On Rotationally Symmetric Hamilton's equation for Kähler-Einstein metrics

Norihito KOISO

Max-Planck-Institut
für Mathematik
5300 Bonn 3
B.R.D.

Department of Mathematics
College of general education
Osaka University
Toyonaka Osaka 560
Japan

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0. Introduction

R.S. Hamilton [H] proved that any riemannian metric g_0 with positive Ricci curvature on a compact 3-dimensional manifold is deformed to an Einstein metric along the equation

$$(0.0.1) \qquad \frac{\partial}{\partial t} g_t = -r_t + \frac{1}{n}\bar{s}_t \cdot g_t \qquad (n = \text{dimension} = 3) ,$$

where r_t denotes the Ricci tensor of g_t, $\bar{s}_t$ the mean value of the scalar curvature. It is weakly generalized to the case on higher dimensional manifolds by S. Nishikawa [N].

Equation (0.0.1) has quite good properties: if the initial riemannian metric g_0 is invariant under a compact group action then so is g_t ; if g_0 is a Kähler metric then so is g_t. In fact, H.D. Cao [Co] proves that any Kähler metric on a compact Kähler manifold with vanishing or negative first Chern class is deformed to a Kähler-Einstein metric along equation (0.0.1).

This result suggests that, even on a compact Kähler manifold with positive first Chern class, the solution of equation (0.0.1) converges to a Kähler-Einstein metric if it exists. The first purpose of this paper is to show that it is true in some special cases due to Y. Sakane [S] and N. Koiso - Y. Sakane [KS] (Theorem 4.2), which contains rotationally symmetric metrics on the 2-dimensional sphere. On the other hand, if the manifold admits no Kähler-Einstein metrics then the solution of equation (0.0.1) can not converge. But it is interesting to see the behaviour of the solution, which is the second purpose (Theorem 5.5). We will see "how Futaki's obstruction obstructs the convergence to an Einstein metric" (Propositions 5.2 and 5.3).

In 1 we will show the long time existence for equation (0.0.1) on compact Kähler manifolds with positive first Chern class by a similar way to [Co]. In 2 we will introduce some manifolds in [KS1] and reduce equation (0.0.1) to a heat equation of one variable. In 3 we will show that the solution converges. In 4 we will see, unsing results in 3, that the metric converges to a Kähler-Einstein metric if it exists. In 5 we will treat the cases without Kähler-Einstein metrics.

1. The long time existence

Let M be an m-dimensional compact complex manifold with positive first Chern class $C_1(M)$. We consider Hamilton's equation

$$(1.0.1)\qquad \frac{\partial}{\partial t}\tilde{g}_t = -\tilde{r}_t + \tilde{g}_t \qquad (\tilde{r}_t \text{ is the Ricci tensor of } \tilde{g}_t)\ ,$$

with an initial Kähler metric $\tilde{g}_0$ in $C_1(M)$. We will show in this section the following

<u>Proposition 1.1</u> The solution of equation (1.0.1) exists for all time.

Since the short time unique existence is known by [H, Theorem 4.2], it suffices to prove some a priori estimates. We will prove it by the same way as the proof of Cao [Co] for the case of vanishing first Chern class. Fix a Kähler metric g in $C_1(M)$ and define real valued functions u_t and f by

$$(1.1.1)\qquad \tilde{g}_t = g + \partial\bar{\partial}u_t\ ,$$

$$(1.1.2)\qquad r = g + \partial\bar{\partial}f\ ,$$

where $\partial\bar{\partial}$ denotes the complex hessian. Then, using a general formula

$$(1.1.3) \qquad r = -\partial\bar{\partial} \log \det g \ ,$$

we see that equation (1.0.1) reduces to

$$(1.1.4) \quad \partial\bar{\partial}\left(\frac{\partial}{\partial t} u_t\right) = \partial\bar{\partial} \log \det \tilde{g}_t - \partial\bar{\partial} \log \det g + \partial\bar{\partial} u_t - \partial\bar{\partial} f \ ,$$

and so

$$(1.1.5) \qquad \frac{\partial}{\partial t} u_t = \log \det \tilde{g}_t - \log \det g + u_t - f \ .$$

From now on, omitting t , u denotes the maximal solution of equation (1.1.5) such that the tensor $\tilde{g}$ is positive definite. Put

$$(1.1.6) \qquad \varphi = \frac{\partial}{\partial t} u - u + f \ .$$

Then we see that

$$(1.1.7) \qquad \det \tilde{g} = e^{\varphi} \det g \ .$$

We will apply S.T. Yau's results [Y] to this equality. (But we refer to also [B].) Since we know the short time existence for equation (1.1.5), it suffices to show the a priori estimates up to third order which may depend on time. We assume that the solution exists only for finite time $[0,T)$ and denote by C_i a priori constants which may depend on T and C_i' constants which do not depend on T .

<u>Lemma 1.2</u> The function $\frac{\partial}{\partial t} u$ is time dependently estimated:

(1.2.1) $$|\frac{\partial}{\partial t} u| \le C_1 ,$$

and so

(1.2.2) $$|u| \le C_2 \quad \text{and} \quad |\varphi| \le C_3 .$$

<u>Proof</u> Put

(1.2.3) $$\Delta = - g^{i\bar{j}} \partial_i \partial_{\bar{j}} , \ \tilde{\Delta} = - \tilde{g}^{i\bar{j}} \partial_i \partial_{\bar{j}} , \ \Box = - \tilde{\Delta} - \frac{\partial}{\partial t} .$$

Differentiating equation (1.1.5) by t , we get

(1.2.4) $$\frac{\partial}{\partial t} (\frac{\partial}{\partial t} u) = \frac{\partial}{\partial t} \log \det \tilde{g} + \frac{\partial}{\partial t} u$$

$$= - \tilde{\Delta} (\frac{\partial}{\partial t} u) + \frac{\partial}{\partial t} u ,$$

and so

(1.2.5) $$\Box (e^{-t} \frac{\partial}{\partial t} u) = 0 .$$

Thus by the maximum principle, the function $e^{-t} \frac{\partial}{\partial t} u$ is bounded. Q.E.D.

<u>Lemma 1.3</u> There exist time dependent positive constants C_1 and C_2 such that

(1.3.1) $$C_1 g < \tilde{g} < C_2 g \quad .$$

<u>Proof</u> Since $\tilde{g}$ is positive definite, we see that $m - \Delta u > 0$. We want to show that Δu is bounded from below. If $m = 1$, then it is a direct consequence of Lemma 1.2 and the equality : $e^{\varphi} = m - \Delta u$. Assume that $m > 1$. Applying [Y. (2.22)] or [B, p. 126] to equality (1.1.7) we see that

(1.3.2) $$\exp\ (C_3' u)\ \tilde{\Delta}\ \{\exp\ (-C_3' u)\ (m - \Delta u)\}$$

$$\leq \Delta\varphi + m^2 \inf R + C_3' m(m - \Delta u) - (C_3' + \inf R)\ \exp\ (-\varphi/(m-1))\ (m - \Delta u)^{\frac{m}{m-1}} \ ,$$

where R is the curvature tensor of g , $\inf R = \inf\ R_{i\bar{i}j\bar{j}}$ and C_3' is a constant such that $C_3' + \inf R > 1$. On the other hand,

(1.3.3) $$\exp\ (C_3' u)\ \frac{\partial}{\partial t}\ \{\exp\ (-C_3' u)\ (m - \Delta u)\}$$

$$= -C_3' (m - \Delta u)\ \frac{\partial}{\partial t}\ u - \frac{\partial}{\partial t}\ \Delta u \ .$$

Therefore we see that

(1.3.4) $$\exp\ (C_3' u)\ \square\ \{\exp(-C_3' u)\ (m - \Delta u)\}$$

$$\geq -\ (\Delta f + m^2 \inf R - m) + (C_3' \ \frac{\partial}{\partial t}\ u - C_3' m - 1)\ (m - \Delta u)$$

$$+ (C_3' + \inf R) \exp(-\varphi/(m-1))(m-\Delta u)^{\frac{m}{m-1}} .$$

At the maximum point of $\exp(-C_3'u)(m-\Delta u)$, we use Lemma 1.2 and get

$$0 \geq -C_4' - C_5(m-\Delta u) + C_6(m-\Delta u)^{\frac{m}{m-1}} , \tag{1.3.5}$$

and so we see that $m - \Delta u \leq C_7$. Therefore

$$\exp(-C_3'u)(m-\Delta u) \leq C_8 , \tag{1.3.6}$$

which holds not only at the maximum point but also for all times in $[0,T)$. Thus

$$|\Delta u| \leq C_9 . \tag{1.3.7}$$

Now we apply Shauder estimate (e.g. [GT, Theorem 8.32]), and get

$$|u_i| \leq C_{10} , \tag{1.3.8}$$

where u_i , $u_{i\bar{j}}$ etc. denote the derivatives. Take a normal coordinate of M so that $u_{i\bar{j}} = 0$ for $i \neq j$ at a point. Then $\Delta u = -\Sigma u_{i\bar{i}}$, $u_{i\bar{i}} + 1 > 0$ and so $u_{i\bar{i}} \leq C_{11}$.

Since

$$\prod_i (1 + u_{i\bar{i}}) = \det \tilde{g} = e^{\varphi} \det g \;, \tag{1.3.9}$$

we see that $\Sigma \log (1 + u_{i\bar{i}}) \leq C_{12}$ and finally that

$$|\log \tilde{g}_{i\bar{i}}| = |\log (1 + u_{i\bar{i}})| \leq C_{13} \;, \tag{1.3.10}$$

which implies (1.3.1). Q.E.D.

<u>Lemma 1.4</u> Put $S = \tilde{g}^{i\bar{j}}\tilde{g}^{k\bar{l}}\tilde{g}^{p\bar{q}}u_{i\bar{l}p}u_{\bar{j}k\bar{q}}$. Then

$$S \leq C_1 \;. \tag{1.4.1}$$

<u>Proof</u> By [B, p. 161] we have

$$\tilde{\Delta} S \leq C(u) \cdot S + F \;, \tag{1.4.2}$$

where

$$\begin{aligned} F = &- \tilde{g}^{i\bar{j}}\tilde{g}^{k\bar{l}}\tilde{g}^{p\bar{q}}(\varphi_{i\bar{l}p}u_{\bar{j}k\bar{q}} + \varphi_{\bar{j}k\bar{q}}u_{i\bar{l}p}) \\ &+ (\tilde{g}^{i\bar{a}}\tilde{g}^{b\bar{j}}\tilde{g}^{k\bar{l}}\tilde{g}^{p\bar{q}} + \tilde{g}^{i\bar{j}}\tilde{g}^{k\bar{a}}\tilde{g}^{b\bar{l}}\tilde{g}^{p\bar{q}} + \tilde{g}^{i\bar{j}}\tilde{g}^{k\bar{l}}\tilde{g}^{p\bar{a}}\tilde{g}^{b\bar{q}})\varphi_{\bar{a}b}u_{i\bar{l}p}u_{\bar{j}k\bar{q}} \end{aligned} \tag{1.4.3}$$

and $C(u)$ is a function depending only on $u_{i\bar{j}}$ (and g). The time derivative $-\frac{\partial}{\partial t} S$ becomes exactly the function F replacing φ by $\frac{\partial}{\partial t} u$. Thus, (1.4.4) $$\square S \geq - C_2 - C_3 \cdot S \;.$$

Moreover, by [B, p. 151] we have

(1.4.5) $$\tilde{\Delta}\Delta u = - \Delta\varphi + \tilde{g}^{i\bar{j}}\tilde{g}^{k\bar{l}} u_{i\bar{l}p} u_{\bar{j}k\bar{p}} + C(u)$$

$$\geq - \Delta\varphi + C_4 \cdot S - C_5 \;,$$

and so

(1.4.6) $$\square \Delta u \leq \Delta\varphi - C_4 \cdot S + C_5 - \Delta \frac{\partial}{\partial t} u$$

$$\leq - C_4 \cdot S + C_6 \;.$$

Let C_7 be a constant such that $C_7 \cdot C_4 > C_2$. Then

(1.4.7) $$\square (S - C_7 \cdot \Delta u) \leq (C_7 \cdot C_4 - C_2) S + C_8$$

$$\leq (C_7 \cdot C_4 - C_2)(S - C_7 \cdot \Delta u) + C_9 \;.$$

(Compare with [Y, (3.4)].) Therefore, by the maximum principle, we see that $S - C_7 \cdot \Delta u \leq C_{10}$ and so $S \leq C_{11}$.

Q.E.D.

Proof of Proposition 1.1 Differentiating equation (1.1.5) we have

$$\square\, u_k = -\tilde{g}^{i\bar{j}}\partial_k g_{i\bar{j}} + g^{i\bar{j}}\partial_k g_{i\bar{j}} - u_k + f_k \ . \tag{1.4.8}$$

We saw that the coefficients of $\square$ and the right hand side are bounded with their space derivatives in finite time $[0,T)$. Therefore Schauder estimate (e.g. [LSU, III Theorem 11.1) allows us to estimate $H^{\alpha,\alpha/2}$ norm of the space derivatives of u_k , where $H^{\alpha,\alpha/2}$ means weighted Hölder continuity counting the time variable as half time of space variables. Thus the coefficients of $\square$ and the right hand side are bounded in $H^{\alpha,\alpha/2}$ norm, and so u_k is bounded in $H^{\alpha+2,\alpha/2+1}$ norm, again by Schauder estimate (e.g. [LSU, IV Theorem 5.1]).

We can repeat this procedure and see that u is bounded in C^∞ norm in finite time. Thus we can extend the solution u over $t = T$, which contradicts to the assumption that T is maximal. Q.E.D.

Finally, we give a sufficient condition for the convergence of a modified solution, which will be used to treat the cases without Einstein metrics. Let V be a holomorphic vector field on M and define a one-parameter family of riemannian metrics $\tilde{h} = \gamma^{-1*}\tilde{g}$, where $\gamma(t) = \exp tV$.

Proposition 1.5 Assume that $\tilde{h}$ converges uniformly to g and is bounded in C^1 norm. Then $\tilde{h}$ converges in C^∞ norm.

Proof Define a function v by $\tilde{h} = g + \partial\bar{\partial}v$. Then it is easy to check that

(1.5.1) $$\partial\bar{\partial}\left(\frac{\partial}{\partial t} v - \log\det\tilde{h}\right)$$

$$= -\partial\bar{\partial}V[v] - \partial\bar{\partial}f + \partial\bar{\partial}v - \partial\bar{\partial}\log\det g - L_V g \ .$$

Since V is holomorphic, there is a C^∞ function p so that $L_V g = \partial\bar{\partial}p$. Thus

(1.5.2) $$\frac{\partial}{\partial t} v - \log\det\tilde{h} = -(f + p + \log\det g) - V[v] + v \ ,$$

modulo constant depending on time. Therefore we can apply the same argument as Proof of Proposition 1.1, moreover we can estimate the C^∞ norm of v_k time-independently.

Assume that $\tilde{h}$ does not converge to g in C^r norm. Then there exists a sequence $\tilde{h}(t_i)$ which converges to some $\tilde{h}(\infty) \neq g$ in C^r norm, which is a contradiction. Q.E.D.

Corollary 1.6 If Δu converges uniformly to 0 , then $\tilde{g}$ converges to g in C^∞ norm.

<u>Proof</u> By the last argument of Proof of Lemma 1.3, we see that $u_{i\bar{j}}$ and u_i converge uniformly to 0 . Remark that the constants C_i in Proof of Lemma 1.4 depend only on the C^0 norm of $u_{i\bar{j}}$. Thus we can estimate the function S time independently and apply Proposition 1.5 with $V = 0$. Q.E.D.

2. Reduction to a heat equation of one variable

First we recall [KS1]. Let (N, g_N) be a compact Kähler--Einstein manifold with $r_N = g_N$ and $\pi : L \to N$ a hermitian holomorphic line bundle. We assume that the eigenvalues of the Ricci form B of L with respect to g_N are constant on N. We put $\overset{\circ}{L} = L \setminus \{0\text{-section}\}$ and consider a compact complex manifold M which contains $\overset{\circ}{L}$ as an open dense subset. Assume that $M \setminus \overset{\circ}{L}$ has two connected components N_1 and N_2 which are closed submanifolds of M of codimension d_1 and d_2, respectively. The indexes i of N_i are chosen so that N_1 (resp. N_2) coincides with the image of $\{0\text{-section}\}$ (resp. $\{\infty\text{-section}\}$) with respect to the continuously extended map: $P(1 \oplus L) \to M$ of the inclusion map: $\overset{\circ}{L} \longrightarrow M$.

We consider Kähler metrics g on M of the form

$$g = ds^2 + (ds \circ J)^2 + \pi^* g_N^s \tag{2.0.1}$$

on $\overset{\circ}{L}$, where s is a function on M depending only on the norm of L and increasing for the norm, and g_N^s a one-parameter family of Kähler metrics on N such that g_N^0 is the Einstein metric g_N. Let $\mathcal{M}$ be the set of all such Kähler metrics g which represent the first Chern class $C_1(M)$. We assume that the set $\mathcal{M}$ is non-empty, in particular that $C_1(M)$ is positive.

[KS1, Theorem 4.1] using g° as a reference Kähler metric.

Q.E.D.

Lemma 2.4 Let $g^\circ \in \mathcal{M}$ and (x°, φ°) the corresponding pair. A Kähler metric g on M is an element of $\mathcal{M}$ if and only if there exists a C^∞ function $h(x^\circ)$ of x° such that $g = g^\circ + \partial\bar{\partial}h$.

Proof Assume that such a function h is given. We put $x = x^\circ + \frac{1}{2} H[h]$ and $\varphi(x) = \varphi^\circ(x^\circ) + \frac{1}{2} H^2[h]$. Then the functions x and φ satisfy the conditions in Lemma 2.3, and so a Kähler metric $g_1 \in \mathcal{M}$ corresponds. By [KS1, Lemmas 1.2 and 1.3], g_1 coincides with g . Conversely, assume that a Kähler metric $g \in \mathcal{M}$ is given and let (x, φ) be the corresponding pair of functions. Since $x = x^\circ$ on $N_1 \cup N_2$, the function $(x - x^\circ)/\varphi^\circ(x^\circ)$ is a C^∞ function of x° . Therefore there is a C^∞ function $h(x^\circ)$ such that $x = x^\circ + \frac{1}{2} H[h]$, and so $\varphi(x) = H[x] = \varphi^\circ(x^\circ) + \frac{1}{2} H^2[h]$. By [KS1, Lemmas 1.2 and 1.3], we see that $g = g^\circ + \partial\bar{\partial}h$. Q.E.D.

Now we define a real number E by

$$\int_{-d_1}^{d_2} x e^{-Ex} Q(x)\, dx = 0 . \tag{2.4.1}$$

Since $Q(x) > 0$ on $(-d_1, d_2)$, such E is unique. Remark that the left hand side of equation (2.4.1) with $E = 0$ gives Futaki's

Now all hypotheses in sections 1 and 2 in [KS1] are satisfied. We have $P^1(\mathbb{C})$ - bundles M over Kähler C - space N with $C_1(N) > 0$ as typical examples of such manifolds, provided that $C_1(M) > 0$.

Let $g \in \mathcal{M}$. Since $\overset{\circ}{L}$ is a $\mathbb{C}^*$ - bundle over N , we can define a holomorphic vector field H on M corresponding to the holomorphic action of $\mathbb{R}^+$ so that $H[s] > 0$ and $\exp 2\pi(JH) = \mathrm{id}_M$. Remark that the function $H[s]$ is a function of s and define a function x on M by $x = \int_0^s H(s)\,ds$. Put $\varphi(x) = H[s]^2 = g(H, H)$ and $Q(x) = \det(\mathrm{id} - x g_N^{-1} B)$. Let [min x, max x] be the range of x .

<u>Lemma 2.1</u> ([KS1, Lemma 2.1]) The function x is a C^∞ function on M . A function $h(x)$ of x is a C^∞ function on M if and only if it is C^∞ as a function of x , i.e., if it extends to a C^∞ function on an open interval containing [min x, max x]. For such a function h , we see that

$$H[h] = \varphi(x) \frac{\partial}{\partial x} h . \tag{2.1.1}$$

<u>Lemma 2.2</u> (1) $[\min x, \max x] = [-d_1, d_2]$. (2) The function $\varphi(x)$ is a C^∞ function of x . It is positive on $(-d_1, d_2)$, vanishes at $x = -d_1$ and d_2 , and its derivative is 2 (resp. -2) at $x = -d_1$ (resp. d_2) . (3) The function $Q(x)$ is positive on $(-d_1, d_2)$ and contains $(1+x/d_1)$ (resp. $(1-x/d_2)$) as a factor of power $d_1 - 1$ (resp. $d_2 - 1$) . (4) If a C^∞

function f on M satisfies the equation $r - g = \partial\bar{\partial}f$, then it is a C^∞ function of x and satisfies the equation

$$(2.2.1) \qquad \frac{\partial}{\partial x}\varphi + 2x + \frac{\varphi}{Q}\frac{\partial}{\partial x}Q + \varphi\frac{\partial}{\partial x}f = 0 .$$

<u>Proof</u> These are shown in section 2 of [KS1], provided that (1) is assumed. When we do not assume (1), we only know that the left hand side of equation (2.2.1) is a constant C and that (1) holds if and only if $C = 0$. However, by the same way as [KS1], we can check that $\partial\bar{\partial}(\iota^* f) = r_N - g_N + \frac{1}{2}CB$, where ι is a section of $\pi : L \longrightarrow N$ such that $s \circ \iota = 0$. Thus under our assumption that $r_N = g_N$ and the eigenvalues of B are constant on N , we see that $C = 0$. Q.E.D.

<u>Lemma 2.3</u> Let $\varphi(y)$ be a function of y which satisfies the properties in Lemma 2.2 (2). Then there exists a continuous function x on M with range $[-d_1, d_2]$ whose restriction on $\mathring{L}$ is a C^1 function depending only on the norm of L and satisfies the equation : $H[x] = \varphi(x)$. Such a function x is automatically C^∞ on M and is unique up to the holomorphic action of $\mathbb{R}^+$. Moreover, there exists a unique Kähler metric $g \in \mathcal{M}$ to which the functions x and φ correspond.

<u>Proof</u> Remark that we assume that $\mathcal{M}$ is non-empty and so we can take an element g° of $\mathcal{M}$. Then the construction of the function x and the metric g can be done by the same way as

obstruction $\int_M H[f]$ for existence of Kähler-Einstein metrics with respect to the holomorphic vector field H ([KS1, (3.1.1)]). Define a function $\varphi^\circ(x)$ by

$$\varphi^\circ(x) = -2Q(x)^{-1}e^{Ex}\int_{-d_1}^{x} xe^{-Ex}Q(x)\,dx \; . \tag{2.4.2}$$

This function satisfies condition (2) in Lemma 2.2 and hence, taking a function x° so that $\varphi^\circ(x^\circ) = H[x^\circ]$, defines a Kähler metric g° on M. Remark that the function φ° satisfies the equation

$$\frac{\partial}{\partial x}\varphi^\circ + 2x + \frac{\varphi^\circ}{Q}\frac{\partial}{\partial x}Q = E\varphi^\circ \; . \tag{2.4.3}$$

Combining with equation (2.2.1), we get

<u>Proposition 2.5</u> ([KS1, Theorem 4.1, 4.2]). The following conditions are equivalent. (1) $E = 0$. (2) g° is an Einstein metric. (3) M admits a Kähler-Einstein metric. (4) Eutaki's obstruction of M vanishes.

Now we solve Hamilton's equation (0.0.1) with an initial Kähler metric $\tilde{g}_0 \in \mathcal{M}$ and denote by $\tilde{g}_t$ the solution. Remark that all $\tilde{g}_t$ are in $\mathcal{M}$ by Lemmas 2.2, 2.3 and 2.4. Thus we can take two coordinate systems of $M \times \mathbb{R}^+$ essentially : (x°, t) and (x_t, t), where x_t corresponds to $\tilde{g}_t$. We denote by D_t (resp. $\frac{\partial}{\partial t}$) the time differential with respect to (x°, t) (resp. (x_t, t)). We will omit t of x_t and φ_t. Hamilton's equation is given by

$$D_t\tilde{g} = -\tilde{r} + \tilde{g} = -\partial\bar{\partial}f_t \ , \tag{2.5.1}$$

where

$$\tilde{g}(H, H) = \varphi(x) \ , \tag{2.5.2}$$

$$(\partial\bar{\partial}f_t)(H, H) = \frac{1}{2}\left(\varphi(x)\frac{\partial}{\partial x}\right)^2 f_t \qquad \text{([KS1, Lemma 1.3])} \ . \tag{2.5.3}$$

And by equation (2.5.1),

$$2D_t\varphi = -\left(\varphi\frac{\partial}{\partial x}\right)^2 f = \varphi\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\varphi + 2x + \frac{\varphi}{Q}\frac{\partial}{\partial x}Q\right) \ . \tag{2.5.4}$$

On the other hand, we see that

$$D_t = D_t x\cdot\frac{\partial}{\partial x} + \frac{\partial}{\partial t} \ , \qquad \left[D_t, \varphi\frac{\partial}{\partial x}\right] = [D_t, H] = 0 \ , \tag{2.5.5}$$

and so

$$\left[D_t x\cdot\frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \varphi\frac{\partial}{\partial x}\right] = 0 \ , \tag{2.5.6}$$

$$\frac{\partial}{\partial t}\varphi = \varphi\frac{\partial}{\partial x}(D_t x) - D_t x\cdot\frac{\partial}{\partial x}\varphi \ , \tag{2.5.7}$$

$$D_t\varphi = \frac{\partial}{\partial t}\varphi + D_t x\cdot\frac{\partial}{\partial x}\varphi = \varphi\frac{\partial}{\partial x}(D_t x) \ , \tag{2.5.8}$$

3. Convergence of the heat equation of one variable.

We continue the discussion in 2. To prove the convergence of ϕ , we need the following

<u>Lemma 3.1</u> $\varphi^\circ(x) - x\frac{\partial}{\partial x}\varphi^\circ(x) > 0$ on $[-d_1, d_2]$.

<u>Proof</u> Put $\xi(x) = xe^{-Ex}Q(x)$ and $\eta(x) = \int_{-d_1}^{x}\xi(x)dx$. Remark that $\frac{\partial}{\partial x}(e^{-Ex}Q\varphi^\circ) = -2\xi$ and $\varphi^\circ = -2e^{Ex}Q^{-1}\eta$.

Therefore,

$$(3.1.1) \quad e^{-Ex}Q(x)(\varphi^\circ - x\frac{\partial}{\partial x}\varphi^\circ) = 2e^{Ex}Q(x)^{-1}(\xi^2 - \eta\frac{\partial}{\partial x}\xi) .$$

Since we know that $\varphi^\circ - x\frac{\partial}{\partial x}\varphi^\circ = 2d_1$ (resp. $2d_2$) > 0 at $x = -d_1$ (resp. d_2) , it suffices to prove that $\xi^2 - \eta\frac{\partial}{\partial x}\xi > 0$ on $(-d_1, d_2)$. Then $\varphi^\circ > 0$ and so $\eta < 0$. Moreover, since $\xi(0) = 0$, $\eta(0) < 0$ and $Q(0) = 1$, we see that $\xi^2 - \eta\frac{\partial}{\partial x}\xi > 0$ at $x = 0$. In the following, we consider only on the interval $(0, d_2)$. Similar proof holds on $(-d_1, 0)$.

Since the function $Q(x)$ is the product of polynomials of first order, the second derivative $\frac{\partial^2}{\partial x^2}\log\xi$ is negative on $(0, d_2)$, which implies that the first derivative $\frac{\partial}{\partial x}\xi$ acrosses with the x-axis at most once. If $\frac{\partial}{\partial x}\xi$ does not across with the x-axis, then $\xi^2 - \eta \frac{\partial}{\partial x} \xi > 0$ on $(0, d_2)$, which completes the proof. Assume that $\frac{\partial}{\partial x} \xi$ acrosses with the x-axis at $x = a$. Then $\xi^2 - \eta \frac{\partial}{\partial x} \xi > 0$ on $(0, a]$ and so we may consider only on the interval (a, d_2).

Thus it suffices to prove that $(\frac{\partial}{\partial x} \xi)^{-1}\xi^2 - \eta < 0$ on (a, d_2), because $\frac{\partial}{\partial x} \xi < 0$. But we see that

$$(3.1.2) \quad \frac{\partial}{\partial x} \{(\frac{\partial}{\partial x} \xi)^{-1}\xi^2 - \eta\} = (\frac{\partial}{\partial x} \xi)^{-2}\xi\{(\frac{\partial}{\partial x} \xi)^2 - \xi \frac{\partial^2}{\partial x^2} \xi\} ,$$

here,

$$(3.1.3) \qquad 0 > \frac{\partial^2}{\partial x^2} \log \xi = \xi^{-2}\{\xi \frac{\partial^2}{\partial x^2} \xi - (\frac{\partial}{\partial x} \xi)^2\} .$$

Therefore, the function $(\frac{\partial}{\partial x} \xi)^{-1}\xi^2 - \eta$ is increasing. Moreover, at $x = d_2$, $\frac{\partial}{\partial x} \xi < 0$, $\xi^2 \geq 0$ and $\eta = 0$. Hence $(\frac{\partial}{\partial x} \xi)^{-1}\xi^2 - \eta < 0$ on (a, d_2). Q.E.D.

<u>Lemma 3.2</u> The function ϕ converges uniformly to 0 in exponential order.

<u>Proof</u> By Lemmas 2.6, 3.1 and equation (2.7.2), we see that the minimum of ϕ is increasing. By Lemma 3.1 we can choose a positive number C smaller than $(\max \varphi^\circ)^{-1} \cdot \min(\varphi^\circ - x \frac{\partial}{\partial x} \varphi^\circ) \cdot (1 + \min \phi_0)$. Then by equation (2.7.2),

we get

$$(3.2.1)\quad 2\varphi^\circ \frac{\partial}{\partial t}(e^{Ct}\phi)$$

$$= \varphi^\circ\varphi \frac{\partial^2}{\partial x^2}(e^{Ct}\phi) - e^{-Ct}(\varphi^\circ \frac{\partial}{\partial x}(e^{Ct}\phi))^2 - 2x\varphi^\circ \frac{\partial}{\partial x}(e^{Ct}\phi)$$

$$- 2\{(\varphi^\circ - x\frac{\partial}{\partial x}\varphi^\circ)(1+\phi) - C\varphi^\circ\}e^{Ct}\phi ,$$

from which we conclude that the function $e^{Ct}\phi$ is bounded by the maximum principle. Q.E.D.

<u>Lemma 3.3</u> The function $\varphi^\circ \frac{\partial}{\partial x}\phi$ converges uniformly to 0 in exponential order.

<u>Proof</u> Put $\xi = \varphi^\circ \frac{\partial}{\partial x}\phi + Cx\phi$, where C is a constant. By definition we see that

$$(3.3.1)\quad 2\varphi^\circ \frac{\partial}{\partial t}\xi = \varphi^\circ \frac{\partial}{\partial x}(2\varphi^\circ \frac{\partial}{\partial t}\phi) - (\frac{\partial}{\partial x}\varphi^\circ - Cx) - 2\varphi^\circ \frac{\partial}{\partial t}\phi .$$

Here,

$$(3.3.2)\quad 2\varphi^\circ \frac{\partial}{\partial t}\phi = \varphi \frac{\partial}{\partial x}\xi - \xi^2 - \{(\frac{\partial}{\partial x}\varphi^\circ - Cx)\phi + \frac{\partial}{\partial x}\varphi^\circ + (C+2)x\}\xi$$

$$- \{(C+2)(\varphi^\circ - x\frac{\partial}{\partial x}\varphi^\circ)(1+\phi) - C(C+2)x^2\}\phi .$$

and so,

$$(3.3.3)\quad \varphi^\circ \frac{\partial}{\partial x}\left(2\varphi^\circ \frac{\partial}{\partial x}\phi\right) = \varphi^\circ\varphi \frac{\partial^2}{\partial x^2}\xi + \left(\frac{\partial}{\partial x}\xi \text{ term}\right) - \left(\frac{\partial}{\partial x}\varphi^\circ - Cx\right)\xi^2$$

$$- \left\{ (C+2)\left(2\varphi^\circ - x\frac{\partial}{\partial x}\varphi^\circ\right) - C(C+2)x^2 + \varphi^\circ \frac{\partial^2}{\partial x^2}\varphi^\circ + (\phi \text{ term})\right\}\xi$$

$$+ (\phi \text{ term}) .$$

Substituting equations (3.3.2) and (3.3.3) into equation (3.3.1), we get

$$(3.3.4)\quad 2\varphi^\circ \frac{\partial}{\partial t}\xi = \varphi^\circ\varphi \frac{\partial^2}{\partial x^2}\xi + \left(\frac{\partial}{\partial x}\xi \text{ term}\right)$$

$$- \left\{ (C+2)\left(2\varphi^\circ - x\frac{\partial}{\partial x}\varphi^\circ\right) + \left(\varphi^\circ \frac{\partial^2}{\partial x^2}\varphi^\circ - 2x\frac{\partial}{\partial x}\varphi^\circ - \left(\frac{\partial}{\partial x}\varphi^\circ\right)^2 + (\phi \text{ term})\right)\right\}\xi$$

$$+ (\phi \text{ term}) .$$

If we remark that $2\varphi^\circ - x\frac{\partial}{\partial x}\varphi^\circ \geq \varphi^\circ - x\frac{\partial}{\partial x}\varphi^\circ > 0$ and choose C sufficiently large, then we can choose a positive constant C_1 so that the function $e^{C_1 t}\xi$ is bounded by a similar way to Proof of Lemma 3.2. We know that ϕ converges to 0, so is $\varphi^\circ \frac{\partial}{\partial x}\phi$.

Q.E.D.

<u>Lemma 3.4</u> The function $\varphi^{-1}\cdot\left(\varphi^\circ \frac{\partial}{\partial x}\phi - 2x\phi\right)$ is bounded.

<u>Proof</u> Put $\xi = \varphi^{-1}\cdot\left(\varphi^\circ \frac{\partial}{\partial x}\phi - 2x\phi\right)$. Remark that ξ is a

C^{∞} function on $[-d_1, d_2]\times[0,\infty)$. Since

$$\frac{\partial}{\partial x}\xi = 2\varphi^{-2}\cdot\frac{\partial}{\partial t}\varphi , \tag{3.4.1}$$

we get

$$2\frac{\partial}{\partial t}\xi = -(\varphi^{\circ}\frac{\partial}{\partial x}\phi - 2x\phi)\frac{\partial}{\partial x}\xi + 2\varphi^{-1}\cdot\frac{\partial}{\partial t}(\varphi^{\circ}\frac{\partial}{\partial x}\phi - 2x\phi) . \tag{3.4.2}$$

On the other hand,

$$\varphi^{\circ}\frac{\partial}{\partial x}\phi - 2x\phi = \frac{\partial}{\partial x}\varphi - (\frac{\partial}{\partial x}\varphi^{\circ} + 2x)\phi . \tag{3.4.3}$$

Therefore,

$$2\frac{\partial}{\partial t}\xi = \varphi\frac{\partial^2}{\partial x^2}\xi + \{-\varphi\xi + 2\frac{\partial}{\partial x}\varphi - (\frac{\partial}{\partial x}\varphi^{\circ}+2x)(1+\phi)\}\frac{\partial}{\partial x}\xi . \tag{3.4.4}$$

In particular, at $x = -d_1, d_2$,

$$2\frac{\partial}{\partial t}\xi(-d_1) = 2(d_1+1)\frac{\partial}{\partial x}\xi(-d_1) , \tag{3.4.5}$$

$$2\frac{\partial}{\partial t}\xi(d_2) = -2(d_2+1)\frac{\partial}{\partial x}\xi(d_2) .$$

Thus the maximum principle completes the proof. Q.E.D.

<u>Lemma 3.5</u> The functions $(\varphi^{\circ})^{-1}\phi$ and $\frac{\partial}{\partial x}\phi$ are bounded.

<u>Proof</u> Put $\eta = (\varphi^\circ)^{-1}\phi$. Since ϕ converges uniformly to 0 , it suffices to prove that η is bounded on a neighbourhood of $x = -d_1, d_2$. In fact, then $\frac{\partial}{\partial x}\phi$ is bounded by Lemma 3.4. Remark that the function ξ in Lemma 3.4 is bounded and so is $(1+\phi)\xi$. But

$$(3.5.1) \qquad (1+\phi)\xi = \varphi^\circ \frac{\partial}{\partial x}\eta + (\frac{\partial}{\partial x}\varphi^\circ - 2x)\eta \ .$$

Thus we can choose a positive constant C_1 so that

$$(3.5.2) \qquad -C_1 - (\frac{\partial}{\partial x}\varphi^\circ - 2x)\eta < \varphi^\circ \frac{\partial}{\partial x}\eta < C_1 - (\frac{\partial}{\partial x}\varphi^\circ - 2x)\eta \ .$$

If we choose a sufficiently small neighbourhood $(a, d_2]$ of $x = d_2$, then we can select a positive constant C_2 so that $\frac{\partial}{\partial x}\varphi^\circ - 2x < -C_2$ on $(a, d_2]$. Therefore, by inequality (3.5.2), if $\eta > C_1C_2^{-1}$ then $\frac{\partial}{\partial x}\eta > 0$, and if $\eta < -C_1C_2^{-1}$ then $\frac{\partial}{\partial x}\eta < 0$. On the other hand, substituting $x = d_2$ into inequality (3.5.2), we see that

$$(3.5.3) \qquad -(d_2+1)^{-1}C_1 < \eta(d_2) < (d_2+1)^{-1}C_1 \ .$$

Thus $|\eta| < \max\{(d_2+1)^{-1}C_1 \ , \ C_1C_2^{-1}\}$ on $(a, d_2]$. We can prove for $[-d_1, b)$ by the same way. Q.E.D.

<u>Corollary 3.6</u> The function $(\varphi^\circ)^{-1}\phi$ converges to 0 in L^1 norm.

4. Convergence to a Kähler-Einstein metric

In this section we assume that Futaki's obstruction of M vanishes, i.e., $E = 0$ in equality (2.4.1). By Proposition 2.5, the function φ° defines a Kähler-Einstein metric. By equation (2.7.1) we know that

$$(4.0.1) \qquad 2D_t x = \varphi^\circ \frac{\partial}{\partial x} \phi - 2x\phi ,$$

and the right hand side converges uniformly to 0 in exponential order by Lemmas 3.2 and 3.3. Therefore the function x converges uniformly to a function x_∞ . Since the function φ also converges to the function φ° , the function x_∞ satisfies the equation $H[x_\infty] = \varphi^\circ(x_\infty)$, and thus the pair $(x_\infty, \varphi^\circ)$ defines a Kähler-Einstein metric by Lemma 2.3. We replace x° by x_∞ so that x converges uniformly to x° . Note that the pair (x°, φ°) corresponds to a Kähler-Einstein metric $g^\circ \in \mathcal{M}$. Since $\varphi(x) \frac{\partial}{\partial x} = \varphi^\circ(x^\circ) \frac{\partial}{\partial x^\circ} = H$, there exists a function $c(t)$ of t such that

$$(4.0.2.) \qquad \int_0^x \frac{dy}{\varphi(y)} = \int_0^{x^\circ} \frac{dy}{\varphi^\circ(y)} + c(t) .$$

Remark that $c(t)$ converges to 0 .

<u>Lemma 4.1</u> The function $\varphi^\circ(x^\circ)^{-1}\varphi(x)$ converges uniformly to 1.

Thus,

$$(4.1.6)\qquad \left|\frac{\varphi(x)}{\varphi^\circ(x^\circ)} - 1\right| \le \left|\frac{\varphi^\circ(x)}{\varphi^\circ(x^\circ)}(1+\phi(x)) - \frac{\varphi^\circ(x)}{\varphi^\circ(x^\circ)}\right| + \left|\frac{\varphi^\circ(x)}{\varphi^\circ(x^\circ)} - 1\right|$$

$$= |\phi(x)|\left|\frac{\varphi^\circ(x)}{\varphi^\circ(x^\circ)}\right| + \frac{|\varphi^\circ(x) - \varphi^\circ(x^\circ)|}{\varphi^\circ(x^\circ)}$$

$$\le |\phi(x)| \frac{\varphi^\circ(x^\circ) + C|x - x^\circ|}{\varphi^\circ(x^\circ)} + \frac{C|x - x^\circ|}{\varphi^\circ(x^\circ)}$$

$$\le |\phi(x)|(1 + 2C\varepsilon) + 2C\varepsilon \ .$$

Q.E.D.

Theorem 4.2. If Futaki's obstruction vanishes, then $\tilde{g}_t \in M$ converges to a Kähler-Einstein metric $\in M$ in C^∞ norm.

Proof We see that

$$(4.2.1)\qquad \Delta u = -\operatorname{tr}_{g^\circ}(\tilde{g} - g^\circ)$$

$$= -\varphi^\circ(x^\circ)^{-1}(\varphi(x) - \varphi^\circ(x^\circ)) \ ,$$

where u and Δ are defined by (1.1.5) and (1.2.3). On the other hand, if h is a C^∞ function of x°, then

$$(4.2.2)\qquad g^\circ(dh, dh) = \frac{1}{2}\varphi^\circ(x^\circ)\left(\frac{\partial}{\partial x^\circ} h\right)^2 \ ,$$

<u>Proof</u> First we see that

$$(4.1.1) \qquad \left| \int_0^x \frac{dy}{\varphi^\circ(y)} - \int_0^{x^\circ} \frac{dy}{\varphi^\circ(y)} \right|$$

$$\leq \left| \int_0^x \frac{dy}{\varphi^\circ(y)} - \int_0^x \frac{dy}{\varphi(y)} \right| + \left| \int_0^x \frac{dy}{\varphi(y)} - \int_0^{x^\circ} \frac{dy}{\varphi^\circ(y)} \right|$$

$$= \left| \int_0^x \frac{1}{1+\phi(y)} \varphi^\circ(y)^{-1} \phi(y)\,dy \right| + |C(t)| \ ,$$

and the last line converges uniformly to 0 by Lemmas 3.2 and 3.6. Put $C = \max \left|\frac{\partial}{\partial y} \varphi^\circ(y)\right|$ and let I be the closed interval between x and x°. If

$$(4.1.2) \qquad \left| \int_0^x \frac{dy}{\varphi^\circ(y)} - \int_0^{x^\circ} \frac{dy}{\varphi^\circ(y)} \right| < \varepsilon \ ,$$

then

$$(4.1.3) \qquad \varepsilon > \left| \int_{x^\circ}^x \frac{dy}{\varphi^\circ(y)} \right| \geq |x - x^\circ| \cdot \min_I \{\varphi^\circ(y)^{-1}\} \ ,$$

and so

$$(4.1.4) \qquad |x - x^\circ| \leq \varepsilon \cdot \max_I \{\varphi^\circ(y)\} \leq \varepsilon \cdot \{\varphi^\circ(x^\circ) + C \cdot |x - x^\circ|\} \ .$$

Therefore, if ε is sufficiently small then

$$(4.1.5) \qquad |x - x^\circ| \leq 2\varepsilon\varphi^\circ(x^\circ) \ .$$

because $g^\circ(H, H) = \varphi^\circ(x^\circ)$ and $H[h] = \varphi^\circ(x^\circ) \dfrac{\partial}{\partial x^\circ} h$. Therefore, if $\dfrac{\partial}{\partial x^\circ} \Delta u$ is bounded, then u is bounded in C^3 norm (up to constant factor). Now,

$$(4.2.3) \quad \frac{\partial}{\partial x^\circ} \Delta u = - \frac{\partial}{\partial x^\circ} \left(\frac{\varphi(x)}{\varphi^\circ(x^\circ)} \right)$$

$$= \frac{\varphi(x)}{\varphi^\circ(x^\circ)} \left\{ \frac{1}{\varphi^\circ(x^\circ)} \left(\frac{\partial}{\partial x} \varphi^\circ(x) - \frac{\partial}{\partial x^\circ} \varphi^\circ(x^\circ) \right) + \frac{\phi(x)}{\varphi^\circ(x^\circ)} + \frac{\varphi^\circ(x)}{\varphi^\circ(x^\circ)} \frac{\partial}{\partial x} \phi(x) \right\},$$

and the last line is bounded by Lemma 4.1, inequality (4.1.4) and Lemma 3.5. Combining equation (4.2.1) with Lemma 4.1, we see that the assumption of Proposition 1.5 with $V = 0$ holds, or more directly, we can apply Corollary 1.6. Q.E.D.

5. Pseudo-convergence to a quasi-Einstein metric

To study the cases when there are no Einstein metrics we give the following

<u>Definition 5.1</u> A riemannian metric g is called a quasi--Einstein metric if there is a vector field V such that $r - \dim^{-1} \bar{s}\, g = L_V g$.

We easily see

<u>Proposition 5.2</u> The solution of Hamilton's equation (0.0.1) whose initial riemannian metric g_0 is a quasi-Einstein metric is given by $g_t = \gamma_t^{-1*} g_0$, where $\gamma_t = \exp tV$. In particular, if g_0 is not Einstein then g_t does not converge.

For the case of Kähler manifolds with positive first Chern class, we get

<u>Proposition 5.3</u> A Kähler metric g in the first Chern class is a quasi-Einstein metric if and only if $r - g = L_V g$ for some holomorphic vector field V . In particular, such a Kähler metric is an Einstein metric if and only if Futaki's obstruction vanishes.

<u>Proof</u> Put $r - g = \partial\bar{\partial} f$. Then by definition of V ,

(5.3.1) $$D_i V_{\bar{j}} + D_{\bar{j}} V_i = D_i D_{\bar{j}} f \ ,$$

(5.3.2) $$D_i V_j + D_j V_i = 0 \ ,$$

where D denotes the covariant derivative. Therefore,

(5.3.3) $$\begin{aligned} - D^k (D_k V_i - D_i V_k) &= - D_k D^k V_i + D_i D^k V_k \\ &= - D_k (D^k D_i f - D_i V^k) + D_i D^k V_k \\ &= D_i (- D_k D^k f + D_k V^k + D^k V_k) = 0 \ , \end{aligned}$$

which implies that $D_i V_j = 0$, i.e., V is holomorphic.

Assume that Futaki's obstruction vanishes. By definition ([F]),

(5.3.4) $$\int V [f] v_g = 0 \ .$$

On the other hand, there is a complex valued function η such that $V_i = D_i \eta$, because the first Chern class is positive and so there are no non-trivial harmonic 1-forms. Substituting it into equality (5.3.1), we get

(5.3.5) $$D_i D_{\bar{j}} (\eta + \bar{\eta} - f) = 0 \ ,$$

and so there is a real valued function v such that $\eta = \frac{1}{2} f + \sqrt{-1}\, v$. We substitute it into equality (5.3.4) and see that $df = 0$. Q.E.D.

Now we come back to the situation of 2 and 3 and assume that Futaki's obstruction of M does not vanish, i.e., $E \neq 0$ in equality (2.4.1). Then we get

<u>Proposition 5.4</u> The Kähler metric g° corresponding to the pair (x°, φ°) is a quasi-Einstein metric but not an Einstein metric.

<u>Proof</u> Put $r^\circ - g^\circ = \partial\bar{\partial} f$. By equalities (2.2.1) and (2.4.3) we see that

$$(5.4.1) \qquad (df)H = -E\varphi^\circ(x^\circ) = -Eg^\circ(H, H) ,$$

i.e., $\operatorname{grad} f = -EH$. Thus $r^\circ - g^\circ = -L_{\frac{1}{2}EH}\, g^\circ$. Q.E.D.

If we solve Hamilton's equation with an initial metric $\tilde{g}_0$, then by equality (2.7.1) and Lemmas 3.2 and 3.3 the function $2D_t x$ does not converge to 0, hence the Kähler metric $\tilde{g}_t$ does not converge. Therefore we analyse the behaviour of the one-parameter family $\gamma_t^{-1*} \tilde{g}_t$ of Kähler metrics, where $\gamma_t = \exp(-\frac{1}{2} E t H)$. Remark that $\gamma_t^{-1*} \tilde{g}_t$ corresponds to the pair $(x_t \circ \gamma_t^{-1}, \varphi_t)$. (See Lemma 2.3).

Theorem 5.5 The family $\gamma_t^{-1*}\tilde{g}_t \in M$ converges to a quasi--Einstein metric $\in M$ in C^∞ norm.

Proof If we can show that $D_t(x_t \circ \gamma_t^{-1})$ converges uniformly to 0 in exponential order, then the proof will be completed by a similar way to 4. By equality (2.7.1) we see that

$$
\begin{aligned}
(5.5.1) \qquad 2D_t(x_t \cdot \gamma_t^{-1}) \circ \gamma_t &= 2D_t x - EH[x] \\
&= 2D_t x - E \cdot \varphi(x) \\
&= \varphi^\circ(x) \frac{\partial}{\partial t} \phi(x) + (E \cdot \varphi^\circ(x) - 2x)\phi(x) - E \cdot (\varphi(x) - \varphi^\circ(x)) \quad ,
\end{aligned}
$$

and the last line converges to 0 by Lemmas 3.2 and 3.3.

Q.E.D.

Remark 5.6 The Kähler metric g° is not extremal in the sence of Calabi [Cl]. A Kähler metric is extremal if and only if the gradient of its scalar curvature is holomorphic. In our case, it is equivalent to the equation

$$
(5.6.1) \qquad \frac{\partial^2}{\partial x^2} (\varphi \frac{\partial}{\partial x} f) = \text{constant},
$$

and by condition (2) in Lemma 2.2 we get a unique solution

$$(5.6.2)\qquad \varphi(x) = -\int_{-d_1}^{x} \{C(d_1 + x)(d_2 - x) + 2x\}Q(x)\,dx/Q(x) \quad ,$$

where the constant C is chosen so that $\varphi(d_2) = 0$. We can easily check that this function φ defines an (extremal) Kähler metric by Lemma 2.3, but it is a quasi-Einstein metric if and only if $E = 0$ and $C = 0$, i.e., if they are Einstein metrics.

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$$(2.5.9) \qquad 2\varphi \frac{\partial}{\partial x} (D_t x) = \varphi \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} \varphi + 2x + \frac{\varphi}{Q} \frac{\partial}{\partial x} Q\right) .$$

But here we know that $D_t x = 0$ and $\frac{\partial}{\partial x} \varphi + 2x + \frac{\varphi}{Q} \frac{\partial}{\partial x} Q$ $= -\varphi \frac{\partial}{\partial x} f = 0$ at $x = -d_1, d_2$. Thus

$$(2.5.10) \qquad 2D_t x = \frac{\partial}{\partial x} \varphi + 2x + \frac{\varphi}{Q} \frac{\partial}{\partial x} Q .$$

Using the function $\varphi^\circ(x)$ defined by (2.4.2), we put

$$(2.5.11) \qquad \phi(x) \doteq \varphi^\circ(x)^{-1}\varphi(x) - 1.$$

Then by Lemma 2.2 (2) we see

<u>Lemma 2.6</u> The function $\phi(x)$ is a C^∞ function such that $\phi(x) = 0$ at $x = -d_1, d_2$ and $1+\phi > 0$ on $[-d_1, d_2]$.

<u>Lemma 2.7</u>

$$(2.7.1) \qquad 2D_t x = \varphi^\circ(x) \frac{\partial}{\partial x} \phi(x) + (E\varphi^\circ(x) - 2x)\phi(x) + E\varphi^\circ(x) ,$$

$$(2.7.2) \qquad 2\varphi^\circ \frac{\partial}{\partial t} \phi$$

$$= \varphi^\circ\varphi \frac{\partial^2}{\partial x^2} \phi - \left(\varphi^\circ \frac{\partial}{\partial x} \phi\right)^2 - 2x\varphi^\circ \frac{\partial}{\partial x} \phi - 2\left(\varphi^\circ - x \frac{\partial}{\partial x} \varphi^\circ\right)(1+\phi)\phi .$$

<u>Proof</u> Equation (2.7.1) is easy to see by equations (2.5.10) and (2.4.3). Substituting it into equation (2.5.7), we get equation (2.7.2). Q.E.D.