GENERALIZED LUNEBURG CANONICAL VARIETIES AND VECTOR FIELDS ON QUASICAUSTICS

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Abstract. This paper studies some aspects of a particular class of bifurcation varieties which are provided by simple and unimodal boundary singularities. Their correspondence to a diffraction theory is established. The generic caustics by diffraction on apertures are derived and their generating families for the corresponding Lagrangian varieties are calculated. It is proved that the quasicaustics associated to simple singularities are smooth hypersurfaces or Whitney's cross-caps. The procedure for calculating the modules of logarithmic vector fields is given, and the minimal sets of the corresponding generators are explicitely calculated. The construction is conducted for the general boundary singularities and the structure of quasicaustics defined by parabolic singularities is investigated.

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1. INTRODUCTION

Let $F : (\mathbf{C}^{n+1} \times \mathbf{C}^p, 0) \to (\mathbf{C}, 0)$ be a germ of a holomorphic function. By $(S, 0) \subset (\mathbf{C}^{n+1}, 0)$ we denote a germ of a some hypersurface in $(\mathbf{C}^{n+1}, 0)$. The quasicaustic $Q(F)$ of F is defined as

$$Q(F) = \{a \in \mathbf{C}^p; F(\bullet, a) \, \text{has a critical point on} \, S\}.$$

Let F represent the distance function from the general wavefront in the presence of an obstacle formed by an aperture (cf. **[18-9]**) with boundary S. The corresponding quasicaustic $Q(F)$ is build up from the rays orthogonal to the given wavefront and touching the boundary of the aperture (see the example of the quasicaustic illustrated in Figure 4). The quasicaustic is a subvariety of the usual caustic (also called the bifurcation set **[6-31]**)

$$\{a \in \mathbf{C}^p; F(\bullet, a) \;\; \text{or} \, F\,|_{S\times\mathbf{C}^p} \, (\bullet, a) \, \text{have a critical point}\},$$

and represents the structure of shadows formed by the common, pecular positions of aperture and incident wavefront.

In this paper we investigate the structure of generic caustics and quasicaustics by diffraction on smooth obstacle curves and apertures (optical instruments). We use for this the classical phase space for general optical instruments, i.e. the space of pairs of rays $(l, \tilde{l})$, where l is an incident ray and $\tilde{l}$ is transformed ray (produced by l and the optical instrument), endowed with the canonical symplectic structure. This space was first introduced by R. K. Luneburg [**21**] in his mathematical theory of optics and then revived by V. Guillemin and S. Sternberg [**13**] in their symplectic approach to various physical theories. To each optical instrument, in the mentioned phase space, there corresponds a Lagrangian subvariety, say A, defining all physical properties (from the point of view of the geometrical theory of optics [**19**]) of the system. So when A is fixed we can obtain all transformed wavefronts by taking the symplectic images $A(L)$ of all Lagrangian subvarieties L of incident rays (i.e. optical sources) . See also [**14**].

The plan of the paper is as follows. In Section 2 we give preliminary results about the basic phase spaces and construct representative examples in the symplectic approach to general optical systems. The geometrical structure of caustics by diffraction on apertures, as well as their generic classification in the case of half line aperture on the plane and half plane aperture in Euclidean three-space, is investigated in Section 3. We compute the normal forms for generating families of the generic canonical varieties in the case of diffraction on smooth curves in Section 4. When considering the caustics by diffraction on apertures, the quasicaustic component becomes important. In Section 5 we generalize the methods for ordinary caustics initiated by J. W. Bruce [7-8] to investigate the structure of logarithmic vector fields on quasicaustics. In Section 6 we derive the generators for the modules of tangent vector fields to the quasicaustics corresponding to simple boundary singularities and prove that they are not free. Finally in Section 7 we analyse the structure of quasicaustics and the reduction of functional moduli in normal forms of Lagrangian pairs.

2. SINGULARITIES IN ACTION OF OPTICAL INSTRUMENTS

Let (M,ω) be the symplectic manifold of all oriented lines in $V \cong \mathbf{R}^3$. We look on$V$ as the configurational space of geometrical optics with refraction index $n : V \to \mathbf{R}$, $n \equiv 1$. (M,ω) is given by the standard symplectic reduction

$$\pi_M : H^{-1}(0) \to M \cong T^*S^2,$$

where the hypersurface $H^{-1}(0)$ is defined by the Hamiltonian

$$H : T^*V \to \mathbf{R}, \qquad H(p,q) := \frac{1}{2}(\| p \|^2 - 1),$$

and π_M is the projection along characteristics of the associated hamiltonian system.

Let (p,q) be coordinates on (T^*V, ω_V), where ω_V is an associated Liouville 2-form. By (U,ω) we denote the local chart on (M,ω) described as an image $\pi_M(H^{-1}(0) \cap \{p_1 > 0\})$ with restricted symplectic form ω. (p,q) form Darboux coordinates on (T^*V,ω_V). In corresponding Darboux coordinates (r,s) on (U,ω) we can write

$$\begin{aligned}(r,s) &= \pi_M(p_2,p_3;q_1,q_2,q_3)\\ &= \left(p_2,p_3;q_2 - \frac{q_1p_2}{\sqrt{1-p_2^2-p_3^2}}, q_3 - \frac{q_1p_3}{\sqrt{1-p_2^2-p_3^2}}\right),\end{aligned}$$

where the unique reduced symplectic structure ω is given by the formula

$$\omega_V \mid_{H^{-1}(0)} = \pi_M^*\omega, \quad \omega \mid_U = \sum_{i=1}^{2} dr_i \wedge ds_i.$$

In the introduced coordinates on M, to each point $(r,s) \in U$ we can uniquely associate the corresponding ray (in parametric form),

$$(q_1,q_2,q_3) = (0,s_1,s_2) + u\left(1, \frac{r_1}{\sqrt{1-r_1^2-r_2^2}}, \frac{r_2}{\sqrt{1-r_1^2-r_2^2}}\right), \quad u \in \mathbf{R}.$$

By the above formula one can translate the concrete optical problems into the language of the phase space (M,ω) and vice versa (cf. [**21-13-26**]).

Let (U,ω), $(\tilde{U},\tilde{\omega})$ be two exemples of the symplectic space of optical rays or its open subsets. Usually these manifolds denote the spaces of incident and transformed rays of an optical instrument (see Figure 1,a,b).

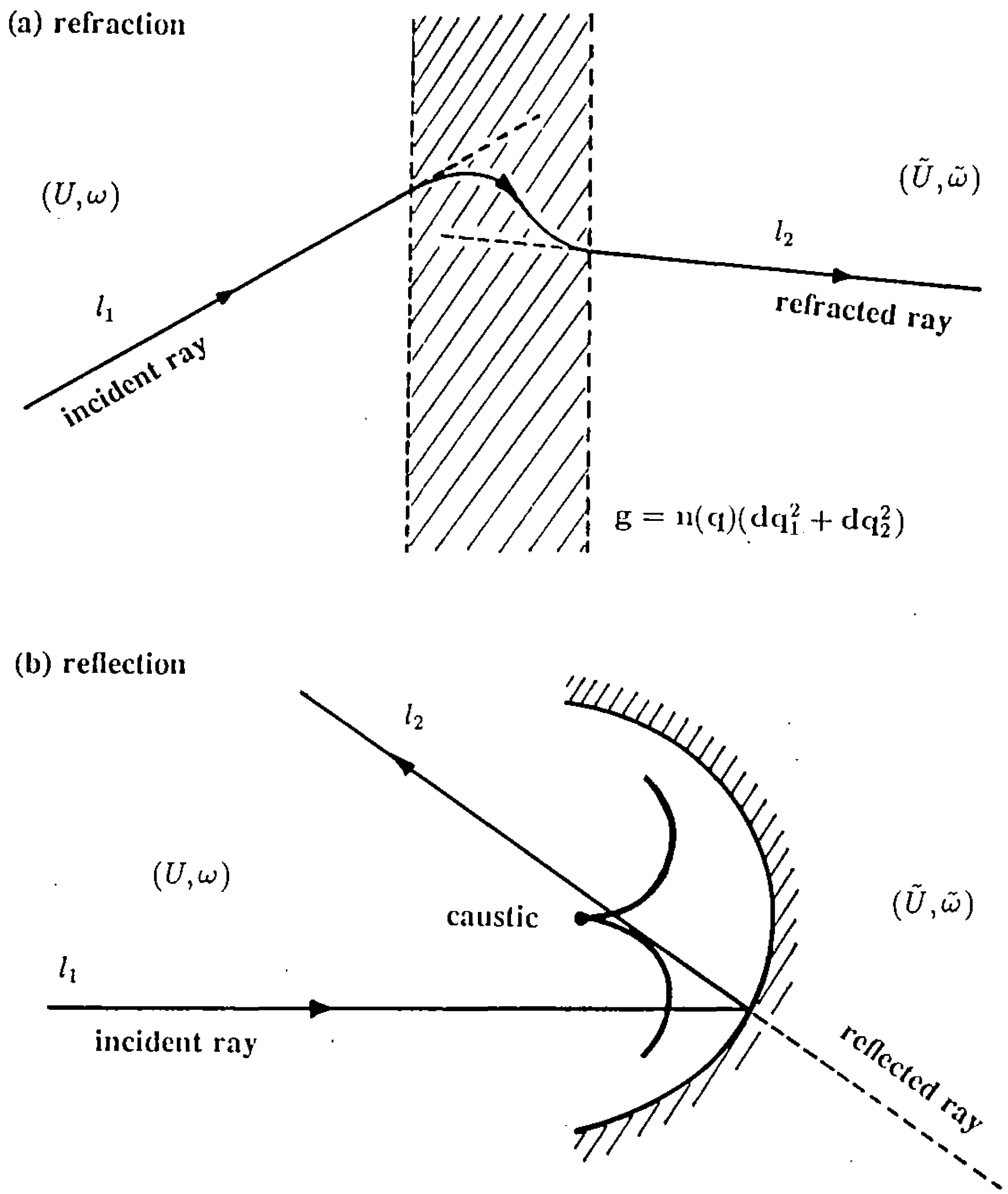

Figure 1.

Definition 2.1. *The phase space of optical instruments is the following product symplectic manifold:*

$$\Pi = (U \times \tilde{U}; \pi_2^* \tilde{\omega} - \pi_1^* \omega),$$

where, $\pi_{1,2} : U \times \tilde{U} \to U, \tilde{U}$ are canonical projections (this was first introduced by R. K. Luneburg [21]).

The process of optical transformation, say reflection, refraction or diffraction, etc. (see Figure 2.) of the incident rays is governed by the subvariety of Π, which is Lagrangian, i.e. it is stratified onto isotropic submanifolds of Π where maximal strata are Lagrangian (cf. [**1-14-16**]).

Definition 2.2. *We define the general optical instrument to be a Lagrangian subvariety of* Π(generalized symplectic relation, [**14-32**]).

REMARK 2.3. It is easily seen that reflecting or refracting optical instruments (cf. [**10**]) correspond to graphs of symplectomorphisms between $(U;\omega)$ and $(\tilde{U},\tilde{\omega})$. But, for example, the diffraction process is described by quite general Lagrangian subvariety of Π (see Figure 2, below, cf. also [**18**]).

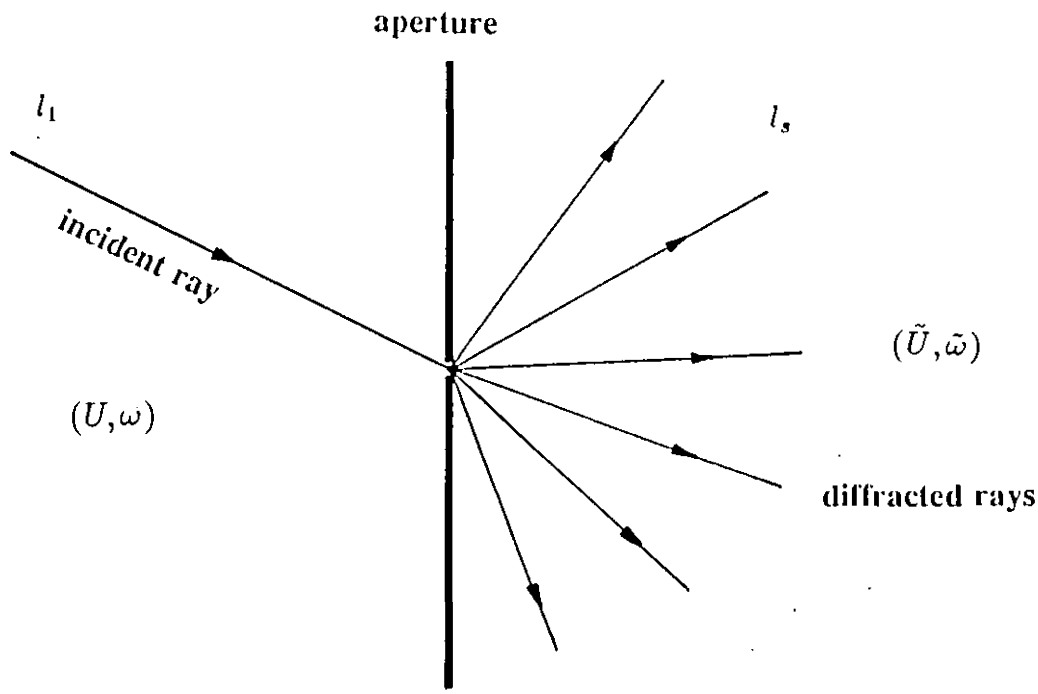

Figure 2.

In fact let $(a,b,x,y,u,v,w) \to F(a,b,x,y,u,v,w)$ be the optical distance function (cf. [**9-28**]) from the wavefront

$$\{z = \varphi(x,y) = \lambda_1 x^2 + \lambda_2 xy + \lambda_3 y^2 + O_3(x,y)\}$$

in the presence of the aperture $\{a \geq 0, \quad z = mb - 1\}$, where $m \geq 0$. If the incident ray goes from $(x, y) = (0, 0)$ to $(a, b) = (0, 0)$ then the transformed rays from $(a, b) = (0, 0)$ to (u, v, w) are given by equation

$$\frac{\partial \bar{F}}{\partial b}(0, u, v, w) = 0,$$
$$\bar{F}(b, x, y, u, v, w) := F(0, b, x, y, u, v, w),$$

(see Figure 3.), which for the distance function

$$F = \left((x-a)^2 + (y-b)^2 + (\varphi(x,y) - mb + 1)^2\right)^{1/2} + \left((u-a)^2 + (v-b)^2 + (w - mb + 1)^2\right)^{1/2}$$

reads

$$m^2u^2 + v^2(m^2 - 1) - 2mv(1 + w) = 0,$$

and

$$v + m(1 + w) \leq 0.$$

These conditions define the half-cone of diffracted rays (see [**18-19**]).

EXAMPLE 2.4. *Reflection from the curve* (see Figure 1.b):
Let the mirror be defined by equation $\{q_1 = 0\}$. Let (U, ω) -the space of incident rays be defined as $\pi_M(H^{-1}(0) \cap \{p_1 > 0\})$ and the corresponding space of reflected rays $\tilde{U} = \pi_M(H^{-1}(0) \cap \{p_1 < 0\})$. Then this reflecting optical instrument is equivalent to the following Lagrangian subvariety of Π,

$$\Pi \supset \{((r, s), (\tilde{r}, \tilde{s})) \in U \times \tilde{U}; r = \tilde{r}, s = \tilde{s}\} =: A$$

and its corresponding generating family (cf. [**34-15-33**]),

$$G(\lambda, s, \tilde{s}) = \lambda(s - \tilde{s}),$$

where $\lambda \in \mathbf{R}$, is a Morse parameter.

In our approach the sources of radiation produce rays in the space denoted by (U, ω). Thus we have

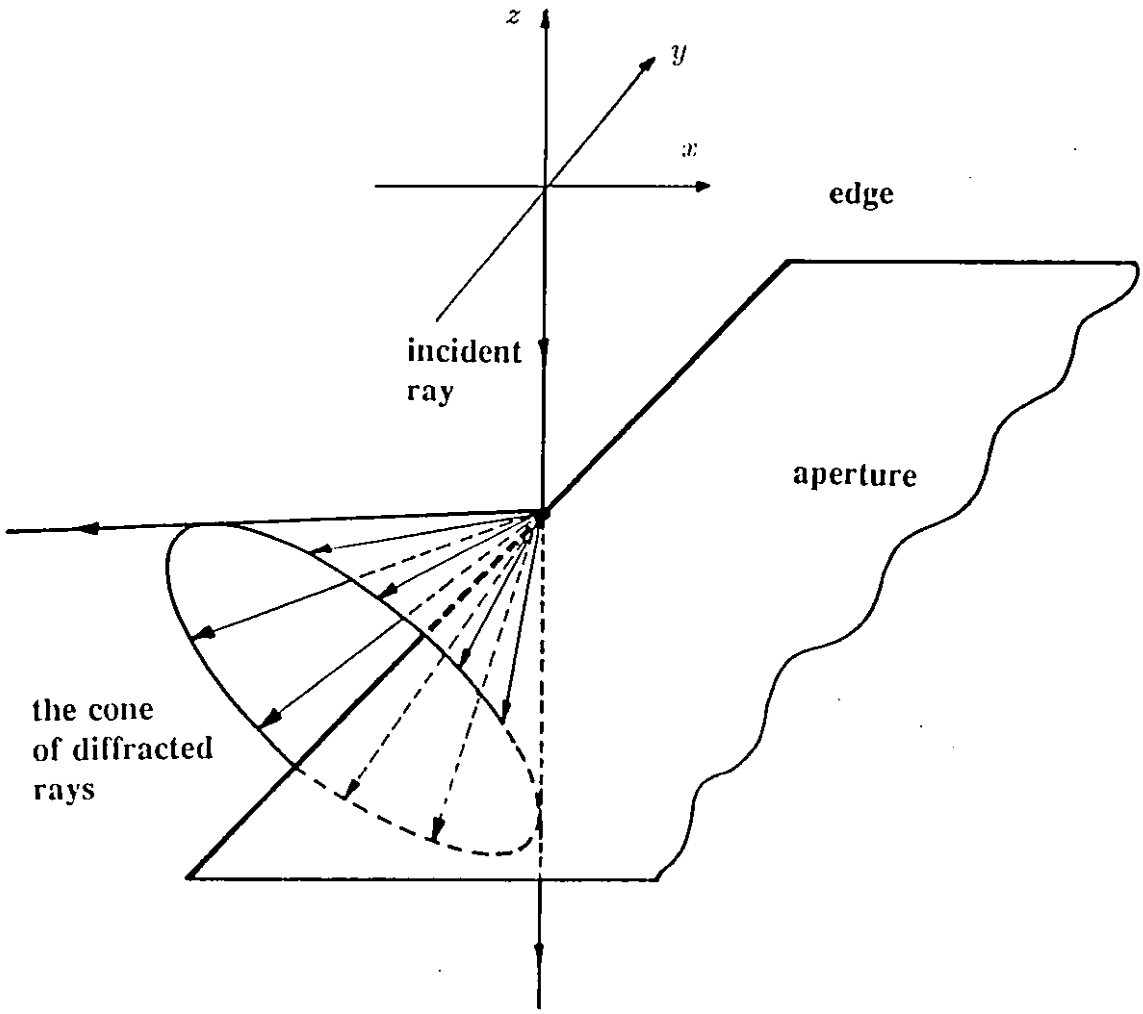

Figure 3.

Definition 2.5. *We define the general source of light as a Lagrangian subvariety $L \subset (U,\omega)$ of the space of incident rays. If $A \subset \Pi$ is an optical instrument , then the transformed system of rays (or equivalently the transformed wavefront , cf.* **[15]** *) is a symplectic image L' of L by means of A, i.e.*

$$L' := A(L) := \{\tilde{p} \in \tilde{U};\, there\ exists\, p \in L\, such\ that\, (p,\tilde{p}) \in A\},$$

which is usually a Lagrangian subvariety of $(\tilde{U},\tilde{\omega})$,(cf. **[14]**).

EXAMPLE 2.6. *Reflection of a parallel beam of rays:*
The beam of parallel rays is given in (U,ω) by $L = \{r = 0\}$ (a point source of light at

infinity). By reflection in the mirror, $x \to (\varphi(x), x) \in \mathbf{R}^2$, $\varphi(0) = \varphi'(0) = 0$, $\varphi''(0) \neq 0$, the canonical variety $A \subset \Pi$ (defining the reflection process) brings into L some focusing property and produces the well known caustic (see Figure 1.b) . The reflected beam of rays $A(L)$ has the form:

$$(\tilde{r}, \tilde{s}) = \left(\frac{2\varphi'(x)}{\varphi'(x)^2 + 1}, x - \frac{\varphi(x)\varphi'(x)(1 + \varphi'(x)^2)^2}{\varphi'(x)^2 - 1} \right).$$

REMARK 2.7. Local genericity of the wavefront produced by $L \subset (U, \omega)$ is preserved during the process of reflection or refraction (cf. [**10**]) because the corresponding canonical variety is a graph of symplectomorphism. Thus the caustics, produced by reflection or refraction are classified by the simple singularities of type A_k, D_k, E_k, (see [**3**]). It may not be so in a diffraction process, where $A \subset \Pi$ is no longer the graph of symplectomorphism. In this case the differentiable structure of L is drastically changed by A and $A(L)$ is no longer smooth. Its singular locus brings a completely new type of caustic responsible for the structure of shadows and halfshadows of an obstacle as well.

3. CAUSTICS AND QUASICAUSTICS BY DIFFRACTION

Let L be a source of light or transformed wavefront in (M,ω). Now we recall the geometric construction which allows us to define caustic or wavefront evolution in V, corresponding to L, (cf. **[1-16]**). Let Ξ be the product symplectic manifold

$$\Xi = (M \times T^*V, \pi_2^*\omega_V - \pi_1^*\omega),$$

where $\pi_{1,2} : M \times T^*V \to M, T^*V$ are the canonical projections. One can check that $\tilde{K} := graph\pi_M \subset \Xi$ is a Lagrangian submanifold of Ξ. Thus there exists its local generating Morse family (cf. **[34]**), say

$$K : \mathbf{R}^k \times \tilde{X} \times V \to \mathbf{R}, (\mu, \tilde{x}, q) \to K(\mu, \tilde{x}, q),$$

where $T^*\tilde{X}$ is an appropriate local cotangent bundle structure (special symplectic structure **[1-16-32]**) on (M,ω). The transformed system of rays forms a Lagrangian subvariety of (T^*V, ω_V) given as an image

$$\tilde{L} = (\tilde{K} \circ A)(L) \subset (T^*V, \omega_V),$$

where $\tilde{K} \circ A \subset \Xi$ is a composition of symplectic relations (cf. **[1-34]**). If

$$G : \mathbf{R}^l \times X \times \tilde{X} \to \mathbf{R}, (\nu, x, \tilde{x}) \to G(\nu, x, \tilde{x}), \quad X, \tilde{X} \cong \mathbf{R}^n,$$

is a generating family for $A \subset \Pi$ and $F : \mathbf{R}^m \times X \to \mathbf{R}, (\lambda, x) \to F(\lambda, x)$ is a generating family for L, then the transformed Lagrangian subvariety $\tilde{L} \subset (T^*V, \omega_V)$ is generated by (not necessary a Morse family),

$$\tilde{F} : \mathbf{R}^{k+l+m+2n} \times V \to \mathbf{R},$$

$$\tilde{F}(\lambda, \nu, \mu, x, \tilde{x}; q) := G(\nu, x, \tilde{x}) + K(\mu, \tilde{x}, q) + F(\lambda, x),$$

where $\mathbf{R}^{k+l+m+2n}$ is a parameter space.

In optical arrangements the source of light is usually a smooth Lagrangian submanifold of (U,ω). Only after the transformation process through an optical instrument does it become singular.

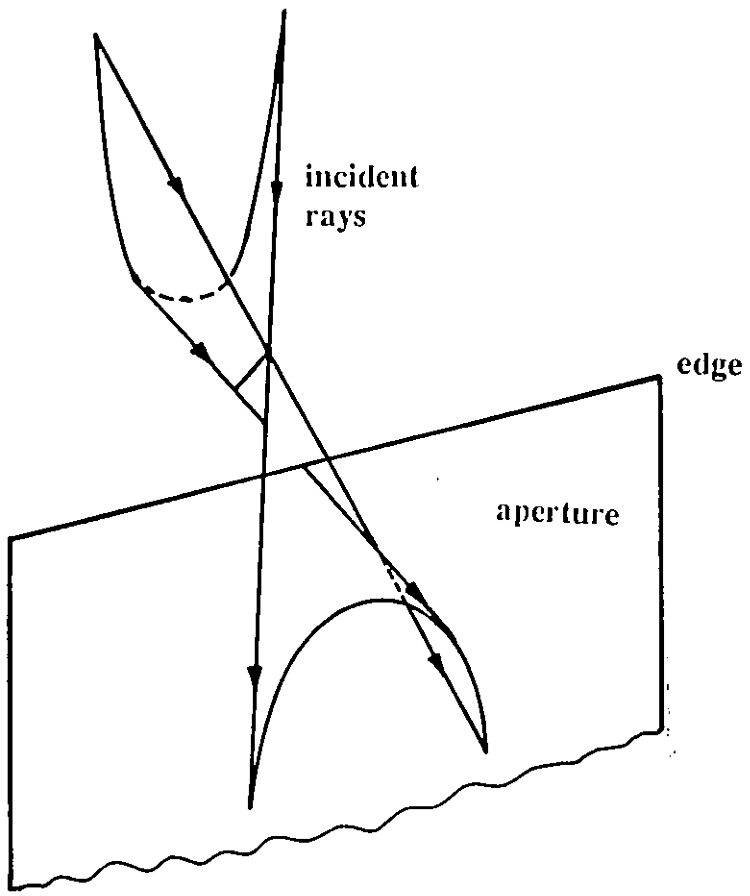

Figure 4.

Definition 3.1. *Let $L \subset (U,\omega)$ be an initial source variety. We define the caustic by an optical instrument $A \subset \Pi$, to be a hypersurface of V formed by two components :*

(1) *Singular values of* $\pi_V \mid_{\tilde{L}-Sing\tilde{L}}$;

(2) $\pi_V(Sing\tilde{L})$,

where $\tilde{L} = (\tilde{K} \circ A)(L)$ *and* $Sing\tilde{L}$ *denotes the singular locus of* $\tilde{L}$.

REMARK 3.2. In reflection or refraction we do not go beyond smooth category of L (at least in this paper) so the associate caustics, in transformed wavefronts $\tilde{L}$, are those realisable by smooth generic sources (cf. [5-10]). Thus in what follows we will be interested in caustics caused by diffraction which will enrich substantially the list of optical events (cf. [4]) and complete the correspondence between singularities of functions and groups generated by reflections (see [28-29]).

Diffracted rays are produced, for example, when an incident ray hits an edge of an impenetrable screen (i.e. an edge of a boundary or interface, cf. [18]). In this case the incident ray produces infinitely many diffracted rays, which have the same angle with the edge as does the incident ray (see Remark 2.3.) This is so if both, incident and diffracted, rays lie in the same medium. Otherwise, the angles between the two rays and the plane normal to the edge are related by Snells law (see [19]). Furthermore, the diffracted ray lies on the opposite side of the normal plane from the incident ray. The edge diffraction is illustrated in Figure 3. That is exactly that all rules and laws of geometrical optics correspond exactly to the lagrangian properties of the corresponding varieties $A \subset \Pi$.

Let I be the diagonal in Π. By Ω we denote the set of oriented lines in (U,ω) which do not intersect the screen. Thus we have

Proposition 3.3. *In the edge diffraction in an arbitrary Euclidean space, the canonical variety $A \subset \Pi$ has two components*

$$A = A^I \cup A^D,$$

where $A^I = \Omega \times \Omega \subset I$ and A^D is a pure diffraction of rays passing through the edge of an aperture, defined in Remark 2.3.

COROLLARY 3.4. Let $L \subset (U,\omega)$ be an incident system of rays. Then the edge diffracted system of rays

$$\tilde{L} = (\tilde{K} \circ A)(L)$$

is a regular intersection (cf. [25]) of two smooth components: $\tilde{L}_1 = (\tilde{K} \circ A^I)(L)$ and $\tilde{L}_2 = (\tilde{K} \circ A^D)(L)$, i.e. $\tilde{L} = \tilde{L}_1 \cup \tilde{L}_2$, $dim\tilde{L}_1 \cap \tilde{L}_2 = dim\tilde{L}_1 - 1$ and $T_x(\tilde{L}_1 \cap \tilde{L}_2) = T_x\tilde{L}_1 \cap T_x\tilde{L}_2$.

Thus we see that the caustic caused by the edge diffraction has a three components:

(1) The caustic of $\tilde{L}_1$, which is a part of the caustic in incident wavefront L.

(2) The caustic, purely by diffraction on the edge, i.e. the caustic of $\tilde{L}_2$.

(3) The image $\pi_V(\tilde{L}_1 \cap \tilde{L}_2)$, of the rays passing exactly through an edge.

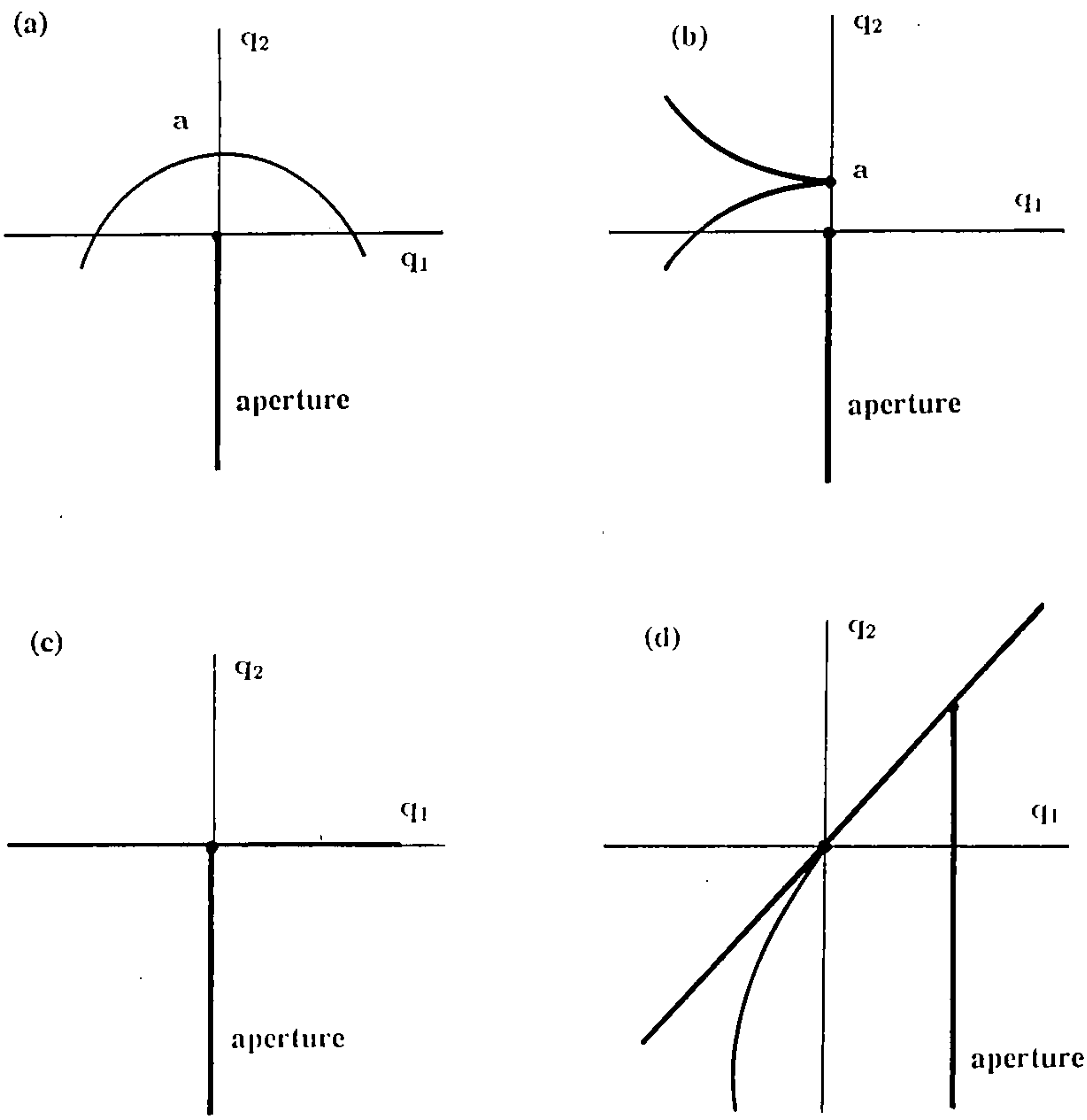

Figure 5.

Definition 3.5. *The set $\pi_V(\tilde{L}_1 \cap \tilde{L}_2) \subset V$ is called the quasicaustic by diffraction on aperture. The rays belonging to quasicaustic which are contained in aperture plane we will call the rays at infinity.*

Usually the quasicaustics describe the structure of shadows and half-shadows in configurational space V (see Figure 4).

Proposition 3.6.

1. *Generic caustics by diffraction on the half line aperture on the plane are diffeomorphic to the* $\tilde{A}_2$, $\tilde{A}_3$, $B_2 \cong C_2$, B_3 *boundary caustics. Normal forms for their generating families as images* $A(L)$ *(or pairs* (A, L) *in general position) are the following:*

$\tilde{A}_2: \ -\frac{1}{3}\lambda^3 + \lambda(q_2 - a) - \frac{1}{2}q_1\lambda^2, \quad a > 0$, *and* $A := \{q_1 = 0, q_2 \leq 0\}$, *(see Figure 5.a),*
$\tilde{A}_3: \ -\frac{1}{4}\lambda^4 + \lambda(q_2 - a) - \frac{1}{2}q_1\lambda^2, \quad a > 0$, *and* $A := \{q_1 = 0, q_2 \leq 0\}$, *(see Figure 5.b),*
$B_2: \ -\frac{1}{2}\lambda^2 + q_2\lambda - \frac{1}{2}q_1\lambda^2, \quad \{\lambda \geq 0\}$, *and* $A := \{q_1 = 0, q_2 \leq 0\}$, *(see Figure 5.c),*
$B_3: \ -\frac{1}{3}\lambda^3 - \frac{1}{2}q_1\lambda^2 + \lambda(q_2 - q_1 a), \quad \{\lambda \geq 0\}$, *and* $A := \{q_1 = 2a, q_2 \leq 2a^2\}, \quad a > 0$, *(see Figure 5.d),*

where λ *is a Morse parameter and* a *is the moduli of common position.*

2. *In generic one-parameter families of caustics by diffraction on the halfline aperture, which do not pass through infinity, the only possible configurations are those ones described in metamorphoses of optical caustics* (see **[11]** ,[5],p.113) *and the additional cases illustrated in Figure 6.a,b,c,d.*

Proof. It is easily seen that $\tilde{K} = graph\pi_M \subset \Xi$ (see (3.1)) is generated locally by

$$K(r, q_1, q_2) = q_2 r - \frac{1}{2}q_1 r^2.$$

The only stable systems of rays $\tilde{K}(L) \subset (T^*V, \omega_V)$ are generated in (M, ω) by $L := \{(r, s); s = -\frac{\partial F}{\partial r}(r)\}$, where
$A_1: \quad F_1(r) = -\frac{1}{2}r^2$,
$A_2: \quad F_2(r) = -\frac{1}{3}r^3$,
$A_3: \quad F_3(r) = -\frac{1}{4}r^4$, (cf. **[5-15]**).

Let the aperture (so $A \subset \Pi$) be defined in its normal form by equations: $q_1 = 0, \quad q_2 \leq 0$. Thus we have the boundary singularities (cf. **[24]**) $A(L)$ defined in $(\tilde{M}, \tilde{\omega})$ by the following generating functions:
$\tilde{A}_1: \quad \tilde{F}_1(\tilde{r}) = -\frac{1}{2}\tilde{r}^2, \ \{\tilde{r} \geq 0\}$,
$\tilde{A}_2: \quad \tilde{F}_2(\tilde{r}) = -\frac{1}{3}\tilde{r}^3, \ \tilde{r} \in \mathbf{R}$,
$\tilde{A}_3: \quad \tilde{F}_3(\tilde{r}) = -\frac{1}{4}\tilde{r}^4, \ \{\tilde{r} \geq 0\}$.

Taking A_i in general position with respect to A we obtain part "1." of Proposition 3.6. Part "2." follows by checking all the possible one-parameter evolutions (where the quasicaustic

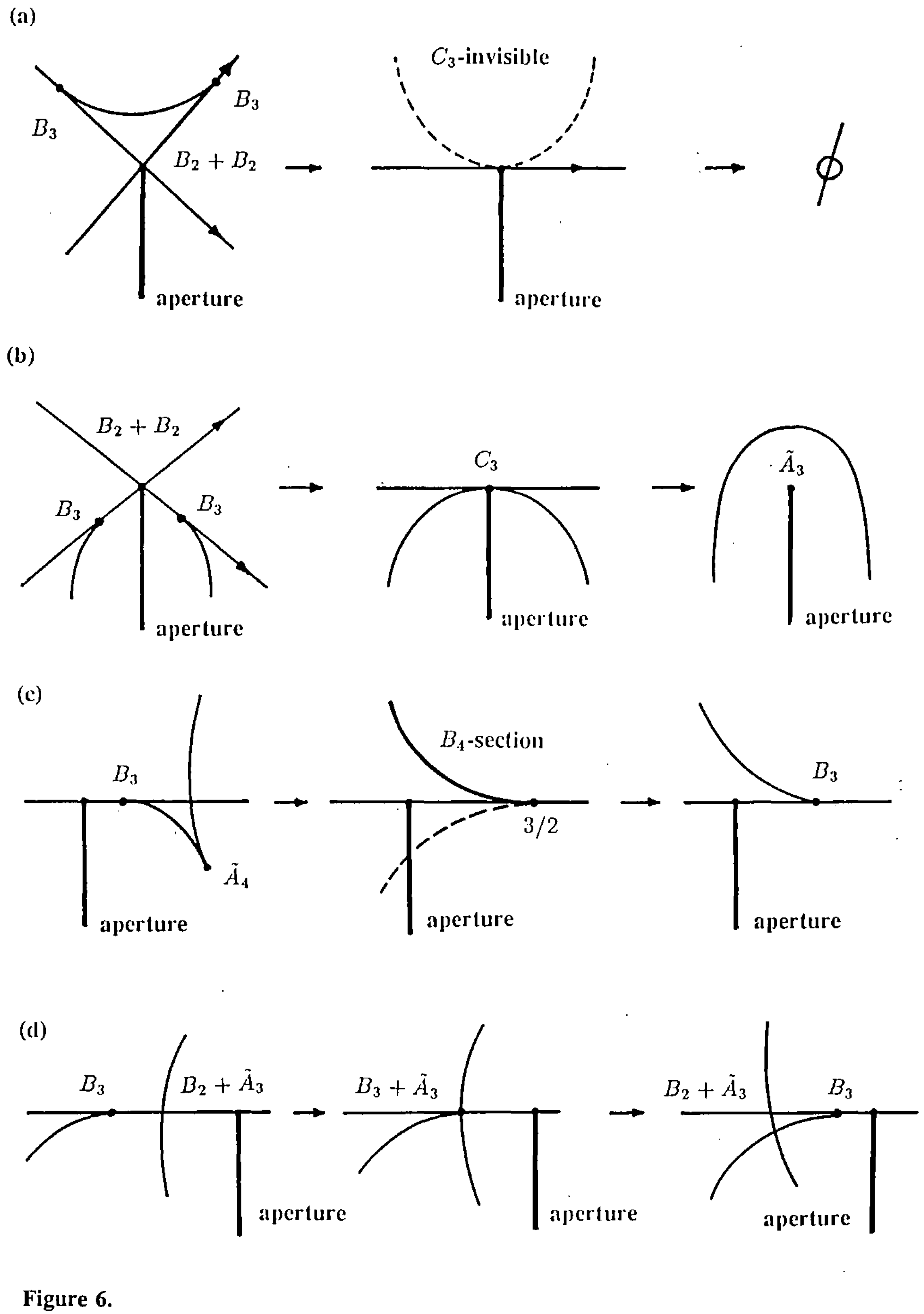
(a)
B_3
B_3
$B_2 + B_2$
aperture
C_3-invisible
aperture
(b)
$B_2 + B_2$
B_3
B_3
aperture
C_3
aperture
$\tilde{A}_3$
aperture
(c)
B_3
$\tilde{A}_4$
aperture
B_4-section
3/2
aperture
B_3
aperture
(d)
B_3
$B_2 + \tilde{A}_3$
aperture
$B_3 + \tilde{A}_3$
aperture
$B_2 + \tilde{A}_3$
B_3
aperture

Figure 6.

is not passing through infinity) of the stable caustic on the plane and in the presence of the half-line aperture. Two possible directions of intersection of A_2-caustic by an edge of aperture give us the cases "a" and "b" in Figure 6. The evolution of an edge of the aperture passing through the ray tangent to the cusp caustic A_3 is illustrated in Figure 6,c. Finally an evolution through the intersection point of $A_2 + A_2$-caustic gives us the case of Figure 6, d. This completes the proof of Proposition 3.6.

□

Looking at the position of quasicaustic in the diffraction problem with a half-plane aperture in $\mathbf{R}^3$ we can eliminate the C_4-boundary caustic. Thus we have

Proposition 3.7. *Generic caustics by diffraction on the half-plane aperture in* $\mathbf{R}^3$ *are diffeomorphic to the* $\tilde{A}_2,\ \tilde{A}_3,\ \tilde{A}_4,\ B_2, B_3,\ B_4,\ F_4$ *boundary caustics.*

REMARK 3.8.

1. For the general linear hyperbolic system of first order (cf. [19]),

$$\mathcal{L}u = u_t + \sum_{\nu=1}^{3} A^\nu \frac{\partial u}{\partial x_\nu} + Bu = 0,$$

where u represents, say in the case of crystal optics, the pair of vectors (E, H) and $\mathcal{L}u = 0$ corresponds to Maxwells equations. In the geometrical optics approximation we obtain an another characteristic equation (eikonal equation),

$$det\left(\Phi_t + \sum_{\nu=1}^{3} A^\nu \frac{\partial \Phi}{\partial x_\nu}\right) = 0$$

for the phase function $\Phi(x,t)$; $u \sim e^{i\omega\Phi(x,t)} a^0(x,t)$. In this case the conical refraction in crystal optics is an example of a Lagrangian variety quite generally situated in the associated phase space (cf. [13-19]).

2. In the edge diffraction on system of apertures (mentioned in [18]) the singularities of the distance function are classified by the singularities on many dimensional corners (see [30]). In very constrained systems of apertures the classification is obtained using the methods of the theory of singularities of functions on singular varieties (cf. [7-28]).

3. The generic quasicaustic in the edge diffraction in $\mathbf{R}^3$, corresponding to the F_4 singularity of the distance function (cf. [11]), is realized geometrically (see Figure 4) when the curve of rays passing through the edge on the incident wavefront is tangent to a constant curvature line on the wavefront. This situation is generic (cf. [5]).

4. DIFFRACTION ON SMOOTH OBSTACLES

Now we can apply an introduced symplectic framework to describe the diffraction on smooth closed surfaces in $\mathbf{R}^3$. The problem is connected to the Riemannian obstacle problem (cf. [**2**]), i.e. determination of geodesics on a Riemannian manifold with smooth boundary. Any geodesic on such manifold is C^1 and consists generically finitely many so-called switchpoints, where geodesic has an initial or end point according to lie in interior part of the manifold or on the boundary. Cauchy uniqueness for manifolds with boundary states that every boundary point (point of an obstacle) has a neighbourhood in which: if two geodesic segments with the same initial point, initial tangent vector and length do not coincide, then one of them has its right endpoint in the interior part of the manifold and is an involutive of the other (in the planar case it lies on an appropriate involute of the obstacle curve). By an involutive of a geodesic γ is meant a beodesic $\gamma\prime$ which has the same initial point, initial tangent vector and length as γ. The reformulation of the above obstacle problem in terms of geometrical optics of diffraction needs to define a surface diffracted ray. A surface diffracted ray is produced when a ray is incident tangentially on a smooth boundary or interface. It is a geodesic on the surface in the metric nds, where n is the refractive index of the medium on the side of the surface containing the incident ray. At every point it sheds a diffracted ray along its tangent (cf. [**18-4**]). A surface diffracted ray is also produced on the second side of an interface by a ray incident from the first side at the critical angle $(arcsin(\frac{n_1}{n_2}))$. In this case at every point it sheds rays back toward the first side at the critical angle . However in what follows we will neglect these rays.

Let us consider an open subset S of an obstacle surface in $\mathbf{R}^3$. By l_1 we denote the initial tangent line to the geodesic segment γ on S. Let l_2 be a tangent line to S. We say that l_2 is subordinate to l_1 with respect to an obstacle S if l_2 (or its piece in $(\mathbf{R}^3, S)$) belongs to the geodesic segment with the same initial point and the same tangent vector as γ has. By simple checking we have the following (cf. [**15**]).

Proposition 4.1. *Let γ be a geodesic flow on S. Then the set*

$$A = \{(l, \tilde{l}) \in \Pi; \tilde{l}\, is\ subordinate\ to\, l\, with\ respect\ to\, S\ and\ geodesic\ flow\, \gamma\}$$

is a Lagrangian subvariety of Π *defining the diffraction process on an obstacle* S.

Now we look for the generic pairs (A, L). At first we consider the planar case.

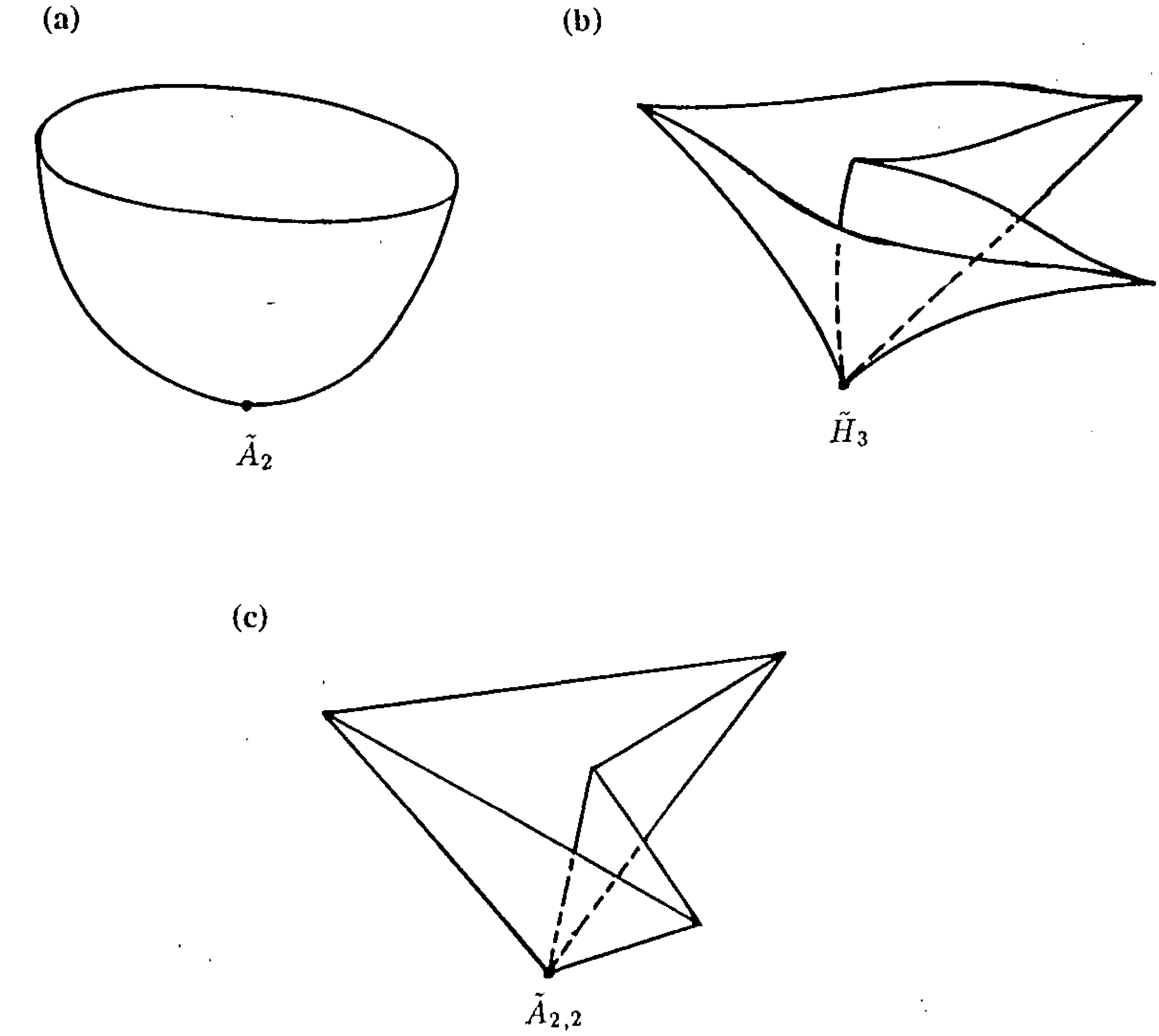

Figure 7.

Proposition 4.2. *For the generic obstacle curve on the plane the only possible canonical varieties $A \subset \Pi$ have the following normal forms of generating families (or functions):*

$\tilde{A}_2$: $\quad G(r, \tilde{r}) = -\frac{1}{12}(r^3 + \tilde{r}^3)$,

(obstacle curve $q_2 = -q_1^2$), see Figure 7,a ,

$\tilde{H}_3$: $\quad G(\lambda_1, \lambda_2, r, \tilde{r}) = \frac{9}{10}(\lambda_1^5 + \lambda_2^5) - r\lambda_1^3 - \tilde{r}\lambda_2^3 + \frac{1}{2}r^2\lambda_1 + \frac{1}{2}\tilde{r}^2\lambda_2$,

(obstacle curve $q_2 = q_1^3$), see Figure 7,b,

$\tilde{A}_{2,2}$: $\quad G(r, \tilde{r}) = \frac{1}{2}(r \mid r \mid + \tilde{r} \mid \tilde{r} \mid)$,

(double tangent), see Figure 7,c.

Proof. Let us take the noninflection point of the generic curve. Parametrically the curve is given as $(q_1, q_2) = (v, -v^2)$, $v \in \mathbf{R}$, and the corresponding family of tangent lines corresponding to the given incident ray has a form $(q_1, q_2) = (0, v^2) + u(1, -2v)$, $u \in \mathbf{R}$. By identification $s = v^2$, $r = \frac{4v}{1+4v^2}$, $\tilde{s} = \bar{v}^2$, $\tilde{r} = \frac{4\bar{v}}{1+4\bar{v}^2}$, where $(v, \bar{v}) \in \mathbf{R}^2$ parametrize the variety A, we obtain the case $\tilde{A}_2$ which corresponds to the cartesian product of two ordinary folds. Taking the inflection point for an obstacle curve, we obtain, in the same way, the following parametrization for $A \subset \Pi$, namely

$$s = -2v^3, \; r = \frac{3v^2}{\sqrt{1+9v^4}}, \; \tilde{s} = -2\bar{v}^3, \; \tilde{r} = \frac{3\bar{v}^2}{\sqrt{1+9\bar{v}^4}}.$$

After straightforward calculations we obtain the generating family for it, denoted by $\tilde{H}_3$. Analogously we obtain the $\tilde{A}_{2,2}$ case (see Figure 8).

□

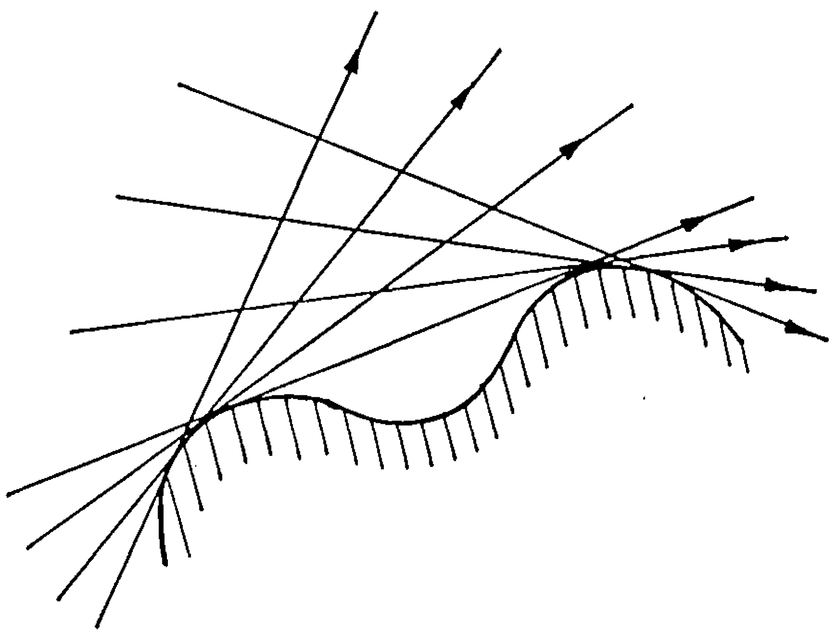

Figure 8.

Corollary 4.3. *For (A, L) in general position we have the following possible stable images $A(L) \subset (\tilde{M}, \tilde{\omega})$,*

$A_2 : \; \tilde{F}_1(\tilde{r}) = -\frac{1}{12}\tilde{r}^3,$

$H_3 : \; \tilde{F}_2(\lambda, \tilde{r}) = \frac{9}{10}\lambda^5 - \tilde{r}\lambda^3 + \frac{1}{2}\tilde{r}^2\lambda,$

$A_{2,2} : \; \tilde{F}_3(\tilde{r}) = \frac{1}{2} \mid \tilde{r} \mid \tilde{r}.$

and the generating families for their corresponding configurational images

$K(A_2):\quad F_1(\lambda, q_1, q_2) = -\frac{1}{12}\lambda^3 + q_2\lambda - \frac{1}{2}q_1\lambda^2,$

$K(H_3):\quad F_2(\lambda_1, \lambda_2, q_1, q_2) = \frac{9}{10}\lambda_1^5 - \lambda_2\lambda_1^3 + \frac{1}{2}\lambda_2^2\lambda_1 + q_2\lambda_2 - \frac{1}{2}q_1\lambda_2^2,$

$K(A_{2,2}):\quad F_3(\lambda, q_1, q_2) = \frac{1}{2}\lambda \mid \lambda \mid + q_2\lambda - \frac{1}{2}q_1\lambda^2,$

(see Figures 9, a. b. c. and also Figures in [**4**]).

Proof. In general position of A and L, only one point of L is tangent to an obstacle curve in the neighbourhood of the considered point of this curve. Hence in calculation of $(\tilde{K} \circ A)(L)$ in all cases ($\tilde{A}_2$,$\tilde{H}_3$, and $\tilde{A}_{2,2}$) it is necessary to put $r = const.$ in generating families of Proposition 4.2.

□

REMARK 4.4.

A. The first, most important, results in obstacle geometry and its correspondence to the structure of singular orbits of H_3 and H_4 group actions, were discovered by Scherbak [**28**]. The aim of the present paper is to show how singular wavefront evolutions appear in general setting of mathematical theory of optics (cf. [**13-15-21**]) and to complete investigations of the caustics and quasicaustics which appear there. As we see the planar obstacle problem is connected to the studies of tangent developables. More degenerated singularities there can be described using the blowing-up construction (cf. [**22**]).

B. The $K(A_{2,2})$ singularity appeared as an adjaicent to the higher singular one (see Figure 10.) in generic one-parameter family of obstacles $q_2 = -\frac{1}{4}q_1^4 + \frac{1}{2}aq_1^2 - \frac{1}{4}a^2$, $a \in \mathbf{R}_+$. i.e.

$$r = -2av_\epsilon - 3\epsilon\sqrt{a}v_\epsilon^2 + (4a^3 - 1)v_\epsilon^3 + O(v_\epsilon^4),$$

$$s = 2\epsilon a^{3/2}v_\epsilon + 4av_\epsilon^2 + 3\epsilon\sqrt{a}v_\epsilon^3 + \frac{3}{4}v_\epsilon^4,$$

$\epsilon = \pm 1$, $v_+ \geq 0$, $v_- \leq 0$.

C. We can see that choosing the special symplectic structure fibered over (p_1, p_2) in the H_3 case, we can investigate only a cuspidal edge of $A(L)$. In fact its generating family

$$F_2'(\lambda, \mu, p) = F_2(\lambda, \mu) - \mu_1 p_1 - \mu_2 p_2,$$

after reduction of μ_1, μ_2, and λ_2 parameters we obtain the generating family for the H_2 singularity,

$$F_2'(\lambda, p) = \frac{9}{10}\lambda^5 - p_2\lambda^3 + \frac{1}{2}p_2^2\lambda$$

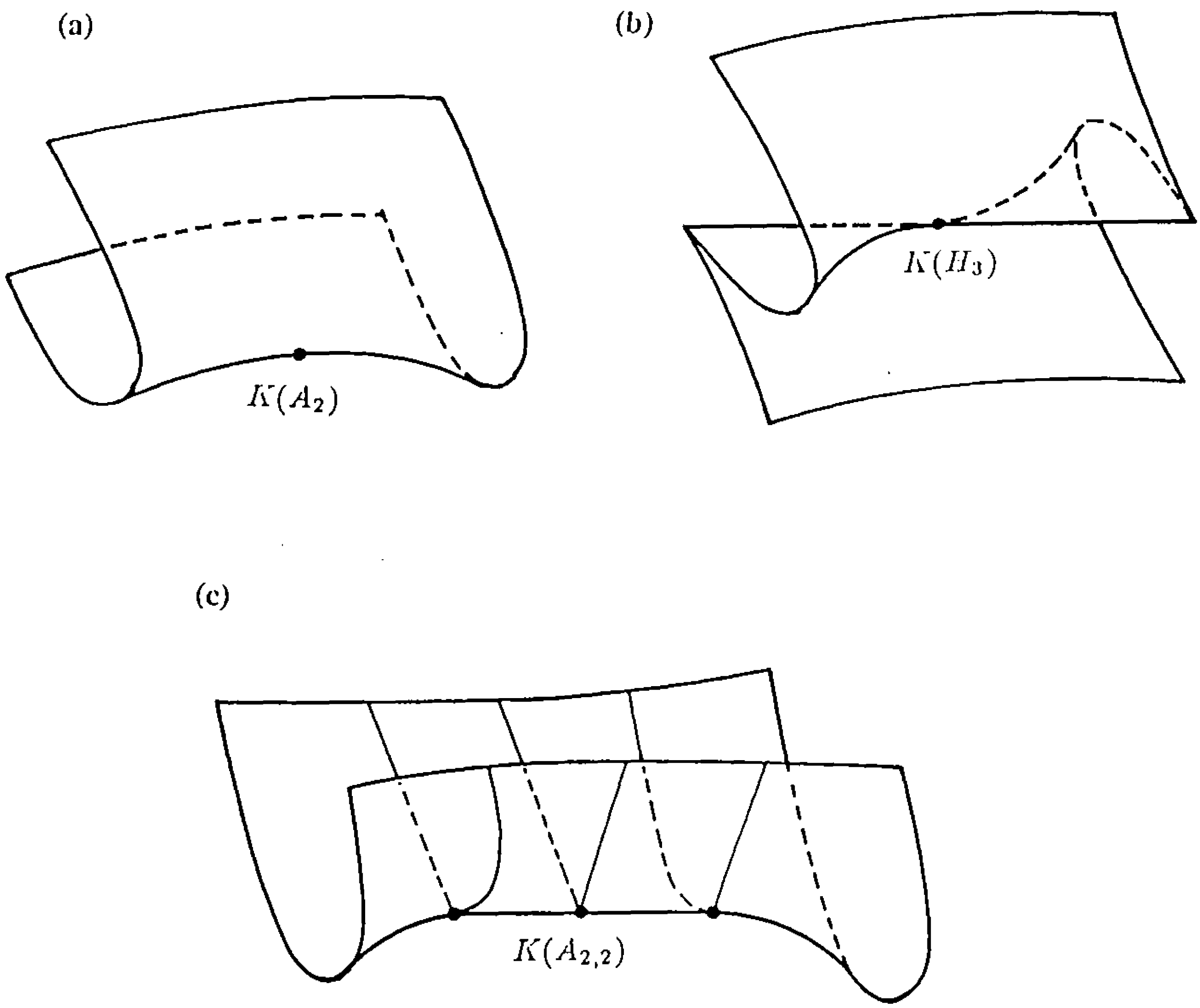

Figure 9.

and its level-sets (wave fronts) as has been written down in Table 2. of [**28**]. That observation is connected with much more general feature of obstacle singular wavefront evolutions. Namely all singularities in obstacle geometry as indicated in Table 2. of [**28**] are generated by the generalized open swallowtails (in $(\tilde{M},\tilde{\omega})$ space) with generating family (see [**14**] p. 106),

$\tilde{A}_{2(k+1)}$:

$$\int_0^\lambda \left(x^{k+1} + \sum_{i=2}^{k+1} \tilde{s}_{i-1} x_{k-i+1} \right)^2 dx.$$

Ξ_l, $(l \geq 1)$, Δ_l, $(l \geq 2)$ (cf. [28])- singular wavefront evolutions are reconstructed from $\tilde{A}_{2(k+1)}$ singularities by specifying an appropriate common generic positions of $A \subset \Pi$ and $\tilde{A}_{2(k+1)} \subset (\tilde{M}, \tilde{\omega})$.

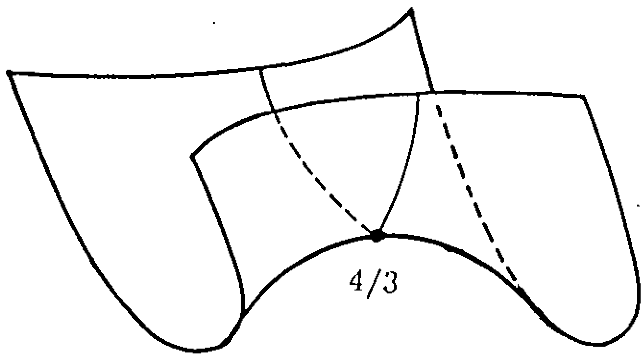

Figure 10.

5. VECTOR FIELDS ON CAUSTICS AND QUASICAUSTICS

As we can see from the preceeding sections, caustics in the wavefront evolution, or in diffracted wavefront on aperture, are defined as bifurcation sets for the corresponding generating family (Morse family **[9-34]**) of functions or the family of functions on manifold with boundary respectively (cf. **[3-24]**). To investigate the structure of these sets and modules of tangent vector fields on them, in what follows, we shall consider the real analytic or holomorphic functions (germs). For the ordinary caustics, defined as the critical values of Lagrange projections (cf. **[3]**) from the Lagrangian submanifolds, which are not necessary fibered by optical rays, the procedure is following (see **[6-31]**):

Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be a holomorphic function of finite codimension , i.e. the dimension of the quotient $\mathcal{O}_{(x)}/J(f)$ as a complex vector space is finite, where $\mathcal{O}_{(x)}$ denotes the ring of holomorphic functions $h : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ and $J(f)$ is the ideal in $\mathcal{O}_{(x)}$ generated by the partial derivatives $\dfrac{\partial f}{\partial x_1}, \ldots, \dfrac{\partial f}{\partial x_n}$. Let $\mathcal{M}_{(x)}$ denote the maximal ideal in $\mathcal{O}_{(x)}$. If $g_1, \ldots, g_p$ is a basis for $\mathcal{M}_{(x)}/J(f)$, then

$$F : (\mathbf{C}^n \times \mathbf{C}^p, 0) \to (\mathbf{C}, 0),$$

$$F(x, a) = f(x) + \sum_{i=1}^{p} a_i g_i(x)$$

is a miniversal unfolding of f (cf. **[23]**).

The caustic of F (or bifurcation set of F, see **[31-7]**) is the following set (germ),

$$B(F) = \{a \in \mathbf{C}^p; F_a \text{ has a degenerate critical point}\}.$$

The set of critical values of $\pi : (\Sigma F, 0) \to (\mathbf{C}^p, 0)$ (π is a canonical projection on the second factor), where $\Sigma F = \{(x, a) \in \mathbf{C}^n \times \mathbf{C}^p; \dfrac{\partial F}{\partial x_1} = \ldots = \dfrac{\partial F}{\partial x_n} = 0\}$, is the caustic. It appeared to be important to know the modules of tangent vector fields to caustics (as well as to wavefronts **[35-3-7]**, which is easier). They are useful in reduction of functional moduli in classification of generic symmetric and nonsymmetric Lagrangian submanifolds (cf. **[17]** ,**[3]** , p. 344). We recall after **[27-6]** some necessary definitions. The set of germs of holomorphic vector fields on $\mathbf{C}^p$, at 0, tangent to the nonsingular part of $B(F)$ is called the set of logarithmic vector fields of $B(F)$ at 0. It is denoted also by $Derlog B(F)$. In **[31-7-8]** (see also **[35-27]**) it was given a general method for computing these vector fields. It was shown there that A_k singularities are the only ones whose module of tangent vector fields to $B(F)$ is free (i.e. caustic is a free divisor [27]). Applying the method used in these papers we investigate the modules of vector fields tangent to the quasicaustics in diffraction on apertures (this is a first step in investigation of the structure of caustics by diffraction).

Let $\mathcal{O}_{(y,x)}$ denote the ring of holomorphic functions $h : (\mathbf{C} \times \mathbf{C}^n, 0) \to (\mathbf{C}, 0)$. The hypersurface $S = \{y = 0\}$ corresponds to the boundary of an aperture. Following the general scheme used in [3] for boundary singularities, we shall consider holomorphic functions $f : (\mathbf{C} \times \mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ of finite codimension, i.e.

$$dim_{\mathbf{C}} \mathcal{O}_{(y,x)} / \Delta(f) < \infty,$$

where $\Delta(f) = \langle y\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n} \rangle$ denotes the ideal in $\mathcal{O}_{(y,x)}$ generated by the partial derivatives $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ and $y\frac{\partial f}{\partial y}$ (cf. [3-20]). Let $g_0, ..., g_{\mu-1}$ form a basis for $\mathcal{O}_{(y,x)}/\Delta(f)$ with $g_0 = 1$ and $g_i \in \mathcal{M}_{(y,x)}$. Then the miniversal deformation , in the category of deformations of functions on manifold with boundary, as a Morse family for the corresponding diffracted Lagrangian variety (cf. [25-16]) is defined as follows

$$F : (\mathbf{C} \times \mathbf{C}^n \times \mathbf{C}^{\mu-1}, 0) \to (\mathbf{C}, 0)$$

$$F(y, x, a) = f(y, x) + \sum_{i=1}^{\mu-1} a_i e_i(y, x).$$

Proposition 5.1. *The caustic (or bifurcation set) by diffraction on aperture, having the generating family $F : (\mathbf{C} \times \mathbf{C}^n \times \mathbf{C}^p, 0) \to (\mathbf{C}, 0)$ (p is not necessary minimal) of functions on manifold with boundary (extended edge) has a three components,*

(1) $B_1(F) = \{a \in \mathbf{C}^p; F(\bullet, \bullet, a)$ *has a degenerate critical point*$\}$,

(2) $B_2(F) = \{a \in \mathbf{C}^p; F(0, \bullet, a)$ *has a degenerate critical point*$\}$,

(3) $Q(F) = \{a \in \mathbf{C}^p; F(\bullet, \bullet, a)$ *has a degenerate critical point on* $S = \{y = 0\}\}$.

Proof. By Corollary 3.4 we have the three isothropic submanifolds defining the system of diffracted rays $\tilde{L}_1$, $\tilde{L}_2$ and $\tilde{L}_1 \cap \tilde{L}_2$. It is easily seen that in terms of generating family - distance function F, the corresponding caustics can be written in forms (1), (2), (3) of the proposition.

□

The set-germ

$$(\Sigma_r F,\, 0) = (\{(x,a) \in \mathbf{C}^n \times C^p; \frac{\partial F}{\partial y} \mid_S = \frac{\partial F}{\partial x_1} \mid_S = \ldots = \frac{\partial F}{\partial x_n} \mid_S = 0\}, 0)$$

we call the restricted critical set.

Using the splitting Lemma (see **[23]**) and the versality property of F we have,

Proposition 5.2.

A. *The restricted critical set $(\Sigma_r F,\, 0)$ is the germ of a smooth manifold of dimension $p-1$.*

B. *The quasicaustic of F, $(Q(F), 0)$ is an image of $(\Sigma_r F, 0)$ by the natural projection $\pi : \Sigma_r F, 0 \to \mathbf{C}^p, 0$ to the second factor.*

The set of logarithmic vector fields of $Q(F)$ at 0 is defined (cf. **[27-6]**) to be the set of germs of holomorphic vector fields on $\mathbf{C}^p$ at 0, tangent to the nonsingular part of $Q(F)$; it is an $\mathcal{O}_{(a)}$-module.

Proposition 5.3. *Let $\xi \in DerlogQ(F)$, then it is π-liftable, i.e. for some germ of a vector field $\tilde{\xi}$, on $\mathbf{C}^n \times \mathbf{C}^p$, tangent to $\Sigma_r F$ at 0 we have*

$$\xi \circ \pi = d\pi \circ \tilde{\xi}.$$

Proof. ξ lifts uniquely by π at every point $a \in \mathbf{C}^p - \Gamma(\pi \mid_{\Sigma_r F})$. Hence ξ lifts to a holomorphic vector field $\tilde{\xi}_1$ on $\mathbf{C}^n \times \mathbf{C}^p$, tangent to $\Sigma_r F$ and defined off a set of codimension 2 in $\mathbf{C}^n \times \mathbf{C}^p$. By Hartogs theorem $\tilde{\xi}_1$ extends to a holomorphic vector field $\tilde{\xi}$ tangent to $\Sigma_r F$.

□

Now using the π-lowerable vector fields $\tilde{\xi}$ tangent to $\Sigma_r F$ we will construct the module $DerlogQ(F)$. Let F be as above, we define the ideal

$$I(F) = \langle \psi(x,a), \frac{\partial \bar{F}}{\partial x_1}(x,a), \ldots, \frac{\partial \bar{F}}{\partial x_n}(x,a) \rangle \mathcal{O}_{(x,a)},$$

where ψ and $\bar{F}$ are given by decomposition

$$F(y,x,a) = F(0,x,a) + y\psi(x,a) + y^2 g(y,x,a), \;\; \bar{F}(x,a) := F(0,x,a).$$

Let $\tilde{\xi} = \sum_{i=1}^{n} \beta_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{p} \gamma_i \frac{\partial}{\partial a_i}$, $\beta_i, \gamma_i \in \mathcal{O}_{(x,a)}$, be the germ of a vector field at $0 \in \mathbf{C}^n \times \mathbf{C}^p$, tangent to $\Sigma_r F$. Then we have

$$\tilde{\xi}(\frac{\partial F}{\partial y}(0,x,a)) \in I(F)$$

and

$$\tilde{\xi}(\frac{\partial F}{\partial x_i}(0,x,a)) \in I(F), \;\; i=1,...,n.$$

For our $F(y,x,a) = f(y,x) + \sum_{i=1}^{\mu-1} a_i g_i(y,x)$ we have

$$\psi(x,a) = \frac{\partial f}{\partial y}(0,x) + \sum_{i=1}^{\mu-1} a_i \frac{\partial g_i}{\partial y}(0,x).$$

So we need

$$\sum_{i=1}^{n} \beta_i \frac{\partial \psi}{\partial x_i} + \sum_{i=1}^{\mu-1} \gamma_i \frac{\partial g_i}{\partial y} \;|_{0\times \mathbf{C}^n} \in I(F)$$

and

$$\sum_{i=1}^{n} \beta_i \frac{\partial^2 \bar{F}}{\partial x_i \partial x_j} + \sum_{i=1}^{\mu-1} \gamma_i \frac{\partial \bar{g}_i}{\partial x_j} \in I(F), \;\; 1 \le j \le n,$$

where $\bar{g}(x) := g(0,x)$. Thus we obtain

Lemma 5.4. *$\tilde{\xi}$ is a lifting of $\xi \in DerlogQ(F)$, $\xi = \sum_{i=1}^{p} \alpha_i(a)\frac{\partial}{\partial a_i}$, if and only if for some $\beta_i \in \mathcal{O}_{(x,a)}$, $(i=1,...,n)$ we have*

$$\sum_{i=1}^{n} \beta_i \frac{\partial \psi}{\partial x_i} + \sum_{i=1}^{\mu-1} \alpha_i \frac{\partial g_i}{\partial y} \;|_{0\times \mathbf{C}^n} \in I(F),$$

$$\sum_{i=1}^{n} \beta_i \frac{\partial^2 \bar{F}}{\partial x_i \partial x_j} + \sum_{i=1}^{\mu-1} \alpha_i \frac{\partial \bar{g}_i}{\partial x_j} \in I(F). \tag{5.1}$$

We have choosen the normal form for F in such a way that the variables $a_\mu, ..., a_p$ $(p \ge \mu - 1)$ do not appear in F. Now following the general scheme used in **[6-8]** for ordinary bifurcation sets, we can propose the procedure for constructing the tangent vector fields to quasicaustics.

By Preparation Theorem (see **[23-3]**), the module

$$\mathcal{O}_{(y,x,a)}/\bar{\Delta}(F),$$

where $\bar{\Delta}(F) = \langle y\frac{\partial F}{\partial y}, \frac{\partial F}{\partial x_1}, ..., \frac{\partial F}{\partial x_n}\rangle \mathcal{O}_{(y,x,a)}$, is a free $\mathcal{O}_{(a)}$-module (see [3]) generated by $1, g_1, ..., g_{\mu-1}$. So for any $h \in \mathcal{O}_{(y,x,a)}$ we can write

$$h(y,x,a) = \beta(y,x,a) y \frac{\partial F}{\partial y}(y,x,a) + \sum_{i=1}^{n} \beta_i(y,x,a)\frac{\partial F}{\partial x_i}(y,x,a)$$

$$+ \sum_{j=1}^{\mu-1} \alpha_j(a) g_j(y,x) + \alpha(a), \tag{5.2}$$

for some $\beta_i \in \mathcal{O}_{(y,x,a)}$, $\alpha_j \in \mathcal{O}_{(a)}$, $\alpha \in \mathcal{O}_{(a)}$.

Proposition 5.5. *Let $h \in \mathcal{O}_{(y,x,a)}$ satisfy*

$$\frac{\partial h}{\partial y}\mid_{0\times\mathbf{C}^n\times\mathbf{C}^p} \in I(F), \quad \frac{\partial h}{\partial x_i}\mid_{0\times\mathbf{C}^n\times\mathbf{C}^p} \in I(F), \ i = 1,...,n.$$

Then the vector field $\xi = \sum_{i=1}^{p} \alpha_i \frac{\partial}{\partial a_i}$, where α_i, $i = 1,...,\mu-1$, are defined in (5.2) and α_i, $i = \mu,...,p$ are arbitrary holomorphic functions from $\mathcal{O}_{(a)}$, is tangent to quasicaustic $Q(F) = \pi(\Sigma_r F)$. Conversely; suppose $\xi = \sum_{i=1}^{p} \alpha_i \frac{\partial}{\partial a_i}$ is tangent to $Q(F)$, then there is some $h \in \mathcal{O}_{(y,x,a)}$ as above with

$$h = \sum_{i=1}^{n} \beta_i \frac{\partial F}{\partial x_i} + \beta y \frac{\partial F}{\partial y} + \sum_{i=1}^{\mu-1} \alpha_i g_i + \alpha,$$

and $\frac{\partial h}{\partial x_i}\mid_{0\times\mathbf{C}^n\times\mathbf{C}^p} \in I(F)$, $\frac{\partial h}{\partial y}\mid_{0\times\mathbf{C}^n\times\mathbf{C}^p} \in I(F)$.

Proof. For derivatives of h we have

$$\frac{\partial h}{\partial y}\mid_{\bar{S}} = \beta\psi\mid_{\bar{S}} + \sum_{i=1}^{n} \frac{\partial \beta_i}{\partial y}\mid_{\bar{S}} \frac{\partial \bar{F}}{\partial x_i} + \sum_{i=1}^{n} \beta_i \mid_{\bar{S}} \frac{\partial \psi}{\partial x_i} + \sum_{i=1}^{\mu-1} \alpha_i \frac{\partial g_i}{\partial y}\mid_{S} \in I(F),$$

$$\frac{\partial h}{\partial x_j}\mid_{\bar{S}} = \sum_{i=1}^{n} \frac{\partial \beta_i}{\partial x_j}\mid_{\bar{S}} \frac{\partial \bar{F}}{\partial x_i} + \sum_{i=1}^{n} \beta_i \mid_{\bar{S}} \frac{\partial^2 \bar{F}}{\partial x_i \partial x_j} + \sum_{i=1}^{\mu-1} \alpha_i \frac{\partial g_i}{\partial x_j}\mid_{S} \in I(F), \ j = 1,...,n,$$

where $\bar{S} = \{(y,x,a) \in \mathbf{C}\times\mathbf{C}^n\times\mathbf{C}^p; y = 0\}$, But, on the basis of assumptions, these conditions are equivalent to (5.1.), so $\sum_{i=1}^{p} \alpha_i \frac{\partial}{\partial a_i}$, is tangent to $Q(F)$. The converse statement is straightforward .

□

We see that the set of all such h with $\frac{\partial h}{\partial y}\mid_{S} \in I(F)$, $\frac{\partial h}{\partial x_i}\mid_{S} \in I(F)$, $1 \le i \le n$ form an $\mathcal{O}_{(a)}$ -module. In fact it is the kernel of the $\mathcal{O}_{(a)}$-module homomorphism,

$$\Phi : \mathcal{O}_{(y,x,a)} \ni h \to \left(\frac{\partial h}{\partial y}, \frac{\partial h}{\partial x_1}, ..., \frac{\partial h}{\partial x_n}\right) \in \left(\frac{\mathcal{O}_{(y,x,a)}}{I(F) + \langle y\rangle \mathcal{M}_{(y,x,a)}}\right)^{n+1}.$$

$\bar{\Delta}(F) \subset I(F) + \langle y\rangle \mathcal{M}_{(y,x,a)}$ and clearly the set of tangent vector fields to $Q(F)$ is a finitely generated $\mathcal{O}_{(a)}$-module.

6. QUASICAUSTICS OF SIMPLE BOUNDARY SINGULARITIES

The simple singularities of functions on the boundary $\{y = 0\}$ of a manifold with boundary were classified in [3], (p.281). Their miniversal unfoldings are:

$\tilde{A}_\mu$: $\pm y \pm x^{\mu+1} + \sum_{i=1}^{\mu-1} a_i x^i, \ \mu \geq 1,$

B_μ : $\pm y^\mu \pm x^2 + \sum_{i=1}^{\mu-1} a_i y^{\mu-i}, \ \mu \geq 2,$

C_μ : $yx \pm x^\mu + \sum_{i=1}^{\mu-1} a_i x^{\mu-i}, \ \mu \geq 2,$

$\tilde{D}_\mu$: $\pm y + x_1^2 x_2 \pm x_2^{\mu-1} + \sum_{i=1}^{\mu-2} a_i x_2^i + a_{\mu-1} x_1, \ \mu \geq 4,$

$\tilde{E}_6$: $\pm y + x_1^3 \pm x_2^4 + a_1 x_1 + a_2 x_2 + a_3 x_2^2 + a_4 x_1 x_2 + a_5 x_1 x_2^2,$

$\tilde{E}_7$: $\pm y + x_1^3 + x_1 x_2^3 + a_1 x_1 + a_2 x_2 + a_3 x_2^2 + a_4 x_1 x_2 + a_5 x_2^3 + a_6 x_2^4,$

$\tilde{E}_8$: $\pm y + x_1^3 + x_2^5 + a_1 x_1 + a_2 x_2 + a_3 x_2^2 + a_4 x_1 x_2 + a_5 x_2^3 + a_6 x_1 x_2^2 + a_7 x_1 x_2^3,$

F_4 : $\pm y^2 + x^3 + a_2 y + a_3 x + a_1 xy.$

Thus we have, after direct checking, the following

Proposition 6.1. *The quasicaustics for simple boundary singularities are:*

$\tilde{A}_\mu, \tilde{D}_\mu, \tilde{E}_k$: $Q(F) = \emptyset,$

B_μ : $Q(F) = \{a \in \mathbf{C}^{\mu-1}; a_{\mu-1} = 0\},$

C_μ : $Q(F) = \{a \in \mathbf{C}^{\mu-1}; a_{\mu-1} = 0\},$

F_4 : $Q(F) = \{a \in \mathbf{C}^3; a_2^2 + \frac{1}{3} a_1^2 a_3 = 0\}$, *(i.e. Whitneys cross-cap, see Figure 4).*

Thus we need to calculate only the module of vector fields tangent to $Q(F_4)$. Let us define the germ, at zero, of the variety $X := Q(F_4) \cup \{a_1 = 0\}$. We see that the vector fields tangent to $(X, 0)$ lie in $DerlogQ(F_4)$.

Proposition 6.2. *The vector fields*

$$V_1 = -\frac{1}{6} a_1^2 \frac{\partial}{\partial a_2} + a_2 \frac{\partial}{\partial a_3},$$

$$V_2 = a_1 \frac{\partial}{\partial a_1} + a_2 \frac{\partial}{\partial a_2},$$

$$V_3 = -\frac{1}{3} a_1 \frac{\partial}{\partial a_1} + \frac{2}{3} \frac{\partial}{\partial a_3},$$

form a free basis for the $\mathcal{O}_{(a)}$-module $DerlogX$.

Before we prove this theorem we need the following

Proposition 6.3. *For corank two boundary singularities $F : (\mathbf{C}\times\mathbf{C}\times\mathbf{C}^p, 0) \to (\mathbf{C}, 0)$, the space of functions $h \in \mathcal{O}_{(y,x,a)}$ reconstructing the $\mathcal{O}_{(a)}$- module of vector fields tangent to quasicaustic $Q(F)$ has a following form*

$$h(y,x,a) = \int_0^x \left(\frac{\partial F}{\partial y}(0,s,a)\psi_1(s,a) + \frac{\partial F}{\partial x}(0,s,a)\psi_2(s,a)\right) ds \\ + y^2\xi(y,x,a),$$

where $\psi_i \in \mathcal{O}_{(x,a)}$, $(i = 1,2)$, $\xi \in \mathcal{O}_{(y,x,a)}$.

Proof. Every function $h \in \mathcal{O}_{(y,x,a)}$ can be written in the form

$$h(y,x,a) = \eta_2(x,a) + y\eta_1(x,a) + y^2\eta(y,x,a),$$

and thus

$$\frac{\partial h}{\partial y}(0,x,a) = \eta_1(x,a), \qquad \frac{\partial h}{\partial x}(0,x,a) = \frac{\partial \eta_2}{\partial x}(x,a).$$

By Proposition 5.5, we can take

$$\eta_1(x,a) \in I(F), \text{and } \eta_2(x,a) = \int_0^x g(s,a)ds, \qquad g \in I(F),$$

obtaining all functions

$$\eta_2(x,a) + y\eta_1(x,a) + y^2\eta(y,x,a)\,(\mathrm{mod}\bar{\Delta}(F)),$$

defining the $\mathcal{O}_{(a)}$-module of vector fields tangent to $Q(F)$. Now we see that,

$$\eta_2(x,a) + y\eta_1(x,a) + y^2\eta(y,x,a) = \eta_2(x,a) + y^2\xi(y,x,a)\,(\mathrm{mod}\langle y\frac{\partial F}{\partial y}, y\frac{\partial F}{\partial x}\rangle\mathcal{O}_{(y,x,a)}),$$

where $\xi \in \mathcal{O}_{(y,x,a)}$. Adding an element of $\langle y\rangle\bar{J}(F)$, ($\bar{J}(F)$ is an ideal of $\mathcal{O}_{(y,x,a)}$ generated by: $\frac{\partial F}{\partial y}, \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}$) does preserve the space of functions and does not affect the resulting vector field.

□

Proof of Proposition 6.2. $I(F_4) = \langle a_1x + a_2, 3x^2 + a_3\rangle\mathcal{O}_{(x,a)}$. By Proposition 6.3, taking $\psi_1, \psi_2, \xi \equiv 1$, we have ,

$$\begin{aligned} h_1(x,a) &= \frac{1}{2}a_1x^2 + a_2x = -\frac{1}{6}a_1^2y + a_2x - \frac{1}{6}a_1a_3\,(\mathrm{mod}\ \bar{\Delta}(F_4)), \\ h_2(x,a) &= y^2 = -a_1xy - a_2y\,(\mathrm{mod}\ \bar{\Delta}(F_4)), \\ h_3(x,a) &= x^3 + xa_3 = -\frac{1}{3}a_1xy + \frac{2}{3}a_3x\,(\mathrm{mod}\ \bar{\Delta}(F_4)). \end{aligned}$$

Then the corresponding V_i belongs to $DerlogQ(F_4)$, $(i = 1,2,3)$. By simple computation we obtain

$$V_1(a_1) = 0, \qquad V_2(a_1) = -a_1, \qquad V_3(a_1) = -\frac{1}{3}a_1,$$

so $V_i \in DerlogX$ as well. We also have

$$det(V_1(a), V_2(a), V_3(a)) = -\frac{1}{3}a_1(a_2^2 + \frac{1}{3}a_3a_1^2)$$

is a reduced equation for $(X, 0)$, so by the results of Saito [27] (see also [6]) we find that $(X, 0)$ is a free divisor.

□

We define the following ideals of $\mathcal{O}_{(y,x)}$ and $\mathcal{O}_{(y,x,a)}$ respectively,

$$\Theta(f) = \langle y\rangle J(f) + \langle\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}\rangle^2 \mathcal{O}_{(y,x)},$$

and

$$\bar{\Theta}(F) = \langle y\rangle \bar{J}(F) + \langle\frac{\partial F}{\partial x_1}, ..., \frac{\partial F}{\partial x_n}\rangle^2 \mathcal{O}_{(y,x,a)}.$$

For determining all fields tangent to the quasicaustic we need the following

Lemma 6.4 *The space $\mathcal{O}_{(y,x)}/\Theta(f)$ is finite dimensional. Its* **C***-basis also generates the quotient space $\mathcal{O}_{(y,x,a)}/\bar{\Theta}(F)$ as an $\mathcal{O}_{(a)}$-module.*

Proof. $\Theta(f) \supset \Delta(f)$ and f is finitely determined as a boundary singularity. Thus $\mathcal{O}_{(y,x)}/\Theta(f)$ is **C**-finite dimensional with the basis $\{g_1, ..., g_N\}$. Let us define the mapping

$$\Psi : (\mathbf{C} \times \mathbf{C}^n \times \mathbf{C}^p, 0) \to (\mathbf{C} \times \mathbf{C}^n \times \mathbf{C}^{\frac{n(n+1)}{2}} \times \mathbf{C}^p, 0),$$

$$\begin{aligned}\Psi(y,x,a) =&(y\frac{\partial F}{\partial y}(y,x,a), y\frac{\partial F}{\partial x_1}(y,x,a), ..., y\frac{\partial F}{\partial x_n}(y,x,a),\\ &\frac{\partial F}{\partial x_i}(y,x,a)\frac{\partial F}{\partial x_j}(y,x,a), a),\end{aligned}$$

with $1 \le i,j \le n$; $i \le j$, and ordered set of pairs (i,j). Thus we have

$$\mathcal{O}_{(y,x,a)}/\Psi^*(\mathcal{M}_{(y,x,a)})\mathcal{O}_{(y,x,a)} \cong \mathcal{O}_{(y,x)}/\Theta(f)\mathcal{O}_{(y,x)}.$$

By the Preparation Theorem (see [23]) an every element h of $\mathcal{O}_{(y,x,a)}$ has the form:

$$\begin{aligned}h(y,x,a) = \sum_{l=1}^{N} &\phi_l(y\frac{\partial F}{\partial y}(y,x,a), y\frac{\partial F}{\partial x_1}(y,x,a), ..., y\frac{\partial F}{\partial x_n}(y,x,a),\\ &\frac{\partial F}{\partial x_i}(y,x,a)\frac{\partial F}{\partial x_j}(y,x,a), a)g_l(y,x).\end{aligned}$$

Thus

$$\mathcal{O}_{(y,x,a)}/\tilde{\Theta}(F) \cong \{\sum_{i=1}^{N} \psi_i(a) g_i(y,x)\}, \qquad \psi_i \in \mathcal{O}_{(a)},$$

which completes the proof of Proposition 6.4.

□

Let $\{g_1, ..., g_N\}$ be a C-basis for $\mathcal{O}_{(y,x)}/\Theta(f)$. In general we have

Proposition 6.5. *Functions $h \in \mathcal{O}_{(y,x,a)}$ which reconstruct the $\mathcal{O}_{(a)}$-module of vector fields tangent to $Q(F)$, can be written as :*

$$h(y,x,a) = \sum_{i=1}^{N} \alpha_i(a) g_i(y,x),$$

where

$$\sum_{i=1}^{N} \alpha_i(a) \frac{\partial g_i}{\partial y}(0,x) \in I(F),$$

$$\sum_{i=1}^{N} \alpha_i(a) \frac{\partial g_i}{\partial x_j}(0,x) \in I(F),$$

$1 \leq j \leq n$.

Proof. By Lemma 6.4, any $h \in \mathcal{O}_{(y,x,a)}$ can be written as

$$h(y,x,a) = \sum_{i=1}^{N} \alpha_i(a) g_i(y,x) + \beta(y,x,a) y \frac{\partial F}{\partial y}(y,x,a) + \sum_{j=1}^{n} \beta_j(y,x,a) y \frac{\partial F}{\partial x_j}(y,x,a)$$
$$+ \sum_{k,l=1}^{n} \beta_{k,l}(y,x,a) \frac{\partial F}{\partial x_k}(y,x,a) \frac{\partial F}{\partial x_l}(y,x,a),$$

where $\alpha_i \in \mathcal{O}_{(a)}, \beta, \beta_j, \beta_{kl} \in \mathcal{O}_{(y,x,a)}$. By simply checking the assumptions of Proposition 5.5, we see that the three last terms in the above formula do not affect on the resulting vector field belonging to $DerlogQ(F)$. This proves Proposition 6.5.

□

Proposition 6.6. *$\mathcal{O}_{(a)}$-module $DerlogQ(F_4)$, i.e. the module of holomorphic vector* *fields tangent to the Whitney's cross-cap, is generated by the following fields:*

$$
\begin{aligned}
V_1 =& -\frac{1}{6}a_1^2\frac{\partial}{\partial a_2} + a_2\frac{\partial}{\partial a_3},\\
V_2 =& a_1\frac{\partial}{\partial a_1} + a_2\frac{\partial}{\partial a_2},\\
V_3 =& -\frac{1}{3}a_1\frac{\partial}{\partial a_1} + \frac{2}{3}a_3\frac{\partial}{\partial a_3},\\
V_4 =& a_2\frac{\partial}{\partial a_1},\\
V_5 =& a_3\frac{\partial}{\partial a_1},\\
V_6 =& a_3\frac{\partial}{\partial a_2},
\end{aligned}
$$

with relations,

$$
\begin{aligned}
0 =& a_2V_5 - a_3V_4,\\
0 =& -a_3^2V_1 - \frac{1}{6}a_1^2a_3V_6 + \frac{3}{2}a_2V_3 + \frac{1}{2}a_1a_3V_4,\\
0 =& a_1(a_3-1)V_3 - a_1^2V_5 - a_2a_1V_6 + a_1a_3V_2.
\end{aligned}
$$

Proof. We have

$$\mathcal{O}_{(y,x)}/\Theta(f) \cong [1, x, y, x^2, x^3, xy].$$

So by Proposition 6.5, all functions $h \in \mathcal{O}_{(y,x,a)}$ reconstructing $DerlogQ(F_4)$ can be written as:

$$
\begin{aligned}
h(y,x,a) =& A_0(a) + (a_3C_1(a) + a_2D_1(a))x + (a_3A_1(a) + a_2B_1(a))y +\\
& (a_3A_2(a) + a_1B_1(a) + a_2B_2(a))xy + \frac{1}{2}(a_3C_2(a) + a_1D_1(a) + a_2D_2(a))x^2\\
& + \frac{1}{3}(3C_1(a) + a_3C_3(a) + a_1D_2(a) + a_2D_3(a))x^3,
\end{aligned}
$$

where A_i, B_j, C_k, D_l, $\in \mathcal{O}_{(a)}$, $i = 1,2,3;$ $j = 1,2;$ $k = 1,2,3;$ $l = 1,2,3$. Thus the general vector field tangent to $Q(F_4)$ can be written as:

$$
\begin{aligned}
&(a_3A_2 + a_1B_1 + a_2B_2 - \frac{1}{3}a_1C_1 - \frac{1}{9}a_1a_3C_3 - \frac{1}{9}a_1^2D_2 - \frac{1}{9}a_1a_2D_3)\frac{\partial}{\partial a_1} +\\
&(a_3A_1 + a_2B_1 - \frac{1}{6}a_1a_3C_2 - \frac{1}{6}a_1^2D_1 - \frac{1}{6}a_1a_2D_2)\frac{\partial}{\partial a_2} +\\
&(\frac{2}{3}a_3C_1 + a_2D_1 - \frac{1}{9}a_3^2C_3 - \frac{1}{9}a_3a_1D_2 - \frac{1}{9}a_2a_3D_3)\frac{\partial}{\partial a_3}.
\end{aligned}
$$

By straightforward calculations we find the generators V_i, $(i = 1, ..., 6)$ and the relations between them.

□

7. ON QUASICAUSTICS OF UNIMODAL BOUNDARY SINGULARITIES

Let us consider the miniversal deformations for parabolic boundary singularities (see [3]),

$F_{1,0}:\quad y^3 + x^3 + a_1 y^2 x + a_2 xy + a_3 y^2 + a_4 y + a_5 x,$

$K_{4,2}:\quad y^2 + x^4 + a_1 y x^2 + a_2 xy + a_3 x^2 + a_4 x + a_5 y,$

$D_{4,1}(= L_6):\quad \frac{1}{2}x_1^2 x_2 + \frac{1}{3}x_2^3 + y x_1 + a_1 y x_2 + \frac{1}{2}a_2 x_2^2 + a_3 y + a_4 x_1 + a_5 x_2,$

where a_1 is a moduli parameter, Milnor number $\mu = 6$ and the boundary: $\{y = 0\}$.

Proposition 7.1. *For parabolic unimodal boundary singularities, the modules of logarithmic vector fields $DerlogQ(F)$, are not free. They can be generated by the following vector fields:*

$DerlogQ(F_{1,0}):$

$$\frac{\partial}{\partial a_1},\quad \frac{\partial}{\partial a_3},\quad a_5\frac{\partial}{\partial a_4},\quad a_5\frac{\partial}{\partial a_2},\quad a_2\frac{\partial}{\partial a_2} + a_4\frac{\partial}{\partial a_4},\quad a_4\frac{\partial}{\partial a_2},$$
$$-a_2\frac{\partial}{\partial a_2} + 2a_5\frac{\partial}{\partial a_5},\quad a_2 a_5\frac{\partial}{\partial a_4},\quad a_2 a_5\frac{\partial}{\partial a_2} + a_5^2\frac{\partial}{\partial a_5},\quad -\frac{1}{6}a_2^2\frac{\partial}{\partial a_4}$$
$$+ a_4\frac{\partial}{\partial a_5},\quad -\frac{1}{9}a_2^2\frac{\partial}{\partial a_2} - \frac{1}{6}a_2 a_4\frac{\partial}{\partial a_4} - \frac{1}{9}a_5 a_2\frac{\partial}{\partial a_5},\quad a_2 a_4\frac{\partial}{\partial a_2} + a_5 a_4\frac{\partial}{\partial a_5}.$$

$DerlogQ(K_{4,2}):$

$$a_1\frac{\partial}{\partial a_1} + a_2\frac{\partial}{\partial a_2} + a_5\frac{\partial}{\partial a_5},\quad a_2\frac{\partial}{\partial a_1} + a_5\frac{\partial}{\partial a_2},\quad a_5\frac{\partial}{\partial a_1},\quad 2a_3\frac{\partial}{\partial a_2} + a_4\frac{\partial}{\partial a_5}$$
$$2a_3\frac{\partial}{\partial a_1} + a_4\frac{\partial}{\partial a_2},\quad a_4\frac{\partial}{\partial a_1},\quad -\frac{1}{6}a_1^2\frac{\partial}{\partial a_2} + \frac{1}{2}a_2\frac{\partial}{\partial a_3} + (a_5 - \frac{1}{6}a_1 a_3)\frac{\partial}{\partial a_4} -$$
$$\frac{1}{12}a_1 a_2\frac{\partial}{\partial a_5},\quad -\frac{1}{4}a_1^2\frac{\partial}{\partial a_1} - (\frac{1}{3}a_1 a_2 + \frac{1}{8}a_1 a_2)\frac{\partial}{\partial a_2} + (a_5 - \frac{1}{4}a_1 a_3)\frac{\partial}{\partial a_3} -$$
$$(\frac{1}{3}a_3 a_2 + \frac{1}{8}a_1 a_4)\frac{\partial}{\partial a_4} - \frac{1}{6}a_2^2\frac{\partial}{\partial a_5},\quad (\frac{1}{5}a_1 K - \frac{1}{8}a_1 a_2)\frac{\partial}{\partial a_1} - (a_1 H + \frac{1}{6}a_1 a_5 +$$
$$\frac{1}{16}a_2^2)\frac{\partial}{\partial a_2} - (\frac{1}{8}a_3 a_2 + \frac{1}{20}a_1 a_4)\frac{\partial}{\partial a_3} + (\frac{1}{20}a_1 a_3^2 - \frac{1}{16}a_4 a_2 - \frac{1}{6}a_3 a_5)\frac{\partial}{\partial a_4}$$
$$-(La_1 + \frac{1}{12}a_2 a_5)\frac{\partial}{\partial a_5},\quad (-10a_1 a_5 + 16a_2 K)\frac{\partial}{\partial a_1} - (5a_2 a_5 + 80a_2 H)\frac{\partial}{\partial a_2}$$
$$-(10a_3 a_5 + 4a_2 a_4)\frac{\partial}{\partial a_3} + (4a_2 a_3^2 - 5a_4 a_5)\frac{\partial}{\partial a_4} - 80La_2\frac{\partial}{\partial a_5},\quad 4a_5 K\frac{\partial}{\partial a_1}$$
$$-20Ha_5\frac{\partial}{\partial a_2} - a_4 a_5\frac{\partial}{\partial a_3} + a_3^2 a_5\frac{\partial}{\partial a_4} - 20La_5\frac{\partial}{\partial a_5},\quad -2a_1\frac{\partial}{\partial a_1} - a_2\frac{\partial}{\partial a_2}$$
$$+2a_3\frac{\partial}{\partial a_3} + 3a_4\frac{\partial}{\partial a_4}, 24K\frac{\partial}{\partial a_1} - (10a_1 a_3 + 120H)\frac{\partial}{\partial a_2} + 9a_4\frac{\partial}{\partial a_3} -$$

$$4a_3^2\frac{\partial}{\partial a_4} - (5a_2a_3 + 120L)\frac{\partial}{\partial a_5}, \quad -6a_1a_3\frac{\partial}{\partial a_1} - (4a_1a_4 + 3a_2a_3)\frac{\partial}{\partial a_2}$$
$$- 6a_3^2\frac{\partial}{\partial a_3} - 7a_4a_3\frac{\partial}{\partial a_4} - 2a_2a_4\frac{\partial}{\partial a_5}, \quad (32a_3K - 10a_1a_4)\frac{\partial}{\partial a_1}$$
$$- (5a_2a_4 + 160Ha_3)\frac{\partial}{\partial a_2} - 18a_3a_4\frac{\partial}{\partial a_3} + (8a_3^3 - 5a_4^2)\frac{\partial}{\partial a_4} - 160a_3L\frac{\partial}{\partial a_5},$$
$$4a_4K\frac{\partial}{\partial a_1} - 20a_4H\frac{\partial}{\partial a_2} - a_4^2\frac{\partial}{\partial a_3} + a_3^2a_4\frac{\partial}{\partial a_4} - 20La_4\frac{\partial}{\partial a_5}.$$

$Derlog(D_{4,1})$:

$$a_1\frac{\partial}{\partial a_1} + a_3\frac{\partial}{\partial a_3}, \quad a_3\frac{\partial}{\partial a_1}, \quad a_2\frac{\partial}{\partial a_1}, \quad a_2\frac{\partial}{\partial a_2} + a_5\frac{\partial}{\partial a_3}, \quad a_5\frac{\partial}{\partial a_1},$$
$$a_2\frac{\partial}{\partial a_3}, \quad 2\frac{\partial}{\partial a_1} - 2a_1\frac{\partial}{\partial a_3} - 3(a_1a_4 - a_3)\frac{\partial}{\partial a_4} - 2(a_2 - a_4)\frac{\partial}{\partial a_5}, \quad \frac{1}{2}a_2a_1\frac{\partial}{\partial a_2}$$
$$- (a_2 + a_1)\frac{\partial}{\partial a_3} + (a_3 + a_5 - a_1a_4)\frac{\partial}{\partial a_4} + (2a_4 - a_2 + a_3a_2)\frac{\partial}{\partial a_5}, \quad -3a_1\frac{\partial}{\partial a_1}$$
$$+ a_2\frac{\partial}{\partial a_2} - a_3\frac{\partial}{\partial a_3} + a_4\frac{\partial}{\partial a_4} + (4a_5 + 3a_1a_4 - 3a_3)\frac{\partial}{\partial a_5}, \quad (6a_1a_2\frac{M+1}{M}$$
$$+ \frac{1}{M}(9a_1^2a_3 - 2a_3))\frac{\partial}{\partial a_1} + (a_5 + a_1a_4 - a_2^2)\frac{\partial}{\partial a_2} + (2a_3a_2 - 2a_4 +$$
$$\frac{1}{M}(6a_1a_5 - 6a_1^2a_4 + 3a_3^2a_1))\frac{\partial}{\partial a_3} - 2a_2a_4\frac{\partial}{\partial a_4} + (8a_4a_3 + 8a_3a_2 - 12a_2a_5$$
$$- 8a_1a_2a_4)\frac{\partial}{\partial a_5}, \quad \frac{2}{M}(2a_1a_3 + a_2)\frac{\partial}{\partial a_1} + (a_4 - a_2^2 + a_5 + \frac{1}{2}a_1a_4)\frac{\partial}{\partial a_2}$$
$$+ (2a_2^2 - 2a_5 - a_1a_4 + \frac{2}{M}(a_5 - a_4a_1 + \frac{1}{2}a_3^2)\frac{\partial}{\partial a_3} + (a_4a_3 + a_3a_2 - 2a_2a_5$$
$$- a_1a_2a_4)\frac{\partial}{\partial a_4} - a_3(2a_2^2 - 2a_5 - a_1a_4)\frac{\partial}{\partial a_5}, \quad (a_1^2 - 2)\frac{\partial}{\partial a_2} - 4a_1\frac{\partial}{\partial a_3}$$
$$+ 2a_3\frac{\partial}{\partial a_4} + 2(a_1a_3 - a_2)\frac{\partial}{\partial a_5}, \quad a_2\frac{\partial}{\partial a_2} + 2a_3\frac{\partial}{\partial a_3} + 4a_4\frac{\partial}{\partial a_4} + 4a_5\frac{\partial}{\partial a_5}.$$

where, $M = 1 + \frac{3}{2}a_1^2$, $K = -\frac{a_2}{2}\frac{2+a_1^2}{4-a_1^2}$, $H = \frac{a_1}{20}(\frac{a_2^2+2a_5a_1}{4-a_1^2} - a_3)$, $L = \frac{a_2}{40}(\frac{2a_5a_1}{4-a_1^2} - a_3)$.

Proof. For the cases $F_{1,0}$ and $K_{4,2}$ the method of the proof is analogous to the one used in the proof of Proposition 6.6. We only show how to prove the case $D_{4,1}$. Here we have

$$\mathcal{O}_{(y,x)}/\theta(f) \cong [1, x_1, x_2, y, x_1^2, x_1x_2, x_2^2, x_2y, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_2^4, x_1x_2^3],$$

where $x = (x_1, x_2)$. So by Proposition 6.5, we have

$$\begin{aligned} h = &\alpha_0 + \alpha_1x_1 + \alpha_2x_2 + \alpha_3y + \alpha_4x_1^2 + \alpha_5x_1x_2 + \alpha_6x_2^2 + \alpha_7x_2y \\ &+ \alpha_8x_1^3 + \alpha_9F(0,x,a) + \alpha_{10}x_1x_2^2 + \alpha_{11}x_2^3 + \alpha_{12}x_2^4 + \alpha_{13}x_1x_2^3. \end{aligned}$$

Now we need

$$\text{(i)}\qquad \frac{\partial h}{\partial x_1}(0,x,a) = \alpha_1 + 2\alpha_4 x_1 + 3\alpha_8 x_1^2 + \alpha_5 x_2 + \alpha_{10} x_2^2 + \alpha_{13} x_2^3 \in I(F)$$

$$\begin{aligned} \frac{\partial h}{\partial x_2}(0,x,a) =& \alpha_2 + 2\alpha_6 x_2 + 3\alpha_{11} x_2^2 + 4\alpha_{12} x_2^3 + \alpha_5 x_1 \\ \text{(ii)}\qquad & + 2\alpha_{10} x_1 x_2 + 3\alpha_{13} x_1 x_2^2 \in I(F). \end{aligned}$$

Using the relations

$$\begin{aligned} x_1 =& - a_1 x_2 - a_3 (\text{mod } I(F)), \\ x_1 x_2 =& - a_4 (\text{mod} I(F)), \\ x_1^2 =& a_1 a_4 - a_3 (\text{mod } I(F)), \\ x_2^2 =& - a_2 x_2 + \frac{1}{2}(a_1 a_4 - a_3) - a_5 (\text{mod} I(F)), \\ x_2^3 =& (a_2^2 - \frac{1}{2} a_1 a_4 - a_5) x_2 + \frac{1}{2} a_1 a_2 a_4 + a_2 a_5 - \frac{1}{2} a_3 a_2 - \frac{1}{2} a_4 a_3 (\text{mod } I(F)), \end{aligned}$$

we can solve (i), (ii) and obtain α_1, α_2, α_3, α_5, α_6; expressed linearly by independent functions α_4, α_8, α_9, α_{10}, α_{11}, α_{12}, α_{13}, with polynomial coefficients. Then using the relations

$$\begin{aligned} x_1^2 =& - 2(x_2^2 + a_1 y + a_2 x_2 + a_5)(\text{mod } \bar{\Delta}(F)), \\ x_1^3 =& - 2(y a_1 + (a_2 - a_4) x_2 - y x_2 + a_5)(\text{mod } \bar{\Delta}(F)), \\ x_1 x_2 =& - y - a_4 (\text{mod } \bar{\Delta}(F)), \\ x_2^3 =& - \frac{3}{2} a_1 x_2 y - a_2 x_2^2 + \frac{1}{2} a_4 x_1 - a_5 x_2 - \frac{1}{2} a_3 y (\text{mod } \bar{\Delta}(F)), \\ x_2^4 =& x_2^2 (a_2^2 - a_5) + x_2 y (\frac{3}{2} a_1 a_2 + \frac{1}{4M}(6 a_1 a_2 + 9 a_3 a_1^2 - 2 a_3)) - \frac{1}{2} a_2 a_4 x_1 \\ & + a_2 a_5 x_2 + \frac{1}{2} y (a_3 a_2 - a_4 + \frac{3}{2M}(2 a_5 a_1 - 2 a_1^2 a_4 + a_1 a_3^2))(\text{mod } \bar{\Delta}(F)), \\ x_1 x_2^3 =& \frac{1}{M}(y x_2 (2 a_1 a_3 + a_2) + y (a_5 - a_4 a_1 + \frac{1}{2} a_3^2) - a_4 x_2^2 (\text{mod } \bar{\Delta}(F)), \\ F(0,x,a) =& \frac{1}{6}(2 a_3 y + 4 a_4 x_1 + a_2 x_2^2 + 4 a_5 x_2)(\text{mod } \bar{\Delta}(F)), \end{aligned}$$

we obtain, after straightforward if messy calculations, the vector fields listed in the proposition.

□

Let $\rho : \mathbf{C}^p \to \mathbf{C}^k$, be a projection on $\mathbf{C}^k \subset \mathbf{C}^p$. We say that $Q(F) \subset \mathbf{C}^p$ is locally equisingular along $\mathbf{C}^k$ near $p_0 \in \mathbf{C}^k$ if for all $p \in \mathbf{C}^k$ near p_0 the pairs $(\rho^{-1}(p), 0)$ and $(\rho^{-1}(p) \cap Q(F), 0)$ are all diffeomorphic. Checking the vector fields listed in Proposition 7.1, we have

Corollary 7.2.

1. *The quasicaustic $Q(F_{1,0})$ is equisingular along the two-dimentional singular locus, parametrized by $\{a_1, a_3\}$.*
2. *The quasicaustic $Q(D_{4,1})$ is equisingular along the two-dimensional singular locus, parametrized by $\{a_1, a_2\}$, where $a_1 \neq \pm\sqrt{2}$.*
   In both cases the fibre $(\rho^{-1}(p) \cap Q(F), 0)$ is diffeomorphic to the Whitney's cross-cap.

The logarithmic vector fields, can be used also for the classification of the generic Lagrangian pairs (L_1, L_2) up to quasicaustic equivalence (cf. **[25-17]**). The singular Lagrangian variety $L_1 \cup L_2$ is provided by generic families of functions on manifold with boundary. In this sense, to determine the germ of Lagrangian pair means to define the generating family of functions on a manifold with boundary (cf. Section 3).

Let $f : (\mathbf{C} \times \mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be a finitely determined boundary singularity. Let $F : (\mathbf{C} \times \mathbf{C}^n \times \mathbf{C}^{\mu-1}, 0) \to (\mathbf{C}, 0)$ be its miniversal unfolding. If $G : (\mathbf{C} \times \mathbf{C}^n \times \mathbf{C}^p, 0) \to (\mathbf{C}, 0)$ is a generating family for Lagrangian pair, then generically G is a pullback from the miniversal unfolding F of the finitely determined germ $f(y, x) = G(y, x, 0)$, i.e.,

$$G(y, x, a) = F(\Phi(y, x, a), \phi(a)) + h(a),$$

where $\Phi : (\mathbf{C} \times \mathbf{C}^n \times \mathbf{C}^p, 0) \to (\mathbf{C} \times \mathbf{C}^n, 0)$ is a family of biholomorphisms, germs preserving the hypersurface $\{y = 0\}$. The pullback $\phi : (\mathbf{C}^p, 0) \to \mathbf{C}^{\mu-1}, 0)$, $\phi \in \mathcal{O}_{(a)}^{\mu-1}$ and $h \in \mathcal{O}_{(a)}$. Thus, analogously to the classification of generic Lagrangian submanifolds (see **[3]**, p.337), the classification of generic Lagrangian pairs is done by specifying the miniversal unfoldings of finitely determined boundary singularities and their generic pullbacks $\phi \in \mathcal{O}_{(a)}^{\mu-1}$.

Let us assume that Lagrangian pairs are modelled on unimodal singularities $f : (\mathbf{C} \times \mathbf{C}^n, 0) \to (\mathbf{C}, 0)$, i.e. the generic generating family with such f has the following prenormal form

$$G : (\mathbf{C} \times \mathbf{C}^n \times \mathbf{C}^p, 0) \to (\mathbf{C}, 0), \qquad p \geq \mu - 2,$$
$$G(y, x, a) = f(y, x) + \sum_{i=1}^{\mu-2} g_i(y, x) a_i + g_{\mu-1}(y, x)\lambda(a),$$

where $g_{\mu-1}(y, x)$ defines the modulus direction.

Generically, the pullback ϕ is transversal to this direction, so $\bar{\lambda} := \lambda \mid_{\{a_1 = \ldots = a_{\mu-2} = 0\}}$: $(\mathbf{C}^{p-\mu+2}, 0) \to (\mathbf{C}, 0)$ is a Morse function. Thus there are possible two generic normal forms for the generating families of Lagrangian pairs of unimodal type :

1. $\lambda(a) = a_{\mu-1}$, when $p > \mu - 2$, and $D\bar{\lambda}(0) \neq 0$,
2. $\lambda(a) = \eta(a_1, ..., a_{\mu-2}) \pm a_{\mu-1}^2 \pm ... \pm a_p^2$, when $D\bar{\lambda}(0) = 0$,
   where $\eta \in \mathcal{O}_{(\bar{a})}$, $(\bar{a} = (a_1, ..., a_{\mu-2}))$ is a functional modulus.

To obtain more information about classifying quasicaustics, we need to introduce a weaker equivalence relation in Lagrangian pairs (cf. **[3-17]** in the case of functional moduli in standard classification of Lagrangian submanifolds). Let

$$G_1(y, x, a) = F(y, x, \phi_1(a)) + f_1(a),$$
$$G_2(y, x, a) = F(y, x, \phi_2(a)) + f_2(a),$$

be two generating families for the corresponding Lagrangian pairs $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. We say that $\mathcal{L}_1, \mathcal{L}_2$ are quasicaustic equivalent if ϕ_1, ϕ_2 are right-left equivalent, i.e.,

$$\phi_1(a) = (\psi \circ \phi_2 \circ \xi)(a),$$

for some biholomorphism $\xi : (\mathbf{C}^p, 0) \to (\mathbf{C}^p, 0)$, and some biholomorphism ψ : $(\mathbf{C}^{\mu-1}, 0) \to (\mathbf{C}^{\mu-1}, 0)$ preserving the quasicaustic $(Q(F), 0)$.

Proposition 7.3. *For unimodal parabolic singularities F, by quasicaustic equivalence, the functional modulus λ can be reduced to zero.*

Proof. On the basis of **[3]**,(p. 343) we need to check only that

$$(*) \qquad \mathcal{M}_{(a)} \subseteq \langle A_1^1(a), ..., A_1^r(a) \rangle \mathcal{O}_{(a)},$$

which implies that

$$\mathcal{M}_{(a)}\mathcal{E}(\phi) \subseteq \phi^*\mathcal{E}(\mu - 1) + T\phi(\mathcal{E}(p))$$

for $\phi : (\mathbf{C}^p, 0) \to (\mathbf{C}^{\mu-1}, 0)$ being in general position to the modulus direction. Here by $\mathcal{E}(\phi)$ we denote the vector fields along ϕ (cf. [23]). Let $\mathcal{E}(\mu-1)$ and $\mathcal{E}(p)$ be the spaces of vector fields on $(\mathbf{C}^{\mu-1}, 0)$ and $(\mathbf{C}^p, 0)$ respectively. This enables us to apply the ordinary homotopic method to eliminate the functional modulus λ. Taking into account the vector fields listed in the Proposition 7.1, $V_l = \sum_{i=1}^{5} A_i^l \frac{\partial}{\partial a_i}$, we immediately have fulfilled $(*)$ for the parabolic singularities.

□

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REFERENCES

[1] R. Abraham, J. E. Marsden, "Foundations of Mechanics", *Benjamin/Cummings, Reading,* 1978.

[2] S. B. Alexander, I. D. Berg, R. I. Bishop, "The riemannian obstacle problem", *Illinois Journal of Mathematics,* **31**, No. 1, (1987), 167-184.

[3] V. I. Arnold, S. M. Gusein-Zade, A. N. Varchenko, "Singularities of Differentiable Maps", Vol. 1, *Birkhauser, Boston,* 1985, (English ed.).

[4] V. I. Arnold, "Singularities in variational calculus", *Itogi Nauki, Contemporary Problems in Mathematics*, **22**, (1983), 3-55, (Russian ed.).

[5] V. I. Arnold, A. B. Givental, "Symplectic geometry", *Itogi Nauki, Contemporary Problems in Mathematics, Fundamental Directions,* Vol. 4, (1985), 5-139, (Russian ed.).

[6] J. W. Bruce, "Vector fields on discriminants and bifurcation varieties", *Bull. London Math. Soc.***17**, (1985), 257-262.

[7] J. W. Bruce, "Functions on discriminants", *J. London Math. Soc.* (2), **30**, (1984), 551-567.

[8] J. W. Bruce, D. L. Fidal, "Vector fields on caustics", *manuscript,* 1988.

[9] J. W. Bruce, S. Janeczko, "Classification of caustics by diffraction", *in preparation.*

[10] J. W. Bruce, P. J. Giblin, C. G. Gibson, "On caustics by reflection", *Topology,***21**, (1982), 179-199.

[11] Yu. V. Chekanov, "Caustics in geometrical optics", *Functional Anal. Appl.* ,**20,** No. 3. (1986), 223-226.

[12] J. Damon, "Deformations of sections of singularities and Gorenstein surface singularities", *American Journal of Math.* **109**, (1987), 695-722.

[13] V. W. Guillemin, S. Sternberg, "Symplectic Techniques in Physics", *Cambridge Univ. Press,* 1984.

[14] S. Janeczko, "Generating families for images of Lagrangian submanifolds and open swallowtails", *Math. Proc. Cambridge Phil. Soc.* **100**, (1986), 91-107.

[15] S. Janeczko, "Singularities in the geometry of an obstacle", *Supplemento ai Rendiconti del Circolo Matematico di Palermo,* Serie II, No. 16, (1987), 71-84.

[16] S. Janeczko, "Covariant symplectic geometry of binary forms and singularities of systems of rays", *Preprint, Max-Planck-Institut für Mathematik, Bonn,* MPI, 1986-40.

[17] S. Janeczko, M. Roberts. "Classification of symmetric caustics", *in preparation.*

[18] J. B. Keller, "Rays, waves and asymptotics", *Bull. of the A.M.S.* **84**, No. 5, (1978), 727-749.

[19] M. Kline, I. W. Kay, "Electromagnetic Theory and Geometrical Optics", *Interscience Publishers, John Wiley and Sons Inc.* , 1965.

[20] E. J. N. Looijenga, "Isolated Singular Points on Complete Intersections", *Cambridge Univ. Press,* 1984.

[21] R. K. Luneburg, "Mathematical Theory of Optics", *University of California Press, Berkeley,* 1964.

[22] D. Mond,"On the tangent developable of a space curve", *Math. Proc. Camb. Phil. Soc.* **91**, (1982), 351-355.

[23] J. Martinet, "Singularities of Smooth Functions and Maps", *Cambridge Univ. Press, Cambridge,* 1982.

[24] V. I. Matov, "Unimodal and bimodal germs of functions on manifold with boundary", *Proc. of the Petrovski Seminar,* **7**, (1981), 174-189.

[25] Nguyên hûu Dùc, Nguyên tiên Dai, "Stabilite de l'interaction geometrique entre deux composantes holonomes simples", *Comptes Rendus Acad. Sci. Paris,* **A291**,(1980), 113-116.

[26] T. Poston, I. Stewart, "Catastrophe Theory and its Applications", *Pitman, San Francisco,* 1978.

[27] K. Saito, "Theory of logarithmic differential forms and logarithmic vector fields", *J. Fac. Sci. Univ. Tokyo Sec. I A*,**27**, (1980), 265-291.

[28] O. P. Shcherbak, "Wave fronts and groups generated by reflections", *Uspekhi Mat. Nauk,* **43**, No. 3,(1988), 125-160.

[29] I. G. Shcherbak, "Focal set of a surface with boundary, and caustics of groups generated by reflections B_k, C_k, and F_4, *Functional Anal. Appl.*, **18,** No. 1. (1984), 84-85.

[30] D. Siersma, "Singularities of functions on boundaries, corners, etc.", *Quart. J. Math. Oxford (2)*, **32**, (1981), 119-127.

[31] H. Terao, "The bifurcation set and logarithmic vector fields", *Math. Ann.* , **263**, (1983), 313-321.

[32] W. M. Tulczyjew, "The Legendre transformation", *Ann. Inst. H. Poincare,* **27**, No. 1, (1977), 101-114.

[33] C. T. C. Wall, "Geometric properties of generic differentiable manifolds", In A. Dold

and B. Eckmann, editors, *Geometry and Topology, pages 707-774, Springer-Verlag , Berlin,* 977, *Lecture Notes in Math.* **597**.

[34] A. Weinstein, "Lectures on Symplectic Manifolds", *C. B. M. S., Conf. Series A. M. S.*, **29**, 1977.

[35] V. M. Zakalyukin, "Bifurcations of wavefronts depending on one parameter", *Funct. Anal. Appl.* ,**10**, (1976), 139-140.