SURFACES WITH DIF≠DEF REAL STRUCTURES

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Abstract. In the article, we study real Campedelli surfaces up to real deformations and exhibit a number of such surfaces which are equivariantly diffeomorphic but not real deformation equivalent.

Introduction

The real DIF=DEF problem is at least as old as the complex one. As in the complex DIF=DEF problem it is a question of interaction between two basic equivalence relations: by diffeomorphisms of real structures, and by deformations of varieties together with real structures.

A *real structure* on a complex surface X is an anti-holomorphic involution $X \to X$. A complex surface supplied with a real structure is called a *real surface*. A *deformation* of surfaces is a proper holomorphic submersion $p : Z \to D$, where Z is a 3-dimensional complex variety and $D \subset \mathbb{C}$ is a unit disk. If Z is real and p is equivariant, the deformation is called real. Two real surfaces X' and X'' are called *deformation equivalent* if they can be connected by a chain $X' = X_0$, $\dots$, $X_k = X''$ so that X_i and X_{i-1} are isomorphic to real fibers of a real deformation.

Under these definitions, *up to a diffeomorphism the real structure is preserved under deformation.* So the problem is in what extent the diffeomorphic type of the real structure determines the deformation type. In fact, the diffeomorphisms provided by deformations preserve the canonical orientation and the canonical class. But following the tradition, we include into the statement of the Dif=Def problem only the orientation preserving hypothesis.

Namely, let call a real surface X to be *quasi-simple* if it is deformation equivalent to any other real surface X' such that, first, X' is deformation equivalent to X as a complex surface, and, second, the

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real structure of X' is diffeomorphic to the real structure of X via an orientation preserving diffeomorphism. Thus, we understand the real DIF=DEF problem as the question *are there non quasi-simple real surfaces*? (Note that in the case of curves the response to such a question is in negative: *any real curve is quasi-simple.* In this, and many other quasi-simplicity results, the orientation preserving hypothesis can be omitted.)

The first quasi-simplicity result belongs to F. Klein and L. Schläfli [13] and concerns real cubic surfaces in the projective 3-space. In fact, the quasi-simplicity holds for many other special classes of surfaces. It is observed for rational surfaces (A. Degtyarev and V. Kharlamov [9]), for real Abelian surfaces (follows from A. Comessatti [4]), for geometrically ruled real surfaces (J.-Y. Welschinger [22]), for real hyperelliptic surfaces (F. Catanese and P. Frediani [3]), for real $K3$-surfaces (follows from V. Nikulin [20]), and for real Enriques surfaces (A. Degtyarev and V. Kharlamov; the quasi-simplicity statement was announced in [8], and the complete list of deformation classes of real Enriques surfaces was obtained in collaboration with I. Itenberg in [6]; note also that quasi-simplicity of hyperelliptic and Enriques surfaces extends to quasi-simplicity of the quotients of Abelian and $K3$-surfaces by certain finite group actions, see [7]).

Whether elliptic surfaces and irrational ruled surfaces quasi-simple is, as far as we know, still an open question.

It was natural to expect that such a simple behaviour would no longer take place for more complicated surfaces, like those of general type. However, probably because of lack of convenient deformation invariants not covered by the differential topology of the real structure, no any example of non quasi-simple real surfaces (or real varieties of higher dimension) was known. The main result of this paper is providing such examples. Namely, we prove that *the Campedelli surfaces* (see the definition in Section 1.1) *are not quasi-simple: there exist real Campedelli sufaces which have diffeomorphic real structures without being deformation equivalent. In these examples, the diffeomorphisms of real structures preserve not only the orientation but the canonical class as well.* (Note that under the canonical orientation the intersection form of Campedelli surfaces is of signature $(1, n)$ with $n = 7 > 1$, so that every diffeomorphism of Campedelli surfaces preserves the canonical orientation.)

Let us notice that existence of non quasi-simple families of surfaces of general type does not prevent certain particular classes of surfaces of general type from being quasi-simple. And examples of quasi-simple real surfaces of general type do exist. One such example is given by real Bogomolov-Miyaoka-Yau surfaces, that is, surfaces covered by a ball in $\mathbb{C}^2$, see [14]. In fact, in [14] it is also shown that there exist diffeomorphic, in fact complex conjugated, Bogomolov-Miyaoka-Yau surfaces which are not real and thus, being rigid, they are not deformation equivalent. These surfaces are counter-examples to the Diff = Deff problem in complex geometry. (Let us notice that in these examples the diffeomorphisms reverse the canonical class.)

The first counter-examples to the Diff = Deff problem in the complex geometry of surfaces belong to Manetti [18]. They are not involving the complex conjugation. Already their existence explains why we need to fix complex deformation class in the definition of quasi-simplicity of real varieties. Moreover, our examples of diffeomorphic but not deformation equivalent real structures are closely related to Manetti's examples. In fact, to establish a diffeomorphism we follow Manetti's approach, and to study the deformation equivalence we use the full description of the Campedelli surfaces given by Miayoka [19].

The paper is organized as follows. In Section 1, we collect essentially known results on complex Campedelli surfaces adapting them to our needs and making emphasis on representing Campedelli surfaces as Galois coverings of $\mathbb{P}^2$. In Section 2, we begin our study of real structures on Campedelli surfaces and give a kind of classification of real structures on such surfaces. Section 3 is devoted to a study of real structures up to diffeomorphisms and up to deformations. In Section 4, we apply the technique developed to construct real surfaces which have diffeomorphic real structures without begin deformation equivalent. Related remarks are collected in Section 5.

1. Moduli space of Campedelli surfaces

1.1. **Campedelli surfaces as branched Galois coverings of the projective plane.** Let X be *a Campedelli surface*, that is, X is a minimal surface of general type which has $p_g = q = 0, K_X^2 = 2$, and $\pi_1(X) = (\mathbb{Z}/2\mathbb{Z})^3$. Denote by $X_{can} = \mathrm{Proj}(\sum_m H^0(X; mK))$ the canonical model of X, by $\tilde{X}$ the universal covering of X, by G_{un} the Galois group of this universal covering, and by $\tilde{X}_{can}$ the canonical model of $\tilde{X}$. Note that $\tilde{X}_{can}$ and X_{can} have at most simple double points as singularities, so that $\tilde{X}_{can}$ is the universal covering of X_{can}. The universal coverings $\tilde{X} \to X$ and $\tilde{X}_{can} \to X_{can}$ have the same Galois group, so that $X_{can} = \tilde{X}_{can}/G_{un}$.

According to [19], Theorem 9, the following statement holds.

Theorem 1.1. *The canonical map imbeds $\tilde{X}_{can}$ in $\mathbb{P}^6$. With respect to suitable homogeneous coordinates $w_0, \dots, w_6$ in $\mathbb{P}^6$, this image of $\tilde{X}_{can}$ is given by equations*

$$w_i^2 = a_i w_0^2 + b_i w_1^2 + c_i w_2^2, \quad a_i, b_i, c_i \in \mathbb{C}, \qquad i = 3, 4, 5, 6, \tag{1}$$

and the group $G_{un} = (\mathbb{Z}/2\mathbb{Z})^3$ acts on $\tilde{X}_{can}$ by diagonal projective transformations: $g^*(w_j) = \pm w_j$ for any $g \in G_{un}$. □

As equations (1) and Theorem 1.1 show, the whole group $\widetilde{G} \simeq (\mathbb{Z}/2\mathbb{Z})^6 \subset PGL(6, \mathbb{C})$ of diagonal involutions ($g^*(w_j) = \pm w_j$ for any $g \in \tilde{G}$) acts on $\tilde{X}_{can}$ and the following statement holds.

Corollary 1.2. *The quotient space $\tilde{X}_{can}/\widetilde{G}$ is isomorphic to $\mathbb{P}^2$ and the quotient map $\tilde{X}_{can} \to \tilde{X}_{can}/\widetilde{G}$ is a Galois covering of $\mathbb{P}^2$ with Galois group $\widetilde{G} \simeq (\mathbb{Z}/2\mathbb{Z})^6$ branched along seven lines given by equations*

$$\begin{array}{ll} z_i = 0, & i = 0, 1, 2, \\ a_i z_0 + b_i z_1 + c_i z_2 = 0, & i = 3, 4, 5, 6, \end{array}$$

where z_0, z_1, z_2 are homogeneous coordinates in $P^2 = \tilde{X}_{can}/\widetilde{G}$.

The canonical model X_{can} of X is a Galois covering of $\mathbb{P}^2$ with Galois group $G \simeq (\mathbb{Z}/2\mathbb{Z})^3$ branched along the same lines. □

Let us underline that in the above statements the choice of the equations and the coverings are not arbitrary.

1.2. **Few basic facts on Galois coverings.** Recall that a Galois covering of a smooth algebraic variety Y is a finite morphism $h : X \to Y$ of a normal algebraic variety X to Y such that the function fields imbedding $\mathbb{C}(Y) \subset \mathbb{C}(X)$ induced by h is a Galois extension. As is well known, a finite morphism $h : X \to Y$ is a Galois covering with Galois group G if and only if G coincides with the group of covering transformations and the latter acts transitively on every fiber of h. Besides, a finite branched covering is Galois if and only if the unramified part of the covering (i.e., the restriction to the complements of the ramification and branch loci) is Galois. In addition, a branched covering is determined up to isomorphism by its unramified part. Moreover, a map of Galois coverings from the unramified part $U_1 \to V_1$ of one branched covering $h_1 : X_1 \to Y_1$ (where $U_1 \subset X_1$ and $V_1 \subset Y_1$) to the unramified part $U_2 \to V_2$ of another one, $h_2 : X_2 \to Y_2$ ($U_2 \subset X_2$ and $V_2 \subset Y_2$), induces a morphism $X_1 \to X_2$ of covering varieties if the extension of the morphism of underlying varieties, $V_1 \to V_2$, to the branch loci is given. Let us recall also that an unramified covering is Galois with Galois group G if and only if it is a covering associated with an epimorphism of the fundamental group of the underlying variety to G, and, in particular, the Galois coverings with abelian Galois group G are in one-to-one correspondence with epimorphisms to G of the first homology group with integral coefficients. All these results are well known and their most nontrivial part can be deduced, for example, from the Grauert-Remmert existence theorem [12].

In what follows we deal with Galois coverings with Galois group $G \simeq (\mathbb{Z}/2\mathbb{Z})^k$. Galois groups are considered up to isomorphism, and two Galois coverings $h_1 : X_1 \to Y$ and $h_2 : X_2 \to Y$ with Galois groups G_1 and G_2 are said to be *equivalent* if there exist a biregular map $f : X_1 \to X_2$ and an isomorphism $F : G_1 \to G_2$ such that $h_2 \circ f = h_1$ and $F(g)f(x) = f(gx)$ for any $x \in X_1$ and $g \in G_1$.

1.3. **Galois coverings of $\mathbb{P}^2$ with Galois group $(\mathbb{Z}/2\mathbb{Z})^k$ branched along seven lines.** Let $\mathcal{L} = L_0 \cup \dots \cup L_6$ be an arrangement of seven distinct numbered lines in $\mathbb{P}^2$. The simple loops $\lambda_i, 0 \leqslant i \leqslant 6$, around the lines L_i generate $H_1(\mathbb{P}^2 \setminus \mathcal{L}, \mathbb{Z}) \simeq \mathbb{Z}^6$. They are subject to the relation

$$\lambda_0 + \dots + \lambda_6 = 0.$$

The natural epimorphism $\widetilde{\varphi} : H_1(\mathbb{P}^2 \setminus \mathcal{L}, \mathbb{Z}) \to H_1(\mathbb{P}^2 \setminus \mathcal{L}, \mathbb{Z}/2\mathbb{Z}) \simeq (\mathbb{Z}/2\mathbb{Z})^6$ defines a particular Galois covering of $\mathbb{P}^2$ branched in $\mathcal{L}$. We call it *universal* and denote by $\widetilde{g} : \widetilde{Y} \to \mathbb{P}^2$. The following statement, which is a straightforward consequence of the general results on branched coverings mentioned in Section 1.2, precises, in particular, at what sense it is universal.

Proposition 1.3. *Galois coverings with Galois group $G \simeq (\mathbb{Z}/2\mathbb{Z})^k$ branched along $\mathcal{L}$ exist if and only if $k \leqslant 6$. Their equivalence classes are in one-to-one correspondence with epimorphisms $H_1(\mathbb{P}^2 \setminus \mathcal{L}) \to G$ considered up to automorphisms of G. If $g : Y \to \mathbb{P}^2$ is a Galois covering with Galois group $G \simeq (\mathbb{Z}/2\mathbb{Z})^k$ branched along $\mathcal{L}$, then there exists a Galois covering $h : \widetilde{Y} \to Y$ such that $\widetilde{g} = g \circ h$.* □

Without loss of generality, we can assume that the universal Galois covering $\widetilde{g} : \widetilde{Y} \to \mathbb{P}^2$ is associated with the epimorphism $\widetilde{\varphi} : H_1(\mathbb{P}^2 \setminus \mathcal{L}, \mathbb{Z}) \to (\mathbb{Z}/2\mathbb{Z})^6$ sending λ_0 to $(1, \dots, 1)$ and λ_i with $1 \leq i \leq 6$ to $(0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the i-th place.

Let (v_1, v_2) be affine coordinates in $\mathbb{C}^2 = \mathbb{P}^2 \setminus L_0$ and $l_i(v_1, v_2) = 0$, $1 \leqslant i \leqslant 6$, be a linear equation of $L_i \cap \mathbb{C}^2$. The function field $K_u = \mathbb{C}(\widetilde{Y})$ of $\widetilde{Y}$ is the abelian extension $\mathbb{C}(\widetilde{Y}) = \mathbb{C}(v_1, v_2, w_1, \dots, w_6)$ of the function field $K = \mathbb{C}(v_1, v_2)$ of $\mathbb{P}^2$ of degree 2^6 determined by $w_i^2 = l_i$, $i = 1, \dots, 6$. (In other words, the pull-back of $\mathbb{P}^2 \setminus L_0$ in $\widetilde{Y}$ is naturally isomorphic to the affine subvariety of $\mathbb{C}^8$ given in affine coordinates $v_1, v_2, w_1, \dots, w_6$ by equations $w_1^2 = l_1, \dots, w_6^2 = l_6$.)

The action of $\gamma = (\gamma_1, \dots, \gamma_6) \in (\mathbb{Z}/2\mathbb{Z})^6$ on K_u is given by

$$\gamma(w^a) = (-1)^{(\gamma,a)} w^a,$$

where for any multi-index $a = (a_1, \dots, a_6)$, $0 \le a_i \le 1$, we put

$$w^a = \prod_{i=1}^{6} w_i^{a_i}.$$

Therefore, $\mathrm{Gal}(K_u/\mathbb{C}(v_1, v_2)) = (\mathbb{Z}/2\mathbb{Z})^6$ and

$$K_u = \bigoplus_{0 \leqslant a_i \leqslant 1} \mathbb{C}(v_1, v_2) w^a$$

is a decomposition of the vector space K_u over $\mathbb{C}(v_1, v_2)$ into a finite direct sum of degree 1 representations of $(\mathbb{Z}/2\mathbb{Z})^6$.

Let $\varphi : H_1(\mathbb{P}^2 \backslash \mathcal{L}, \mathbb{Z}) \to (\mathbb{Z}/2\mathbb{Z})^k$ be an epimorphism given by $\varphi(\lambda_i) = (a_{i,1}, \dots, a_{i,k})$, where $a_{0,j} + \dots + a_{6,j} \equiv 0 \bmod 2$ for every $j = 1, \dots, k$, and let $g : Y \to \mathbb{P}^2$ be the Galois covering associated with φ. This covering is branched in the union $\mathcal{L}^{\varphi}$ of lines $L_i \subset \mathcal{L}$ with $\varphi(\lambda_i) \neq 0$. The epimorphism φ factors through a unique epimorphism $\psi : (\mathbb{Z}/2\mathbb{Z})^6 \to (\mathbb{Z}/2\mathbb{Z})^k$, so that, by Proposition 1.3, the covering g factors through a unique Galois covering $h : \widetilde{Y} \to Y$. The latter determines the inclusion $h^* : \mathbb{C}(Y) \to K_u$ of the function field $\mathbb{C}(Y)$ of Y into the function field $K_u = \mathbb{C}(\widetilde{Y})$. Clearly, $\mathrm{Gal}(K_u/h^*(\mathbb{C}(Y))) = \ker \psi$, the field $h^*(\mathbb{C}(Y))$ coincides with the subfield $K_{\varphi} = \mathbb{C}(v_1, v_2, u_1, \dots, u_k)$ of K_u, where

$$u_j = w_1^{a_{1,j}} \cdot \dots \cdot w_6^{a_{6,j}}, \tag{2}$$

and

$$\mathrm{Gal}(K_u/K_{\varphi}) = \{ (\gamma_1, \dots, \gamma_6) \in (\mathbb{Z}/2\mathbb{Z})^6 \mid \sum_{i=1}^{6} a_{i,j}\gamma_i \equiv 0\,(2),\ 1 \le j \le k \}.$$

1.4. **Resolution of singularities of Y.** By construction, Y is a normal surface with isolated singularities. The singular points of Y can appear only over an r-fold point of $\mathcal{L}^{\varphi}$ with $r \geqslant 2$, i.e., over a point belonging to exactly r lines $L_{i_1}, \dots, L_{i_r} \in \mathcal{L}$ with $\varphi(\lambda_{i_k}) \neq 0$, $1 \leqslant k \leqslant r$.

Lemma 1.4. (see, f.e., ([15]) *If $p = L_{i_1} \cap L_{i_2}$ is a 2-fold point of $\mathcal{L}^{\varphi}$ and $\varphi(\lambda_{i_1}) \neq \varphi(\lambda_{i_2})$, then Y is non-singular at each point of $g^{-1}(p)$.* $\square$

We say that an r-fold point $p_{i_1,\dots,i_r}$ of $\mathcal{L}^\phi$ is *a non-branch point with respect to* φ if $\sum_{j=1}^r \varphi(\lambda_{i_j})=0$.

To resolve the singularities of Y, we start from a suitable blow-up of $\mathbb{P}^2$. First, we blow up all the 2-fold non-branch points and all the r-fold points of $\mathcal{L}^\varphi$ with $r\geq 3$. Second, for each pair $(p_{i_1,\dots,i_r},k)$ such that $p_{i_1,\dots,i_r}$ is an r-fold point of $\mathcal{L}^\varphi$ and $\sum_{j=1}^r \varphi(\lambda_{i_j})=\varphi(\lambda_{i_k})$, we effectuate a blow-up with center at the intersection point of the strict transform of L_{i_k} with the exceptional divisor $E_{i_1,\dots,i_r}$ blown-up over $p_{i_1,\dots,i_r}$ at the first series of the blow-ups. The resulting combination of the blow-ups is denoted by $\sigma:\hat{\mathbb{P}}^2\to\mathbb{P}^2$.

By $L_i'\subset\hat{\mathbb{P}}^2$ we denote the strict transform of L_i, by E_p' with $p=p_{i_1,\dots,i_r}$ the strict transform of $E_{i_1,\dots,i_r}$, by $E_{p,\,i_k}$ the exceptional curves of the second series of the blow-ups, and by $\varepsilon_p,\varepsilon_{p,\,i_k}\in H_1(\hat{\mathbb{P}}^2\setminus\sigma^{-1}(\mathcal{L}^\varphi),\mathbb{Z})=H_1(\mathbb{P}^2\setminus\mathcal{L}^\varphi,\mathbb{Z})$ simple loops around E_p' and $E_{p,\,i_k}$, respectively.

The identification $H_1(\hat{\mathbb{P}}^2\setminus\sigma^{-1}(\mathcal{L}^\varphi),\mathbb{Z})=H_1(\mathbb{P}^2\setminus\mathcal{L}^\varphi,\mathbb{Z})$ composed with φ provides an epimorphism $\hat\varphi: H_1(\hat{\mathbb{P}}^2\setminus\sigma^{-1}(\mathcal{L}^\varphi),\mathbb{Z})\to(\mathbb{Z}/2\mathbb{Z})^k$. Let consider the associated Galois covering $f:X\to\hat{\mathbb{P}}^2$.

Lemma 1.5. ([15]) *Let $p=L_{i_1}\cap\dots\cap L_{i_r}$ be an r-fold point of $\mathcal{L}^\varphi$, $r\geqslant 2$. Then,*

(i) $\varepsilon_p=\lambda_{i_1}+\dots+\lambda_{i_r}$,

(ii) $\varepsilon_{p,\,i_k}=\lambda_{i_k}+\sum_{j=1}^{r}\lambda_{i_j}$,

(iii) $\varphi(\varepsilon_{p,\,i_k})=0$.

□

The following theorem is a straightforward consequence of Lemmas 1.4 and 1.5.

Theorem 1.6. *The Galois coverings f and g are included in the commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\ \nu\ } & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \hat{\mathbb{P}}^2 & \xrightarrow[\ \sigma\ ]{} & \mathbb{P}^2 \end{array}$$

in which $\nu:X\to Y$ is a resolution of singularities of Y. □

Lemma 1.7. *Suppose that the Galois group of the covering $Y\to\mathbb{P}^2$ is $(\mathbb{Z}/2\mathbb{Z})^3$. Then, a point $q\in Y$ situated over an r-fold point $p=$*

$p_{i_1,\dots,i_r} = L_{i_1}\cap\dots\cap L_{i_r}$ of $\mathcal{L}^{\phi}$ is not a canonical singular point (that is, q is not an A-D-E-singularity) if, and only if, either $r>3$ or $r=3$ and p is not a branch point of φ.

Proof. To determine the type of a singular point we look at its resolution provided by $\nu : X\to Y$, see Theorem 1.6.

If $r=2$ and $\varphi(\lambda_1)\neq\varphi(\lambda_2)$, then, by Lemma 1.4, each point $q\in g^{-1}(p)$ is a nonsingular point of Y. If, by contrary, $\varphi(\lambda_1)=\varphi(\lambda_2)$, then the covering $f: X\to\hat{\mathbb{P}}^2$ is not branched at E'_p and it splits over E'_p into four copies of a Galois double covering of $\mathbb{P}^1$ branched at two points, so that each of the four points $q\in g^{-1}(p)$ is replaced in the resolution by a rational curve with self-intersection number $\frac{(-1)\cdot 8}{4}=-2$. Hence, in this case all the four points are of type A_1.

If $r=3$ and p is a non-branch point, then up to a coordinate change in G we have $\varphi(\lambda_{i_1})=(1,0,0)$, $\varphi(\lambda_{i_2})=(0,1,0)$, and $\varphi(\lambda_{i_3})=(1,1,0)$. Therefore, $f^{-1}(E'_p)$ is a disjoint union of two rational curves C_1 and C_2 with self-intersection $\frac{(-1)\cdot 8}{2}=-4$. Hence, the singular points $q\in g^{-1}(p)$ are not canonical.

Now, let suppose that $r=3$, p is a branch point, and $\varphi(\lambda_{i_1})$, $\varphi(\lambda_{i_2})$, $\varphi(\lambda_{i_3})$ are pairwise distinct (note that for a branch point the latter assumption is equivalent to $\sum_{j=1}^{3}\varphi(\lambda_{i_j})\neq\varphi(\lambda_{i_k})$ for any $1\leqslant k\leqslant 3$). Then, after a coordinate change in G we may suppose that $\varphi(\lambda_{i_1})=(1,0,0)$, $\varphi(\lambda_{i_2})=(0,1,0)$, $\varphi(\lambda_{i_3})=(0,0,1)$. Therefore, for E'_p we get a Galois covering over $E'_p=\mathbb{P}^1$ with Galois group $(\mathbb{Z}/2\mathbb{Z})^2$ and three branched points, so that $f^{-1}(E'_p)$ is a rational curve with self-intersection $\frac{(-1)\cdot 8}{4}=-2$, and, hence, the singular point $q=g^{-1}(p)$ is of type A_1.

Next, let treat the case when $r\geqslant 3$ and there is at least one k such that $\sum_{j=1}^{r}\varphi(\lambda_{i_j})=\varphi(\lambda_{i_k})$. Then: p is a branch point, $\sigma^{-1}(p)=E'_p+\sum_{j=1}^{s}E'_{p,k_j}$ where $(E'_p)^2=-(s+1)$ and $(E'_{p,k_1})^2=\dots=(E'_{p,k_s})^2=-1$; E'_p is a branch curve of f, but $E'_{p,k_1},\dots,E'_{p,k_s}$ are not branch curves of f. Therefore, each of $f^{-1}(E_{p,k_j})$, $1\leqslant j\leqslant s$, splits into a disjoint union of four (-2)-curves, while $f^*(E'_p)=2C_1+\dots+2C_{2^n}$, where 2^n is the index in G of the subgroup $G_{i_1,\dots,i_r}$ generated by $\varphi(\lambda_{i_1}),\dots,\varphi(\lambda_{i_r})$ and $C_1,\dots,C_{2^n}$ are copies of a Galois covering of E'_p of degree 2^{2-n} (recall that $\deg f=8$) branched at $r-s$ points. Thus, for each $i=1,\dots,2^n$ we have

$$\begin{aligned}&(C_i^2)_X=-2^{1-n}(s+1),\\ &g(C_i)=2^{-n}(r-s)-2^{2-n}+1=2^{-n}(r-s-4)+1,\end{aligned}$$

where $0 \leq n \leq 2$. If $g^{-1}(p)$ consists of canonical singularities, then $(C_i^2)_X = -2$ and $g(C_i) = 0$. Therefore

$$\begin{aligned} &2^{1-n}(s+1) = 2, \\ &2^{-n}(r-s-4)+1 = 0. \end{aligned}$$

The only solutions are $n = 1, s = 1, r = 3$ and $n = 2, s = 3, r = 3$. In the first subcase $g^{-1}(p)$ splits in two A_3-singularities, and in the second one, it splits in four D_4-singularities.

The only remaining case is when $r \geqslant 4$ and $\sum_{j=1}^r \varphi(\lambda_{i_j}) \neq \varphi(\lambda_{i_k})$ whatever is $1 \leqslant k \leqslant r$. Then $f^{-1}(E'_p)$ splits into a number of copies of a 2^m-sheeted Galois covering $C \to \mathbb{P}^1 = E'_p$ branched at r points, where $m \geq 1$. By the Hurwitz formula,

$$g(C) = 2^{m-2}r - 2^m + 1 \geq 1.$$

Hence the singular points $q \in g^{-1}(p)$ are not canonical. $\square$

Lemma 1.8. *Suppose that the Galois group of the covering $Y \to \mathbb{P}^2$ is $(\mathbb{Z}/2\mathbb{Z})^3$ and that the line arrangement $\mathcal{L} = L_0 \cup \dots \cup L_6$ have no r-fold singular points with $r \geq 4$. If $\varphi(\lambda_i) \neq 0$ for any $0 \leqslant i \leqslant 6$ and there are two distinct lines L_{i_1} and L_{i_2} with $\varphi(\lambda_{i_1}) = \varphi(\lambda_{i_2})$, then $p_g(X) \neq 0$.*

Proof. By (2), Y can be given by equations

$$u_j^2 = \prod l_i(v_1, v_2)^{a_{i,j}}, \qquad j = 1, 2, 3,$$

where $(a_{i,1}, a_{i,2}, a_{i,3}) = \varphi(\lambda_i)$. Since φ is an epimorphism to $(\mathbb{Z}/2\mathbb{Z})^3$, there are at most four lines with equal values of φ. Hence, up to renumbering of lines and acting on φ by an automorphism of $(\mathbb{Z}/2\mathbb{Z})^3$ there are four cases to consider:

- (four equal values) $\varphi(\lambda_1) = \varphi(\lambda_2) = \varphi(\lambda_3) = \varphi(\lambda_4) = (1, 0, 0)$, $\varphi(\lambda_5) = (0, 1, 0)$, and $\varphi(\lambda_6) = (0, 0, 1)$;
- (three equal values) $\varphi(\lambda_1) = \varphi(\lambda_2) = \varphi(\lambda_3) = (1, 0, 0)$, $\varphi(\lambda_4) = (0, 1, 0)$, and $\varphi(\lambda_5) = (0, 0, 1)$.
- (two pairs of equal values) $\varphi(\lambda_1) = \varphi(\lambda_2) = (1, 0, 0)$, $\varphi(\lambda_3) = \varphi(\lambda_4) = (0, 1, 0)$, and $\varphi(\lambda_5) = (0, 0, 1)$.
- (one pair of equal values) $\varphi(\lambda_1) = \varphi(\lambda_2) = (1, 0, 0)$, $\varphi(\lambda_3) = (0, 1, 0)$, $\varphi(\lambda_4) = (0, 0, 1)$, while $\varphi(\lambda_i)$ with $i \in \{0, 5, 6\}$ are distinct from each other and distinct from $(1, 0, 0), (0, 1, 0), (0, 0, 1)$.

In the first three cases the function $u = u_1u_2u_3 \in \mathbb{C}(Y)$ satisfies the following equation

$$u^2 = l_1(v_1, v_2) \dots l_5(v_1, v_2) l_6(v_1, v_2)^a, \tag{3}$$

where $a = 0$ or 1 (in the first case, $a = 1$). Such an equation defines a double covering $Z \to \mathbb{P}^2$ branched in six lines ($L_1, \dots, L_6$ if $a = 1$

and $L_1, \dots, L_5, L_0$ if $a = 0$). Since the line arrangement has no r-fold points with $r \geq 4$, Z has only canonical singularities, and therefore it is a $K3$-surface, and hence it has $p_g(Z) = 1$. The inequality $p_g(X) \geq 1$ follows now from the existence of a dominant rational map from X to Z.

To complete the proof, let us notice that the fourth case is impossible. Indeed, it is impossible to satisfy the relation $\varphi(\lambda_0) + \varphi(\lambda_5) + \varphi(\lambda_6) = (0, 1, 1)$, by three distinct elements among $(1, 1, 0)$, $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 1)$. □

1.5. **Campedelli surfaces as Galois coverings branched over Campedelli arrangements.** Let $\mathcal{L}$ be a line arrangement in $\mathbb{P}^2$ consisting of seven distinct lines L_α labeled by the non-zero elements $\alpha \in (\mathbb{Z}/2\mathbb{Z})^3$. We call such a labeled arrangement $\mathcal{L}$ a *Campedelli line arrangement* if it has neither r-fold points with $r \geq 4$ nor triple points $p_{\alpha_1,\alpha_2,\alpha_3} = L_{\alpha_1} \cap L_{\alpha_2} \cap L_{\alpha_3}$ with $\alpha_1 + \alpha_2 + \alpha_3 = 0$. We say that a Campedelli line arrangement $\mathcal{L} = \sum L_\alpha$ is obtained from a Campedelli line arrangement $\mathcal{L}' = \sum L'_\alpha$ by means of *renumbering* of lines if there is an automorphism $\tau \in \mathrm{Aut}(\mathbb{Z}/2\mathbb{Z})^3$ such that $L_\alpha = L'_{\tau(\alpha)}$ for any $\alpha \in (\mathbb{Z}/2\mathbb{Z})^3 \setminus \{0\}$.

Given a Campedelli line arrangement $\mathcal{L}$, one can consider the Galois covering $Y(\mathcal{L}) \to \mathbb{P}^2$ with Galois group $(\mathbb{Z}/2\mathbb{Z})^3$ branched in $\mathcal{L}$ and defined by the epimorphism $\varphi : H_1(\mathbb{P}^2 \setminus \mathcal{L}, \mathbb{Z}) \to (\mathbb{Z}/2\mathbb{Z})^3$ given by $\varphi(\lambda_\alpha) = \alpha$. We call this covering *the Galois covering branched over a Campedelli arrangement* $\mathcal{L}$. Clearly, a renumbering of a Campedelli arrangement leads to an equivalent covering.

Theorem 1.9. *For any Campedelli surface X there exists a Campedelli line arrangement $\mathcal{L}$ such that $X_{can} = Y(\mathcal{L})$.*

Proof. By Corollary 1.2, given a Campedelli surface X there exists an arrangement $\mathcal{L}$ of seven distinct lines in $\mathbb{P}^2$ such that X_{can} is a $(\mathbb{Z}/2\mathbb{Z})^3$-Galois covering of $\mathbb{P}^2$ branched in $\mathcal{L}$. Since X_{can} has only canonical singularities, Lemma 1.7 implies that $\mathcal{L}$ have no neither any r-fold point with $r \geq 4$ nor any 3-fold point which is not a branch point. Now Lemma 1.8 applies and shows that $\mathcal{L}$ is a Campedelli arrangement. □

The following, converse, statement is proved in [17].

Theorem 1.10. ([17]) *For any Campedelli line arrangement $\mathcal{L}$ the surface $Y(\mathcal{L})$ is isomorphic to the canonical model of a Campedelli surface.* □

If a Campedelli line arrangement $\mathcal{L}$ has no triple points, then by Lemma 1.4, the surface $Y(\mathcal{L})$ is nonsingular (so that it is itself a Campedelli surface, $X = X_{can}$) and it can be imbedded as a complete intersection into the weighted projective space

$$\mathbb{P}^9_w = \mathbb{P}^9(1,1,1,2,2,2,2,2,2,2)$$

with three weight-1 coordinates z_i, $i = 0,1,2$, and seven weight-2 coordinates u_α, $\alpha \in (\mathbb{Z}/2\mathbb{Z})^3 \setminus \{0\}$. Namely, in accordance with what was seen in subsection 1.3, $Y(\mathcal{L})$ is isomorphic to a surface in $\mathbb{P}^9_w$ given by

$$\begin{aligned}
u^2_{(1,0,0)} &= l_{(1,0,0)} l_{(1,1,0)} l_{(1,0,1)} l_{(1,1,1)}\\
u^2_{(0,1,0)} &= l_{(0,1,0)} l_{(1,1,0)} l_{(0,1,1)} l_{(1,1,1)}\\
u^2_{(0,0,1)} &= l_{(0,0,1)} l_{(0,1,1)} l_{(1,0,1)} l_{(1,1,1)}\\
u^2_{(1,1,0)} &= l_{(1,0,0)} l_{(0,1,0)} l_{(1,0,1)} l_{(0,1,1)}\\
u^2_{(1,0,1)} &= l_{(1,0,0)} l_{(0,0,1)} l_{(1,1,0)} l_{(0,1,1)}\\
u^2_{(0,1,1)} &= l_{(0,1,0)} l_{(0,0,1)} l_{(1,0,1)} l_{(1,1,0)}\\
u^2_{(1,1,1)} &= l_{(1,0,0)} l_{(0,1,0)} l_{(0,0,1)} l_{(1,1,1)}.
\end{aligned} \tag{4}$$

where $l_\alpha(z_0, z_1, z_2) = 0$ are linear equations of $L_\alpha \subset \mathcal{L}$ in $\mathbb{P}^2$.

Note that u_α satisfy the following relations

$$\begin{aligned}
u_{(1,1,0)} &= \frac{u_{(1,0,0)} u_{(0,1,0)}}{l_{(1,1,0)} l_{(1,1,1)}}, \quad u_{(1,0,1)} = \frac{u_{(1,0,0)} u_{(0,0,1)}}{l_{(1,0,1)} l_{(1,1,1)}},\\
u_{(0,1,1)} &= \frac{u_{(0,1,0)} u_{(0,0,1)}}{l_{(0,1,1)} l_{(1,1,1)}}, \quad u_{(1,1,1)} = \frac{u_{(1,0,0)} u_{(0,1,0)} u_{(0,0,1)}}{l_{(1,1,0)} l_{(1,0,1)} l_{(0,1,1)} l_{(1,1,1)}}.
\end{aligned} \tag{5}$$

Note also that if $\mathcal{L}'$ is obtained from $\mathcal{L}$ by a renumbering of the lines $\mathcal{L}$ given by an automorphism $\tau \in \mathrm{Aut}(\mathbb{Z}/2\mathbb{Z})^3$, then this renumbering (in order to save the form of the equations in (4)) defines the renumbering of u_α by the automorphism τ^{-1}.

1.6. **Moduli space of the Campedelli surfaces.** In this section, we identify the moduli space of Campedelli surfaces with the moduli space of Campedelli line arrangements. Here and further, we apply to Campedelli surfaces the following general property of minimal surfaces of general type: their isomorphisms (respectively, automorphisms) are in a natural bijection with the isomorphisms (respectively, automorphisms) of their canonical models.

As above, let a Galois covering $g : Y(\mathcal{L}) \to \mathbb{P}^2$ with Galois group $G \simeq (\mathbb{Z}/2\mathbb{Z})^3$ be branched along a Campedelli line arrangement $\mathcal{L} = \sum L_\alpha$, where the sum is taken over all $\alpha \in G$, $\alpha \neq 0$, and be determined by an epimorphism $\varphi : H_1(\mathbb{P}^2 \setminus \mathcal{L}, \mathbb{Z}) \to G$ such that $\varphi(\lambda_\alpha) = \alpha$. Denote by $X = X(\mathcal{L})$ the minimal nonsingular model of $Y(\mathcal{L})$ constructed in subsection 1.4. Since $\mathcal{L}$ has neither r-fold points with $r \geq 4$ nor triple points $p_{\alpha_1,\alpha_2,\alpha_3} = L_{\alpha_1} \cap L_{\alpha_2} \cap L_{\alpha_3}$ with $\alpha_1+\alpha_2+\alpha_3 = 0$, this construction reduces to the composition $\sigma : \hat{\mathbb{P}}^2 \to \mathbb{P}^2$ of the blow-ups with centers at all the 3-fold points of $\mathcal{L}$ followed by the covering $f : X(\mathcal{L}) \to \hat{\mathbb{P}}^2$ induced by the lift $\hat{\varphi}$ of φ.

Denote by f_σ the composition $f_\sigma = \sigma \circ f : X(\mathcal{L}) \to \mathbb{P}^2$.

Lemma 1.11. ([17]) *The bicanonical system $|2K_X|$ of $X = X(\mathcal{L})$ is equal to $|f_\sigma^* L|$, where $L \subset \mathbb{P}^2$ is a line in $\mathbb{P}^2$.* □

The next Lemma is a straightforward corollary of Proposition 1.3.

Lemma 1.12. *Let $\mathcal{L}_1 = \sum_{i=1}^7 L_{1,\alpha_i}$ and $\mathcal{L}_2 = \sum_{i=1}^7 L_{2,\beta_i}$, $\alpha_i, \beta_i \in G = (\mathbb{Z}/2\mathbb{Z})^3$, $\alpha_i, \beta_i \neq 0$, be two Campedelli line arrangements in $\mathbb{P}^2$ such that $L_{1,\alpha_i} = L_{2,\beta_i}$ for $i = 1, \dots, 7$. Then the Galois coverings $Y(\mathcal{L}_1) \to \mathbb{P}^2$ and $Y(\mathcal{L}_2) \to \mathbb{P}^2$ are equivalent if, and only if, $\mathcal{L}_1$ can be obtained from $\mathcal{L}_2$ by means of renumbering of lines.* □

Theorem 1.13. *Let $X_{1,can} = Y(\mathcal{L}_1)$ and $X_{2,can} = Y(\mathcal{L}_2)$ be two Galois coverings $g_i : X_{i,can} \to \mathbb{P}^2$ branched over Campedelli line arrangements $\mathcal{L}_1$ and $\mathcal{L}_2$. If $X_{1,can}$ and $X_{2,can}$ are isomorphic, then any isomorphism $\nu : X_{1,can} \to X_{2,can}$ can be included in a commutative diagram*

$$\begin{tikzcd} X_{1,can} \arrow[r, "\nu"] \arrow[d, "g_1"] & X_{2,can} \arrow[d, "g_2"] \\ \mathbb{P}^2 \arrow[r, "\psi"'] & \mathbb{P}^2 . \end{tikzcd}$$

Proof. Consider the resolutions $X_i = X(\mathcal{L}_i)$ of $X_{i,can} = Y(\mathcal{L}_i)$, the associated morphisms $f_i : X_i \to \hat{\mathbb{P}}^2$, and the composed morphisms $f_{\sigma,i} = \sigma \circ f_i : X_i \to \mathbb{P}^2$. As it was mentioned above, since X_i are minimal surfaces of general type, any isomorphism between their canonical models, $X_{1,can} \to X_{2,can}$ lifts uniquely to an isomorphism $X_1 \to X_2$, and vice versa. Thus, for given isomorphism $\nu : X_1 \to X_2$, it is sufficient to find a projective transformation ψ such that $\psi \circ f_{\sigma,1} = f_{\sigma,2} \circ \nu$. Moreover, the latter relation would follow from the corresponding relation between the induced maps of the function fields: $\nu^* \circ f_{\sigma,2}^* = f_{\sigma,1}^* \circ \psi^*$.

As for any Campedelli surface, the torsion subgroup $\mathrm{Tors}(X_i)$ of $H^2(X_i, \mathbb{Z})$ is 2-torsion and isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$. Given any $\alpha \in \mathrm{Tors}\,(X_i)$, $\alpha \neq 0$, the linear system $|K_{X_i} + \alpha|$ is non-empty as it follows from Serre duality,

$$\dim H^2(X_i, \mathcal{O}_{X_i}(K_{X_i} + \alpha)) = \dim H^0(X_i, \mathcal{O}_{X_i}(\alpha)) = 0,$$

and the Riemann-Roch theorem. Hence, there exists at least one effective divisor $D_\alpha \in |K_{X_i} + \alpha|$, and $2D_\alpha \in |2K_{X_i}|$. Since X_i are minimal surfaces of general type, we have $\dim H^0(X_i, \mathcal{O}(2K_{X_i})) = K_{X_i}^2 + 1 = 3$. On the other hand, $\dim H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(L)) = 3$, where L is a line in $\mathbb{P}^2$, while by Lemma 1.11 we have $|2K_{X_i}| = |f_\sigma^*(L)|$. Finally, $|2K_{X_i}| = f_{\sigma,i}^*(|L|)$ and $D \in |K_{X_i} + \alpha|$ for some $\alpha \in \mathrm{Tors}(X_i)$ if and only if $2D = f_{\sigma,i}^*(\widetilde{L})$ for some $\widetilde{L} \in |L|$.

The only lines $\widetilde{L} \in |L|$ for which the divisors $f_{\sigma,i}^*(\widetilde{L})$ are divisible by 2 are the seven branch lines belonging to $\mathcal{L}_i$. Hence, they give all the different torsion elements and can be relabeled by the torsion elements so that $\mathcal{L}_i = \sum L_{i,\alpha}$, where the sum is taken over the nonzero torsion elements, and $\frac{1}{2} f_{\sigma,i}^*(L_{i,\alpha}) = D_{i,\alpha} \in |K_{X_i} + \alpha|$. (Note that this labeling of lines may not coincide with the initial one.)

Let $\nu : X_1 \to X_2$ be an isomorphism. It induces an isomorphism of torsion groups, $\nu^* : \mathrm{Tors}\,(X_2) \to \mathrm{Tors}\,(X_1)$, and isomorphisms of linear systems,

$$\nu^* : H^0(X_2, \mathcal{O}_{X_2}(K_{X_2} + \alpha)) \to H^0(X_1, \mathcal{O}_{X_1}(K_{X_1} + \nu^*(\alpha)))$$

for each $\alpha \in \mathrm{Tors}\,(X_2)$. Therefore, $\nu^*(D_{2,\alpha}) = D_{1,\nu^*(\alpha)}$ for any $\alpha \in \mathrm{Tors}(X_2)$, $\alpha \neq 0$, and we get

$$\nu^*(f_{\sigma,2}^*(L_{2,\alpha_1} - L_{2,\alpha_2})) = \nu^*(2D_{2,\alpha_1} - 2D_{2,\alpha_2}) = 2D_{1,\nu^*(\alpha_1)} - 2D_{1,\nu^*(\alpha_2)} = f_{\sigma,1}^*(L_{1,\nu^*(\alpha_1)} - L_{1,\nu^*(\alpha_2)})$$

for any non zero $\alpha_1, \alpha_2 \in \mathrm{Tors}\,(X_2)$. Since any rational function is defined uniquely up to multiplication by a constant by its divisors of zeros and poles, it implies the existence of a system of constants c_{α_1,α_2} such that

$$\nu^*(f_{\sigma,2}^*(\frac{l_{2,\alpha_1}(v_1,v_2)}{l_{2,\alpha_2}(v_1,v_2)})) = c_{\alpha_1,\alpha_2} f_{\sigma,1}^*(\frac{l_{1,\nu^*(\alpha_1)}(v_1,v_2)}{l_{1,\nu^*(\alpha_2)}(v_1,v_2)}), \tag{6}$$

where v_1, v_2 are affine coordinates in $\mathbb{P}^2$ and $l_{2,\alpha}, l_{1,\beta}$ are linear equations of the corresponding lines. Since the functions $f_{\sigma,i}^*(\frac{l_{i,\alpha_1}(v_1,v_2)}{l_{i,\alpha_2}(v_1,v_2)}))$ generate the subfields $f_{\sigma,i}^*(\mathbb{C}(\mathbb{P}^2))$ of $\mathbb{C}(X_i)$, the relations (6) imply the existence of a projective transformation $\psi : \mathbb{P}^2 \to \mathbb{P}^2$ such that $f_{\sigma,1}^* \circ \psi^* = \nu^* \circ f_{\sigma,2}^*$. □

Corollary 1.14. *If $X = X(\mathcal{L})$, where $\mathcal{L}$ is a generic Campedelli line arrangement, then $Aut(X) = Gal(Y(\mathcal{L}) \to \mathbb{P}^2) \simeq (\mathbb{Z}/2\mathbb{Z})^3$.* □

Denote by $\mathcal{P} = \mathbb{P}^2 \times \cdots \times \mathbb{P}^2$ the product of seven copies of the projective plane. We consider each factor in this product as the dual projective plane, so that elements of each factor are lines in the initial

$\mathbb{P}^2$. In addition, we numerate the factors of $\mathcal{P}$ by the non-zero elements $\alpha \in G = (\mathbb{Z}/2\mathbb{Z})^3$. Let D be the union of all diagonals in $\mathcal{P}$,

$$\begin{aligned} T_3 = \{ \ & \mathcal{L} \in \mathcal{P} \mid \exists\, \alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3} \text{ such that} \\ & \alpha_{i_1} + \alpha_{i_2} + \alpha_{i_3} = 0 \text{ and } L_{\alpha_{i_1}} \cap L_{\alpha_{i_2}} \cap L_{\alpha_{i_3}} \neq \emptyset \}, \end{aligned}$$

$$\begin{aligned} T_4 = \{ \ & \mathcal{L} \in \mathcal{P} \mid \exists\, \alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}, \alpha_{i_4} \text{ such that} \\ & L_{\alpha_{i_1}} \cap L_{\alpha_{i_2}} \cap L_{\alpha_{i_3}} \cap L_{\alpha_{i_4}} \neq \emptyset \}. \end{aligned}$$

The group $\mathrm{PGL}(2, \mathbb{C}) \times \mathrm{Aut}\, G$ acts on $\mathcal{P} \setminus (D \cup T_3 \cup T_4)$ as follows: $\mathrm{PGL}(2, \mathbb{C})$ acts in a usual way on each factor of $\mathcal{P}$, and the elements h of $\mathrm{Aut}\, G$ permute the factors, $h : \mathbb{P}^2_\alpha \to \mathbb{P}^2_{h(\alpha)}$.

The following theorem is a consequence of Lemma 1.12 and Theorems 1.9, 1.10, and 1.13.

Theorem 1.15. *The moduli space $\mathcal{M}$ of the Campedelli surfaces is isomorphic to the quotient space*

$$(\mathcal{P} \setminus (D \cup T_3 \cup T_4))/(PGL(2, \mathbb{C}) \times Aut G).$$

□

Note that, as a result, all Campedelli surfaces are deformation equivalent.

2. Real Campedelli surfaces

2.1. **An extension of the automorphism group.** For any complex space X, denote by $\mathrm{Kl} = \mathrm{Kl}(X)$ the group of holomorphic and anti-holomorphic bijections $X \to X$. Recall that, by definition, an anti-holomoprhic map $X \to X$ can be seen as a holomorphic map $X \to \overline{X}$, where $\overline{X}$ states for the complex conjugate to X.

Note (*cf.* subsection 1.6) that for any minimal surface X of general type the groups $\mathrm{Kl}(X)$ and $\mathrm{Kl}(X_{can})$ are naturally isomorphic. In what follows we identify them as soon as it does not lead to a confusion.

Clearly, if Kl contains at least one anti-holomorphic element, the holomorphic elements form in Kl a subgroup $\mathrm{Aut} = \mathrm{Aut}(X)$ of index 2. In other words, there is a short exact sequence $1 \to \mathrm{Aut} \to \mathrm{Kl} \to H \to 1$, where $H \simeq \mathbb{Z}/2$ or 1. We denote by $\mathrm{kl} : \mathrm{Kl} \to H$ the homomorphism of this sequence.

The real structures on X are the elements $c \in \mathrm{Kl}(X)$ such that $\mathrm{kl}(c) \neq 1$ and $c^2 = \mathrm{id}$. Two real structures, c_1 and c_2 are called *equivalent* (or *isomorphic*) if there exists $h \in \mathrm{Aut}(X)$ such that $h \circ c_2 = c_1 \circ h$. Recall that on the projective plane $\mathbb{P}^2$ (as well as on any projective space of even dimension) any two real structures are equivalent by a projective transformation.

2.2. **A criteria of existence of real structures on Campedelli surfaces.** Given Campedelli surface $X = X(\mathcal{L})$ associated with a Campedelli line arrangement $\mathcal{L}$, we consider the composed map $f_\sigma = \sigma \circ f : X \to \mathbb{P}^2$ and say that $c_X \in \mathrm{Kl}(X)$ is *lifted* from $\mathbb{P}^2$ if there exists $c_P \in \mathrm{Kl}(\mathbb{P}^2)$ such that the following diagram is commutative

$$\begin{tikzcd} X \arrow[r, "c_X"] \arrow[d, "f_\sigma"] & X \arrow[d, "f_\sigma"] \\ \mathbb{P}^2 \arrow[r, "c_P"'] & \mathbb{P}^2. \end{tikzcd}$$

Theorem 2.1. *For any Campedelli line arrangement $\mathcal{L}$, every $c_X \in$ Kl(X) is lifted from $\mathbb{P}^2$. In particular, if X has a real structure c_X, then there exists a real structure c_P on $\mathbb{P}^2$ such that $c_P \circ f_\sigma = f_\sigma \circ c_X$.*

Proof. If $c_X \in \mathrm{Aut}(X)$, then c_X is lifted from $\mathbb{P}^2$ by Theorem 1.13. Let $c_X \in \mathrm{Kl}(X)$ and $c_X \notin \mathrm{Aut}(X)$. Then $c_X : X \to \overline{X}$ is a holomorphic isomorphism. Consider the complex conjugated covering $\overline{f}_\sigma : \overline{X} \to \overline{\mathbb{P}^2}$. By Theorem 1.13, there is a holomorphic isomorphism $c_P : \mathbb{P}^2 \to \overline{\mathbb{P}^2}$ which makes commutative the following diagram

$$\begin{tikzcd} X \arrow[r, "c_X"] \arrow[d, "f_\sigma"] & \overline{X} \arrow[d, "\overline{f}_\sigma"] \\ \mathbb{P}^2 \arrow[r, "c_P"'] & \overline{\mathbb{P}^2}. \end{tikzcd}$$

To get the last statement, it is sufficient to notice that $c_P^2 = \mathrm{id}$ if $c_X^2 = \mathrm{id}$. □

Corollary 2.2. *For any Campedelli line arrangement $\mathcal{L} \subset \mathbb{P}^2$, the Campedelli surface $X = X(\mathcal{L})$ admits a real structure if, and only if, for a suitably chosen real structure c_P of $\mathbb{P}^2$ the (labeled) Campedelli line arrangement $\mathcal{L}$ is real, that is, there exists an automorphism (renumbering) $\tau : (\mathbb{Z}/2\mathbb{Z})^3 \to (\mathbb{Z}/2\mathbb{Z})^3$ such that $c_P(L_\alpha) = L_{\tau(\alpha)}$ for each $\alpha \in (\mathbb{Z}/2\mathbb{Z})^3$, $\alpha \neq 0$.*

Proof. In the case of a real arrangement, to lift c_P it is sufficient to notice that c_P (as any real structure on $\mathbb{P}^2$) has a whole real projective plane of fixed points, to pick such a fixed point in the complement of the arrangement, and to identify the unbranched points of X_{can} with classes of pathes issued from the fixed point. So that c_P and the identification define c properly acting on X_{can}. A renumbering induced by a transformation of $\mathbb{P}^2$ is a homomorphism, since it factors through the induced action on $H_1(\mathbb{P}^2 \setminus \mathcal{L}, \mathbb{Z})$. □

2.3. **Real Campedelli line arrangements.** The Galois group $G = \mathrm{Gal}(X/\hat{\mathbb{P}}^2) \simeq (\mathbb{Z}/2\mathbb{Z})^3$ is a subgroup of $\mathrm{Aut}(X)$. As it follows from Theorem 2.1, G is a normal subgroup of $\mathrm{Kl}(X)$, and in addition, by Corollary 2.2, $c(L_\alpha) = L_{c\alpha c^{-1}}$ for any $\alpha \in G$ and $c \in \mathrm{Kl}(X)$.

Proposition 2.3. *Let $\mathcal{L}$ be a Campedelli line arrangement which is real with respect to some real structure $c_P : \mathbb{P}^2 \to \mathbb{P}^2$. Then either $\mathcal{L}$ consists of seven real lines or it consists of three real lines and two pairs of complex conjugated lines. Respectively, c_P acts on the labeling of $\mathcal{L}$ either identically or not.*

Proof. The homomorphism $\alpha \in G = (\mathbb{Z}/2\mathbb{Z})^3 \mapsto c\alpha c^{-1} \in G = (\mathbb{Z}/2\mathbb{Z})^3$, where c is the real structure on X, is an involution, and, as any involution on a $\mathbb{Z}/2$-vector space, it splits into irreducible 1- and 2-dimensional components. In dimension 3, there are only two possibilities, either the involution is trivial or it contains a 2-dimensional irreducible component, that is an involution interchanging two generators. In the first case, all α are fixed, and hence all the lines are real. In the second case, there are three and only three fixed elements, and hence three and only three real lines. □

Let call a Campedelli line arrangement $\mathcal{L}$ *purely real* if it consists of seven real lines and *mixed real* if it consists of three real lines and two pairs of complex conjugated lines.

Given a real structure c_X, denote by $\mathrm{Kl}(X, c_X)$ the subgroup of $\mathrm{Kl}(X)$ generated by G and c_X. If $X = X(\mathcal{L})$ and $\mathcal{L}$ is real with respect to a real structure c_P on $\mathbb{P}^2$, then the subgroup $\mathrm{Kl}(X, c_X)$ does not depend on the choice of a lift c_X of c_P and we denote it by $\mathrm{Kl}(X, c_P)$. Note that for a generic real Campedelli line arrangement $\mathcal{L}$ it holds $\mathrm{Aut}X(\mathcal{L}) = G$, so that $\mathrm{Kl}(\mathrm{X}) = \mathrm{Kl}(X, c_X)$ for any c_X.

Proposition 2.4. *Let $X = X(\mathcal{L})$ be a Campedelli surface associated with a Campedelli line arrangement $\mathcal{L}$ which is real with respect to c_P. Then:*

- (i) *if $\mathcal{L}$ is a purely real line arrangement, then $Kl(X, c_P) \simeq (\mathbb{Z}/2\mathbb{Z})^4$, and if $\mathcal{L}$ is a generic purely real line arrangement, then there are exactly eight different real structures on X;*
- (ii) *if $\mathcal{L}$ is a mixed real line arrangement, then $Kl(X, c_P) \simeq \mathbb{H} \times (\mathbb{Z}/2\mathbb{Z})$, where $\mathbb{H}$ is the quaternion group of order eight, and if $\mathcal{L}$ is a generic mixed real line arrangement, then there are exactly four different real structures on X.*

Proof. Pick a real point $p \in \mathbb{P}^2 \setminus \mathcal{L}$ and consider a real structure $c \in \mathrm{Kl}(X)$ which is a lift of c_P from $\mathbb{P}^2$ to X and have fixed points over p. If all the lines are real, then $c\alpha c^{-1} = \alpha$ for any $\alpha \in G$ (indeed, since $c = \mathrm{id}$ at each point of the G-orbit over p, the relation $c\alpha c^{-1} = \alpha$ holds at the points of this G-orbit, and, hence, it holds everywhere).

If there are only three real lines in the arrangement, then in a suitable basis e_1, e_2, e_3 of G the (renumbering) involution $\alpha \mapsto c\alpha c^{-1}$ acts as $e_1 \mapsto e_2$ and $e_3 \mapsto e_3$. Therefore, in the latter case, $\mathrm{Kl}(X, c_P)$ splits in a direct sum of $\mathbb{Z}/2$ generated by e_3 with a non-commutative group of order 8 generated by e_1, e_2, and c.

Since for a generic arrangement it holds $\mathrm{Kl(X)} = \mathrm{Kl}(X, c_X)$, the statements concerning the generic cases follow now from enumerating anti-involutions in $\mathrm{Kl}(X, c_P) \simeq (\mathbb{Z}/2\mathbb{Z})^4$ and, respectively, $\mathrm{Kl}(X, c_P) \simeq \mathbb{H} \times (\mathbb{Z}/2\mathbb{Z})$. $\square$

2.4. **Purely real Campedelli line arrangements.** Let $\mathcal{L} = \cup L_\alpha$ be a Campedelli line arrangement which is purely real with respect to a real structure $c_P : \mathbb{P}^2 \to \mathbb{P}^2$. Choose homogeneous coordinates (z_0, z_1, z_2) in $\mathbb{P}^2$ such that c_P turns in the standard complex conjugation

$$c_P(z_0, z_1, z_2) = (\overline{z}_0, \overline{z}_1, \overline{z}_2).$$

Then each of the lines $L_\alpha \in \mathcal{L}$, $\alpha \in G \setminus \{0\}$, is given by equation

$$a_{\alpha,0} z_0 + a_{\alpha,1} z_1 + a_{\alpha,2} z_2 = 0$$

with real coefficients, $a_{\alpha,i} \in \mathbb{R}$.

Consider the set $\mathbb{RP}^2 = \{(z_0, z_1, z_2) \mid \ z_i \in \mathbb{R}\}$ of real points of $\mathbb{P}^2$. If $\mathcal{L}$ has no triple points, then $\mathcal{L}$ divides $\mathbb{RP}^2$ into twenty two n-gons P_i, $i = 1, \dots, 22$, $3 \le n \le 7$. The collection $(m_3, \dots, m_7)$, where m_n is the number of n-gons P_i, is called the *type* of $\mathcal{L}$.

The following description of topology of the inverse image of P_i in the associated Campedelli surface $X(\mathcal{L})$ is a straightforward consequence of the construction of ramified coverings.

Proposition 2.5. *For any polygon P_i of a purely real Campedelli line arrangement $\mathcal{L}$ without triple points, its inverse image $f^{-1}(P_i) \subset X(\mathcal{L})$ is a two-manifold and it is homeomorphic to the following quotient of $P_i \times G$, $G = (\mathbb{Z}/2\mathbb{Z})^3$: the points (a, β) and (b, γ) are identified if $a = b \in L_\alpha$ where $\gamma = \beta + \alpha$.* $\square$

A triangle P_i bounded by L_{α_1}, L_{α_2}, and L_{α_3} is said to have *linear (in)dependent sides*, if α_1, α_2, α_3 are linear (in)dependent.

Corollary 2.6. *For any n-gon P_i of a purely real Campedelli line arrangement $\mathcal{L}$ without triple points,* (*i*) *the Euler characteristic of* $f^{-1}(P_i)$ *is equal to* $8-2n$*;*

(*ii*) $f^{-1}(P_i)$ *is the disjoint union of two copies of* $\mathbb{RP}^2$*, if* $n=3$ *and the triangle* P_i *has linear depended sides;*

(*iii*) $f^{-1}(P_i)$ *is the two-dimensional sphere, if* $n=3$ *and the triangle* P_i *has linear independent sides;*

(*iv*) $f^{-1}(P_i)$ *is connected, if* $n=4$*, and it is orientable if, and only if,* $\alpha_1+\dots+\alpha_4=0$*, where* α_j *are the labels of the sides* L_{α_j} *of* P_i*;*

(*v*) $f^{-1}(P_i)$ *is a connected non-orientable two-manifold, if* $n\geq 5$*.*

Proof. The Euler characteristic $e(f^{-1}(P_i))$ is equal to

$$e(f^{-1}(P_i)) = 8-4n+2n = 8-2n$$

according to the cellular decomposition given by Proposition 2.5.

Let $L_{\alpha_1},\dots,L_{\alpha_n}$ be the sides of P_i. Consider a subgroup $G_{P_i} = \langle\alpha_1,\dots,\alpha_n\rangle$ of G generated by $\alpha_1,\dots,\alpha_n$. As it follows from Proposition 2.5, the number of connected components of $f^{-1}(P_i)$ coincides with the index of G_{P_i} in G. On the other hand, since $n>2$, either G_{P_i} coincides with G or it is a subgroup of index 2, and in the latter case, P_i is a triangle with linear dependent sides. Therefore, $f^{-1}(P_i)$ is connected except in the case of triangles with linear dependent sides and, moreover, if P_i is a triangle with linear dependent sides, then $f^{-1}(P_i)$ consists of two connected components.

If $n=3$, then $e(f^{-1}(P_i))=2$. Hence, if P_i is a triangle with linear independent sides, then $f^{-1}(P_i)$ is the 2-sphere, and if P_i is a triangle with linear dependent sides, then $f^{-1}(P_i)$ is the disjoint union of two copies of $\mathbb{RP}^2$.

Let $n\geq 4$. Then, P_i has three successive sides whose indices $\alpha_1,\alpha_2,\alpha_3$ are linear independent. After renumbering we can assume that $\alpha_1 = (1,0,0)$, $\alpha_2=(0,1,0)$, and $\alpha_3=(0,0,1)$. Following Proposition 2.5, perform a partial gluing of eight copies $P_\beta=(P_i,\beta)$ of P_i as is depicted in Fig. 1 (in Fig. 1, we denote the union of sides $L_{\alpha_4}\cup\dots\cup L_{\alpha_n}$ by $\widetilde{L}_\alpha$).

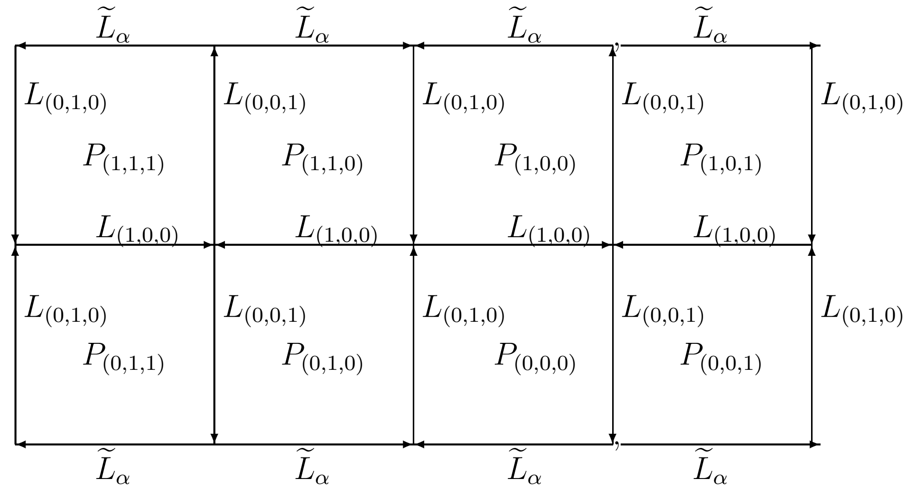


Fig. 1

Let $n = 4$. Then, for $\widetilde{L}_\alpha = L_{\alpha_4}$ there are four cases: either $\alpha_4 = (1,1,0)$, or $\alpha_4 = (1,0,1)$, or $\alpha_4 = (0,1,1)$, or $\alpha_4 = (1,1,1)$. It is easy to see from Fig. 1 that $f^{-1}(P)$ is non-orientable in the first three cases and it is orientable in the last case.

Let, finally, $n \geq 5$. Then, $\widetilde{L}_\alpha = L_{\alpha_4} \cup \dots \cup L_{\alpha_n}$ and at least one of $\alpha_4, \dots, \alpha_n$, say α_j, has to be equal to either $(1,1,0)$, or $(1,0,1)$, or $(0,1,1)$. Therefore, the gluing of $P_{(0,0,0)}$ and P_{α_j} along L_{α_j} gives rise to non-orientability of $f^{-1}(P)$. $\square$

Consider a real structure $c_X : X(\mathcal{L}) \to X(\mathcal{L})$ which is a lift of c_P. According to Proposition 2.4, c_X commutes with every element of G. Therefore, for any P_i, $1 \leqslant i \leqslant 22$, there exists one and only one $g_i \in G$ such that $c_X(x) = g_i(x)$ for any $x \in X$ for which $f_\sigma(x) \in P_i$. Using the same identification of G with $(\mathbb{Z}/2\mathbb{Z})^3$ which we have already fixed introducing the labeling of $\mathcal{L}$, $\varphi : H_1(\mathbb{P}^2 \setminus \mathcal{L}, \mathbb{Z}) \to (\mathbb{Z}/2\mathbb{Z})^3$, we put

$$g_i = (g_{i,1}, g_{i,2}, g_{i,3})$$

and for each P_i introduce sign-triples

$$Sign(P_i) = Sign_i = (sign_{i,1}, sign_{i,2}, sign_{i,3}),$$

where, by definition, $sign_{i,k} = (-1)^{g_{i,k}}, 1 \leqslant k \leqslant 3$. When we renumber the lines in $\mathcal{L}$ by means of an automorphism $h : (\mathbb{Z}/2\mathbb{Z})^3 \to (\mathbb{Z}/2\mathbb{Z})^3$ the labels g_i of P_i transform in $h(g_i)$; in particular, the signs $Sign_i$ of the polygons P_i equal to $Sign_i = (+,+,+)$ (corresponding to $g_i = 0$) remain unchanged under any renumbering. We call a polygon P_i to be *positive* if its signs are $Sign_i = (+,+,+)$.

The labels $Sign_i$ satisfy the following transition rule:

$$sign_{i,k} = (-1)^{a_k} sign_{j,k} \tag{7}$$

if P_i and P_j have a common side on L_α, $\alpha = (a_1, a_2, a_3)$. In particular, if one of $Sign_i$ is given, then it determines all the other.

Let us notice that we switch from g_i to $Sign_i$ by two reasons: first, it allows us to distinguish more easily (say, on Figures) a labeling of lines, $L_\alpha \mapsto \alpha$, from a labeling g_i of polygons P_i; second, these signs have a natural meaning described below (and are convenient in use).

Namely, to give an equivalent description of the above sign-labeling, let consider the embedding of $Y(\mathcal{L})$ into $\mathbb{P}^9_w$ given by equations (4) and the products

$$\begin{array}{l} l_{(1,0,0)}l_{(1,1,0)}l_{(1,0,1)}l_{(1,1,1)}, \\ l_{(0,1,0)}l_{(1,1,0)}l_{(0,1,1)}l_{(1,1,1)}, \\ l_{(0,0,1)}l_{(1,0,1)}l_{(0,1,1)}l_{(1,1,1)} \end{array} \tag{8}$$

participating in the first three equations (see subsection 1.5 for notations related with $\mathbb{P}^9_w$). As any homogeneous form of even degree with real coefficients, each of the products has a well defined sign at any point of $\mathbb{RP}^2$, where the product is nonzero. In particular, all the three products have well defined signs at the interior of each of P_i, $1 \leqslant i \leqslant 22$. Clearly, for each P_i the triple of signs ordered in accordance with the appearance of the products in (8) is equal to $Sign(P_i)$ determined by the real structure induced on $Y(\mathcal{L})$ by the standard complex conjugation in $\mathbb{P}^9_w$, $z_k \mapsto \bar{z}_k$ and $u_\alpha \mapsto \bar{u}_\alpha$. (Any real structure on $Y(\mathcal{L})$ lifts to a real structure on $X(\mathcal{L})$ and such a lift is unique, cf. subsection 2.1.)

By Proposition 2.4, there are eight and only eight distinct real structures c_X which are lifts of c_P. Let show that each of them can be induced by a suitable diagonal real structure on $\mathbb{P}^9_w$, where by a diagonal real structure on $\mathbb{P}^9_w$ we mean a real structure given by $z_k \mapsto \bar{z}_k$ and $u_\alpha \mapsto \epsilon_\alpha \bar{u}_\alpha$ with $\epsilon_\alpha = \pm 1$. Note that such a real structure c_ϵ preserves $Y(\mathcal{L})$ if, and only if, each of the equations (5) is preserved. In particular, there are eight and only eight real diagonal structures which preserve $Y(\mathcal{L})$ and they are determined by an arbitrary choice of ϵ_α with $\alpha = (1,0,0), (0,1,0)$, and $(0,0,1)$. We denote by

$$c_{\epsilon_{(1,0,0)},\epsilon_{(0,1,0)},\epsilon_{(0,0,1)}} : X(\mathcal{L}) \to X(\mathcal{L})$$

the real structures thus obtained. Each of them is a lift of c_P, since they all transform z_k in $\bar{z}_k$.

As is easy to check, the sign-triple $Sign'_i = (sign'_{i,1}, sign'_{i,2}, sign'_{i,3})$ of P_i defined by $c_{\epsilon_{(1,0,0)},\epsilon_{(0,1,0)},\epsilon_{(0,0,1)}}$ is equal to $(\epsilon_1 sign_{i,1}, \epsilon_2 sign_{i,2}, \epsilon_3 sign_{i,3})$, which in its turn is equal to the triple of sings of the homogeneous forms

$$\begin{array}{l} \epsilon_{(1,0,0)} l_{(1,0,0)} l_{(1,1,0)} l_{(1,0,1)} l_{(1,1,1)}, \\ \epsilon_{(0,1,0)} l_{(0,1,0)} l_{(1,1,0)} l_{(0,1,1)} l_{(1,1,1)}, \\ \epsilon_{(0,0,1)} l_{(0,0,1)} l_{(1,0,1)} l_{(0,1,1)} l_{(1,1,1)}. \end{array} \tag{9}$$

In what follows, a line arrangement $\mathcal{L}$ equipped with one of these **eight sign-labelings** is called *equipped* (*by signs*).

The sign-equipment of a (labelled) pure real Campedelli arrangement contains a complete information on the real structure, as the following proposition shows.

Proposition 2.7. *Let Campedelli line arrangements $\mathcal{L}$ and $\mathcal{L}'$ be pure real with respect to real structures $c_P : \mathbb{P}^2 \to \mathbb{P}^2$ and $c'_P : \mathbb{P}^2 \to \mathbb{P}^2$. A real structure $c : X(\mathcal{L}) \to X(\mathcal{L})$ lifting c_P and a real structure $c' : X(\mathcal{L}') \to X(\mathcal{L}')$ lifting c'_P are equivalent if, and only if, there exist a homomorphism $h : (\mathbb{Z}/2\mathbb{Z})^3 \to (\mathbb{Z}/2\mathbb{Z})^3$ and a projective transformation $H : \mathbb{P}^2 \to \mathbb{P}^2$ such that*

$$c'_P \circ H = H \circ c_P, \quad \phi' \circ H_* = h \circ \phi$$

(here $\phi : H_1(\mathbb{P}^2 \setminus \mathcal{L}; \mathbb{Z}) \to (\mathbb{Z}/2\mathbb{Z})^3$ and $\phi' : H_1(\mathbb{P}^2 \setminus \mathcal{L}'; \mathbb{Z}) \to (\mathbb{Z}/2\mathbb{Z})^3$ are the labelings participating in definition of $\mathcal{L}$ and $\mathcal{L}'$), and

$$Sign'(H(P_i)) = (-1)^{h(g_i)},$$

where

$$(-1)^{g_i} = Sign P_i.$$

Proof. It follows from Theorem 1.13 and the definition of the sign-triples (recall that one sign-triple determines all the other). □

Proposition 2.8. *The eight real structures $c_{\epsilon_{(1,0,0)},\epsilon_{(0,1,0)},\epsilon_{(0,0,1)}}$ are distinct. If $\mathcal{L}$ has no triple points, these eight real structures are the only reals structures of $X(\mathcal{L})$.*

Proof. There exist points in $Y(\mathcal{L})$ where all the three coordinates z_0, z_1, z_2 are real and all the three coordinates $u_{(1,0,0)}, u_{(0,1,0)}, u_{(0,0,1)}$ are nonzero. The real structures $c_{\epsilon_{(1,0,0)},\epsilon_{(0,1,0)},\epsilon_{(0,0,1)}}$ with different $(\epsilon_{(1,0,0)}, \epsilon_{(0,1,0)}, \epsilon_{(0,0,1)})$ act differently on such a point. It implies the first statement.

Now, assume that $\mathcal{L}$ has no triple points and consider two real structures, $c : X(\mathcal{L}) \to X(\mathcal{L})$ lifting c_P and $c' : X(\mathcal{L}) \to X(\mathcal{L})$ lifting c'_P. Assume that $\mathcal{L}$ is pure real with respect to c_P.

Let show, first, that $\mathcal{L}$ is pure real with respect to c'_P as well. Suppose that $\mathcal{L}$ is mixed real with respect to c'_P. Then, c'_P preserves three lines in $\mathcal{L}$ and two points which are intersections of conjugated lines in $\mathcal{L}$. On the other hand, c_P preserves all the lines in $\mathcal{L}$. Therefore, $c_P \circ c'_P$ acts trivially on three generic lines and two points outside them. Hence, $c_P \circ c'_P = \mathrm{id}$, so that $c_P = c'_P$. Which is a contradiction.

If $\mathcal{L}$ is pure real with respect to both c'_P and c_P, then the same argument implies that $c_P = c'_P$. Thus, c and c' differ by a Galois transformation. $\square$

Remark 2.9. *As follows from Propositions 2.7 and 2.8, if $\mathcal{L}$ is a pure real Campedelli arrangement without nontrivial projective automorphisms, then the eight real structures $c_{\epsilon_{(1,0,0)},\epsilon_{(0,1,0)},\epsilon_{(0,0,1)}}$ are non-equivalent to each other and they represent all the real structures on $X(\mathcal{L})$.*

Lemma 2.10. *For any choice of ϵ_α with $\alpha = (1,0,0)$, $(0,1,0)$, and $(0,0,1)$, the real point set $X_{\mathbb{R}} = Fix\, c$, $c = c_{\epsilon_{(1,0,0)},\epsilon_{(0,1,0)},\epsilon_{(0,0,1)}}$, is*

$$X_{\mathbb{R}} = \bigcup_{Sign_i=(+,+,+)} f^{-1}(P_i),$$

where $Sign_i$ are the sign-triples defined by c. $\square$

Assume that $\mathcal{L}$ has no triple points (in fact, one can treat in a similar way the degenerate cases, but we do not need it). Let P_{i_0} be a n-gon. For each its side and for each its vertex, there is one and only one polygon $P_i, i \neq i_0$, intersecting P_{i_0} along this side or, respectively, only at the vertex. Inspecting the sides and the vertices along the border of P_{i_0}, we obtain a sequence of polygons

$$(P_{i_1}, P'_{i_2}, \dots, P_{i_{2n-1}}, P'_{i_{2n}}),$$

where $P'_{\_}$ are the polygons adjacent to the vertices. Let us and associate with P_{i_0} an integer sequence $A_{i_0} = (n_{i_1}, n'_{i_2}, \dots, n_{i_{2n-1}}, n'_{i_{2n}})$, where n_{i_j} and n'_{i_j} state for the number of sides of P_{i_j} and, respectively, P'_{i_j}. The sequence A_{i_0} is called the *adjacency type* of P_{i_0}. The adjacency type is defined up to cyclic permutation and reversing the order.

Let finally $\mathcal{L}$ be equipped by signs and let $P_{i_1}, \dots, P_{i_k}$ be the set of positive polygons. The unordered collection $A(\mathcal{L}) = (A_{i_1}, \dots, A_{i_k})$, where A_{i_j} is the adjacency type of P_{i_j}, is called the *adjacency type of positive polygons.*

Lemma 2.11. *If $\mathcal{L}$ is a purely real Campedelli line arrangement without triple points, then any its sign-equipment contains at least seven different labels $Sign_i$.*

Proof. The arrangement $\mathcal{L}$, as any arrangement without triple points consisting of $\geqslant 5$ lines, defines at least five triangles P_i. Through a simple counting of edges and cells, it implies that in the case of seven lines there is a n-gon P_i with $n \geq 5$.

If P_i is a $\geqslant 6$-gon, then P_i and six polygons having a common side with P_i have all different triples of signs, as it follows from the transition rule (7).

Let P_i be a 5-gon bounded by $L_{i_1}, L_{i_3}, L_{i_5}, L_{i_7}$, and L_{i_9}, and let $(P_{i_1}, P'_{i_2}, \dots, P_{i_9}, P'_{i_{10}})$ be its sequence of adjacent polygons. As in the proof of Lemma 2.5, we can assume (maybe, after renumbering of lines and a cyclic permutation of adjacent polygons; note that a renumbering may change the sign-equipment but preserve distinct the distinct sign-triples) that $\alpha_{i_1} = (1,0,0)$, $\alpha_{i_3} = (0,1,0)$, $\alpha_{i_5} = (0,0,1)$ and

$$\alpha_{i_7}, \alpha_{i_9} \in \{(1,1,0), (1,0,1), (0,1,1), (1,1,1)\}.$$

By the transition rule (7), the sign-triples of P_i and its adjacent polygons form the set $\{(-1)^a Sign_i,\ a \in A\}$, where $A = \{0, \alpha_{i_1},\ \alpha_{i_3},\ \alpha_{i_5},\ \alpha_{i_7},\ \alpha_{i_9},\ \alpha_{i_1} + \alpha_{i_3},\ \alpha_3 + \alpha_{i_5},\ \alpha_{i_5} + \alpha_{i_7},\ \alpha_{i_7} + \alpha_{i_9},\ \alpha_{i_9} + \alpha_{i_1}\}$. We have $(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (0,1,1) \in A$, that is, A consists of at least six elements. If α_7 or α_9 is equal to $(1,0,1)$ or $(1,1,1)$, then A consists of at least seven elements. Otherwise, if $\{\alpha_7, \alpha_9\} = \{(1,1,0), (0,1,1)\}$, again A consists of at least seven elements, since in this case $(1,1,0) + (0,1,1) = (1,0,1) \in A$. $\square$

The following proposition follows from Lemma 2.10 and Lemma 2.11.

Proposition 2.12. *Let $\mathcal{L}$ be a purely real Campedelli line arrangement without triple points. For each real structure $c_{\epsilon_{(1,0,0)}, \epsilon_{(0,1,0)}, \epsilon_{(0,0,1)}}$ on $X = X(\mathcal{L})$ except, possibly, one, its real points set is non-empty.*

2.5. **Mixed real Campedelli line arrangements.** Let $\mathcal{L} = \cup L_\alpha$ be a Campedelli line arrangement which is mixed real with respect to a real structure $c_P : \mathbb{P}^2 \to \mathbb{P}^2$. Choose homogeneous coordinates (z_0, z_1, z_2) in $\mathbb{P}^2$ such that c_P turns in

$$c_P(z_0, z_1, z_2) = (\overline{z}_0, \overline{z}_1, \overline{z}_2).$$

Then, up to a renumbering and a real projective transformation, the lines $L_{(1,1,0)}$, $L_{(1,1,1)}$, and $L_{(0,0,1)}$ are given by equations $z_0 = 0$, $z_1 = 0$, and $z_2 = 0$, while the lines $L_{(1,0,0)}$, $L_{(0,1,0)}$, $L_{(1,0,1)}$, and $L_{(0,1,1)}$ are given by equations

$$a_{\alpha,0} z_0 + a_{\alpha,1} z_1 + a_{\alpha,2} z_2 = 0,$$

where $a_{(1,0,0),j} = \overline{a}_{(0,1,0),j}$ and $a_{(1,0,1),j} = \overline{a}_{(0,1,1),j}$ for any $j = 0, 1, 2$ (cf., the proof of Proposition 2.4).

As above, consider the set $\mathbb{RP}^2 = \{(z_0 : z_1 : z_2) \mid z_i \in \mathbb{R}\}$ of real points of $\mathbb{P}^2$. A mixed real Campedelli line arrangement $\mathcal{L}$ intersect $\mathbb{RP}^2$ along three distinct real lines $L_{\alpha,\mathbb{R}} = L_\alpha \cap \mathbb{RP}^2$, $\alpha = (0,0,1), (1,1,1)$, and $(1,1,0)$, and at two distinct real points $p_1 = L_{(1,0,0)} \cap L_{(0,1,0)}$ and $p_2 = L_{(1,0,1)} \cap L_{(0,1,1)}$. We call the points p_1 and p_2 the *vertices* of $\mathcal{L}$. The vertices can not belong to $L_{1,1,0}$, but it may happen that one of them (or both together) belong to $L_{1,1,1} \cup L_{0,0,1}$. There is a renumbering which exchange the indices of p_1 and p_2.

Denote by $l_\alpha = a_{\alpha,0}z_0 + a_{\alpha,1}z_1 + a_{\alpha,2}z_2$, $\alpha \in G \setminus \{0\}$, the above linear forms defining L_α. Put $q_1 = l_{(1,0,0)}l_{(0,1,0)}$ and $q_2 = l_{(1,0,1)}l_{(01,1)}$. Note that q_1 and q_2 have real coefficients. Moreover, $q_1 \geqslant 0$ and $q_2 \geqslant 0$ at each point of $\mathbb{RP}^2$.

The Campedelli surface $X = X(\mathcal{L})$ is given in $\mathbb{P}^9_w$ by equations

$$\begin{aligned}
u^2_{(1,0,0)} &= l_{(1,0,0)}l_{(1,0,1)}z_0z_1, \\
u^2_{(0,1,0)} &= l_{(0,1,0)}l_{(0,1,1)}z_0z_1, \\
u^2_{(0,0,1)} &= q_2z_1z_2, \\
u^2_{(1,1,0)} &= q_1q_2, \\
u^2_{(1,0,1)} &= l_{(1,0,0)}l_{(0,1,1)}z_0z_2, \\
u^2_{(0,1,1)} &= l_{(0,1,0)}l_{(1,0,1)}z_0z_2, \\
u^2_{(1,1,1)} &= q_1z_1z_2.
\end{aligned} \tag{10}$$

It inherits a real structure $c_{++} : X \to X$ from the real structure on $\mathbb{P}^9_w$ defined by $z_k \mapsto \bar{z}_k$ and $u_{(i,j,k)} \mapsto \bar{u}_{(j,i,k)}$.

In accordance with Proposition 2.4, there are three more real structures on X (only three, if the arrangement has no a nontrivial projective automorphism) which are obtained from c_{++} by composing it with Galois automorphisms. Namely, they are

$$c_{-+} = g_{(1,0,0)}c_{++}g_{(1,0,0)}, \quad c_{+-} = g_{(0,0,1)}c_{++}, \quad c_{--} = g_{(0,0,1)}c_{-+}, \tag{11}$$

where $g_{(1,0,0)}, g_{(0,0,1)} \in \mathrm{Gal}(X/\mathbb{P}^2)$ are defined as follows:

$$g_{(1,0,0)}u_{(i,j,k)} = (-1)^i u_{(i,j,k)}, \ g_{(0,0,1)}u_{(i,j,k)} = (-1)^k u_{(i,j,k)}.$$

In particular, one can notice that up to conjugation by automorphisms of X this list of four real structures reduces to two conjugacy classes represented, respectively, by $c_+ = c_{++}$ and $c_- = c_{+-}$.

We subdivide mixed real arrangements $\mathcal{L}$ having no triple points in three following types. The lines $L_{\alpha,\mathbb{R}}$, $\alpha = (0,0,1), (1,1,1), (1,1,0)$, divide $\mathbb{RP}^2$ into four triangles P_i, $i = 1, \dots, 4$, as it is depicted in Fig.2, where the axe $x = 0$ is the line $L_{(1,1,1),\mathbb{R}}$, the axe $y = 0$ is the line $L_{(0,0,1),\mathbb{R}}$, while the line $L_{(1,1,0),\mathbb{R}}$ is put at infinity. Using renumberings which transform $(1,0,0)$ in $(1,0,0)$, $(0,1,0)$ in $(0,1,0)$, and $(0,0,1)$ in

$(1,1,1)$ together with linear transformations $x \mapsto \pm x$, $y \mapsto \pm y$, we can and will assume that $p_1 \in P_1$ and p_2 belongs either to P_1 (**Type I**), or to P_2 (**Type II**), or to P_3 (**Type III**).

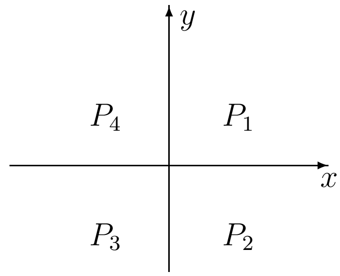


Fig. 2

Such a normalization makes the products $l_{(1,1,0)}l_{(1,1,1)} = z_0z_1$ and $l_{(1,1,0)}l_{(0,0,1)} = z_0z_2$ to be positive on P_1 (and on P_3) and, in particular, fixes a choice of c_+. Under this convention, c_- becomes the real structure induced by $z_k \mapsto \bar{z}_k$ and $u_{(i,j,k)} \mapsto \bar{u}_{(j,i,k)}$ on the copy of X which is given by

$$\begin{aligned} u^2_{(1,0,0)} &= l_{(1,0,0)})l_{(1,0,1)}v_1, \\ u^2_{(0,1,0)} &= l_{(0,1,0)}l_{(0,1,1)}v_1, \\ u^2_{(0,0,1)} &= -q_2v_1v_2. \end{aligned} \tag{12}$$

Lemma 2.13. *Let $\mathcal{L}$ be a mixed real Campedelli line arrangement without triple points. Suppose that P_i and $c_\pm$ are labelled as above. Then, for any $i = 1,\dots,4$,*

- (*i*) *$f^{-1}(P_i)$ is a disjoint union $P_{i,1} \cup P_{i,2}$ of two connected non-orientable two-manifolds,*
- (*ii*) *the Euler characteristic of $P_{i,j}$, $j = 1,2$, is equal to $1-2n$, where n is the number of vertices $\{p_1,p_2\}$ belonging to P_i,*
- (*iii*) *the real point set $X_{\mathbb{R}} = Fix\, c$, $c = c_\pm$, is*

$$X_{\mathbb{R}} = P_{i,1} \cup P_{i+2,1},$$

where $i = 1$ if $c = c_+$ and $i = 2$ if $c = c_-$.

Proof. It is similar to the proof of Lemma 2.6. The only difference is that here inside P_i we have vertices p_1, p_2 which are (simple) branching points of the projection $\widetilde{P}_{i,j} \to P_i$. $\square$

Remark 2.14. *A Campedelli line arrangement can be purely real with respect to one real structure and mixed real with respect to another one. More precisely, a Campedelli line arrangement $\mathcal{L}$ is simultaneously purely real and mixed real if and only if (maybe after renumbering of the lines) there are coordinates (z_0,z_1,z_2) in $\mathbb{P}^2$ such that: $z_i = 0$,*

$i = 0, 1, 2$, is an equation respectively of $L_{(1,1,0)}$, $L_{(1,1,1)}$, $L_{(0,0,1)}$; the lines $L_{(1,0,0)}$ and $L_{(0,1,0)}$ are given by $a_1z_1 + (a_0z_0 \pm z_2) = 0$, and the lines $L_{(1,0,1)}$ and $L_{(0,1,1)}$ are given by $b_1z_1 + (b_0z_0 \pm z_2) = 0$ for some non-zero a_0, a_1, b_0, $b_1 \in \mathbb{R}$.

3. Diffeomorphisms and deformations of real Campedelli surfaces

3.1. **Deformations and smoothing of A_1-singularities.** By a real Morse-Lefschetz perturbation of a real surface with A_1-singularities we mean a complex three-manifold Z with a real structure $c : Z \to Z$ equipped with a proper holomorphic map f from Z to the unit disc $D \subset \mathbb{C}$ respecting the real structures on Z and $D \subset \mathbb{C}$ and such that: all the fibers of f, except the fiber over 0, are (compact) nonsingular surfaces; the fiber over 0 contains only isolated singular points $O_1, \dots, O_k$, and the quadratic form of f at each of the singular points is non-degenerate. The fibers $f^{-1}(t)$ are denoted by X_t, so that the singular fiber $f^{-1}(0)$ is denoted by X_0. The real structure $c : X_0 \to X_0$ lifts to a unique real structure $c : \tilde{X}_0 \to \tilde{X}_0$ where $\tilde{X}_0$ is the minimal desingularization of X_0. According the definition of the deformation equivalence of real surfaces, for all $t \in \mathbb{R}, t \neq 0$, of the same sign the real surfaces (X_t, c) are of the same real deformation type. If $O_j, 1 \leqslant j \leqslant k$, is real then we pick a small (Milnor) ball $B_j \subset Z$ around O_j and, for every small real $t \neq 0$, speak on the local Euler characteristic of $X_{t,\mathbb{R}}$ which means the Euler characteristic of the intersection of the real part of X_t with B_j.

Such Morse-Lefschetz perturbations will arise by reversing of triangle of real Campedelli line arrangements, see subsection 3.2.

Lemma 3.1. *Let (Z, f, c) be a real Morse-Lefschetz perturbation of a real surface with A_1-singularities. If for $t' \neq 0$ of certain sign, at each real singular point $O_j \in X_0$ the local Euler characteristic of $X_{t',\mathbb{R}}$ is 0, then $(X_{t'}, c)$ with t' of this sign are real deformation equivalent to $(\tilde{X}_0, c)$.*

Proof. Introduce an auxiliary real one-parametric family by making the base change which substitutes u^2 instead of t if t' is positive, and $-u^2$ otherwise. The total space of this family has A_1-singularities at $O_1, \dots, O_k \in X_0$ and it has no any other singular point. Blowing up the total space at the A_1-singularities we respect the real structure, replace each of the singular points by a quadric, and resolve both the singular points of the family and the singular points of X_0. At each real point O_j, the blown-up quadric is real, and the two families of generating lines on this real quadric are real if, and only if, the local

Euler characteristic of $X_{t,\mathbb{R}}$ with $t = t'$ is 0. Pick a real family of lines at each of real O_j and conjugated families of lines at each pair of conjugated O_j. As is known, a contraction of any family of lines gives a smooth family. The contraction of the chosen families is real and thus provides a real deformation equivalence between $(\tilde{X}_0, c)$ and $(X_{t'}, c)$. □

Remark 3.2. *If (Z, f, c) is a Morse-Lefschetz perturbation of a real surface with complex conjugated (non real) A_1-singularities, then all (X_t, c) with real $t \neq 0$ are real deformation equivalent to each other.*

3.2. **Reversings of triangles.** Let $\mathcal{L}$ be an equipped purely real Campedelli line arrangement, see subsection 2.4, and let $P_{i_0} \subset \mathbb{RP}^2$ be a triangle whose sides are L_{α_1}, L_{α_2}, L_{α_3} belonging to $\mathcal{L}$. A modification depicted in Fig. 3 which turns $\mathcal{L}$ into an equipped purely real Campedelli line arrangement $\mathcal{L}'$ is called the *reversing of triangle* P_{i_0}. By definition, the sign-triples $\mathrm{Sign}'_i = \mathrm{Sign}(P'_i)$ with $i \neq i_0$ coincide with $\mathrm{Sign}_i = \mathrm{Sign}(P_i)$, while, in accordance with the transition rule (7), $\mathrm{Sign}'_{i_0} = (\mathrm{sign}'_{i_0,1}, \mathrm{sign}'_{i_0,2}, \mathrm{sign}'_{i_0,3})$ is determined by $\mathrm{Sign}_{i_0} = (\mathrm{sign}_{i_0,1}, \mathrm{sign}_{i_0,2}, \mathrm{sign}_{i_0,3})$ as follows:

$$\mathrm{sign}'_{i_0,j} = (-1)^{a_j} \mathrm{sign}_{i_0,j},$$

where $(a_1, a_2, a_3) = \alpha_1 + \alpha_2 + \alpha_3$.

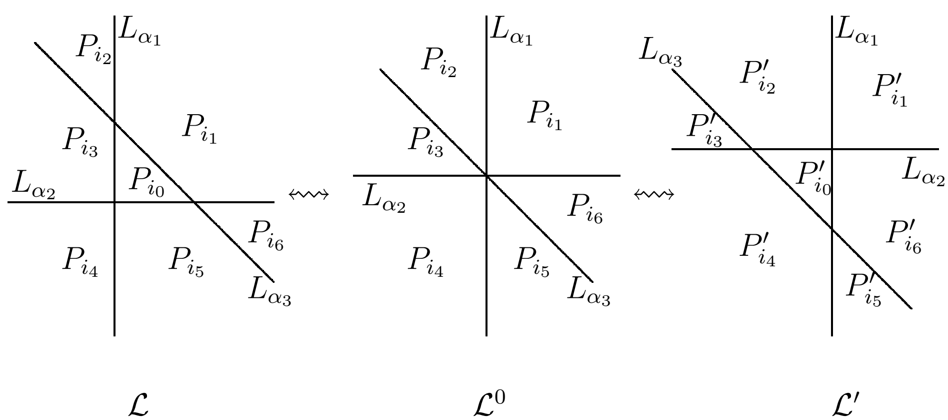


Fig. 3

Remark 3.3. *If the sides of P_{i_0} are linear dependent, then: $Sign_{i_j} = Sign_{i_{j+3}}$ for $j = 1, 2, 3$; $Sign'_{i_0} = Sign_{i_0}$; and $Sign_{i_1}$, $Sign_{i_2}$, and $Sign_{i_3}$ are pairwise distinct. In the case of linear dependent sides, $\mathcal{L}^0$ is not a Campedelli arrangement.*

If the sides of P_{i_0} are linear independent, then all the triples Sign_{i_j}, $j = 0, 1, \dots, 6$, are pairwise distinct and Sign'_{i_0} is the complementary element in the set of all triples of signs. In the case of linear independent sides, $\mathcal{L}^0$ is a Campedelli line arrangement and the canonical model $X(\mathcal{L}^0)$ of a Campedelli surface X^0 has a unique A_1-singular point over the triple point (the point to which P_{i_0} degenerates). The local Euler characteristic of $X(\mathcal{L})$ at this singular point is 0 if, and only if,

$$(+,+,+) \notin \{Sing_{i_k}\}_{k=0,2,4,6}.$$

Respectively, the local Euler characteristic of $X(\mathcal{L}')$ at this singular point is 0 if, and only if, $(+,+,+) \notin \{Sing'_{i_k}\}_{k=0,2,4,6}$. The last condition is equivalent to $(+,+,+) \in \{Sing_{i_k}\}_{k=0,2,4,6}$; in particular, if the local Euler characteristic is equal to 0 for one of $\mathcal{L}$ and $\mathcal{L}'$, it is not equal to 0 for the other one, and vise versa.

3.3. **Reduction to generic deformations.**

Lemma 3.4. *Suppose that (Z, f, c) is a real deformation such that all the fibers except X_0 have nonsingular canonical models, while the canonical model of X_0 is a surface with A_1-singularities. Then, at each singular point $O_j \in Z$ which is real the local Euler characteristic of $X_{t,\mathbb{R}}$, $t \neq 0$ is 0.*

Proof. The deformation (Z, f, c) is a simultaneous resolution of the singularities of the family constituted of the canonical models X_t^{can} of X_t and regarded over the same base. Hence, for each small real t the local Euler characteristics of $X_{t,\mathbb{R}}$ coincide with the local Euler characteristics of the resolution of the singular points. The latter characteristics are 0 in the case of A_1-singularities, whatever are the real forms of the singularities. □

Lemma 3.5. *Let (Z, f, c) be a real deformation of Campedelli surfaces. For any real $t' \in D$, there exist a real neighborhood $U \subset D$ of t' and a real family $\mathcal{L}_t, t \in U$, of Campedelli line arrangements in a real projective plane $(\mathbb{P}^2; c_P)$ such that $X_t = X(\mathcal{L}_t)$ and $c_t = c|_{X_t}$ are lifts of c_P.*

Proof. Consider the relative bi-canonical bundle $2K|_{Z/D}$. Its restriction to any fiber X_t is the bi-canonical bundle of X_t. The space of sections of such a restriction is of dimension three, and the sections determine a finite map to $\mathbb{P}^2$ representing X_t as $X(\mathcal{L}_t)$, where $\mathcal{L}_t$ is the branching locus of this map, see the proof of Theorem 1.13. Since the space of sections is of constant dimension, any three sections generating the bi-canonical bundle of $X_{t'}$ extend to three sections generating $2K|_{Z/D}$ at least over a small neighborhood of t'. By theorem 2.1, the three sections of the bi-canonical bundle of $X_{t'}$ can be chosen real with respect to a real structure c_P of $\mathbb{P}^2$, and then it remains to average their extensions by c and pick a sufficiently small equivariant neighborhood of t'. $\square$

Proposition 3.6. *Let (X_1, c_1) and (X_2, c_2) be two deformation equivalent real Campedelli surfaces associated respectively with Campedelli line arrangements $\mathcal{L}_1$ and $\mathcal{L}_2$. If $\mathcal{L}_1$ is purely real, then $\mathcal{L}_2$ is also purely real. If they are purely real and have no triple points, then their sign-equipments in $\mathbb{RP}^2$ are homeomorphic, so that, in particular, $\mathcal{L}_1$ and $\mathcal{L}_2$ have the same type and the same adjacency type of positive polygons.*

Note that this statement implies that a deformation can not provide any reversing of triangle.

Proof. By Lemma 3.5, a chain of real deformations connecting (X_1, c_1) and (X_2, c_2) is a result of a choice of a chain of real families of Campedelli line arrangements $\mathcal{L}_t$. We look at $\mathcal{L}_t$ with real values of t. It gives a chain of real Campedelli line arrangements connecting $\mathcal{L}_1$ and $\mathcal{L}_2$. Campedelli line arrangements have at worse triple points. Therefore, the number of real lines in an arrangement is not changing in a chain of real deformations. It proves the first statement.

Now assume that $\mathcal{L}_1$ and $\mathcal{L}_2$ are purely real and have no triple points. The triple points on intermediate arrangements $\mathcal{L}_t$ appear and disappear independently. Their appearance and disappearance befalls by triangle half-reversings: contracting and reappearing of triangles like in Fig. 3. The half-reversing provided by a reappearing triangle should turn back the local combinatorial structure and the local sign-equipment, since, on one hand, according to Lemma 3.4 the local input to the Euler characteristic of the real part should be 0 for both types of half-reversings, and, on the other hand, as we observed already in subsection 3.2, such an input due to a contracting triangle P_{i_0} (or, respectively, to a turning back triangle P'_{i_0}) is 0 if, and only if, $(+,+,+) \notin \{Sing_{i_k}\}_{k=0,2,4,6}$ (respectively, $(+,+,+) \notin \{Sing'_{i_k}\}_{k=0,2,4,6}$). Finally, it implies that replacing the straight lines $L_{i,\mathbb{R}}$ by suitable flexible lines one can connect $\mathcal{L}_1$ and $\mathcal{L}_2$ by a continuous family of equipped flexible configurations in $\mathbb{RP}^2$ without triple points. $\square$

3.4. **Smoothing of $T(-4)$ singularities.** By *a real smoothing* of a real surface (M,c) we mean any real fiber of a real flat family of surfaces $Z\to D$ over the unit disc D (where the real structure on D is given by the usual complex conjugation) such that $(X_0,c)=(M,c)$ and X_t is nonsingular for any $t\in D, t\neq 0$. A singular point of a surface is called $T(-4)$-*singularity* if its germ is isomorphic to the $(\mathbb{Z}/2\mathbb{Z})^2$-Galois covering of the germ $(\mathbb{C}^2,0)$ branched in three lines $L_{\alpha_1}\cup L_{\alpha_2}\cup L_{\alpha_3}$ through 0 with a $(\mathbb{Z}/2\mathbb{Z})^2$ labeling $\{\alpha_i\}_{i=1,2,3}$ such that $\alpha_1+\alpha_2+\alpha_3=0$. We speak on *a real surface with non real* $T(-4)$-*singularities*, if all the singular points of the surface are $T(-4)$-singularities and neither of the singular points is real.

Theorem 3.7. *Any two real smoothings (M_1,c) and (M_2,c) of a real surface (M,c) with non real $T(-4)$ singularities have diffeomorphic real structures.*

Proof. The pairs (M_1,c) and (M_2,c) are obtained from (M,c) by removing c-invariant Milnor neighborhoods $U_j\cup c(U_j)$ of the each pair of conjugated singularities followed by a c-invariant gluing of some standard pieces $N_j\cup N_{\bar j}$, $N_j=(N,j)$ and $N_{\bar j}=(N,\bar j)$, instead of $U_j\cup c(U_j)$ by means of some boundary diffeomorphisms $\phi_j:\partial N\to\partial U_j, \phi_{\bar j}:\partial N\to\partial c(U_j)$ such that $c\circ\phi_j=\phi_{\bar j}$ (so that c acts on $N_j\cup N_{\bar j}$ by $(x,j)\mapsto(x,\bar j)$). As is shown, for example, in [18], the result of gluing of the half of these pieces, say $\cup N_j$, gives diffeomorphic four-manifolds $M_1\setminus\bigcup_j N_{\bar j}$ and $M_2\setminus\bigcup_j N_{\bar j}$ (in fact, ∂N is a lens space $L(4,1)$ and the existence of such a diffeomorphism follows from a corresponding Bonahon theorem, see [1]). Now, it remains to extend such a diffeomorphism Φ to $M_1\to M_2$ by symmetry, that is by taking $\Phi(x)=(c\circ\Phi)(x)$ for each $x\in(N,\bar j)$. □

The following corollary is straightforward.

Corollary 3.8. *Let $\mathcal{L}$ and $\mathcal{L}'$ be two equipped real Campedelli line arrangements related by a reversing of triangle P_{i_0}. Suppose that the sides of P_{i_0} are linear dependent and that $Sign_{i_j}\neq(+,+,+)$ for $j=0,1,\dots,6$, where P_{i_j}, $j=1,\dots,6$, are the polygons adjacent to P_{i_0}. Then the real Campedelli surfaces $X(\mathcal{L})$ and $X(\mathcal{L}')$ have diffeomorphic real structures.* □

Remark 3.9. *In Theorem 3.7 and Corollary 3.8 the diffeomorphism carrying one real structure into another preserves both the orientation and the canonical class.*

Proof. The orientation and the canonical class of $M = X_0$, $M_1 = X_\epsilon$, and $M_2 = X_{-\epsilon}$ are determined by the complex structure. The identification of $M \setminus \cup U_j$ with the corresponding pieces of $X_{\pm\epsilon}$ is given by Morse-Lefschetz diffeomorphisms. Since the complex structure on X_t depends continuously on $t \neq 0$, the Morse-Lefschetz diffeomorphisms preserve the complex orientation and the canonical class. Therefore, Φ restricted to $M_1^{ext} = M_1 \setminus \bigcup_j (N_j \cup N_{\bar{j}}) \to M_2^{ext} = M_2 \setminus \bigcup_j (N_j \cup N_{\bar{j}})$ preserves the complex orientation and transforms $K(M_2^{ext})$ into $K(M_1^{ext})$. It remains to notice that the homomorphisms $H^2(M_i, \mathbb{Z}) \to H^2(M_i^{ext}; \mathbb{Z})$ induced by inclusions $M_i^{ext} \subset M_i$ are injective. Indeed, by Poincaré-Lefschetz duality, $H^2(M_i, M_i^{ext}; \mathbb{Z}) = H_2(\bigcup_j (N_j \cup N_{\bar{j}}); \mathbb{Z})$, while $H_2(\bigcup_j (N_j \cup N_{\bar{j}}); \mathbb{Z}) = 0$, since each (of pairwise disjoint) N_j and $N_{\bar{j}}$ is homotopy equivalent to $\mathbb{RP}^2$, see Corollary 2.6. $\square$

3.5. **Classification of mixed real Campedelli line arrangements up to real deformations.** Let $\mathcal{L}$ be a Campedelli line arrangement which is mixed real with respect to a real structure $c_P : \mathbb{P}^2 \to \mathbb{P}^2$. We say that a real Campedelli surface (X, c_X), where $X = X(\mathcal{L})$ and c_X is a lift of c_P, has the *type* $J_\pm$, where $J = I$, II, or III, if: $\mathcal{L}$ is without triple points; it has the type J; and $c_X = c_\pm$ (see subsection 2.5 for notation I, II, III and $c_\pm$).

Theorem 3.10. *There are exactly five different types of deformation equivalent real Campedelli surfaces (X, c) associated with mixed real Campedelli line arrangements. Arrangements of type II_+ represent the same deformation type as the arrangements of type II_-. The other four deformation types are provided by arrangements of types $I_\pm$ and $III_\pm$.*

Proof. According to Proposition 3.6, $X(\mathcal{L})$ and $X(\mathcal{L}')$ are not deformation equivalent if $\mathcal{L}$ is a purely real Campedelli arrangement and $\mathcal{L}'$ is a mixed real one. By Lemma 3.1, if $\mathcal{L}$ has triple points, the surface (X, c_X) is real deformation equivalent to a surface associated with a mixed real Campedelli line arrangement without triple points. Therefore, there exist at most six types of deformation non-equivalent real Campedelli surfaces corresponding to six types of mixed real Campedelli arrangements: $I_\pm$, $II_\pm$, and $III_\pm$. To distinguish them, notice that a real deformation of a real Campedelli surface (X, c_X) is simultaneously a $H = \mathrm{Kl}(X, c_P)$-deformation, in a sense that not only the action of c_X but the action of the whole group H extends to the total space of the deformation. Moreover, since the Galois group $G \subset H$ preserves each fiber of the deformation, the real deformation of (X, c_X)

is simultaneously a real deformation for each of the other real structures contained in H.

In the case of mixed real Campedelli arrangements, H is a quaternion group (see Proposition 2.4), it contains four distinct real structures, and they split in two conjugacy classes $c_\pm$ (see subsection 2.5). Finally, the topological type of the unordered pair of two-manifolds $(\mathrm{Fix} c_+, \mathrm{Fix} c_-)$ is invariant under real deformations of (X, c_X). Lemma 2.13 implies this invariant distinguishes the types $I_\pm$, II_+, and $III_\pm$.

To finish the proof, let us show that the types II_+ and II_- are deformation equivalent. Up to deformation equivalence, we can assume that the vertices p_1 and p_2 of an arrangement $\mathcal{L}$ of type II have projective coordinates $(1,1,1)$ and, respectively, $(1,1,-1)$. Moreover, we can assume that $l_{(1,0,0)}l_{(0,1,0)} = (z_1 - z_0)^2 + (z_2 - z_0)^2$ and, respectively, $l_{(1,0,1)}l_{(0,1,1)} = (z_1 - z_0)^2 + (z_2 + z_0)^2$. Then, the diagonal transformation $z_0 \mapsto z_0$, $z_1 \mapsto z_1$, $z_2 \mapsto -z_2$ gives rise (see equations (10)) to an equivalence between the real structures c_- and c_+ after renumbering $(1,0,0) \mapsto (1,0,1)$, $(0,1,0) \mapsto (0,1,1)$, and $(0,0,1) \mapsto (0,0,1)$. $\square$

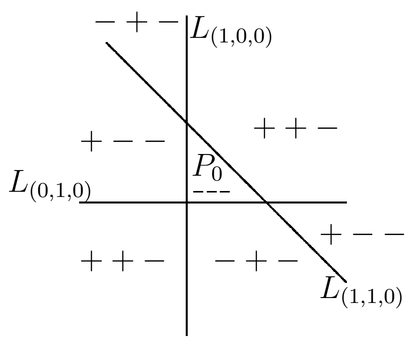
4. "Dif $\neq$ Def"

In this Section we give several examples of diffeomorphic real Campedelli surfaces which are not deformation equivalent.

4.1. **Example of a pair of diffeomorphic, but deformation non-equivalent real surfaces.**

Example 4.1. *Two real Campedelli surfaces which have diffeomorphic real structures but which are not real deformation equivalent.*

Let $\mathcal{L}$ be a purely real Campedelli line arrangement defined by the sides of a 7-gone P_1, that is an arrangement of type $(7,14,0,0,1)$. Label the lines in a way that three consecutive sides of P_1 be labelled by $(1,0,0),(1,1,0)$, and $(0,1,0)$. Then, we equip $\mathcal{L}$ by signs so that the triangle P_0 having a common side with P_1 along $L_{(1,1,0)}$ has the signs $Sign_0 = (-,-,-)$. Recall that this choice can be extended to a sign-equipment of $\mathcal{L}$ following transition rule (7), see a fragment of $\mathcal{L}$ in Fig. 4.



$\mathcal{L}$

Fig. 4

Let $\mathcal{L}'$ be a sign-equipped arrangement obtained by the reversing the triangle P_0. This arrangement is of type type $(7, 13, 1, 1, 0)$. By Corollary 3.8, the real Campedelli surfaces $X(\mathcal{L})$ and $X(\mathcal{L}')$ have diffeomorphic real structures, and by Proposition 3.6 they are not deformation equivalent.

4.2. **Example of eight diffeomorphic, but pair-wise deformation non-equivalent real surfaces.**

Example 4.2. *Eight real Campedelli surfaces* (X_1, c_1), *…*, (X_8, c_8) *which have diffeomorphic to each other real structures and which are pairwise non-deformation equivalent.*

Similar to Example Example 4.1, we search for equipped purely real Campedelli line arrangements $\mathcal{L}_i$, $i = 1, \dots, 8$, such that, first, they are related be sequences of reversings of non-positive triangles with linear dependent sides, and, second, they differ by their types or the adjacency types of their positive polygons. Then, by Theorem 3.7, the real Campedelli surfaces $X(\mathcal{L}_i)$ have diffeomorphic real structures, and by Corollary 3.6, they are pairwise non-deformation equivalent.

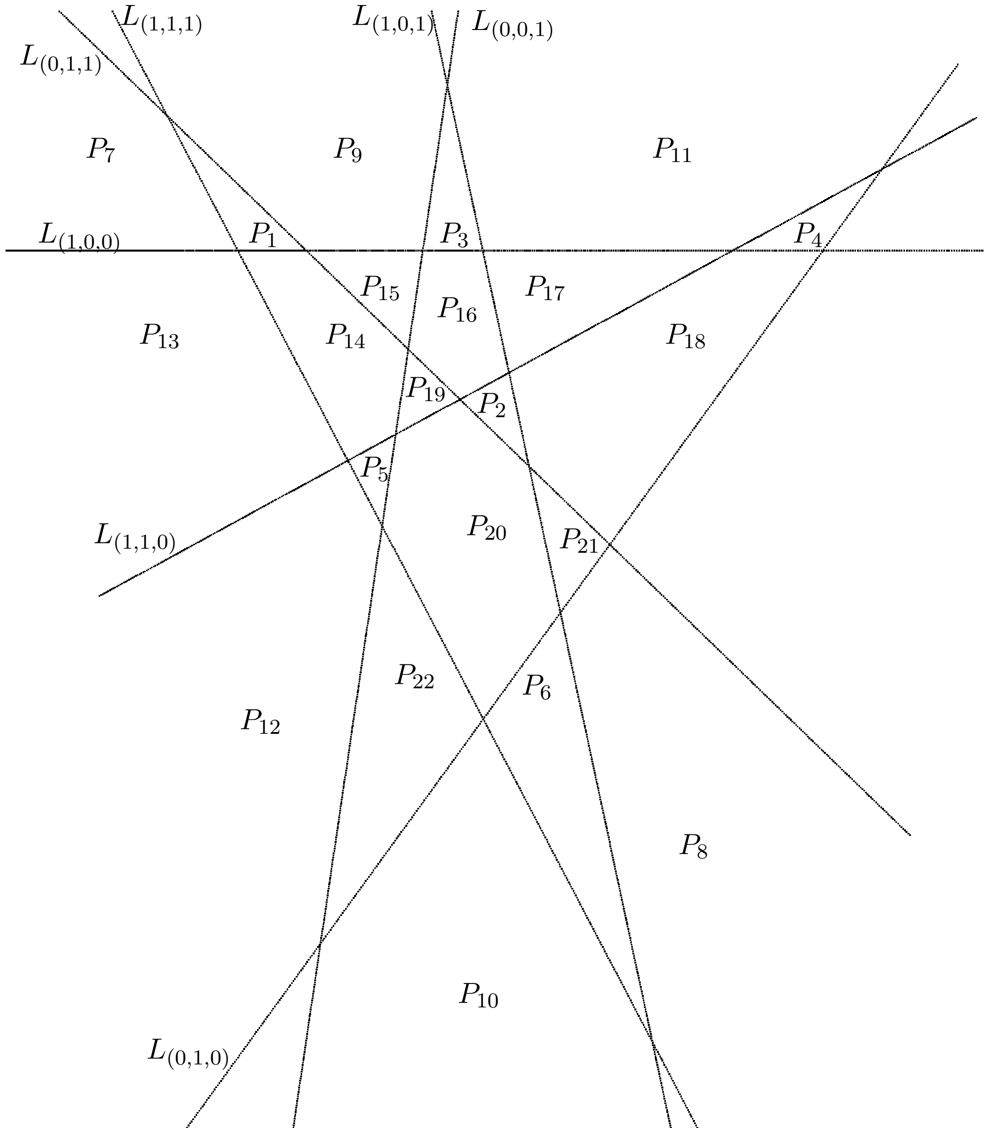


Fig. 5

To construct such arrangements, start from a purely real Campedelli line arrangement $\mathcal{L}_{(0,0,0,0)}$ of type $(11, 5, 5, 1, 0)$ depicted in Fig 5. This arrangement has six pairwise disjoint triangles $P_1, \dots, P_6$. Each of them has linear dependent sides. The number of sides for each of $P_7, \dots, P_{22}$ is given in Table $(0, 0, 0, 0)_1$.

P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
4	4	5	4	5	4	4	5

P_{15}	P_{16}	P_{17}	P_{18}	P_{19}	P_{20}	P_{21}	P_{22}
3	5	3	5	3	6	3	3

Table $(0,0,0,0)_1$

Sign the triangle P_1 by $(+,+,+)$ and extend this choice to a sign-equipment of $\mathcal{L}_{(0,0,0,0)}$ following the transition rule (7). Then, as is easy to check, $\mathcal{L}_{(0,0,0,0)}$ has only two positive polygons, namely P_1 and P_2.

To insure a possibility to perform independent reversings of the four triangles $P_3, \dots, P_6$ it is sufficient to consider $\mathcal{L}_{(0,0,0,0)}$ as a small perturbation of a degenerate configuration shown in Fig. 6. Now, it remains to select the reversings and to count for each configuration its type and the adjacency type of its positive polygons.

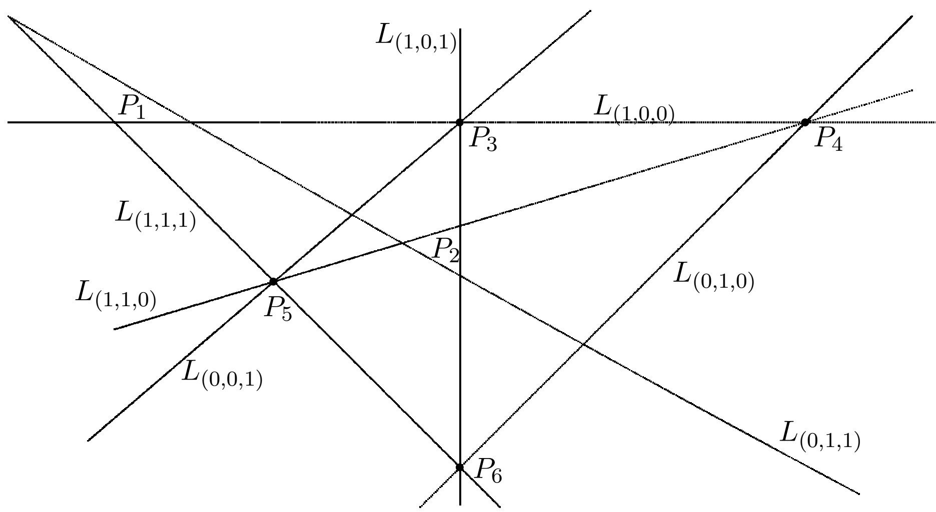


Fig. 6

Before, for convenience in further computations, we collect in Table $(0,0,0,0)_2$ the adjacency types of the triangles $P_1, \dots, P_6$ of $\mathcal{L}_{(0,0,0,0)}$.

P_1	$(4_7, 4'_8, 5_9, 3'_{15}, 5_{14}, 4'_{13})$		P_2	$(5_{16}, 3'_{17}, 5_{18}, 3'_{21}, 6_{20}, 3'_{19})$
P_3	$(5_9, 4'_{10}, 5_{11}, 3'_{17}, 5_{16}, 3'_{15})$		P_4	$(5_{11}, 4'_{12}, 4_{13}, 4'_7, 5_{18}, 3'_{17})$
P_5	$(4_{12}, 4'_{13}, 5_{14}, 3'_{19}, 6_{20}, 3'_{22})$		P_6	$(6_{20}, 3'_{21}, 4_8, 5'_9, 4_{10}, 3'_{22})$

Table $(0,0,0,0)_2$

(Here, we include in the adjacency type of the triangle P_i, $i = 1, \dots, 6$, the indices of the adjacent polygons. For example, in the adjacency type $(4_7, 4'_8, 5_9, 3'_{15}, 5_{14}, 4'_{13})$ of P_1 the pattern 4_7 points out that the polygon P_7 having four sides has a common side with the triangle P_1.)

The adjacency type of the positive polygons of $\mathcal{L}_{(0,0,0,0)}$ is equal to

$$A_{(0,0,0,0)} = ((4, 4', 5, 3', 5, 4'), (5, 3', 5, 3', 6, 3')).$$

Perform in $\mathcal{L}_{(0,0,0,0)}$ the reversing of P_3. We obtain a new equipped purely real Campedelli line arrangement. We denote it by $\mathcal{L}_{(1,0,0,0)}$ and we keep to denote its polygons (denoted by P'_i in subsection 3.2) by P_i. To count its invariants, we notice, first, that the adjacency type of P_3 changes as follows:

$$(5_9, 4'_{10}, 5_{11}, 3'_{17}, 5_{16}, 3'_{15}) \mapsto (4'_9, 5_{10}, 4'_{11}, 4_{17}, 4'_{16}, 4_{15}).$$

After that, we adjust the number of sides of $P_9, P_{10}, P_{11}, P_{17}, P_{16}, P_{15}$ given in Tables $(0, 0, 0, 0)_1$ and $(0, 0, 0, 0,)_2$ and obtain the tables for $\mathcal{L}_{(1,0,0,0)}$: Table $(1, 0, 0, 0)_1$ and Table $(1, 0, 0, 0)_2$.

P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
4	4	4	5	4	4	4	5
P_{15}	P_{16}	P_{17}	P_{18}	P_{19}	P_{20}	P_{21}	P_{22}
4	4	4	5	3	6	3	3

Table $(1, 0, 0, 0)_1$

P_1	$(4_7, 4'_8, 4_9, 4'_{15}, 5_{14}, 4'_{13})$		P_2	$(4_{16}, 4'_{17}, 5_{18}, 3'_{21}, 6_{20}, 3'_{19})$
P_3	$(4'_9, 5_{10}, 4'_{11}, 4_{17}, 4'_{16}, 4_{15})$		P_4	$(4_{11}, 4'_{12}, 4_{13}, 4'_7, 5_{18}, 4'_{17})$
P_5	$(4_{12}, 4'_{13}, 5_{14}, 3'_{19}, 6_{20}, 3'_{22})$		P_6	$(6_{20}, 3'_{21}, 4_8, 4'_9, 5_{10}, 3'_{22})$

Table $(1, 0, 0, 0)_2$

We conclude that: $\mathcal{L}_{(1,0,0,0)}$ has the type $(9, 9, 3, 1, 0)$; it contains two and only two positive polygons, namely P_1 and P_2; and the adjacency type of its positive polygons is equal to

$$A_{(1,0,0,0)} = ((4, 4', 4, 4', 5, 4'), (4, 4', 5, 3', 6, 3')).$$

Perform in $\mathcal{L}_{(1,0,0,0)}$ the reversing of P_4. Denote the new equipped purely real Campedelli line arrangement by $\mathcal{L}_{(1,1,0,0)}$ and proceed as before. As a result, we obtain two tables for $\mathcal{L}_{(1,1,0,0)}$: Table $(1, 1, 0, 0)_1$ and Table $(1, 1, 0, 0)_2$.

P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
5	4	4	5	3	5	3	5

P_{15}	P_{16}	P_{17}	P_{18}	P_{19}	P_{20}	P_{21}	P_{22}
4	4	5	4	3	6	3	3

Table $(1,1,0,0)_1$

P_1	$(5_7, 4'_8, 4_9, 4'_{15}, 5_{14}, 3'_{13})$		P_2	$(4_{16}, 5'_{17}, 4_{18}, 3'_{21}, 6_{20}, 3'_{19})$
P_3	$(4'_9, 5_{10}, 3'_{11}, 5_{17}, 4'_{16}, 4_{15})$		P_4	$(3'_{11}, 5_{12}, 3'_{13}, 5_7, 4'_{18}, 5_{17})$
P_5	$(5_{12}, 3'_{13}, 5_{14}, 3'_{19}, 6_{20}, 3'_{22})$		P_6	$(6_{20}, 3'_{21}, 4_8, 4'_9, 5_{10}, 3'_{22})$

Table $(1,1,0,0)_2$

We conclude that: $\mathcal{L}_{(1,1,0,0)}$ has the type $(11,5,5,1,0)$; it contains two and only two positive polygons, namely P_1 and P_2; and the adjacency type of its positive polygons is equal to

$$A_{(1,1,0,0)} = ((5,4',4,4',5,3'),(4,5',4,3',6,3')).$$

Perform in $\mathcal{L}_{(1,1,0,0)}$ the reversing of P_3. Denote the new equipped purely real Campedelli line arrangement by $\mathcal{L}_{(0,1,0,0)}$ and proceed as before. As a result, we obtain two tables for $\mathcal{L}_{(0,1,0,0)}$: Table $(0,1,0,0)_1$ and Table $(0,1,0,0)_2$.

P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
5	4	5	4	4	5	3	5

P_{15}	P_{16}	P_{17}	P_{18}	P_{19}	P_{20}	P_{21}	P_{22}
3	5	4	4	3	6	3	3

Table $(0,1,0,0)_1$

P_1	$(5_7, 4'_8, 5_9, 3'_{15}, 5_{14}, 3'_{13})$		P_2	$(5_{16}, 4'_{17}, 4_{18}, 3'_{21}, 6_{20}, 3'_{19})$
P_3	$(5_9, 4'_{10}, 4_{11}, 4'_{17}, 5_{16}, 3'_{15})$		P_4	$(4'_{11}, 5_{12}, 3'_{13}, 5_7, 4'_{18}, 4_{17})$
P_5	$(5_{12}, 3'_{13}, 5_{14}, 3'_{19}, 6_{20}, 3'_{22})$		P_6	$(6_{20}, 3'_{21}, 4_8, 5'_9, 4_{10}, 3'_{22})$

Table $(0,1,0,0)_2$

We conclude that: $\mathcal{L}_{(0,1,0,0)}$ has the type $(11,5,5,1,0)$; it contains two and only two positive polygons, namely P_1 and P_2; and the adjacency type of its positive polygons is equal to

$$A_{(0,1,0,0)} = ((5,4',5,3',5,3'),(5,4',4,3',6,3')).$$

Perform in $\mathcal{L}_{(0,1,0,0)}$ the reversing of P_5. Denote the new equipped purely real Campedelli line arrangement by $\mathcal{L}_{(0,1,1,0)}$ and proceed as before. As a result, we obtain two tables for $\mathcal{L}_{(0,1,1,0)}$: Table $(0,1,1,0)_1$ and Table $(0,1,1,0)_2$.

P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
5	4	5	4	4	4	4	4
P_{15}	P_{16}	P_{17}	P_{18}	P_{19}	P_{20}	P_{21}	P_{22}
3	5	4	4	4	5	3	4

Table $(0,1,1,0)_1$

P_1	$(5_7,4'_8,5_9,3'_{15},4_{14},4'_{13})$		P_2	$(5_{16},4'_{17},4_{18},3'_{21},5_{20},4'_{19})$
P_3	$(5_9,4'_{10},4_{11},4'_{17},5_{16},3'_{15})$		P_4	$(4'_{11},4_{12},4'_{13},5_7,4'_{18},4_{17})$
P_5	$(4'_{12},4_{13},4'_{14},4_{19},5'_{20},4_{22})$		P_6	$(5_{20},3'_{21},4_8,5'_9,4_{10},4'_{22})$

Table $(0,1,1,0)_2$

We conclude that: $\mathcal{L}_{(0,1,1,0)}$ has the type $(8,10,4,0,0)$; it contains two and only two positive polygons, namely P_1 and P_2; and the adjacency type of its positive polygons is equal to

$$A_{(0,1,1,0)} = ((5,4',5,3',4,4'),(5,4',4,3',5,4')).$$

Perform in $\mathcal{L}_{(0,1,1,0)}$ the reversing of P_4. Denote the new equipped purely real Campedelli line arrangement by $\mathcal{L}_{(0,0,1,0)}$ and proceed as above. As a result, we obtain two tables for $\mathcal{L}_{(0,0,1,0)}$: Table $(0,0,1,0)_1$ and Table $(0,0,1,0)_2$.

P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
4	4	5	4	5	3	5	4
P_{15}	P_{16}	P_{17}	P_{18}	P_{19}	P_{20}	P_{21}	P_{22}
3	5	3	5	4	5	3	4

Table $(0,0,1,0)_1$

P_1	$(4_7, 4'_8, 5_9, 3'_{15}, 4_{14}, 5'_{13})$		P_2	$(5_{16}, 3'_{17}, 5_{18}, 3'_{21}, 5_{20}, 4'_{19})$
P_3	$(5_9, 4'_{10}, 5_{11}, 3'_{17}, 5_{16}, 3'_{15})$		P_4	$(5_{11}, 3'_{12}, 5_{13}, 4'_7, 5_{18}, 3'_{17})$
P_5	$(3'_{12}, 5_{13}, 4'_{14}, 4_{19}, 5'_{20}, 4_{22})$		P_6	$(5_{20}, 3'_{21}, 4_8, 5'_9, 4_{10}, 4'_{22})$

Table $(0, 0, 1, 0)_2$

We conclude that: $\mathcal{L}_{(0,0,1,0)}$ has the type $(10, 6, 6, 0, 0)$; it contains two and only two positive polygons, namely P_1 and P_2; and the adjacency type of its positive polygons is equal to

$$A_{(0,0,1,0)} = ((4, 4', 5, 3', 4, 5'), (5, 3', 5, 3', 5, 4')).$$

Perform in $\mathcal{L}_{(0,0,1,0)}$ the reversing of P_3. Denote the new equipped purely real Campedelli line arrangement by $\mathcal{L}_{(1,0,1,0)}$ and proceed as before. As a result, we obtain two tables for $\mathcal{L}_{(1,0,1,0)}$: Table $(1, 0, 1, 0)_1$ and Table $(1, 0, 1, 0)_2$.

P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
4	4	4	5	4	3	5	4
P_{15}	P_{16}	P_{17}	P_{18}	P_{19}	P_{20}	P_{21}	P_{22}
4	4	4	5	4	5	3	4

Table $(1, 0, 1, 0)_1$

P_1	$(4_7, 4'_8, 4_9, 4'_{15}, 4_{14}, 5'_{13})$		P_2	$(4_{16}, 4'_{17}, 5_{18}, 3'_{21}, 5_{20}, 4'_{19})$
P_3	$(4'_9, 5_{10}, 4'_{11}, 4_{17}, 4'_{16}, 4_{15})$		P_4	$(4_{11}, 3'_{12}, 5_{13}, 4'_7, 5_{18}, 4'_{17})$
P_5	$(3'_{12}, 5_{13}, 4'_{14}, 4_{19}, 5'_{20}, 4_{22})$		P_6	$(5_{20}, 3'_{21}, 4_8, 4'_9, 5_{10}, 4'_{22})$

Table $(1, 0, 1, 0)_2$

We conclude that: $\mathcal{L}_{(1,0,1,0)}$ has the type $(8, 10, 4, 0, 0)$; it contains two and only two positive polygons, namely P_1 and P_2; and the adjacency type of its positive polygons is equal to

$$A_{(1,0,1,0)} = ((4, 4', 4, 4', 4, 5'), (4, 4', 5, 3', 5, 4')).$$

Finally, perform in $\mathcal{L}_{(1,0,1,0)}$ the reversing of P_6, denote the new equipped purely real Campedelli line arrangement $\mathcal{L}_{(1,0,1,1)}$ and, once more, proceed as before. As a result, we obtain two tables for $\mathcal{L}_{(1,0,1,1)}$: Table $(1, 0, 1, 1)_1$ and Table $(1, 0, 1, 1)_2$.

P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
4	3	5	4	4	3	5	4

P_{15}	P_{16}	P_{17}	P_{18}	P_{19}	P_{20}	P_{21}	P_{22}
4	4	4	5	4	4	4	5

Table $(1,0,1,1)_1$

P_1	$(4_7, 3'_8, 5_9, 4'_{15}, 4_{14}, 5'_{13})$		P_2	$(4_{16}, 4'_{17}, 5_{18}, 4'_{21}, 4_{20}, 4'_{19})$
P_3	$(5'_9, 4_{10}, 4'_{11}, 4_{17}, 4'_{16}, 4_{15})$		P_4	$(4_{11}, 3'_{12}, 5_{13}, 4'_7, 5_{18}, 4'_{17})$
P_5	$(3'_{12}, 5_{13}, 4'_{14}, 4_{19}, 4'_{20}, 5_{22})$		P_6	$(4'_{20}, 4_{21}, 3'_8, 5_9, 4'_{10}, 5_{22})$

Table $(1,0,1,1)_2$

We conclude that: $\mathcal{L}_{(1,0,1,1)}$ has the type $(8,10,4,0,0)$; it has two and only two positive polygons, namely P_1 and P_2; and the adjacency types of its positive polygons is equal to

$$A_{(1,0,1,1)} = ((4,3',5,4',4,5'),(4,4',5,4',4,4')).$$

The results obtained show that each two of the eight constructed arrangements either have different types or if their types coincide, they have different adjacency types of their positive polygons.

4.3. **Mixed real arrangements.**

Example 4.3. *Real Campedelli surfaces of types I_- and III_+ have diffeomorphic real structures, while they are not deformation equivalent.*

Indeed, let (X, c) be a real Campedelli surface of type III_+ associated with a mixed real Campedelli line arrangement of type III. We can move the vertex p_2 so that it goes from the triangle P_3 to P_1 through the line at infinity, $L_{(1,1,0)}$. Theorem 3.7 applies and shows that the real structures c and c_1 are diffeomorphic.

On the other hand, by Theorem 3.10, real surfaces of type I_- are not deformation equivalent to real surfaces of type III_+. □

In fact, in the case of mixed real types one can get a complete answer to the Dif≠Def. As it follows from the next theorem and Theorem 3.10, the number of Dif classes is four, and the number of Def classes is five.

Theorem 4.4. *The real structures of types $I_\pm$, II_+, and III_- are pairwise non-diffeomorphic.*

Proof. As it follows from Lemma 2.13, their real points sets have different topological types. □

5. Final remarks

5.1. **A pre-maximal surface.** Recall that a real surface (X, c) is called an $(M-k)$-surface, if $\dim H_*(X_{\mathbb{R}}; \mathbb{Z}/2\mathbb{Z}) = \dim H_*(X_{\mathbb{C}}; \mathbb{Z}/2\mathbb{Z} \bigotimes) - 2k$.

One can show that there are no M- and $(M-1)$-surfaces among real Campedelli surfaces. As seems for us, the following $(M-2)$-surface is of certain interest.

Let $\mathcal{L}$ be the equipped purely real Campedelli line arrangement depicted in Fig. 7. Its type is $(7, 14, 0, 0, 1)$ and it has three and only three positive polygons: the 7-gon P_1 and two quadrangles, P_2 and P_3, with the sides $L_{(1,1,0)}$, $L_{(0,1,1)}$, $L_{(0,0,1)}$, and $L_{(1,1,1)}$ for P_2, and $L_{(0,1,0)}$, $L_{(1,0,0)}$, $L_{(0,1,1)}$, and $L_{(1,0,1)}$ for P_3.

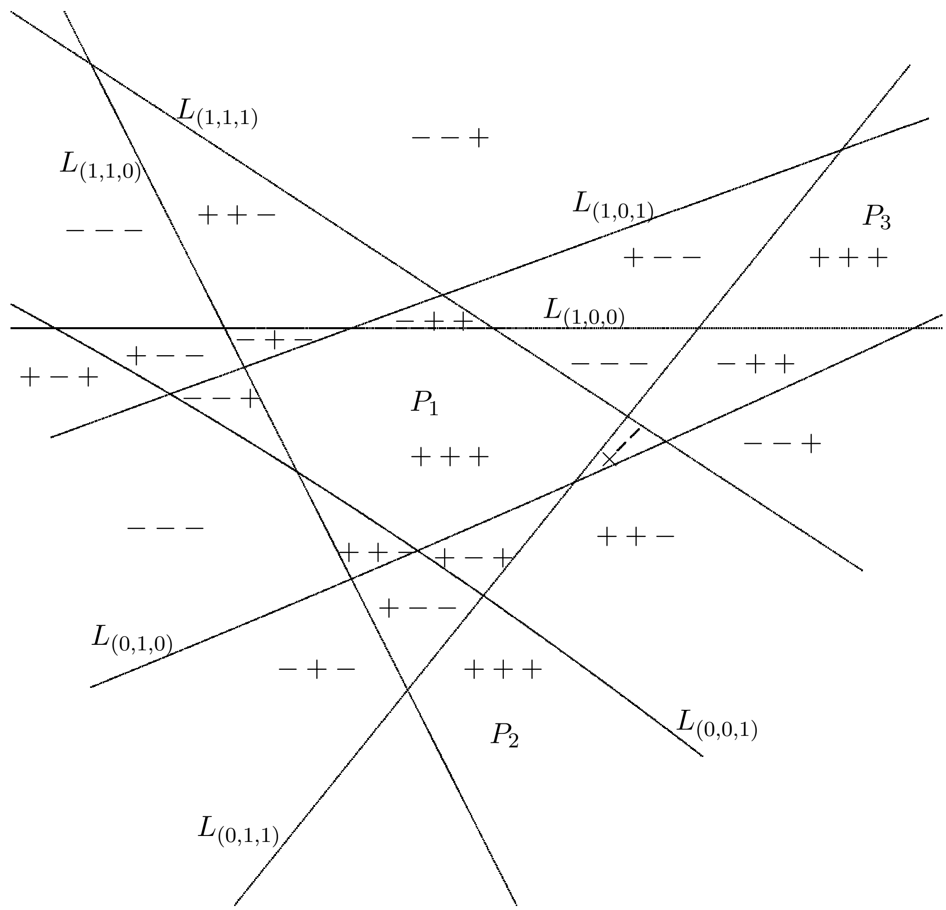


Fig. 7

Consider the Campedelli surface $X = X(\mathcal{L})$ with its real structure $c = c_{+++}$. As it follows from Corollary 2.6, the real part $X_{\mathbb{R}}$ of (X, c) consists of three connected components: the one over P_1 is a connected sum of eight real projective planes (the Euler characteristic -6), the one over P_2 is a Klein bottle, and the one over P_3 is a torus. It may be interesting to notice that, in accordance with the Smith-Thom inequality, $\dim H_*(X_{\mathbb{R}}; \mathbb{Z}/2\mathbb{Z}) = 10 + 4 + 4 < 22 = \dim H_*(X_{\mathbb{C}}; \mathbb{Z}/2\mathbb{Z})$, while the ordinary Betti numbers of $X_{\mathbb{R}}$ surpass those of $X_{\mathbb{C}}$: $\dim H_*(X_{\mathbb{R}}; \mathbb{Q}) = 8 + 2 + 4 = 14 > 10 = \dim H_*(X_{\mathbb{C}}; \mathbb{Q})$.

5.2. **Bad reversings of triangles.** Let us show that the hypothesis on the signs in Corollary 3.8 is essential: without it, there is no local equivariant diffeomorphism between the real Campedelli surfaces $X(\mathcal{L})$ and $X(\mathcal{L}')$, where $\mathcal{L}$ and $\mathcal{L}'$ are related by a reversing of triangle as in Corollary 3.8. For example, in the less evident case, when $X(\mathcal{L})$ (and thus $X(\mathcal{L}')$ as well) has a real component over the triangle, to prove the nonexistence of a local equivariant diffeomorphism one can argue in the following way. We need to compare the quotients by the complex conjugation of the Galois $(\mathbb{Z}/2\mathbb{Z})^2$-coverings of a small ball around the triple point branched in, respectively, $\mathcal{L}$ and $\mathcal{L}'$ (recall that $\alpha_1 + \alpha_2 + \alpha_3 = 0$). More precisely, the boundaries of these quotients are naturally identified, and the question is about possibility to extend this identification to the interior. In fact, this is exactly one of the questions treated in [21] in an equivalent form. As it follows from [21], the extension *does not exist* if and only if the four-manifold M obtained by sewing of the quotients along the boundary has the same homology as the four-sphere S^4. Observing that there is a loop on the real projective plane lying over the vanishing triangle in one half of M linked with the real projective plane lying over the vanishing triangle in the other half (it is sufficient to consider a loop over one of the sides of the vanishing triangle), and applying the Poincaré-Lefschetz duality to $H_i(M, M^-)$ ($M^{\pm}$ stand for the halves of M) in the exact sequence

$$\to H_2(M, M^-) \to H_1(M^-) \to H_1(M) \to H_1(M, M^-) \to,$$

one can easily deduce that $H_*(M) = H_*(S^4)$.

5.3. **Moves of class T.** Theorem 3.7 (and its Corollary 3.8) are based on smoothings of $T(-4)$-singularities. The latters constitute the simplest example of the so-called *singularities of class* T. These singularities, introduced by J. Kollár and N. J. Shepherd-Barron in [16], play crucial role in Manetti's examples [18]: as is proved in [18], smoothings of such singularities provide diffeomorphic surfaces. As a consequence, the statement and the proof of Theorem 3.7 extend word-by-word to real smoothings of real surfaces with arbitrary non real singularities of class T.

5.4. **On the number of deformation classes.** According to Proposition 3.6, the set of deformation classes of real Campedelli surfaces splits into two disjoint subsets: deformation classes of real surfaces associated with mixed real Campedelli line arrangements, and those of real surfaces associated with purely real Campedelli line arrangements. By Theorem 3.10, the first subset contains only five elements. Let us show that the other one contains more than hundread elements.

We base our count on Proposition 2.7 (and Proposition 3.6), which implies that if $X(\mathcal{L}_1)$ and $X(\mathcal{L}_2)$ are real deformation equivalent, where $\mathcal{L}_i$, $i = 1, 2$, are equipped purely real Campedelli line arrangements without triple points, then, after a change of the labels in $\mathcal{L}_2$ and the change of the sign-equipment corresponding to it by a renumbering isomorphism $h : (\mathbb{Z}/2\mathbb{Z})^3 \to (\mathbb{Z}/2\mathbb{Z})^3$, it is possible to find a homeomorphism $\lambda : \mathbb{RP}^2 \to \mathbb{RP}^2$ which transforms $\mathcal{L}_1 \cap \mathbb{RP}^2$ in $\mathcal{L}_2 \cap \mathbb{RP}^2$ and preserves the labels and the sign-equipments.

Consider an arrangement $\mathcal{L}$ of seven real lines which has no triple points and is of type $(11, 5, 5, 1, 0)$. It has $7!$ distinct labelings turning it in purely real Campedelli line arrangements, and, for each labeling, there are the eight distinct sign-equipments. Any homeomorphism of $\mathbb{RP}^2$ preserving $\mathcal{L} \cap \mathbb{RP}^2$ should preserve $L_{(1,0,0)} \cap \mathbb{RP}^2$ and the six-gon P_{20} (see Fig. 5). It is easy to see that, up to isotopy fixing $\mathcal{L} \cap \mathbb{RP}^2$, there is only one such homeomorphism, except identity. Since the order of the group $\mathrm{Aut}G$ of $G = (\mathbb{Z}/2\mathbb{Z})^3$ is equal to $7 \cdot 6 \cdot 4 = 168$, we find that there are at least $\frac{7! \cdot 8}{(7 \cdot 6 \cdot 4) \cdot 2} = 120$ distinct deformation classes of real Campedelli surfaces $X = X(\mathcal{L})$, where $\mathcal{L}$ is of the type $(11, 5, 5, 1, 0)$.

In fact, the number of deformation classes is even bigger. Indeed, similar arguments show that at least 120 more deformation classes of real Campedelli surfaces are given by $X = X(\mathcal{L})$, where $\mathcal{L}$ are of the type $(9, 9, 3, 1, 0)$. In addition, as is known (see [11], [5], and [23]) there are nine other deformation classes (of seven other types) of purely real arrangements of seven lines without triple points. Note also that two such arrangements are deformation equivalent if, and only if, they are homeomorphic, see [11] (proof is found in [10]). Similarly, a self-homeomorphism of an equipped purely real arrangement of seven lines without triple points should be isotopic to a projective automorphism, which would imply, according to Corollary 3.6, that the number of deformation classes of purely real Campedelli surfaces is the same as the number of deformation classes of equipped purely real Campedelli arrangements without triple points.

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