Complex Rays Method and Resurgent Analysis

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Abstract

The usage of the method of geometrical optics for constructing short-wave expansions in diffraction problems is a classical one. This method has the most simple form for investigation of the wave field in the light region where the corresponding rays are real ones. Application of the geometrical optics approximation in the *deep shadow region* is much more complicated task. This is connected with the fact that not all (complex) rays which appear in this situation do contribute to the asymptotic expansion. Therefore, one needs to give a criteria of selecting *active rays* (that is, those who really contribute to the asymptotics).

In this paper we propose the procedure of investigation of this phenomenon which leads to the explicit selection rule for active rays based on the so-called *resurgent analysis method* (see [1]).

Introduction

This paper is aimed at the asymptotic investigation of wave fields in the framework of the geometrical optics approximation including exponentially decreasing fields (which can occur, for example, when considering the wave field in shadow regions). It is well-known that such an investigation can be performed with the help of the so-called *complex rays method* which encounters not only the rays of geometrical optics going along the real space but also those rays which come to the real space along the complex space (see [2] - [10], and others).

However, when using the complex rays method one can obtain that not all complex rays coming to the given point of the (real) physical space will contribute in the asymptotic expansion of the wave field at this point. So the problem arises to give a rule of selection of those complex rays which contribute to the asymptotic expansion of the wave field, say, in shadow regions.

Let us discuss in more detail the statement of the problem.

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1 The wave field with a circular caustic.

To make the discussion more clear, we shall consider first the well-known example of the wave field, that is the wave field with a circular caustic (see Figure 1). Such field is a solution to the Helmholz equation

$$\Delta u(x,y,k) + k^2 n^2(x,y) u(x,y,k) = 0 \tag{1}$$

where $n(x,y)$ is an optical density of the considered medium. This field can be described as a field corresponding to the congruence of geometrical optics rays tangent to a given circle; to be definite, we consider a circle with the unit radius

$$x^2 + y^2 = 1$$

in the two-dimensional real plane $\mathbf{R}^2$ with coordinates (x,y).

The above mentioned congruence of rays can be described by equations

$$\begin{cases} x = q + pt, \\ y = -p + qt, \\ p^2 + q^2 = 1, \end{cases} \tag{2}$$

where (p,q) is the vector tangent to the considered ray, and t is a natural parameter along this ray. Let us investigate how many rays of geometrical optics come through the given point (x,y) of the physical space. To begin with, let us consider the region $x^2 + y^2 \geq 1$. Solving equations (2) with respect to p, q, and t we obtain two different

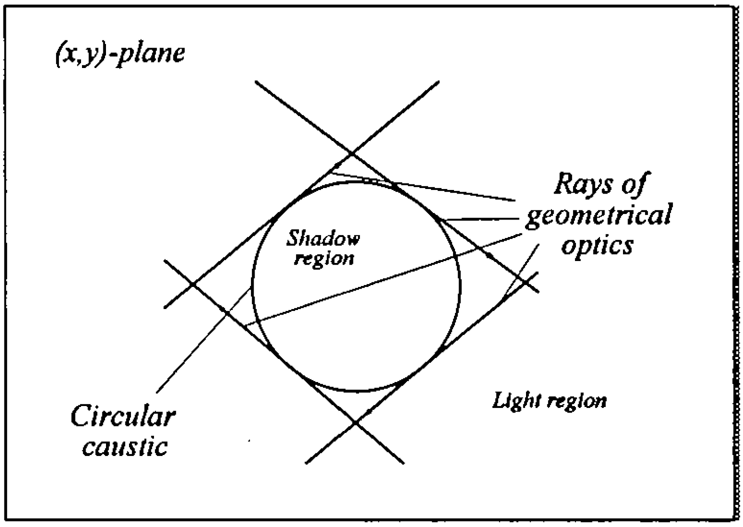

Figure 1: Circular caustic (ray picture).

solutions

$$\begin{cases} p = \dfrac{-y \pm x\sqrt{x^2+y^2-1}}{x^2+y^2}, \\ q = \dfrac{x \pm y\sqrt{x^2+y^2-1}}{x^2+y^2}, \\ t = \pm\sqrt{x^2+y^2-1}, \end{cases} \tag{3}$$

where signs in all the formulas must be chosen in one and the same way. So, the asymptotics of the wave field in the considered region is

$$u(x,y,k) = e^{ikS^{+}(x,y)} \sum_{j=0}^{\infty} k^{-j} a_j^{+}(x,y) + e^{ikS^{-}(x,y)} \sum_{j=0}^{\infty} k^{-j} a_j^{-}(x,y). \tag{4}$$

In the last formula k is a wave number, $S^{(\pm)}(x,y)$ are actions (eikonals) corresponding to the rays (3) and $a_j^{(\pm)}(x,y)$ are amplitude functions. As usual, actions $S^{(\pm)}(x,y)$ can be found with the help of an integral

$$S^{\pm}(x,y) = \int_{(x_0,y_0)}^{(x,y)} p\,dx + q\,dy, \tag{5}$$

(where (x_0,y_0) is some fixed point in the plane $\mathbf{R}^2$ which is chosen, for convenience, on the caustic $x^2+y^2=1$) with p and q given by (3). Thus, as we have already mentioned, in the light region the asymptotics of the wave field is given by the sum (4) of two terms each corresponding to a ray of geometrical optics coming to the considered point.

On the contrary, if we consider the shadow region $x^2 + y^2 \leq 1$, we shall obtain quite different situation. Actually, there is again exactly two rays (3) coming through this point, but now these rays are *complex* ones and, as a consequence, the values of actions corresponding to these rays by formula (5) are complex ones. More exactly, the actions $S^{(\pm)}(x, y)$ will be complex-conjugate to each other and, hence, the second term on the right in (4) will have exponential growth as $k \to \infty$. For physical reasons, such term cannot be included into the asymptotic expansion of the wave field and the expansion in the shadow region will contain only the exponentially decreasing term

$$u(x, y, k) = e^{ikS^{+}(x,y)} \sum_{j=0}^{\infty} k^{-j} a_j^{+}(x, y)$$

corresponding to one of the complex rays passing through the point (x, y).

Thus, in the shadow region one of the complex rays (3) contributes to the asymptotic expansion (we shall call it an *active* ray) and the other does not (the *passive* ray). Certainly, for wave fields with more complicated geometry of rays the criteria for selecting active rays will be of more complicated nature than simple reasons of exponential growth. In particular, the selection of rays is performed not only with the help of (evident) property of exponential growth as $k \to +\infty$, but also with the help of the more refined criteria connected with the Stokes phenomenon. One of the aims of this paper is the explicit formulation of such criteria.

2 Geometrical description

In this Section we introduce a geometrical object which helps to visualize all the variety of rays which *can* contribute to asymptotic expansion of the wave field corresponding to the given congruence of rays of geometrical optics. To do this, we remark that, as it was shown in the previous section, not only the rays themselves but also the values of actions carried by these rays are of importance for constructing the asymptotic expansions. To include this values into our consideration, we shall 'lift' rays of geometrical optics to the three-dimensional space $\mathbf{R}^3$ with coordinates (s, x, y) with the help of the formula

$$s = S(x, y),$$

where $S(x, y)$ is the action carried by the corresponding ray (this function can be computed with the help of formula (5) where the integral can be considered as the

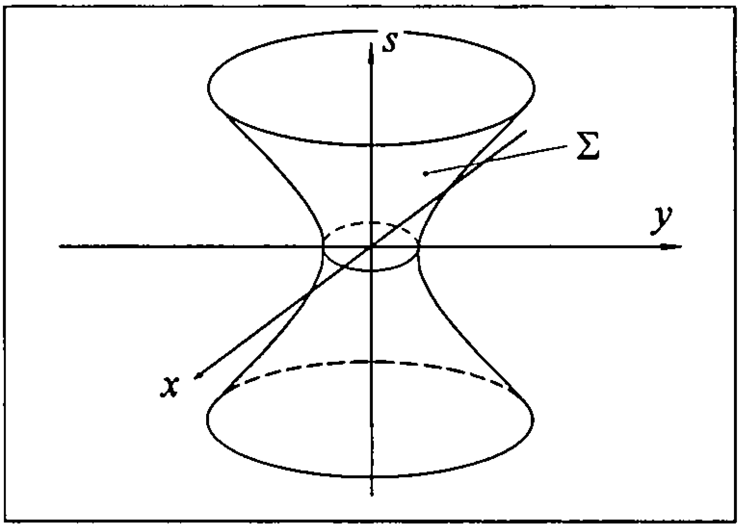

Figure 2: Circular caustic (the surface Σ).

integral along the ray). The union of all rays included into the congruence in question form (after the described lifting) a surface in the space $\mathbf{R}^3$ which we denote by Σ. We remark that the projection of Σ on the physical space $\mathbf{R}^2_{(x,y)}$ is exactly the light region for the considered congruence of rays (the example of such surface corresponding to the wave field with circular caustic is drawn on Figure 2).

Mathematically this situation can be described with the help of the Lagrangian manifold which is formed by trajectories of the Hamilton system determined by the rays included into the congruence in question. More exactly, for describing the asymptotic expansion of the given wave field one must fix the action on this Lagrangian manifold, thus considering a *Legendre manifold* rather than Lagrangina one. This Legendre manifold is a submanifold in the space of 1-jets of the space $\mathbf{R}^2$ with coordinates (x, y, p, q, s) and is determined by equation $s = S(\alpha)$ where α is a point of the Lagrangian manifold and $S(\alpha)$ is the above mentioned action. The concrete value of the action can be determined, in particular, by a choice of a *base point* on the considered Lagrangian manifold (cf. formula 5 above). We remark that the action can be a multivalued function on the Lagrangian manifold and, hence, the projection of thus constructed Legendre manifold on the initial Lagrangian one will be a covering, not a diffeomorphism. Thus, unlike the Lagrangian manifold, which is a graph of the differential of the action, the Legendre manifold is a graph of the action itself.

Now we can give the geometrical description of all *possible* contributions to the asymptotic expansion of the considered wave field. To encounter all the contributions at some point (x, y), one have to intersect the surface Σ by the vertical (that is, parallel to the s-axis) line coming through this point. Each point of intersection will correspond to some term of the form

$$e^{ikS(x,y)} \sum_{j=0}^{\infty} (ik)^{-j} a_j(x, y) \tag{6}$$

of the asymptotic expansion of the wave field in question. We remark that, if we consider the *real* situation, every point of intersection will correspond to some contribution to the asymptotic expansion, so the set of intersection points is in one-to-one correspondence with the set of terms (6) of the asymptotic expansion at the given point (x, y).

Let us turn our mind to consideration of (not only real, but) complex rays. In this case the space $\mathbf{R}^3$ must be replaced by the corresponding *complex* space $\mathbf{C}^3$; we shall denote the coordinates in this space by the same letters (s, x, y) which now must be considered as complex numbers.

Remark 1 The nesessity of complexification along the s-axis is quite evident from the above considerations; clearly one should also complexify the variables x and y to include complex rays into consideration.

Remark 2 Clearly, for the described complexification to be possible, one should require that the ray congruence in question is an analitic one.

So, let us consider the complexification of the above introduced surface Σ in the space $\mathbf{C}^3$. To avoid the complicated notation, we denote this complexification by the same letter Σ. Now for any point $(x, y) \in \mathbf{C}^2$ (even for real values of x and y) consider the intersection $\Sigma_{(x,y)}$ of the surface Σ with the vertical (complex) line coming through this point.

From the first glance it seems that, similar to the real case, all the actions obtained with the help of this procedure (and, as a consequence, all complex rays originating this actions) will really contribute into the asymptotic expansion. However, the above example shows (and it is well-known in physics), that the situation is quite different. Namely, *only some* of complex rays (different for different points (x, y)) do contribute to the asymptotic expansion. Thus, unlike the real case, Σ is *not* a geometrical object adequate to the asymptotic expansion.

Thus, there arises a problem of constructing an adequate geometrical object. It is convenient to solve this problem in terms of the *Borel transform* of the wave field in question. Namely, the above constructed surface Σ can be treated as the set of singularities of the Borel image of the wave field.

Now the resurgent analysis gives us an adequate description of the needed geometrical object, and the corresponding *Laplace transform* (the return to the physical space) supplies us with the asymptotic expansion including the needed complex rays.

3 Resurgent analysis of the wave field

Consider the Borel transform of the function $u(x, y, k)$ with respect to the variable k

$$U(s, x, y) = \mathcal{B}[u(x, y, k)] = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int_A^\infty e^{-iks} u(x, y, k)\, dk, \tag{7}$$

where A is an arbitrary positive number (this integral is uniquely defined up to an entire function). The following affirmation is valid.

Theorem 1 *Each point of the set Σ is a singular point of the function $U(s, x, y)$.*

The proof of this Theorem is based on the following observations. First of all, the function $u(x, y, k)$ describing the considered wave is clearly a solution of the Helmholz equation (1) Applying the Borel transform to equation (1), we obtain the following equation for the function $U(s, x, y)$:

$$\Delta U(s, x, y) + n^2(x, y) \frac{\partial^2 U(s, x, y)}{\partial s^2} = 0. \tag{8}$$

It can be easily verified that the characteristics for the obtained equation with respect to smoothness exactly coincide with rays of geometrical optics of the Helmholz equation (1).

More naturally (and, maybe, more exactly) one should speak about $\partial/\partial s$-*characteristics* of equation (8). It is known (see, for example, [11], [12]) that $\partial/\partial s$-characteristics (similar to any other characteristics) are connected with some (special) quantization. In the considered case this quantization is

$$x \mapsto x, \ \ p \mapsto \left(\frac{\partial}{\partial s}\right)^{-1} \frac{\partial}{\partial x},$$

such that the operator corresponding to the Hamiltonian $H(x, p)$ is $H\left(x, \left(\frac{\partial}{\partial s}\right)^{-1} \frac{\partial}{\partial x}\right)$ (for the details see [12]).

Later on, the straightforward computation shows that if the function $u(x, y, k)$ has a term of the asymptotic expansion of the form (6) then the corresponding function

$U(s,x,y)$ has the singularity of the type[1]

$$U(s,x,y) \simeq \frac{1}{2\pi i}\left[\frac{a_0(x,y)}{s-S(x,y)} + \ln(s-S(x,y))\sum_{j=0}^{\infty}\frac{(s-S(x,y))^j}{j!}a_{j+1}(x,y)\right] \quad (9)$$

at point $s = S(x,y)$.

Actually, since $\mathcal{B}[1] = \frac{1}{2\pi i s}$ and

$$\mathcal{B}\circ k = -i\frac{\partial}{\partial s}\circ\mathcal{B},$$

expansion (6) will be transformed by $\mathcal{B}$ into the expansion

$$e^{S(x,y)\partial/\partial_s}\sum_{j=0}^{\infty}\left(\frac{\partial}{\partial s}\right)^{-j}a_j(x,y)\frac{1}{s}$$

which coincides with expansion (9).

So, at least for real values of (x,y) lying over the light region the function $U(s,x,y)$ has a singularity at any point of the surface Σ. Since the surface Σ in the complex space was constructed simply as the analytic continuation of the corresponding real surface, we arrive at the conclusion that any point of the surface Σ is a singular point of the function $U(s,x,y)$.

Thus, the constructing a wave field corresponding to the given congruence of rays of geometrical optics including exponentially decreasing terms can be (at least theoretically) performed with the help of the following procedure.

1. Construction of the complex surface $\Sigma \in \mathbf{C}^3_{(s,x,y)}$ corresponding to the given congruence of rays.
2. Construction of a solution to equation (8) having Σ as a set of its singularities.
3. Computation of the wave field $u(x,y,k)$ as the inverse (Laplace) transform of the above constructed function $U(s,x,y)$.

Let us try to understand, how active and passive (complex) rays of geometrical optics will be selected during this process.

Clearly, it cannot happen earlier than at the third stage since the function $U(s,x,y)$ computed at the second stage encounters all the rays as its singularities. The only place where the mentioned selection can happen is the third stage of the above described procedure. Thus, this procedure can be performed in the process of analytic continuation of the integral

$$u(x,y,k) = \int_{\Gamma(x,y)} e^{iks}U(s,x,y)\,ds \quad (10)$$

[1]Singularities of such type are called *simple singularities* in the resugent analysis.

where $\Gamma(x,y)$ defines some relative homology class in the complex plane $\mathbf{C}_s$ modulo $\operatorname{Im} s = +\infty$.

The investigation of the analytic continuation of integrals of the type (10), usual in the resurgent functions theory, shows that any ray can change its properties (to be passive or active) exactly at points of the *Stokes surfaces*, that is, surfaces in the space $\mathbf{C}^2_{(x,y)}$ determined by the relation

$$\operatorname{Re} S_j(x,y) = \operatorname{Re} S_l(x,y)$$

for any $j \neq l$. Here $\{S_j(x,y),\ j = 1,2,\ldots\}$ are branches of a ramifying analytic function $S(x,y)$ such that the equation of Σ is

$$\Sigma = \{(s,x,y) |\ s = S(x,y)\}.$$

In case when we know all information about the Riemannian surface of the function $U(s,x,y)$ the choice of active (complex) rays for the wave field (10) is quite simple. Namely, to select active rays one should deform the integration contour $\Gamma(x,y)$ in (10) along the direction of positive imaginary axis. As a result, integral (10) would be represented as a sum of integrals[2]

$$u(x,y,k) = \sum_j \int_{\Gamma_j(x,y)} e^{iks} U(s,x,y)\, ds \tag{11}$$

over special contours $\Gamma_j(x,y)$ encircling *some* of ramification points of the function $U(s,x,y)$. The set of ramification points in the sum (11) is determined by the contour $\Gamma(x,y)$ as well as by the structure of the Riemannian surface of the function $U(s,x,y)$ (see Figure 3). We recall that all these points are in one-to-one correspondence with all (real and complex) rays of geometrical optics as well as with all possible terms of asymptotic expansion of the function $u(x,y,k)$ at the given point (x,y).

Now we notice that, as it was already mentioned (see formula (9) above), each integral

$$\int_{\Gamma_j(x,y)} e^{iks} U(s,x,y)\, ds \tag{12}$$

on the right in (11) has an asymptotic expansion of the form (6) provided that the singularity of U at the point $S_j(x,y)$ has the form (9). Therefore, (11) gives rise to the

[2]The convergence of integrals on the right in (11) is not of importance because even if these integrals diverge they can be correctly defined modulo functions decreasing more rapidly than any exponential; see [1].

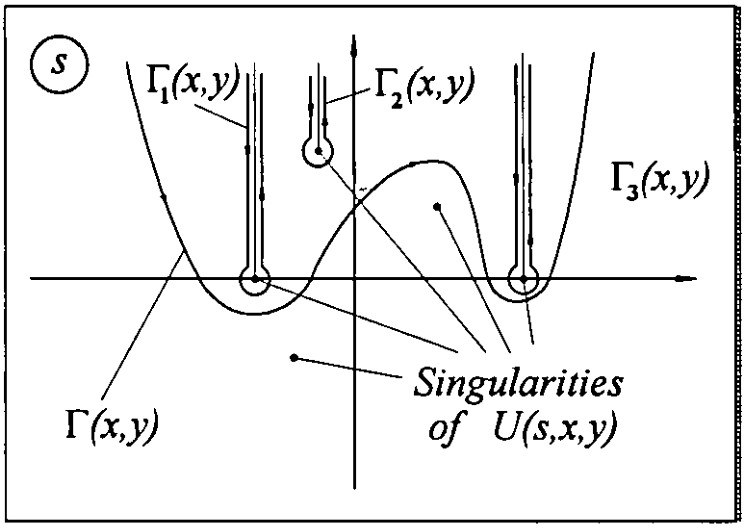

Figure 3: Decomposition of the contour.

asymptotic expansion

$$u(x,y,k) = \sum_j e^{ikS_j(x,y)} \sum_{j=0}^{\infty} (ik)^{-l} a_{lj}(x,y), \tag{13}$$

which is the (exact) WKB expansion of the function $u(x,y,k)$. This shows that *active rays correspond* (at the point (x,y)) *exactly to the points of singularity of the function* $U(s,x,y)$ *involved into the right-hand part of* (14). Thus, we have obtained some form of the needed selection procedure.

We emphasize that for performing the above mentioned deformation procedure one should have the exact knowledge about the global structure of the Riemannian surface of the function U. Unfortunately, the global investigation of the mentioned Riemannian structure is rather a difficult task and it is hard to compute it explicitely even in the simplest cases and, hence, it is difficult to realize directly the selection procedure in the above mentioned form. Therefore, one needs to construct some computational procedure allowing to determine the set of active rays (or, what is the same, the set of singularities of the function $U(s,x,y)$ involved into the decomposition (11)). Such procedure is connected with the so-called *resurgent equations* (see, for example, [1], [13], [14]), and we shall describe its application to the considered problem in the following Section.

4 Resurgent equations

Let us briefly recall here main notions of resurgent analysis needed for formulation of a computational rule for activity of rays of geometrical optics.

We remind (see [1]) that functions $u(x,y,k)$ representable in the form (11) with an endlessly continuable[3] in s function $U(s,x,y)$ under the integral sign are called *resurgent functions* in the variable k.

Let us investigate in more detail the structure of asymptotic expansion (13) for the given function $u(x,y,k)$. As it can be seen from the above considerations, we have two possible represntations of terms of this asymptotic expansion. Namely, each its term can be represented both in the form (6) (k-representation, which is the basic one in the asymptotic analysis) and in the form (12) (s-representation, which is, as we shall see below, the main computational tool). Since k-representation is more or less clear, let us turn our mind to the investigation of the s-representation. First of all, we notice that each term

$$\int\limits_{\Gamma_j(x,y)} e^{iks}U(s,x,y)\,ds \tag{14}$$

on the right in (11) depends on the function $U(s,x,y)$ up to terms holomorphic in a neighbourhood of the corresponding singular point $s_j = S_j(x,y)$. Therefore, expression (14) can be treated as a Laplace transform of the element of quotient space of endlessly continuable functions modulo functions holomorphic in a neighbourhood of the point s_j. These elements are determined by the function $U(s,x,y)$ with the given value of (x,y) at its points of singularity and are called *microfunctions* with support at the point s_j. The space of microfunction supported at a point s will be denoted by $\mathcal{M}_s$.

Let us introduce two functional spaces which we shall use for the description of asymptotic expansions of the form (13) under the assumption that the solution $U(s,x,y)$ to equation (8) is fixed.

We remark first that the different choice of the integration contours in (10) leads to different solutions to Helmholz equation. Since the choice of the integration contour is crucial for the selection rule, it is natural to consider the space As of asymptotic expansions (13) for all functions $u(x,y,k)$ constructed in such a way. Clearly, this space contains only linear combinations of expansions

$$u_j(x,y,k) = e^{ikS_j(x,y)}\sum_{l=0}^{\infty}(ik)^{-l}a_{lj}(x,y) \tag{15}$$

[3]An endlessly continuable function is a ramifying analytic function with discrete set of singularities on its Riemannian surface (the exact definitions can be found, for example, in [1], [13]).

each determined by some singular point $s_j = S_j(x,y)$ of the function $U(s,x,y)$. Later on, not all combinations of such terms can play the role of an asymptotic expansion. Actually, for (13) to be an asymptotic expansion of some function, it is nesessary that it contains only a finite number of terms having less decay than any given exponential. Thus, the set of points s_j corresponding to the given asymptotic expansion (13) must have a finite intersection with any half-plane $\operatorname{Im} s < A$.

The second of the two above mentioned spaces is simply the s-representation of the first one. We recall that each term of the form (15) corresponds in the s-representation (via formula (12)) to the microfunction $U_{s_j}(s,x,y)$ determined by the analytic function $U(s,x,y)$ at its singular point $s_j = S_j(x,y)$. Therefore, the asymptotics of the form (13) corresponds to the formal[4] sum of microfunctions

$$\sum_j U_j(s,x,y) \tag{16}$$

such that the intersection of the set of singular points involved in this sum with any half-plane $\operatorname{Im} s < A$ is finite. We denote by $\mathcal{M}$ the space of formal sums (16) satisfying the above requirement.

To complete the discussion on the introduced functional spaces, we remark that the Laplace transform determines an isomorphism

$$\mathcal{L} : \mathcal{M} \to As$$

between the two above introduced spaces.

Up to this moment the dependence of functions in question on the parameters (x,y) was inessential. Now we turn our mind to the investigation of this dependence.

As it can be seen from the above considerations, decomposition (11) is determined by the decomposition of the homology class $h(x,y)$ defined by the integration contour $\Gamma(x,y)$ involved in formula (10) over the special basis $\{h_j(x,y) | \, j = 1,2,\ldots\}$ of the homology group. This basis is determined by the set of special contours

$$\{\Gamma_j(x,y) | \, j = 1,2,\ldots\}.$$

What will happen with this decomposition during the process of analytic continuation in variables (x,y)? Clearly, this decomposition will be regular at all points (x,y) except for the points of topological rebuilding of the considered basis. Such rebuilding takes place exactly at those points of $\mathbf{C}^2_{(x,y)}$ at which

$$\operatorname{Re} S_j(x,y) = \operatorname{Re} S_l(x,y)$$

[4]This sum is a formal one since one cannot consider the sum of microfunctons supported at different points s of the complex plane $\mathbf{C}$ (unlike the corresponding asymptotic expansions for which the corresponding sum is well-defined).

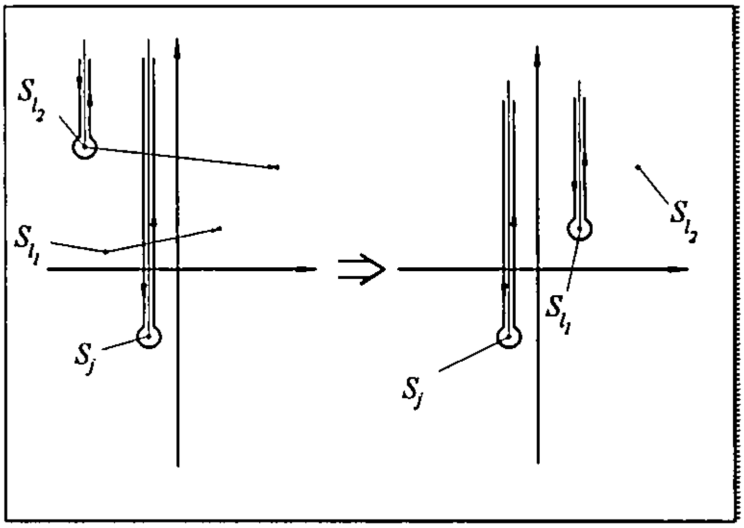

Figure 4: Topological rebuilding of the integration contour. When points S_{l_1} and S_{l_2} intersect the integration contour, the ray corresponding to S_{l_1} becomes passive, and the ray corresponding to S_{l_2} becomes active.

for some $j \neq l$ (see Figure 4). The set of the above mentioned points of rebuilding forms a surface in $\mathbf{C}^2_{(x,y)}$ named *Stokes surface* of the function $u(x, y, k)$ in question. So, we see that the *decomposition* (11) *is changed by jump when the point* (x, y) *intersects the Stokes surface* in the process of analytic continuation. Thus, the points of Stokes surfaces are exactly the points at which the property of geometrical optics rays to be passive or active can be changed.

The connected regions bounded by the Stokes surface will be called *Stokes regions.*

Remark 3 We emphasize that, though the decomposition (11) and, hence, the asymptotic expansion of the considered function, changes by jump at points of the Stokes surface, the function $u(x, y, k)$ itself *remains analytic* at these points. Opposite, if we consider a function $u(x, y, k)$ having one and the same asymptotic expansion in different Stokes regions, this function will have jumps at points of the Stokes surface.

The jumps of the decomposition (11) at points of the Stokes surface can be described by the so-called connection homomorphism [5]

$$\tau \,:\, \mathcal{M} \to \mathcal{M}.$$

[5]In fact, the space $\mathcal{M}$ possesses a natural structure of the algebra with respect to the convolution and the connection homomorphism is an algebra homomorphism of this algebra. Below we shall not use the mentioned algebraic structure in the explicit way.

To describe the exact meaning of this homomorphism we first note that at points of the Stokes surface the definition of the Laplace transform of the microfunction is an ambiguous one. Actually, if a point (x, y) lies on the Stokes surface, there exists at least one pair of points $s = S_j(x, y)$ and $s = S_l(x, y)$ lying on one and the same vertical line in the complex plane $\mathbf{C}_s$. Then the vertical ray emanated from the lower point (suppose, for definiteness, that this point is $s = S_j(x, y)$) along the direction of the positive imaginary axis will clearly intersect the upper one. In this situation the integration contour $\Gamma_j(x, y)$ involved into integral (14) is not uniquely determined: this contour can encircle singular points either from the left or from the right. To determine the directon of encircling of each point of singularity met by the integration contour, one has to find out does this point approach the integration contour from the left or from the right when the point (x, y) approaches the Stokes surface from its negative side[6]. Clearly, in the first case the corresponding point must be encircled by the integration contour from the right, and in the second case it must be encircled from the left. The case when the point (x, y) approaches the Stokes surface from its positive side can be considered in a similar way. We denote the two Laplace transforms of the microfunction $U_j(s, x, y)$ supported at the point $s = S_j(x, y)$ with the two above described determinations of the integrating contour by $\mathcal{L}_-$ and $\mathcal{L}_+$, correspondingly.

Now the definition of the connection homomorphism can be written down with the help of the formula

$$\mathcal{L}_+ [\tau U] = \mathcal{L}_- [U]$$

for any element $U \in \mathcal{M}$.

The above considerations show that to describe the selection rule for geometrical optics rays one must describe the Stokes phenomenon, that is, compute the connection homomorphism at any point of the corresponding Stokes surface. However, as it was shown above (see Figure 3 and the corresponding explanations), the direct computation of the Stokes phenomenon (with the help of direct computation of asymptotic expansions of the considered function in different Stokes regions) requires the knowledge of the global structure of the Riemannian surface of the function $U(s, x, y)$, and it is a very complicated problem to find out this global structure for the given ray congruence.

Thus, there arises a problem of working out the apparatus which allows one to compute the connection homomorphism without explicit use of the global structure of the Riemannian surface of the function U. Such an apparatus is based on the notion of the so-called resurgent equations.

The notion of resurgent equations is based on the following observation.

Let us consider a set of *focal points* of the function $u(x, y, k)$, that is, the set of

[6]We suppose that some orientation is chosen and fixed on the Stokes surface of the considered functions. If this orientation is changed, the connection homomorphism τ will be replaced by its inverse.

points (x, y) for which the relation

$$S_j(x,y) = S_l(x,y)$$

takes place for some $j \neq l$. These points are, as a rule, ramification points of the action $S(x,y)$ and, hence, the asymptotic expansion of the wave field $u(x,y,k)$ is ramified on the set of focal poins. From the other hand, the wave field $u(x,y,k)$ itself, being a solution of the Helmholtz equation, is a real-analytic function and, hence, have no singularities at focal points. Therefore, the asymptotic expansion of the function $u(x,y,k)$ must be a univalued function in neighbourhood of focal points provided that we have encountered the Stokes phenomenon during the computations.

Let us try to write down the formulas expressing the above observation.

Let (x_0, y_0) be some focal point of the function $u(x,y,k)$ and let l be a loop surrounding the set of focal points in a neighbourhood of (x_0, y_0) having transversal intersection with the Stokes surface[7]. Denote by $A_1, \dots, A_N$ the points of intersection of the loop l with the Stokes surface enumerated in the order they appear on l (see Figure 5). For convenience we suppose that the origin of the loop l coincides with the point A_1 and that the orientation of the Stokes surface is chosen in such a way that the loop l goes from the negative to the positive side of this surface at each point A_j. Denote by $U_1, \dots, U_N$ the elements of $\mathcal{M}$ corresponding to the function $u(x,y,k)$ via relation (11) at points $A_1, \dots, A_N$ if we use a decomposition determined by the *positive side* of the Stokes surface. Then it is evident that the univaluedness of the function $u(x,y,k)$ along l is equivalent to the following system of relations:

$$U_{j+1} = \tau\left[\mathcal{A}_j U_j\right],\ j = 1, \dots N, \tag{17}$$

where $\mathcal{A}_j$ stands for the operator of analytic continuation along the loop l from the point A_j to the point A_{j+1} (we identify the point A_{N+1} with A_1). The system of relations (17) is exactly the *system of resurgent equations* written in one of the possible forms (later we shall see another representation of this system).

At the moment the method of obtaining the needed informaton from system (17) remains unclear. In other words, to investigate the Stokes phenomenon we must present the method of solving this system. To do this, we shall try to rewrite this system in a little bit more pleasent form.

To begin with, we shall examine the connection homomorphism τ in more detail. Clearly, it is sufficient to describe the action of the connection homomorphism on the generators of the space $\mathcal{M}$. These generators are single microfunctions U^j supported

[7]We remark that the set of focal points is an analytic set in the space $\mathbf{C}^2$ of *complex* codimension 1 unlike the Stokes surface which is a variety of *real* codimension 1.

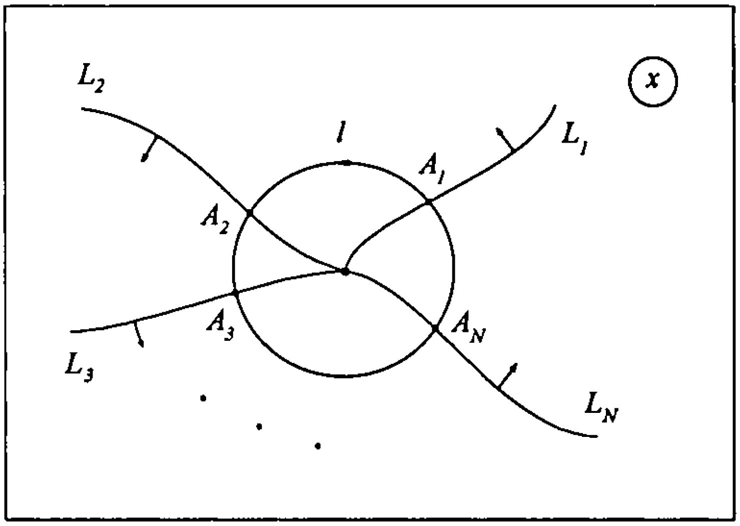

Figure 5: Illustration to the construction of resurgent equations

at different points $s_j = S_j(x, y)$ of singularity of the function $U(s, x, y)$ in question. The above considerations show that the formula

$$\tau U^j = U^j + \sum_l U^l$$

takes place, where the sum is taken over all microfunctions U^l determined by the function $U(s, x, y)$ at points lying on the cut originated from $S_j(x, y)$ and going along the direction of positive imaginary axis[8]. So, we can introduce a 'difference' operator

$$\begin{aligned} &\delta : \mathcal{M} \to \mathcal{M}, \\ &\delta U^j = \tau U^j - U^j \end{aligned} \tag{18}$$

corresponding to the 'shift' operator τ. The operator δ is an operator of *strictly negative* order in the sense that it takes a microfunction supported at $S_j(x, y)$ to the sum of microfunctions with their supports posited strictly above the point $S_j(x, y)$ in the complex plane $\mathbf{C}_s$. Actually, it is clear that the Laplace transform of the obtained sum of microfunctions will be of less exponential order than the Laplace transform of the initial microfunction.

[8]As a consequence of the latter formula we obtain that the continuation of the Laplace transform of the microfunctions U^j across the Stokes surface from its negative to its positive side leads to a functions with asymptotic expansion $u_j + \sum_l u_l$, where the functions u_j are given by (15).

Now we can rewrite system (17) of resurgent equations in terms of the operator (18):

$$U_{j+1} = \mathcal{A}_j U_j + \delta\left[\mathcal{A}_j U_j\right],\ j = 1, \ldots N. \tag{19}$$

The obtained system of equations contains the 'difference' operator δ, thus being an analogue of systems of difference equations in the usual analysis. However, from the usual analysis it is well-known that difference equations are much less convenient for usage than differential ones. Therefore, there arises an idea to write down the system of resurgent equations as, in some sense, a system of 'differential' equations. It occurs, that this can be done with the help of a new notion of the derivative (the so-called *alient derivative*) which is related to the 'shift' operator τ in the same way as usual derivative is related to the standard shift operator

$$T_h f(x) \stackrel{\text{def}}{=} f(x+h).$$

Namely, having the relation

$$T_h = e^{hd/dx}$$

as a model, one can define the alient derivative Δ by the relation

$$\tau = e^{\Delta}, \text{ or } \Delta \stackrel{\text{def}}{=} \ln \tau = \ln(1+\delta).$$

The verification of the correctness of this definition as well as the presentation of the full theory of alient derivatives (alient differential calculus) are out of the framework of this paper (see, for example [15], [1]). However, in the next Section we shall try to illustrate these notions on the simplest model of resurgent functions of the Airy type.

5 Example: resurgent functions of Airy type

In this Section we shall consider functions of variables (x, k) where $x \in \mathbf{C}$ and $k \to \infty$ is a large parameter. We say that a resurgent function $u(x,k)$ is a *resurgent function of the Airy type* if the set of singularities of its Borel transform is given by

$$s = S_{\pm}(x) = \pm\frac{2}{3}x^{\frac{3}{2}}. \tag{20}$$

Clearly, the only focal point for such a function in the x-plane is the origin $x = 0$. Hence, we can use the unit circle $x = e^{i\varphi}$, $\varphi \in [0, 2\pi]$ as the loop l for the construction of the system of resurgent equations described in the previous Section.

To visualize our considerations, we shall use the so-called *illumination diagram* ([1]). We say that one singularity point (say, $S_+(x)$) *illuminates* the other if this other point

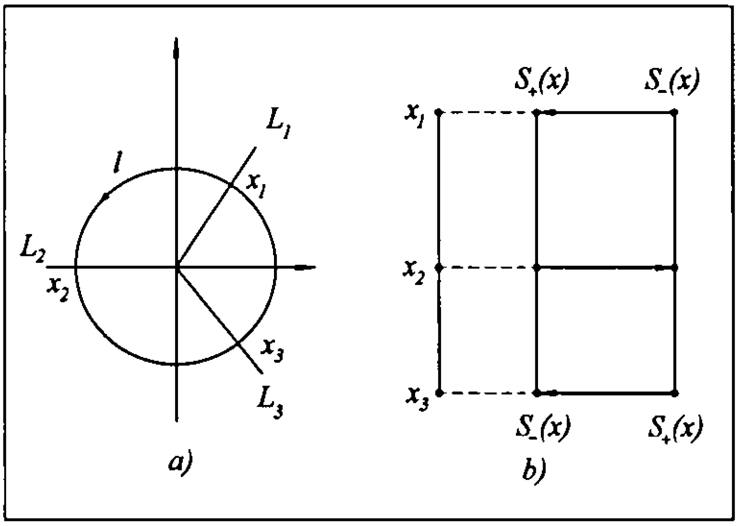

Figure 6: Resurgent function of the Airy type.

lies on the ray originated from $S_+(x)$ along the direction of the positive imaginary axis. Thus, the point x belongs to the Stokes surface if and only if one of the singular points (20) illuminates the another. Evidently, this is possible if and only if $\varphi = \pi$ or $\varphi = \pm\pi/3$. Thus, the Stokes lines for a function fo Airy type are such as it is drawn on Figure 6 a) (the loop l and its intersections with these Stokes lines are also shown there).

The situation which takes place at points of intersection of the loop l with the Stokes lines is schematically shown by the illumination diagram (illumination diagram; see Figure 6 b)). The very left vertical line on this diagram represents the loop l, and the two subsequent vertical lines represent the two points (20) when the point x moves along l. Since the points $S_\pm(x)$ change their places three times when tracing along the loop l, the lower and the upper endpoints of these lines must be identified as it is shown on Figure 6 b).

Later on, the horizontal lines on the diagram represent points of intersection of the loop l with the Stokes lines, and the horizontal arrow coming from one point of singularity to another denotes that the first singularity point illuminates the second.

Now one can see that each arrow involved into the illumination diagram gives rise to exactly one resurgent equation of the type (19).

First of all, we note that in the considered case the space $\mathcal{M}$ is at each point $x \neq 0$ a two-dimensional complex space with the two microfunctions $U^\pm$ supported at points (20)as generators. The corresponding space in the k-representation consists of asymptotic expansions of the form

$$u(x,k) = e^{ikS_+(x)} \sum_{j=0}^{\infty} (ik)^{-j} a_j^+(x) + e^{ikS_-(x)} \sum_{j=0}^{\infty} (ik)^{-j} a_j^-(x)$$

Now, equating the components of elements from $\mathcal{M}$ in both sides of (19), and denoting the components of U_j at singularity points $S^{\pm}(x_j)$ by $U_j^{\pm}$, we come to the following equations:

$$\begin{cases} U_1^+ = \mathcal{A} U_3^- + \delta \left(\mathcal{A} U_3^+ \right), \\ U_1^- = \mathcal{A} U_3^+, \end{cases} \tag{21}$$

at point x_1 of intersection of the loop l with the first Stokes line L_1 (we have taken into account the above mentioned identifications);

$$\begin{cases} U_2^- = \mathcal{A} U_1^- + \delta \left(\mathcal{A} U_1^+ \right), \\ U_2^+ = \mathcal{A} U_1^+, \end{cases} \tag{22}$$

at point x_2, and

$$\begin{cases} U_3^+ = \mathcal{A} U_2^+ + \delta \left(\mathcal{A} U_2^- \right), \\ U_3^- = \mathcal{A} U_2^-, \end{cases} \tag{23}$$

at point x_3.

One can exclude the microfunctions U_1^-, U_2^+, and U_3^- corresponding to the illuminating points from equations (21) – (23) thus obtaining the following system of equations for the components U_1^+, U_2^-, and U_3^+ corresponding to the illuminated points:

$$\begin{cases} U_1^+ = \mathcal{A}^2 U_2^- + \delta \left(\mathcal{A} U_3^+ \right), \\ U_2^- = \mathcal{A}^2 U_3^+ + \delta \left(\mathcal{A} U_1^+ \right), \\ U_3^+ = \mathcal{A}^2 U_1^+ + \delta \left(\mathcal{A} U_2^- \right). \end{cases}$$

Now we notice that for the considered geometry of singular points of the function $U(s,x,y)$ one has $\delta^2 U_j^{\pm} = 0$ for all $j = 1,2,3$ and, hence,

$$\Delta U_j^{\pm} = \ln(1+\delta) U_j^{\pm} = \delta U_j^{\pm}.$$

Thus, the latter system of resurgent equations can be rewritten in the form of system of alient differential equations:

$$\begin{cases} U_1^+ = \mathcal{A}^2 U_2^- + \Delta \left(\mathcal{A} U_3^+ \right), \\ U_2^- = \mathcal{A}^2 U_3^+ + \Delta \left(\mathcal{A} U_1^+ \right), \\ U_3^+ = \mathcal{A}^2 U_1^+ + \Delta \left(\mathcal{A} U_2^- \right). \end{cases} \tag{24}$$

The obtained system of resurgent equations being a 'differential' one has one great disadvantage. The matter is that it contains the operator $\mathcal{A}$ of analytic continuation which is not convenient in use. We shall try to exclude this operator from the system.

Besides, we emphasize that for any given function $U(s,x)$ with (20) as ramification points one can choose a set of microfunctions U_1^+, U_2^-, and U_3^+ determined by singular points of the function $U(s,x)$ in question in such a way that the corresponding resurgent function $u(x,k)$ is univalued in a neighborhood of the origin and, hence, the resurgent equations (24) are valid. To do this, it suffices to define the function $u(x,k)$ as the integral of the form (10) with the integration contour $\Gamma(x)$ encircling both ramification points of $U(s,x)$. Then the decomposition of the obtained function will give us the required microfunctions which satisfy system (24). However, if we require that the analytic continuation of the Laplace transform of any microfunction determined by the function $U(s,x)$ at some point x (say, $x = x_1$) is a univalued function, then resurgent system (24) imposes some restrictions on the function $U(s,x)$ in question.

To be definite, let us consider the system of resurgent equations for the microfunction corresponding to the recessive component of $u(x,k)$ at the point x_1. To do this, we set $U_3^+ = 0$ in (24). Then we arrive at the following system of resurgent equations:

$$\begin{cases} U_1^+ &= \mathcal{A}^2 U_2^-, \\ U_2^- &= \Delta\left(\mathcal{A}U_1^+\right), \\ 0 &= \mathcal{A}^2 U_1^+ + \Delta\left(\mathcal{A}U_2^-\right). \end{cases}$$

Excluding the microfunction U_2^- from the obtained system, we obtain

$$\begin{cases} \mathcal{A}^{-2}U_1^+ &= \Delta\left(\mathcal{A}U_1^+\right), \\ \Delta\left(\mathcal{A}^{-1}U_1^+\right) &= -\mathcal{A}^2 U_1^+. \end{cases}$$

Taking into account that $U_d \overset{\text{def}}{=} \mathcal{A}^{-1}U_1^+$ and $U_r \overset{\text{def}}{=} \mathcal{A}^2 U_1^+$ are dominant and recessive components of the function $U(s,x,y)$ at point x_3, respectively, one can obtain a system of alient differential equations at one and the same point x_3:

$$\begin{cases} \left(\mathcal{A}\Delta\mathcal{A}^{-1}\right)U_r &= U_d, \\ \Delta U_d &= -U_r. \end{cases}$$

Taking into account that the operator $\mathcal{A}\Delta\mathcal{A}^{-1}$ being applied to the function U_r can be treated as the alient derivative of U_r at the point $S_+(x_3)$, we can rewrite the latter system in the following form (which does not contain the operator $\mathcal{A}$ of analytic continuation):

$$\begin{cases} \Delta U_r = U_d, \\ \Delta U_d = -U_r. \end{cases} \tag{25}$$

Now we shall illustrate the procedure of *solving* the obtained system of resurgent equations. This procedure is based on the strong analogy between alient differential equations and usual ordinary differential equation. Namely, we shall use the following result.

Consider an alient differential equation of the form

$$\begin{cases} \Delta F = A_1(x)\,F + B_1(x)\,G, \\ \Delta G = A_2(x)\,F + B_2(x)\,G, \end{cases} \tag{26}$$

where F and G are dominant and recessive components of some resurgent function whose Borel transform has singularities at most at two points in the complex plane $\mathbf{C}_{\bullet}$. Then the following affirmation is valid (see [14]).

Theorem 2 *Let* (F_1, G_1) *and* (F_2, G_2) *be two solutions to system* (26) *such that*

$$D \stackrel{\text{def}}{=} \begin{vmatrix} F_1 & F_2 \\ G_1 & G_2 \end{vmatrix} = F_1G_2 - F_2G_1$$

is an invertible element in $\mathcal{M}_0$. *Then the general solution to* (26) *is given by*

$$\begin{pmatrix} F \\ G \end{pmatrix} = C_1 \begin{pmatrix} F_1 \\ G_1 \end{pmatrix} + C_2 \begin{pmatrix} F_2 \\ G_2 \end{pmatrix}$$

where C_1 *and* C_2 *are constants of resurgence, that is elements* $C_1, C_2 \in \mathcal{M}_0$ *such that* $\Delta C_j = 0$.

Now we take into account that the classical Airy function $Ai(x,k)$, that is, the solution of the following differential equation

$$y'' + k^2 xy = 0$$

is a univalued resurgent function with singularities of its Borel transform at points (20). Hence, its dominant and recessive components U_d^1 and U_r^1 satisfy system (25). Clearly, the same is true for the dominant and recessive components U_d^2 and U_r^2 of the derivative $\partial Ai(x,k)/\partial x$. It can be shown that the corresponding Wronskian does not vanish, and, hence, Theorem 2 is applicable. Thus, we arrive at the following result.

Any resurgent function of the Airy type can be represented in the form

$$u(x,k) = C_1(x,k)\,Ai(x,k) + C_2(x,k)\,\frac{\partial Ai(x,k)}{\partial x},$$

where $C_1(x,k)$ and $C_2(x,k)$ are constants of resurgence, that is, the Stokes phenomenon for these functions is trivial. Thus, the following affirmation is valid

Corollary 1 *Stokes phenomenon for any resurgent function of Airy type is exactly the same as for the Airy functon itself.*

Remark 4 The considerations of this example (as well as of the two subsequent examples) can be carried out in the similar manner for the case when the resurgent structure of the functions in question can be reduced to the form (20) with the help of a holomorphic variable change.

6 Refraction of an electromagnetic wave on the ionosphere layer

In this Section we illustrate the above described computational procedure on a rather simple example. Let us consider a plane electromagnetic wave

$$u_0(x,y,k) = e^{ik(px+qy)}$$

which falls from the left on an ionosphere layer with optical density depending only on x and given by

$$n^2(x,y) = n^2(x) = \begin{cases} x^2 - a^2, & |x| \le \sqrt{a+1}, \\ 1, & |x| \ge \sqrt{a+1}. \end{cases}$$

The potential $u(x,y,k)$ of the resulting wave field satisfies the Helmholz equation

$$\Delta_{x,y} u + k^2 n^2(x) u = 0 \tag{27}$$

completed by the corresponding radiation condition, that is, by the requirement that the difference

$$u(x,y,k) - u_0(x,y,k)$$

is a wave propagating in the direction inclined to the angle less than $\pi/2$ to the direction of the negative part of the axis x.

One can search for the solution of problem (27) in the form

$$u(x,y,k) = e^{ikqy} v(x,k) \tag{28}$$

Substituting (28) into (27) we obtain the following equation for the function $v(x,k)$:

$$\frac{d^2 v}{dx^2} + k^2 n_1^2(x) v = 0,$$

where $n_1^2(x)$ is the *reduced optical density*:

$$n_1^2(x) = n^2(x) - q^2 \tag{29}$$

So, the considered two-dimensional problem can be reduced to an one-dimensional problem for Helmholz equation with the new optical density. We remark that the

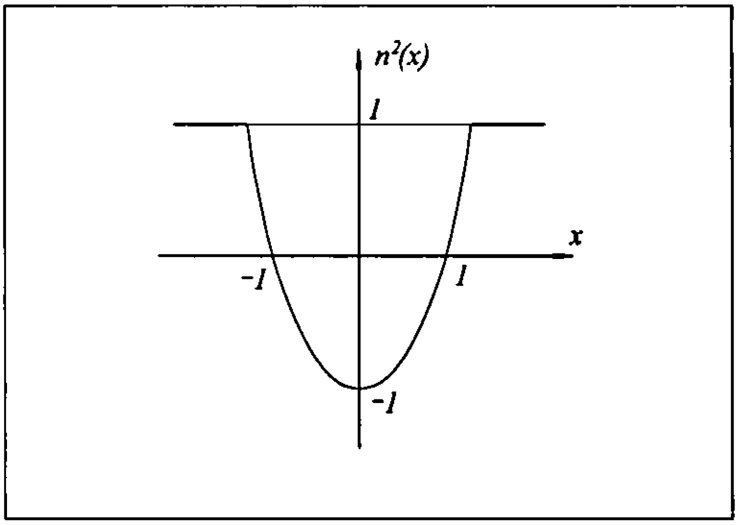

Figure 7: Reduced optical density.

reduced optical density $n_1^2(x)$ can be *negative* in some regions even in the case when the initial optical density $n^2(x)$ is everywhere positive.

To avoid complicated notation, we shall consider a model problem similar to the above described reduced problem. Namely, let us consider an one-dimensional wave $u\,(x,k)$ propagating through the media with optical density given by the expression (see Figure 7)

$$n^2\,(x) = \begin{cases} x^2 - 1, & |x| \le \sqrt{2}, \\ 1, & |x| \ge \sqrt{2}. \end{cases} \tag{30}$$

We suppose that the wave falls upon the ionosphere layer from the side of the negative values of x. Then from the physical reason it is evident that in the light region (that is, for $x < -1$) there exist the incoming and the reflected waves, in the shadow region there exists a transmitted wave with exponentially small amplitude, and in the region $|x| \le 1$ which intransparent for real rays of geometrical optics the field will exponentially decrease from its left boundary $x = -1$ to its right boundary $x = 1$.

In the investigation below we do not take care of the reflection from the angle points $x = \pm\sqrt{2}$ of the function $n^2\,(x)$ given by (30) and, hence, we can carry out all our considerations in the region $|x| \le 2$ where the function $n^2\,(x)$ is an analytic one.

To begin with, let us compute the resurgent structure of the considered wave field, that is, the equation of the above introduced surface Σ. As it follows from the above considerations, to do this we must just compute the action. Due to the Hamilton-Jacobi equation we have

$$S(x) = \int p(x)\, dx,$$

where the function $p(x)$ is determined by the equation

$$p^2 - n^2(x) = 0.$$

Hence, the action $S(x)$ is given by the relation

$$S(x) = \int \sqrt{x^2 - 1}\, dx = \frac{1}{2} x \sqrt{x^2 - 1} - \frac{1}{2} \ln \left(x + \sqrt{x^2 - 1} \right)$$

up to an additive constant. Thus, the equation of the surface Σ reads

$$\Sigma = \left\{ (s, x) | \;\; s = \frac{1}{2} x \sqrt{x^2 - 1} - \frac{1}{2} \ln \left(x + \sqrt{x^2 - 1} \right) \right\}. \tag{31}$$

For $x > 1$ the intersection of Σ with the (complex) line $x = \text{const}$ forms a lattice with the step $i\pi$ originated by two real points corresponding to the two signs of the square root in (31) if we choose real values of the logarythm (see Figure 9 a) below). As we have seen above, each point of intersection of the surface Σ with the line $x = \text{const}$ corresponds to exactly one (real or complex) ray of the geometrical optics coming through the considered point x. Thus, we see that there is an infinite number of rays coming through each point of the physical space; clearly, this is true also for all other regions on the x-space (see Figures 9 — 10).

As we have seen above, the computation of the Stokes phenomenon (and, as a consequence, the selection of active rays in each Stokes region) is determined by the behavior of points $s = S_j(x)$ near focal points of the wave field in question. Therefore, our nearest task is to investigate this behavior.

Let us consider first the focal point $x = 1$. We put

$$x = 1 + \varepsilon$$

thus introducing the small parameter ε near the considered focal point. The straightforward computations show that

$$S(1 + \varepsilon) = \frac{2\sqrt{2}}{3} \varepsilon^{3/2} + O\left(\varepsilon^2\right). \tag{32}$$

for any integer $k \in \mathbf{Z}$. Similar, in a neighbourhood of the second focal point we obtain

$$S(-1 + \varepsilon) = \frac{i\pi}{2} - \frac{2\sqrt{2}}{3} (-\varepsilon)^{3/2} + O\left(\varepsilon^2\right) + i\pi k, \; k \in \mathbf{Z}.$$

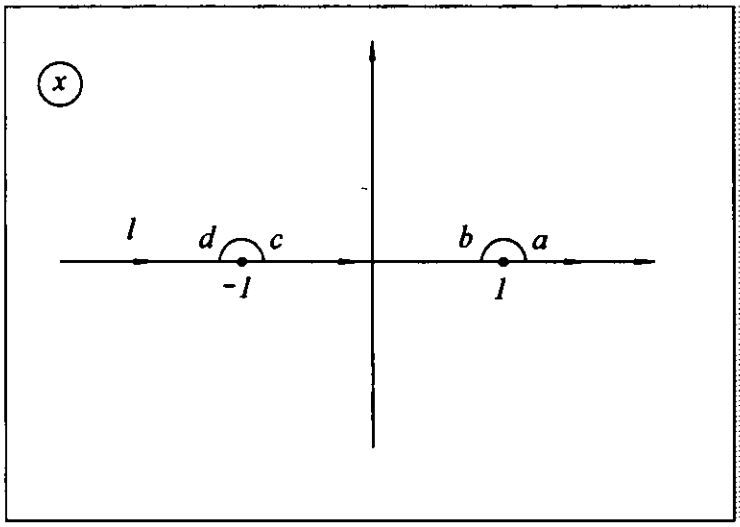

Figure 8: Path of analytic continuation.

It is not hard to show that there exists a change of variable x such that the function $v(x,k)$ becomes a function of the Airy type in a neighbourhood of each its focal point. Now Remark 4 shows that the Stokes phenomenon for the considered function in a neighbourhood of any of its focal points is equivalent to that of the Airy function. This fact will be used in the subsequent computations.

Let us proceed with the constructing of the asymptotic expansion of the wave field $v(x,k)$. To do this we shall begin with a single wave in the region $x > 1$ propagating in the direction of the positive real axis (the transmitted wave) and shall continue analytically the corresponding function along the path l shown on Figure 8.

Remark 5 Since we consider the wave field as a resurgent function satisfying the corresponding resurgent equations, the function $v(x,k)$ is univalued in a neighbourhood of each its focal point. Hence, the particular form of the path l chosen for the analytic continuation process is not of importance, since the result will be just the same.

The function $v(x,k)$ corresponding to a single wave propagating in the direction of the positive real axis can be represented as the Laplace transform of the microfunction supported at the right real point of singularity of the function $V(s,x)$ (see Figure 9 a)). In other words, this function is

$$v(x,k) = \int_{\Gamma(x)} e^{iks} V(s,x)\, ds \tag{33}$$

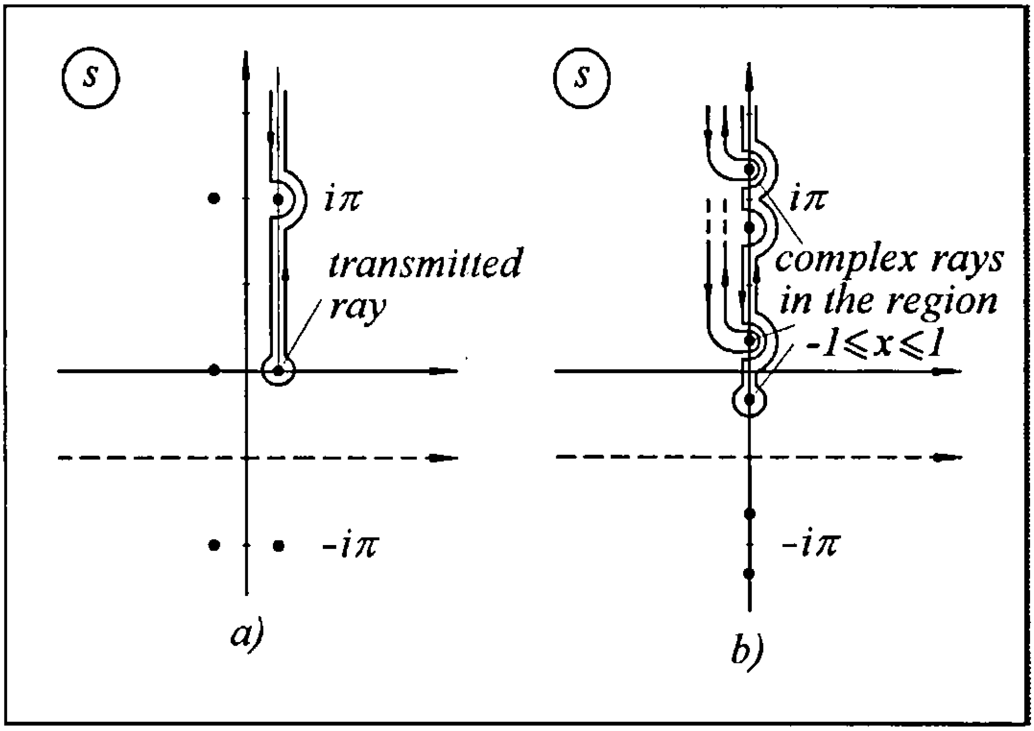

Figure 9: Passing along l from a to b.

(cf. (10)), where $\Gamma(x)$ is a standard contour emanated from the above mentioned point of singularity in the direction of the positive part of the imaginary axis (see Figure 9 a); the geometrical situation on this Figure corresponds to values of $x > 1$ lying sufficiently close to 1).

Now we perform the analytic continuation of the wave (33) along the half-circle going along the path l from the point a to the point b. Equation (32) shows that when the point x is tracing this half-circle, the corresponding points $s = S_j(x)$ will rotate to the angle $3\pi/2$ counterclockwise. Taking into account the fact that any point of singularity which intersects the integration contour extracts from it exactly one integrating contour[9], one can see that the result of the analytic continuation will be represented in the form of the sum of integrals of the type (33) taken over the contours shown on Figure 9 b).

To simplify our considerations we shall consider only the dominant components of the asymptotic expansion of the wave field $v(x,k)$. Therefore, the upper integration contour shown on Figure 9 b) will not be of importance for us.

Later on, when the point x moves along the part of the path l coming from the point b to the point c, the corresponding points $s = S_j(x)$ will move along the imaginary axis and, as a result, these points will come to a neighbourhood of the points $i(\pi/2 + \pi k)$ as it is shown on Figure 10 b). One can perform the analytic continuation along the half-circle coming from c to d exactly in the same way as it was done above for the half-circle ab (certainly, in this process we have to use the fact, that the considered function is of the Airy type in a neighbourhood of $x = -1$). The result of the analytic continuation is shown on Figure 10 b). Thus, we have constructed the analytic continuation of the function given by (33) to all regions on the real axis $\mathbf{R}_x$ (clearly, in the region $|x| \leq \sqrt{2}$ which is of interest for us).

We remark that, as this can be seen from Figures 9 and 10 the constructed function has exponential growth in all the regions of $\mathbf{R}_x$ except for $x > 1$ and, hence, have no physical sense. To overcome this difficulty one can just shift all the above obtained resurgent structure to the value $i\pi/2$ in the complex plane $\mathbf{C}_s$ (on Figures 9 and 10 this shift is showh by the new real axis drawn as a dashed line).

These considerations complete the procedure of selection of the active complex rays. Namely, the active rays in each of the three regions

$$x < -1, \quad -1 < x < 1, \quad \text{and} \quad x > 1$$

correspond exactly to those points $s = S_j(x)$ which carry the integration contours. In particular, in the region $x < -1$ we have rays corresponding to the incoming and

[9]This follows from the above mentioned fact that the Stokes phenomenon for the considered function near each of its focal points is equivalent to that of Airy function.

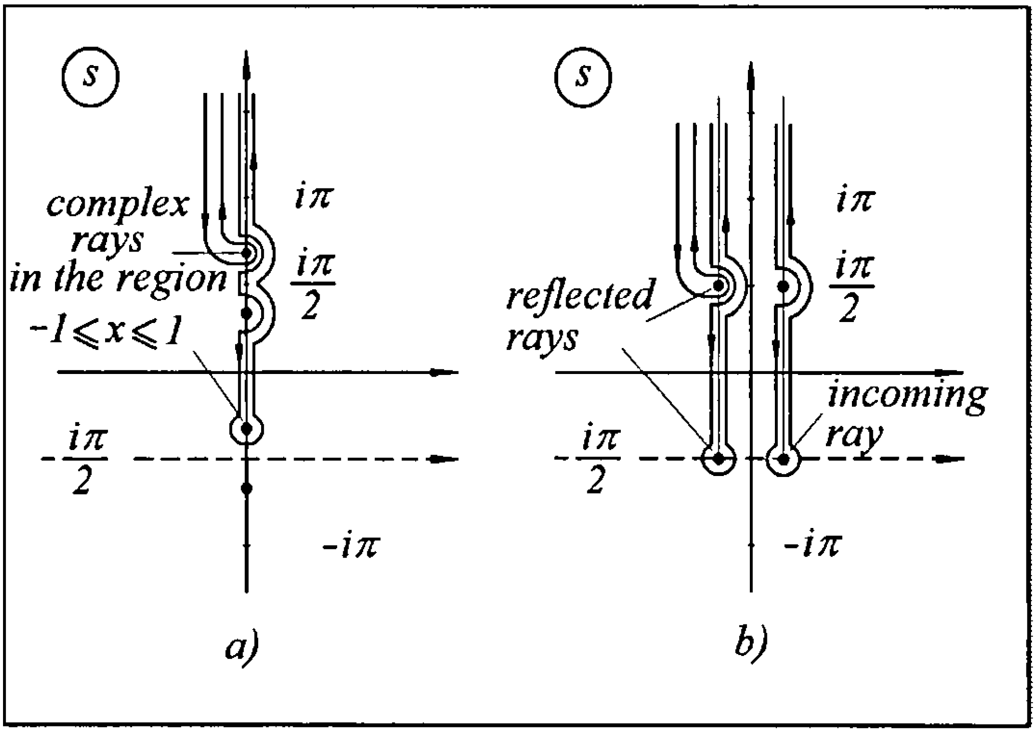

Figure 10: Passing along l from c to d.

reflected waves (see Figure 10 b)), in the region $x > 1$ we have one ray corresponding to the transmitted wave (see Figure 9 a)), and in the region $-1 < x < 1$ we have rays, corresponding to exponentially decreasing waves existing in the non transparent part of the considered media (see Figures 9 b) and 10 a)).

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