{M}_v$ is $N = \delta - \delta_{\min}(X)$.*

Proof. If X is stable but non-locally-free, then X^{**} is stable by 1.2 and $-\chi(X^{**}) = -\chi(X) - h^0(Q_X) < -\chi(X)$, whence X isn't minimal. This proves 1.

The assertion 2 is proved in [1], but we give a simple argument in our notation. Exceptional bundles are stable [4]. Let E be exceptional bundle with $v_E = (r, c_1, \delta)$, F be stable sheaf with $v_F = (r, c_1, \delta')$, then $\dim \mathcal{M}_E = 0$ and

$$\dim \mathcal{M}_F = \dim \mathcal{M}_F - \dim \mathcal{M}_E = \chi(E,E) - \chi(F,F) = 2r(\delta' - \delta) \geqslant 0\,.$$

Thus, $\delta' \geqslant \delta$.

Now let us prove 3. Suppose X is a stable sheaf with $v_X = v$; then using (15), we have: $-\chi(X^{**}) = \delta - h^0(Q_X) \geqslant \delta_{\min}(v)$, thus $p = h^0(Q_X) \leqslant N$. On the other hand, the maximal value $p = N$ is achieved: one can take a stable bundle E with $v_E = (r, c_1, \delta_{\min}(v))$; then the kernel of any epimorphism $E \twoheadrightarrow Q$, where $r(Q) = c_1(Q) = 0$ and $h^0(Q) = N$, is a stable sheaf $X' \in \mathcal{M}_v^{(N)}$. $\square$

3.4 Lemma. *Let X and F be stable sheaves such that at least one of them isn't minimal and $0 < \mu(F) - \mu(X) < 3$. Then $\mathrm{Ext}^1(F,X) \neq 0$.*

Proof. Suppose X isn't minimal, then there is a stable sheaf X' with the same rank and c_1 as X and $-\chi(X') < -\chi(X)$. We have: $\mu(X) = \mu(X')$, this implies that $\mathrm{Hom}(F,X') = \mathrm{Ext}^2(F,X') = 0$, whence $-\chi(F,X') = h^1(F,X') \geqslant 0$. Comparing the expressions for $\chi(F,X)$ and $\chi(F,X')$ by (8), we obtain:

$$\chi(F,X) - \chi(F,X') = r(F)(\chi(X) - \chi(X')) < 0\,.$$

Thus, $-\chi(F,X) = h^1(F,X) > 0$. The case of non-minimal F is quite similar. $\square$

Observe that for minimal sheaves X and F having invariants satisfying the inequality in the assumption of lemma, $\mathrm{Ext}^1(X,F)$ can be either trivial or not. Let us give some examples.

3.5 Example 1) $X = E_n$, $F = E_{n+1}$. We have: $\mathrm{Ext}^1(F,X) = 0$.

2) $X = E_n$, $n \geqslant 0$, F is the cokernel of a non-zero map $E_n \to E_{n+1}$ (remind that $h^0(E_n, E_{n+1}) = 3$). From the proof of 5.3 it follows that F is minimal. Clearly, we have non-trivial extension

$$0 \longrightarrow X \longrightarrow E_{n+1} \longrightarrow F \longrightarrow 0\,,$$

hence $h^1(F,X) \neq 0$. It is easily shown that $\mathrm{Ext}^1(F,X)$ is one-dimensional in this example.

In the examples 1) and 2) at least one of moduli spaces $\mathcal{M}_v$ and $\mathcal{M}_u$ is zero-dimensional. The next example is less trivial.

3) Let X and F be defined from the exact triples:

$$0 \longrightarrow E_n \longrightarrow E_{n+1} \longrightarrow X \longrightarrow 0\,, \qquad 0 \longrightarrow E_{n-1} \longrightarrow E_n \longrightarrow F \longrightarrow 0\,,$$

where $n \geqslant 2$. Using the equality $\chi(E_{i+1}, E_i) = 0$, it is easy to calculate that $-\chi(F,X) = h^1(F,X) = 3$ ($\chi(E_{n-1}, E_{n+1}) = 8$ follows from (5)).

Note that in all the examples above one has $\begin{vmatrix} r(X) & c_1(X) \\ r(F) & c_1(F) \end{vmatrix} = 1$.

The following result describes the image of Φ and necessary and sufficient conditions for emptiness of $\widetilde{\mathcal{M}}_{v\to w}$.

3.6 Theorem. *Suppose that the triples of integers $v=(r,c_1,\delta)$ and $w=(r',c_1',\delta')$ satisfy (1), $0<r<r'$ and the moduli spaces $\mathcal{M}_v$ and $\mathcal{M}_w$ are non-empty. Put $N=\delta-\delta_{\min}(v)$, $p_0=\max\{0,\delta-\delta'+\delta_{\min}(w-v)\}$, $v_0=(r,c_1,\delta-p_0)$ and $u_0=w-v_0$. Then the image of the map $\Phi:\widetilde{\mathcal{M}}_{v\to w}\longrightarrow\mathcal{M}_v$ is described as follows:*
1) If $p_0<N$, then $\operatorname{im}\Phi$ is the union of strata $\mathcal{M}_v^{(p)}$, where $p_0\leqslant p\leqslant N$.
2) If $p_0=N$ and $\chi(u_0,v_0)\neq 0$, then $\operatorname{im}\Phi=\mathcal{M}_v^{(p_0)}$.
3) If $p_0=N$ and $\chi(u_0,v_0)=0$, or $p_0>N$, then the moduli space $\widetilde{\mathcal{M}}_{v\to w}$ is empty.

Proof. Suppose $X\in\mathcal{M}_v^{(p)}$, this means exactly that $h^0(Q_X)=p$.

First let us show that if $X\in\operatorname{im}\Phi$ then $p_0\leqslant p\leqslant N$. According to 2.1 and 2.2, any non-zero map $\varphi:X\longrightarrow Y$ induces a commutative diagram (12). We have: $p\geqslant h^0(T)=\chi(T)$. Since

$$\chi(T)=\chi(Y)-\chi(X)-\chi(F')=-\delta'+\delta-\chi(F')\,,$$

we get: $p\geqslant\delta-\delta'-\chi(F')$. The sheaf F' is stable and has invariants $v_{F'}=w-v-v_T$, hence

$$-\chi(F')\geqslant\delta_{\min}(w-v-v_T)=\delta_{\min}(w-v)$$

(remind that $v_T=(0,0,-h^0(T))$ and $\delta_{\min}$ depends on the two first coordinates only). Therefore, $p\geqslant\delta-\delta'+\delta_{\min}(w-v)$. By the definition, $p\geqslant 0$, so, $p\geqslant p_0$. From assertion 3 of Proposition 3.3 it follows that $p\leqslant N$.

In particular, we have proved that for $p_0>N$ the moduli space $\widetilde{\mathcal{M}}_{v\to w}$ is empty.

Now assume that $p_0\leqslant p\leqslant N$. Consider a subsheaf $T\subset Q_X$ such that $p_0\leqslant h^0(T)$. We have:

$$\delta_{\min}(w-v)\leqslant\delta'-\delta+h^0(T)\,.$$

Thus, the moduli space $\mathcal{M}(w-v-v_T)$ is non-empty. Take a stable sheaf $F'\in\mathcal{M}(w-v-v_T)$. We have: $-\chi(F')=\delta'-\delta+h^0(T)$, and F' is minimal iff $-\chi(F')=\delta_{\min}(w-v)$. This is equivalent to the equality $h^0(T)=\delta-\delta'+\delta_{\min}(w-v)$. Therefore, F' is minimal if and only if $h^0(T)=p_0$ since $h^0(T)$ is non-negative.

Let X_T be the stable sheaf constructed via $T\hookrightarrow Q_X$ as described in 2.3. We have: $X_T^{**}\cong X^{**}$, and X^{**} is minimal whenever $p=N$ because

$$-\chi(X^{**})=\delta-p\geqslant\delta-N=\delta_{\min}(v)\,.$$

Consider the following cases.

1) If $p_0<N$, then for $p<N$ the sheaf X^{**} isn't minimal and for $p=N$ choose $T\subset Q_X$ such that $h^0(T)>p_0$ i.e. F' isn't minimal. By 3.4, we have: $h^1(F',X^{**})>0$, so from 2.5 it follows that there is a stable sheaf Y being represented by an extension

$$0\longrightarrow X_T\xrightarrow{\varphi_T}Y\longrightarrow F'\longrightarrow 0\,.$$

Clearly the composition map $X\hookrightarrow X_T\xrightarrow{\varphi_T}Y$ is non-zero, so $X\in\operatorname{im}\Phi$.

2) If $p_0=p=N$, then the both sheaves, X^{**} and F', are minimal. For the case $\chi(u_0,v_0)=\chi(F',X^{**})<0$ we can apply the above reasoning with $T=Q_X$ and $X_T=X^{**}$. In the case $\chi(u_0,v_0)=\chi(F',X^{**})=0$ we have: $\operatorname{Ext}^1(F',X^{**})=0$. We claim that $\operatorname{Hom}(X,Y)=0$ for all $Y\in\mathcal{M}_Y$, otherwise a non-trivial morphism $\varphi:X\to Y$ induces a diagram of the form (12), in which F' is minimal, $T=Q_X$, and $X_T=X^{**}$ is also minimal. But then from the diagram it follows that $Y=F'\oplus X^{**}$, this contradicts the stability. $\square$

3.7 Corollary. The following statements are equivalent: (i) $\mathcal{M}_v^{(0)}$ is in the image of Φ; (ii) Φ is surjective; (iii) the moduli space $\mathcal{M}(w-v)$ is non-empty.

Clearly, (i) and (ii) are equivalent to the equality $p_0 = 0$, i.e. to the inequality $\delta' - \delta \geqslant \delta_{\min}(w-v)$, which is equivalent to (iii). $\square$

3.8 Proposition. *Assume that* $X \in \mathcal{M}_v^{(N)}$, $Y \in \mathcal{M}_w$ *and* $w-v$ *is minimal. If for* $v' = (r, c_1, \delta_{\min}(v))$ *one has* $\chi(w-v, v') = 0$, *then the cokernel* F *of any non-trivial morphism* $\varphi : X \longrightarrow Y$ *has a torsion.*

Proof. Suppose F has no torsion, then F is stable by 2.1. We have: $v_F = w - v$, so F is minimal and therefore locally free. The condition $X \in \mathcal{M}_v^{(N)}$ means exactly that X^{**} is minimal, in particular, $v' = v_{X^{**}}$. We have: $h^1(F, X^{**}) = -\chi(w-v, v') = 0$, whence it follows that the map $\mathrm{Hom}(F, Q_X) \longrightarrow \mathrm{Ext}^1(F, X)$ is an isomorphism. Therefore, Y isn't stable according to Proposition 2.4. This contradiction completes the proof. $\square$

To conclude this section, we derive from the proved statements some property of the bound of invariants of stable sheaves on $\mathbb{P}^2$, which can be regarded as a kind of non-strict local convexity. Of course, the better understanding of this bound can be obtained from remarkable work [1], where the description of the bound is given in terms of another coordinates (slope and discriminant).

3.9 Proposition. *Let* $v = (r, c_1, \delta)$ *and* $u = (r'', c_1'', \delta'')$ *be two minimal vectors such that* $\begin{vmatrix} r & c_1 \\ r'' & c_1'' \end{vmatrix} = 1$ *and* $-\chi(u, v) > 0$. *Then the following inequality holds:*

$$\delta_{\min}(v+u) \leqslant \delta_{\min}(v) + \delta_{\min}(u)\,. \tag{20}$$

Proof. The moduli spaces $\mathcal{M}(v)$ and $\mathcal{M}(u)$ are non-empty by the definition, so let us consider stable sheaves X and F such that $v_X = v$ and $v_F = u$. These are locally free sheaves because of minimality. Using 1.1, we have: $h^1(F, X) = -\chi(u, v) > 0$, hence, there is a non-trivial extension of the form (11), where Y has the invariants $v_Y = v + u$ and is stable according to 2.4. Therefore,

$$\delta_{\min}(v+u) \leqslant -\chi(Y) = -\chi(X) - \chi(F) = \delta_{\min}(v) + \delta_{\min}(u)$$

$\square$

It is easy to show that if v and u don't correspond to exceptional bundles, then the both values $-\chi(v+u, v)$ and $-\chi(u, v+u)$ are positive, so one can apply the proposition to the pairs of vectors (v, w) and (w, u), where $w = (r + r'', c_1 + c_1'', \delta_{\min}(v+u))$.

4 Fibers of the morphism Φ

In this section we apply technical statements of Section 2 to studying the preimages of points in $\mathcal{M}_v$ under the map $\Phi : \widetilde{\mathcal{M}}_{v\to w} \longrightarrow \mathcal{M}_v$, where $v = (r, c_1, \delta)$ and $w = (r', c_1', \delta')$ satisfy (1) and $r < r'$. Recall that $\widetilde{\mathcal{M}}_{v\to w}$ parametrizes the triples $(X, Y, [\varphi])$, where $X \in \mathcal{M}_v$, $Y \in \mathcal{M}_w$ and $[\varphi] \in P(\mathrm{Hom}(X, Y))$. The map Φ takes a triple $(X, Y, [\varphi])$ to X.

Put $u = w - v$. Fix a sheaf $X \in \mathcal{M}_v$ and consider two cases.

I. X is locally free.

According to 2.1 and 2.2, any non-trivial morphism $\varphi : X \longrightarrow Y$ defines exact triple

$$0 \longrightarrow X \xrightarrow{\varphi} Y \longrightarrow F \longrightarrow 0\,, \tag{11}$$

where F is stable. Hence, the pair

$$(Y, \varphi)\,, \quad \text{where} \quad Y \in \mathcal{M}_w\,, \quad 0 \neq \varphi \in \mathrm{Hom}(X, Y) \tag{21}$$

defines a pair

$$(F, e)\,, \quad \text{where} \quad F \in \mathcal{M}_u \quad \text{and} \quad 0 \neq e \in \mathrm{Ext}^1(F, X) \tag{22}$$

Conversely, any such a pair (F, e) determines an exact triple of the form (11), in which Y is stable by 2.4, i.e. the pair (F, e) gives a pair (Y, φ) of the form (21).

Consider natural equivalence relations on the sets of pairs (21) and (22): $(Y, \varphi) \sim (Y', \varphi')$ whenever there is an isomorphism $\alpha : Y \to Y'$ such that $\varphi' = \alpha \circ \varphi$, and $(F, e) \sim (F', e')$ iff there is an isomorphism $\beta : F \longrightarrow F'$ such that $\tilde{\beta}(e') = e$, where $\tilde{\beta} : \mathrm{Ext}^1(F', X) \longrightarrow \mathrm{Ext}^1(F, X)$ is induced by β.

It is easily shown by standard arguments of homological algebra that the correspondence between the sets of pairs (21) and (22) preserves the equivalence relations. Since a stable sheaf has endomorphisms only of the form $\lambda \cdot \mathrm{id}$ ([14]), the pairs (Y, φ) and (Y, φ') are equivalent if and only if $\varphi' = \lambda\varphi$, $\lambda \neq 0$. Also, the pairs (F, e) and (F, e') are equivalent iff $e' = \mu e$, $\mu \neq 0$. Therefore, the set of pairs $(Y, [\varphi])$, where $Y \in \mathcal{M}_w$ and $[\varphi] \in P(\mathrm{Hom}(X, Y))$ is identified with projectivization of "bundle over $\mathcal{M}_u$ with the fiber $\mathrm{Ext}^1(F, X)$", which is constructed by usual way via universal family.

From (1) it follows that $r(F) = r' - r$ and $c_1(F) = c_1' - c_1$ are coprime, hence, as it was pointed in the introduction, $\mathcal{M}_u$ is smooth and admits a universal family $\mathcal{F} \longrightarrow \mathcal{M}_u \times \mathbb{P}^2$. Let $\pi : \mathcal{M}_u \times \mathbb{P}^2 \to \mathbb{P}^2$ be the projection. Consider the sheaf

$$\mathcal{E}_X = (R^1\pi_*\mathcal{F} \otimes \pi^* X^*(-3))^*\,.$$

Note that $h^1(F, X) = -\chi(F, X)$ by 1.1, hence $\mathcal{E}_X$ is locally free. Using the isomorphisms $H^1(X^*(-3) \otimes F)^* = \mathrm{Ext}^1(X(3), F)^* = \mathrm{Ext}^1(F, X)$, one can interpret $\mathcal{E}_X$ as vector bundle on $\mathcal{M}_F$ with the fiber $\mathrm{Ext}^1(F, X)$ at a point F. Therefore, we obtain the following

4.1 Proposition. *The preimage $\Phi^{-1}(X)$ of a locally free sheaf X coincides with projectivization of the bundle $\mathcal{E}_X$ on $\mathcal{M}_u$.* $\square$

Note that the preimage $\Phi^{-1}(\mathcal{M}_v^{(0)})$ of the moduli space of bundles can be described globally. Denote by π_3, π_2 and π_1 the projections of $\mathcal{M}_v^{(0)} \times \mathcal{M}_u \times \mathbb{P}^2$ on $\mathcal{M}_v^{(0)} \times \mathcal{M}_u$, $\mathcal{M}_v^{(0)} \times \mathbb{P}^2$ and $\mathcal{M}_u \times \mathbb{P}^2$ respectively. Let $\mathcal{X} \longrightarrow \mathcal{M}_v^{(0)} \times \mathbb{P}^2$ be restriction of the universal family, then according to the construction described above we obtain the following result.

4.2 Theorem. *Suppose that $v = (r, c_1, \delta)$ and $w = (r', c_1', \delta')$ satisfy (1), $r < r'$ and the moduli spaces $\mathcal{M}_v$, $\mathcal{M}_w$ and $\mathcal{M}_u = \mathcal{M}_{w-v}$ are non-empty. Then $\Phi^{-1}(\mathcal{M}_v^{(0)}) = P(\mathcal{E})$, where $\mathcal{E} = R^1\pi_{3*}(\pi_2^*\mathcal{X}^* \otimes \pi_1^*\mathcal{F} \otimes \omega_{\pi_3})$ is a vector bundle over $\mathcal{M}_v^{(0)} \times \mathcal{M}_u$ with the fiber* $\mathrm{Ext}^1(F, X)$.

4.3 Corollary. $\dim \Phi^{-1}(\mathcal{M}_v^{(0)}) = 1 - \chi(w, w) + \chi(v, u)$.

Proof. From the theorem it follows that the dimension of $\Phi^{-1}(\mathcal{M}_v^{(0)})$ is equal to

$$\dim \mathcal{M}_v^{(0)} + \dim \mathcal{M}_u - \chi(F,X) - 1 = 1 - \chi(v,v) - \chi(u,u) - \chi(u,v)\,.$$

Besides, $w = v + u$, this implies the required formula. □

II. X is non-locally-free.

We can assume that $X \in \mathcal{M}_v^{(p)}$, where $p > 0$. Proposition 2.1 implies that any non-trivial morphism $\varphi : X \longrightarrow Y$ induces the exact triple (11), but unlike in the previous case the sheaf F can have a torsion. Herewith, by 2.2, the torsion subsheaf $T = T(F) \subset F$ is a subsheaf of Q_X. Hence, we have the diagram (12), in which F' is stable. Note that the sheaf X_T is stable also by Lemma 1.2 because it has no torsion being a subsheaf of a stable sheaf Y. So, any pair of the form (21) defines the following data

- a subsheaf $T \overset{i}{\hookrightarrow} Q_X$, which determines stable sheaf X_T as described in 2.3
- a stable sheaf $F' \in \mathcal{M}(u - v_T) =: \mathcal{M}'$
- a cocycle $e \in \mathrm{Ext}^1(F', X_T) \setminus \mathrm{Hom}(F', Q_{X_T})$

Here we identify the space $\mathrm{Hom}(F', Q_{X_T})$ with its image under the inclusion $\mathrm{Hom}(F', Q_{X_T}) \hookrightarrow \mathrm{Ext}^1(F', X_T)$ induced by the exact triple

$$0 \longrightarrow X_T \longrightarrow X_T^{**} \longrightarrow Q_{X_T} \longrightarrow 0\,.$$

Note that $v_T = (0, 0, -h^0(T))$, hence subtracting this vector adds $h^0(T)$ to the third component leaving the rank and c_1 unchanged.

Conversely, this data defines two left columns and the row containing F' of (12), hence, the whole diagram is restored. This diagram gives the pair (Y, φ) of type (21), where $\varphi = \varphi' \circ \psi$.

We leave for the reader checking that the correspondence between the pairs (Y, φ) of type (21) and the data described above preserves natural equivalence relations. Thus, we obtain the following statement.

4.4 Proposition. *The preimage of non-locally-free sheaf X admits the following description*

$$\Phi^{-1}(X) = \bigcup_{T \subset Q_X} \left(\bigcup_{F' \in \mathcal{M}(u - v_T)} P(\mathrm{Ext}^1(F', X_T)) \setminus P(\mathrm{Hom}(F', Q_{X_T})) \right)\,.$$

5 The case of moduli spaces $\mathcal{M}_n(\delta)$

Throughout this section we assume that $v = (r_n, -r_{n-1}, \delta)$ and $w = (r_{n+1}, -r_n, \delta')$, where $n \geqslant 0$ and the sequence $\{r_n\}_{n \in \mathbb{Z}}$ is defined by relations (2). In other words, $\mathcal{M}_v = \mathcal{M}_n(\delta)$ and $\mathcal{M}_w = \mathcal{M}_{n+1}(\delta')$. We reformulate here the description of the image of $\Phi : \widetilde{\mathcal{M}}_{v \to w} \longrightarrow \mathcal{M}_v$ given in 3.6 and prove that for almost all values of δ' the map $\Psi : \widetilde{\mathcal{M}}_{v \to w} \longrightarrow \mathcal{M}_w$ is non-surjective.

First of all we prove formula (3). Denote the determinant in the left hand side by Δ_n. We have: $r_1 = 2$ and $\Delta_0 = 1$. Besides,

$$\begin{vmatrix} r_{n-1} & r_n \\ r_n & r_{n+1} \end{vmatrix} = \begin{vmatrix} 3r_n - r_{n+1} & r_n \\ 3r_{n+1} - r_{n+2} & r_{n+1} \end{vmatrix} = \begin{vmatrix} -r_{n+1} & r_n \\ -r_{n+2} & r_{n+1} \end{vmatrix} = \begin{vmatrix} r_n & r_{n+1} \\ r_{n+1} & r_{n+2} \end{vmatrix}\,.$$

Thus, $\Delta_n = \Delta_{n+1}$ for all n and (3) is proved.

5.1 Lemma. *The sequence $\{r_n\}$ is increasing for $n \geqslant 0$ and decreasing for $n \leqslant -1$. In particular, all $r_n > 0$.*

Proof. Clearly, the sequence is symmetric w.r. to the "middle between r_{-1} and r_0", i.e. $r_n = r_{-n-1}$, so, it is enough to show that $r_{n+1} > r_n > 0$ for $n \geqslant 0$. If $n = 0$, then this inequality certainly holds: $r_1 = 2$, $r_0 = 1$. Using the induction hypothesis, we have: $r_{n+2} = 3r_{n+1} - r_n > 2r_{n+1} > r_{n+1}$, $n > 0$. □

5.2 Corollary. 1) $0 < r_{n-1} < 3r_n$ for all n; 2) $r_{n+1} > 2r_n$ for $n > 0$. □

The first inequality implies that $-3 < \frac{-r_{n-1}}{r_n} < 0$, hence, for any $X \in \mathcal{M}_n(\delta)$ by 1.1 we have: $-\chi(X) = h^1(X)$.

5.3 Lemma. *1.* $\delta_{\min}(v) = \delta_{\min}(w) = 0$. *2.* $\delta_{\min}(w - v) = \left\{ \begin{array}{rl} 0 & \textit{for } n < 0 \\ -1 & \textit{for } n = 0 \end{array} \right.$.

Proof. Suppose $X \in \mathcal{M}_v = \mathcal{M}_n(\delta)$, then $\delta = h^1(X) \geqslant 0$. On the other hand, the exceptional bundle E_n is stable and has $-\chi(E_n) = 0$. This proves 1.

Consider any non-trivial morphism $\varphi : E_n \longrightarrow E_{n+1}$ (remind that $h^0(E_n, E_{n+1}) = 3$). From 2.1 and 2.2 it follows that φ is an embedding and $F_0 = \operatorname{coker} \varphi$ is a stable sheaf with the vector of invariants $v_{F_0} = (r_{n+1} - r_n, -r_n + r_{n-1}, 0)$. We have: $\mu(E_{n+1}) < \mu(F_0) = \frac{-r_n + r_{n-1}}{r_{n+1} - r_n} < 0$ for $n > 0$ and $\mu(F_0) = 0$ for $n = 0$. From 1.1 we obtain in the former case that for any semi-stable sheaf F with $\mu(F) = \mu(F_0)$ one has: $-\chi(F) = h^1(F) \geqslant 0$, and the assertion follows from the equality $h^1(F_0) = 0$. For $n = 0$ we have: $v_{F_0} = (1, 0, 0)$. The trivial rank 1 bundle is exceptional, therefore, minimal, and one has $v_{\mathcal{O}_{\mathbb{P}^2}} = (1, 0, -1)$. This completes the proof. □

Note that any torsion free sheaf F_0 such that $v_{F_0} = (1, 0, 0)$ is nothing else but the ideal sheaf of a point $x \in \mathbb{P}^2$. Indeed, the reflexive span $F_0^{**} = \mathcal{O}_{\mathbb{P}^2}$ and $c_2(F_0) = h^0(Q_{F_0}) = 1$. Therefore, $\mathcal{M}(1, 0, 0) = \mathbb{P}^2$, this is a partial case of the following statement proved in [2]:

$$\mathcal{M}(r_{n+1} - r_n, -r_n + r_{n-1}, 0) \cong \mathbb{P}^2 .$$

Lemma 5.3 permits to give a simpler formulation of Theorem 3.6 in our case. We have: $p_0 = \max\{0; \delta - \delta'\}$ and $N = \delta$, hence always $p_0 \leqslant N$. Thus, we have two cases:

1) $p_0 = 0$, $\delta \leqslant \delta'$, $v_0 = v = (r_n, -r_{n-1}, \delta)$, $u_0 = w - v = (r_{n+1} - r_n, -r_n + r_{n-1}, \delta' - \delta)$;

2) $p_0 = \delta - \delta' > 0$, $v_0 = (r_n, -r_{n-1}, \delta')$, $u_0 = (r_{n+1} - r_n, -r_n + r_{n-1}, 0)$.

The equality $p_0 = N$ is possible only in the second case for $\delta' = 0$, herewith, $-\chi(u_0, v_0) = 1 + (r_{n+1} - r_n)\delta'$ is always positive. Therefore, Theorem 3.6 has the following form.

5.4 Theorem. *Suppose $v = (r_n, -r_{n-1}, \delta)$, $w = (r_{n+1}, -r_n, \delta')$, and $n \geqslant 0$; then the map $\Phi : \widetilde{\mathcal{M}}_{v \to w} \to \mathcal{M}_v$ is surjective for $\delta \leqslant \delta'$ and* $\operatorname{im} \Phi = \bigcup_{p=\delta-\delta'}^{\delta} \mathcal{M}_v^{(p)}$ *for $\delta > \delta'$. In particular, if $\delta = \delta'$, then* $\operatorname{im} \Phi = \mathcal{M}_v^{(\delta)}$. □

5.5 Proposition. *Suppose $v = (r_n, -r_{n-1}, \delta)$ and $w = (r_{n+1}, -r_n, \delta')$; then for $n = 0$, $\delta' \geqslant 3$, for $n = 1$, $\delta' \geqslant 2$ and for $n \geqslant 2$, $\delta' \geqslant 1$ the map $\Psi : \widetilde{\mathcal{M}}_{v \to w} \longrightarrow \mathcal{M}_w$ isn't surjective.*

Proof. The idea is as follows: we represent $\operatorname{im}\Psi \cap \mathcal{M}_w^{(0)}$ as a union of some schemes and calculate dimensions of their preimages under Ψ. The largest dimension turns to be strictly less than $\dim \mathcal{M}_w^{(0)}$ for the specified values of n and δ'.

Let $z = (X, Y, [\varphi])$ be closed point in $\widetilde{\mathcal{M}}_{v\to w}$ such that $\Psi(z) \in \mathcal{M}_w^{(0)}$. This means that Y is locally free and $\varphi : X \longrightarrow Y$ is a non-trivial morphism. Put as usual $F = \operatorname{coker}\varphi$. By 2.8 we have: $Q_X = T(F) =: T$ and there is diagram (12), where $X_T = X^{**}$. It follows that φ is a composition of the canonical embedding $X \hookrightarrow X^{**}$ and a non-trivial map $\hat{\varphi} : X^{**} \longrightarrow Y$. Thus, we have:

$$\operatorname{im}\Psi \cap \mathcal{M}_w^{(0)} = \bigcup_{k=0}^{\delta} \Lambda_k\,, \quad \Lambda_k = \Psi_k(\Phi_k^{-1}(\mathcal{M}_{v_k}^{(0)}))\,,$$

where $v_k = (r_n, -r_{n+1}, k)$ and Φ_k, Ψ_k are the maps from $\widetilde{\mathcal{M}}_{v_k\to w}$ similar to Φ and Ψ:

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_{v_k\to w} & \xrightarrow{\Psi_k} & \mathcal{M}_w \\ {\scriptstyle\Phi_k}\downarrow & & \\ \mathcal{M}_{v_k} & & \end{array} \qquad \begin{array}{ccc} (\hat{X}, Y, [\hat{\varphi}]) & \xmapsto{\Psi_k} & Y \\ {\scriptstyle\Phi_k}\downarrow & & \\ \hat{X} & & \end{array}$$

By 4.3, we have:

$$\dim \Phi_k^{-1}(\mathcal{M}_{v_k}^{(0)}) = 1 - \chi(w, w) + \chi(v_k, w - v_k) = 2r_{n+1}\delta' + 2 - r_n(\delta' - k) - (r_{n+1} - r_n)k\,.$$

This is linear function on k. For $n \geqslant 1$ we have:

$$\max_{0\leqslant k\leqslant\delta} \dim \Phi_k^{-1}(\mathcal{M}_{v_k}^{(0)}) = \dim \Phi_0^{-1}(\mathcal{M}_{v_0}^{(0)}) = 2r_{n+1}\delta' + 2 - r_n\delta'\,.$$

For $n = 1$, $\delta' \geqslant 2$ and for $n \geqslant 2$, $\delta' \geqslant 1$ this maximum is less than $\dim \mathcal{M}_w = 2r_{n+1}\delta'$, hence $\operatorname{im}\Psi \cap \mathcal{M}_w^{(0)}$ doesn't coincide with $\mathcal{M}_w^{(0)}$ and Ψ isn't surjective.

For $n = 0$ we have: $\dim \Phi_k^{-1}(\mathcal{M}_{v_k}^{(0)}) = 3\delta' + 2$, this value is independent of k and is less than $\dim \mathcal{M}_w = 4\delta'$ whenever $\delta' \geqslant 3$, in this case Ψ is also non-surjective. $\square$

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