GALOIS SYMBOL OF TRANSCENDENTAL EXTENSION FIELDS

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ABSTRACT. In this paper, we find the relationship between Milnor's K-group and Galois cohomology. As like the result of Merkuriev and Suslin, we can show that for a purely transcendental extension field $F(T)$ over a certain field F, there is an canonical isomorphism $K_q^M F(T)/n \simeq H^q(F(T), \mu_n^{\otimes q})$.

1. INTRODUCTION

Let K be an arbitrary field. Then, the q-th Milnor K-group is defined by

$$\begin{aligned} K_0^M K &= \mathbb{Z}, \\ K_1^M K &= K^\times, \\ K_q^M K &= \overbrace{K^\times \otimes \cdots \otimes K^\times}^{q \text{ times}} \Big/ J_q \quad (q \geqq 2), \end{aligned}$$

where J_q is the subgroup of $K^\times \otimes \cdots \otimes K^\times$ which is generated by elements $x_1 \otimes \cdots \otimes x_q$ such that $x_i + x_j = 1$ for some i and j $(i \neq j)$.

On the other hand, one can relate Milnor K-groups to Galois cohomology groups via the following homomorphism which is induced by the cup-product pairing :

$$\begin{aligned} F^\times \times \cdots \times F^\times &\xrightarrow[\text{canonical}]{} H^1(F, \mu_n) \times \cdots \times H^1(F, \mu_n) \\ &\xrightarrow[\text{cup-product}]{} H^q(F, \mu_n^{\otimes q}). \end{aligned}$$

we shall denote the homomorphism induced by the above multilinear map by $h^q_{F,n}$.

In [3], Merkuriev and Suslin have proved the following interesting, remarkable and useful theorem :

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Theorem 1.1 ([3, Theorem 11.5]). *Let K be an arbitrary field. And let n be a positive integer which is invertible on K. Then, the following homomorphism is an isomorphism :*

$$h^2_{K,n}\colon K_2K/n\,K_2K \simeq H^2(K, \mu_n^{\otimes 2}). \tag{1.1}$$

Here μ_n is the multiplicative group of n-th root of unity.

Added to this, in [2], Kato has considered the similar problem about complete discrete valuation fields. And his results leads us to an affirmative answer in the case that a complete discrete valuation field considered is an n-dimensional local field. (In fact, he has shown more general and profound facts, which contain, of course, the above mentioned facts.)

Thus, it seems to be natural to ask whether or not the homomorphism

$$h^q_{F,n}\colon K_q^M F/n \to H^q(F, \mu_n^{\otimes q})$$

is bijective for a given field F. This problem itself is interesting and exciting to solve. However, this relation between Milnor K-groups and Galois cohomology groups is very useful and indispensable to study a given specified field. For example, the many results in higher dimensional class field theory have been proved by using this property of Galois symbols essentially.

In this paper, we shall prove the following theorem :

Theorem 1.2 (Theorem 3.6). *Let F be a field, T be an indeterminate, and n be a non-negative integer which is invertible on F. Assume that, for any finite extension field E of F and arbitrary non-negative integers q, the homomorphism*

$$h^q_{E,n}\colon K_q^M E/nK_q^M E \to H^q(E, \mu_n^{\otimes q})$$

is bijective.

Then, the homomorphism

$$h^q_{F(T),n}\colon K_q^M F(T)/nK_q^M F(T) \to H^q(F(T), \mu_n^{\otimes q})$$

is also bijective.

The above theorem gives us a large amount of fields such that the homomorphisms $h^q_{F,n}$ are always bijective. For example, we can show :

Proposition 1.3 (Proposition 4.2). *Let F be a field which is a one of the listed below :*

(1) *an algebraically closed field,*
(2) *a real closed field,*
(3) *a finite field,*

(4) *an n-dimensional local field,*
(5) *an algebraic number field.*

And, moreover, let T be an indeterminate and n be a natural number which is invertible on F.

Then, for each non-negative integer q, the following homomorphism is bijective :

$$h^q_{F(T),n} : K^M_q F(T)/n \to H^q(F(T), \mu_n^{\otimes q}).$$

Furthermore, as a bi-product, we can also show :

Corollary 1.4 (Corollary 4.4). *Let F be a one of fields listed above, $T_1, \cdots, T_m$ be indeterminates, and n be a natural number which is invertible on F.*

Then, the homomorphism

$$h^3_{F(T_1,\cdots,T_m),n} : K^M_3 F(T_1, \cdots, T_m)/n \to H^3(F(T_1, \cdots, T_m), \mu_n^{\otimes 3})$$

is bijective for any non-negative integers m.

These results are immediate consequence from Theorem 3.6 and the methods which are used in its proof. This fact tells us how our main theorem, Theorem 3.6, is powerful and useful.

Notations and Convention . For an arbitrary field F, we denote its separable closure in its fixed algebraic closure by F^{sep}.

For an arbitrary scheme X, we denote the set of its closed points by X^0.

Let X be a scheme. A sheaf $\mathcal{F}$ on X stands for a sheaf with respect to étale topology of X. Moreover, unless contrary is explicitly stated, cohomology groups $H^q(X, \mathcal{F})$ means étale cohomology groups. Especially, for a field F, we denote $H^q(\mathrm{Spec}(F), \mathcal{F})$ by $H^q(F, \mathcal{F})$. As is well-known, it coincides with Galois cohomology groups.

2. Exact Sequence

In this section, we shall prove the next proposition, which will be used in the later sections.

Proposition 2.1. *Let F be an arbitrary field and T be an indeterminate. And let ℓ be a prime number which is invertible on $F(T)$. Then, for any positive integers r and q, the following sequence is exact and split :*

$$0 \to H^q(F, \mu_\ell^{\otimes r}) \to H^q(F(T), \mu_\ell^{\otimes r}) \to \bigoplus_v H^{q-1}(F(v), \mu_\ell^{\otimes(r-1)}) \to 0, \tag{2.1}$$

where $v \in \mathrm{Spec}(F[T])^0$, and $F(v)$ is the residue field at v.

First, in order to prove Proposition 2.1, we shall show the following lemma.

Lemma 2.2. *Under the conditions and notations in Proposition 2.1, the following sequence is split and exact for any positive integers q.*

$$0 \to H^q(F, \mu_\ell) \to H^q(F(T), \mu_\ell) \to \bigoplus_v H^{q-1}(F(v), \mathbb{Z}/\ell\mathbb{Z}) \to 0, \tag{2.2}$$

Proof of Lemma 2.2. For simplicity we denote $\mathrm{Spec}(F[T])$ by X and $F(T)$ by K. Added to this, let $g\colon \mathrm{Spec}(K) \to X$ and $i_v\colon \mathrm{Spec}(F(v)) \to X$ be the canonical morphisms of schemes. By easy argument on the result explained in [4, Chap. II, §3, Example 3.9], we see that the following sequence of sheaves on $X_{\text{ét}}$ is exact and split :

$$0 \to \mathbb{G}_{m,X} \to g_*\mathbb{G}_{m,K} \to \bigoplus_v i_{v*}\mathbb{Z} \to 0. \tag{2.3}$$

From the above sequence, taking étale cohomology groups, we obtain the following short exact sequences :

$$0 \to H^q(X, \mathbb{G}_{m,X}) \to H^q(X, g_*\mathbb{G}_{m,K}) \to \bigoplus_v H^q(X, i_{v*}\mathbb{Z}) \to 0.$$

As in [4, Chap. III, §2, Example 2.22], however, we know that there exist isomorphisms as follows :

$$H^q(X, g_*\mathbb{G}_{m,K}) = H^q(K, \mathbb{G}_m)$$
$$H^q(X, i_{v*}\mathbb{Z}) = H^{q-1}(F(v), \mathbb{Q}/\mathbb{Z}).$$

Then we observe that the following sequence is exact and split :

$$0 \to H^q(X, \mu_\ell) \to H^q(F(T), \mu_\ell) \to \bigoplus_v H^{q-1}(F(v), \mathbb{Z}/\ell\mathbb{Z}) \to 0,$$

Therefore, we only have to prove $H^q(X, \mu_\ell) = H^q(F, \mu_\ell)$. But this is easily obtained from the next lemma. This completes the proof of the lemma. □

Lemma 2.3. *For an arbitrary field F, let $X = \mathrm{Spec}(F[T])$. Then,*

$$H^q(X, \mu_\ell) = H^q(F, \mu_\ell).$$

Proof. Consider the following Hochschild-Serre spectral sequence :

$$H^p(F, H^q(\bar{X}, \mu_\ell)) \Longrightarrow H^{p+q}(X, \mu_\ell),$$

where $\bar{X} = \mathrm{Spec}(F^{\mathrm{sep}}[T])$. By [4, Chap. VI, §7, Theorem 7.2], we already know $\mathrm{cd}_\ell(\bar{X}) \leq \dim \bar{X} = 1$. Therefore, we observe that the above spectral sequence degenerates and obtain the following long exact sequence of cohomologies :

$$\cdots \to H^p(F, \mu_\ell) \to H^p(X, \mu_\ell) \to H^{p-1}(F, H^1(\bar{X}, \mu_\ell)) \to \cdots .$$

On the other hand, we know that the following sequence is exact :

$$\cdots \to \Gamma(\bar{X}, \mathcal{O}_{\bar{X}})^\times \xrightarrow{\ell} \Gamma(\bar{X}, \mathcal{O}_{\bar{X}})^\times \to H^1(\bar{X}, \mu_\ell) \to \mathrm{Pic}(\bar{X}) \to \cdots .$$

From the definition of $\bar{X}$, we obtain $\mathrm{Pic}(\bar{X}) = 0$. Moreover, since $\Gamma(\bar{X}, \mathcal{O}_{\bar{X}}) = F^{\mathrm{sep}}$, we see $H^1(\bar{X}, \mu_\ell) = 0$. This complete the proof of the lemma. □

Proof of Proposition 2.1. We shall prove the proposition by induction on r.

In the case $r = 1$, we have already proved in Lemma 2.2.

Assume $r > 1$. In the case that the field F contains a primitive ℓ-th root of unity ζ_ℓ, we know $\mu_\ell^{\otimes r} \simeq \mu_\ell^{\otimes (r-1)}$. Hence, by the assumptions of induction, the proposition is valid in this case.

In the case that F does not contain ζ_ℓ, consider the following spectral sequence :

$$H^s(E/F, H^t(E, \mu_\ell^{\otimes(r-1)})) \Longrightarrow H^{s+t}(F, \mu_\ell^{\otimes r}),$$

where $E = F(\zeta_\ell)$. Since $[E\colon F]$ is relatively prime to ℓ, we have that, for each integer $s > 0$, $H^s(E/F, H^t(E, \mu_\ell^{\otimes(r-1)})) = 0$. Therefore, we obtain

$$H^s(F, \mu_\ell^{\otimes r}) = H^s(E, \mu_\ell^{\otimes(r-1)})^{\mathrm{Gal}(E/F)}.$$

Similarly, we can show

$$H^s(F(T), \mu_\ell^{\otimes r}) = H^s(E(T), \mu_\ell^{\otimes(r-1)})^{\mathrm{Gal}(E/F)}.$$

Hence, we only have to prove

$$\bigoplus_v H^s(F(v), \mu_\ell^{\otimes r}) = \left(\bigoplus_w H^s(E(w), \mu_\ell^{\otimes(r-1)})\right)^{\mathrm{Gal}(E/F)}.$$

But, it is easily obtained by the following spectral sequence in the same manner :

$$H^s(E/F, H^t(\bar{X}, \bigoplus_w i_{w*}\mu_\ell^{\otimes(r-1)}) \Longrightarrow H^{s+t}(X, \bigoplus_v i_{v*}\mu_\ell^{\otimes r}).$$

where $\bar{X} = \mathrm{Spec}(E[T])$ and $X = \mathrm{Spec}(F[T])$. From the above argument, the sequence in the case that the field considered contains ζ_ℓ is exact and split. Hence, taking the fixed part of $\mathrm{Gal}(E/F)$, we can obtain our desired exact and split sequence. □

3. Galois Symbols

First, we shall prove the following proposition. It seems to be well-known for experts.

Theorem 3.1. *Let F be a field and ℓ be a prime number which is invertible on F. Assume that the field F contains a primitive ℓ-th root of unity. Furthermore, assume that, for any non-negative integers q, the homomorphism*

$$h^q_{F,\ell}\colon K^M_q F/\ell K^M_q F \to H^q(F, \mu_\ell^{\otimes q})$$

is bijective.

Then, for any non-negative integers q and n, the homomorphism

$$h^q_{F,\ell^n}\colon K^M_q F/\ell^n K^M_q F \to H^q(F, \mu_{\ell^n}^{\otimes q})$$

is also bijective.

The above theorem is an easy consequence of the following two criteria.

Proposition 3.2. *Let F be an arbitrary field, and ℓ be a prime number which is invertible on F. Assume that, for any non-negative integer q, the homomorphism*

$$h^q_{F,\ell}\colon K^M_q F/\ell K^M_q F \to H^q(F, \mu_\ell^{\otimes q})$$

is surjective.

Then, for any non-negative integers q and n, the homomorphism

$$h^q_{F,\ell^n}\colon K^M_q F/\ell^n K^M_q F \to H^q(F, \mu_{\ell^n}^{\otimes q})$$

is also surjective.

Proposition 3.3. *Let F be an arbitrary field, and ℓ be a prime number which is invertible on F. Assume that the field F contains a primitive ℓ-root of unity. Moreover, assume that, for any non-negative integers q, the homomorphism*

$$h^q_{F,\ell}\colon K^M_q F/\ell K^M_q F \to H^q(F, \mu_\ell^{\otimes q})$$

is bijective

Then, for any non-negative integers q and n, the homomorphism

$$h^q_{F,\ell^n}\colon K^M_q F/\ell^n K^M_q F \to H^q(F, \mu_{\ell^n}^{\otimes q})$$

is also injective

Lemma 3.4. *Let F be a field, T be an indeterminate, and ℓ be a prime number which is invertible on F. Assume that, for each finite extension field E of F, the homomorphism*

$$h^q_{E,\ell}\colon K^M_q E/\ell \to H^q(E, \mu_\ell^{\otimes q})$$

is bijective.

Then, the homomorphism

$$h^q_{F(T),\ell}\colon K^M_q F(T)/\ell \to H^q(F(T), \mu_\ell^{\otimes q})$$

is also bijective.

Proof. The lemma is an easy consequence from the following commutative diagram :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K^M_q F/\ell & \longrightarrow & K^M_q F(T)/\ell & \longrightarrow & \bigoplus_v K^M_{q-1} F(v)/\ell & \longrightarrow & 0 \\
 & & \downarrow\simeq & & \downarrow & & \downarrow\simeq & & \\
0 & \longrightarrow & H^q(F, \mu_\ell^{\otimes q}) & \longrightarrow & H^q(F(T), \mu_\ell^{\otimes q}) & \longrightarrow & \bigoplus_v H^{q-1}(F(v), \mu_\ell^{\otimes(q-1)}) & \longrightarrow & 0.
\end{array}$$

Here the upper horizontal sequence is exact and split by [5, Theorem 2.3], and the lower horizontal sequence is also exact and split by Proposition 2.1. Furthermore, note that the right and left vertical arrows are bijective by the assumptions of the lemma. ☐

The next proposition is a one of applications of the bijectivity of Galois symbols, which will be also used in order to prove our main theorem.

Proposition 3.5. *Let F be a field, T be an indeterminate, and ℓ be a prime number which is invertible on F. Assume that, for all finite extensions E of F and arbitrary non-negative integers q and n, the homomorphisms*

$$h^q_{E,\ell^n}\colon K^M_q E/\ell^n \to H^q(E, \mu_{\ell^n}^{\otimes q})$$

are bijective.

Then, the following sequence is exact and split :

$$0 \to H^q(F, \mu_{\ell^n}^{\otimes q}) \to H^q(F(T), \mu_{\ell^n}^{\otimes q}) \to \bigoplus_v H^{q-1}(F(v), \mu_{\ell^n}^{\otimes(q-1)}) \to 0$$

for each non-negative integers q and n.

Proof. First, assume F contains a primitive ℓ-th root of unity ζ_ℓ. Then, by Lemma 3.4 and Theorem 3.1, we know that the homomorphism

$$h^q_{F(T),\ell^n}\colon K^M_q F(T)/\ell^n \to H^q(F(T), \mu_{\ell^n}^{\otimes q})$$

is bijective, for each non-negative integers q and n. Noting the following commutative diagram :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K^M_q F/\ell^n & \longrightarrow & K^M_q F(T)/\ell^n & \longrightarrow & \bigoplus_v K^M_{q-1} F(v)/\ell^n & \longrightarrow & 0 \\
 & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \\
0 & \longrightarrow & H^q(F, \mu_{\ell^n}^{\otimes q}) & \longrightarrow & H^q(F(T), \mu_{\ell^n}^{\otimes q}) & \longrightarrow & \bigoplus_v H^{q-1}(F(v), \mu_{\ell^n}^{\otimes(q-1)}) & \longrightarrow & 0,
\end{array}$$

where the upper horizontal sequence is exact by [5, Theorem 2.3] and all vertical arrows are bijective. Then, the exactness and splitness of the lower horizontal sequence are easily proved by elementary diagram chasing.

Next, let $E = F(\zeta_\ell)$. And, consider, for example, the next spectral sequence :

$$H^s(E/F, H^t(E, \mu_{\ell^n}^{\otimes q})) \Longrightarrow H^{s+t}(F, \mu_{\ell^n}^{\otimes q}).$$

Since $[E\colon F]$ is relatively prime to ℓ^n, we observe

$$H^s(E/F, H^t(E, \mu_{\ell^n}^{\otimes q})) = 0 \quad (s > 0).$$

Therefore, we obtain

$$H^q(F, \mu_{\ell^n}^{\otimes q}) = H^q(E, \mu_{\ell^n}^{\otimes q})^{\mathrm{Gal}(E/F)}.$$

Thus, we can reduce the exactness and splitness in general case to the case that the field F contains a primitive ℓ-th root of unity. This completes the proof of the proposition. □

The following theorem is the main result in this paper. The proof of the theorem below is similar to the one of Lemma 3.4. Therefore, we omit it.

Theorem 3.6. *Let F be a field, T be an indeterminate, and n be a non-negative integer which is invertible on F. Assume that, for any finite extension field E of F and arbitrary non-negative integers q, the homomorphism*

$$h^q_{E,n}\colon K_q^M E/nK_q^M E \to H^q(E, \mu_n^{\otimes q})$$

is bijective.

Then, the homomorphism

$$h^q_{F(T),n}\colon K_q^M F(T)/nK_q^M F(T) \to H^q(F(T), \mu_n^{\otimes q})$$

is also bijective.

4. Miscellaneous Applications

In this section, we shall present some applications of the results proved in the previous section.

First of all, we shall find a field which satisfies the assumptions of Theorem 3.6. The following lemma is well-known for experts.

Lemma 4.1. *Let F be a field which is a one of the listed below :*

(1) *an algebraically closed field,*
(2) *a real closed field,*
(3) *a finite field,*

(4) *an n-dimensional local field,*
(5) *an algebraic number field.*

And, let n be a natural number which is invertible on F.

Then, for any finite extension field E of F, the homomorphism

$$h^q_{E,n}\colon K^M_q E/n \to H^q(E, \mu_n^{\otimes q})$$

is bijective.

The above lemma assures us of the existence of fields which satisfy the assumptions in theorem 3.6. Thus, we obtain the following proposition.

Proposition 4.2. *Let F be a field which is a one of the listed in Lemma 4.1. And, moreover, let T be an indeterminate and n be a natural number which is invertible on F.*

Then, for each non-negative integer q, the following homomorphism is bijective :

$$h^q_{F(T),n}\colon K^M_q F(T)/n \to H^q(F(T), \mu_n^{\otimes q}).$$

The next is not a direct application of the result in the previous section. Since it is, however, an easy exercise about the results and the methods employed in the previous section, we put it together with the other applications.

Proposition 4.3. *Let F be a field and T be an indeterminate. And let n be a natural number which is invertible on F. Assume that the homomorphism*

$$h^3_{F,n}\colon K^M_3 F/n \to H^3(F, \mu_n^{\otimes 3})$$

is bijective.

Then, the homomorphism

$$h^3_{F(T),n}\colon K^M_3 F(T)/n \to H^3(F(T), \mu_n^{\otimes 3})$$

is also bijective.

Proof. Consider the following commutative diagram :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K^M_3 F/n & \longrightarrow & K^M_3 F(T)/n & \longrightarrow & \bigoplus_v K^M_2 F(v)/n & \longrightarrow & 0 \\
 & & \downarrow\simeq & & \downarrow & & \downarrow\simeq & & \\
0 & \longrightarrow & H^3(F, \mu_n^{\otimes 3}) & \longrightarrow & H^3(F(T), \mu_n^{\otimes 3}) & \longrightarrow & \bigoplus_v H^2(F(v), \mu_n^{\otimes 2}) & \longrightarrow & 0,
\end{array}$$

where the upper horizontal sequence is exact by [5, Theorem 2.3] and the lower horizontal sequence is also exact by the proof of Proposition 3.5. From the assumption of the proposition, the left vertical arrow is an isomorphism. Furthermore, by [3, Theorem 11.5], note also that the right vertical arrow is always an isomorphism. Therefore, we can obtain the proposition by using five lemma. □

Corollary 4.4. *Besides the conditions and notations in the previous proposition, let $T_1, \cdots, T_m$ be indeterminates.*

Then, the homomorphism

$$h^3_{F(T_1,\cdots,T_m),n}: K_3^M F(T_1,\cdots,T_m)/n \to H^3(F(T_1,\cdots,T_m), \mu_n^{\otimes 3})$$

is bijective for any non-negative integers n and m.

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