ON THE COHOMOLOGY OF GALOIS GROUPS DETERMINED BY WITT RINGS

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Abstract. Let F denote a field of characteristic different from two. In this paper we describe the mod 2 cohomology of a Galois group $\mathcal{G}_F$ (called the W-group of F) which is known to essentially characterize the Witt ring WF of anisotropic quadratic modules over F. We show that $H^*(\mathcal{G}_F, \mathbb{F}_2)$ contains the mod 2 Galois cohomology of F and that its structure will reflect important properties of the field. We construct a space X_F endowed with an action of an elementary abelian group E such that the computation of the cohomology of $\mathcal{G}_F$ reduces to calculating the equivariant cohomology $H^*_E(X_F, \mathbb{F}_2)$. For the case of a field which is not formally real this amounts to computing the cohomology of an explicit Euclidean space form, an object which is interesting in its own right. We provide a number of examples and a substantial combinatorial computation for the cohomology of the universal W-groups.

1. Introduction

Although there has been substantial recent activity in the area of finite group cohomology, it has hardly involved any interactions with Galois theory. This seems rather surprising in light of the recent major developments in field theory, such as Voevodsky's proof of the Milnor Conjecture [46]. Moreover, cohomological methods in field theory use comparatively little from available techniques in the cohomology of finite groups. An explanation for this is the fact that interesting Galois groups are usually far too complicated to be analyzed with basic cohomological methods. An obvious compromise would be to identify quotient groups which still retain substantial field-theoretic information but which are accessible via methods in the cohomology of groups.

In [38], Minac and Spira introduced a relatively simple Galois group $\mathcal{G}_F$ associated to a field F of characteristic different from 2, known as the W-group of F. They showed that, up to a minor technical condition, this group will characterize the Witt ring WF of anisotropic quadratic modules over F (see [33] for background). Hence these groups would seem to be ideal candidates for a fruitful cohomological analysis, where ideas and methods from Galois theory could be successfully combined with well-established techniques from the cohomology of groups. In this paper we show that this is indeed the case:

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after showing that $\mathcal{G}_F$ can be intrinsically defined as a central extension of an elementary abelian 2-group by an elementary abelian 2-group using field-theoretic data, we prove that its mod 2 cohomology not only contains the Galois cohomology but that its qualitative structure reflects properties of the field. We also provide a topological method for computing $H^*(\mathcal{G}_F, \mathbb{F}_2)$, reducing matters in many instances to determining the cohomology of a compact Euclidean space form, hence allowing us to apply methods from homotopy theory, group actions and combinatorics in our analysis.

Before stating our main results, we recall some basic definitions. Throughout this paper, F will denote a field of characteristic *different from two.* This point is emphasized here as it will be an unstated hypothesis in most of the results in the rest of this paper. We denote the quadratic closure of F by F_q and the associated Galois group by G_F. The W-group of F is defined (see [38]) as the quotient group

$$\mathcal{G}_F = G_F/G_F^4[G_F^2, G_F].$$

A fairly simple description of the intermediate extension $F \subset F^{(3)} \subset F_q$ such that $\mathcal{G}_F = \mathrm{Gal}(F^{(3)}/F)$ is given in §2. Let $\Phi(\mathcal{G}_F) \subset \mathcal{G}_F$ denote its Frattini subgroup, then it is an elementary abelian 2-group and $\mathcal{G}_F$ can be expressed as a central extension

$$1 \to \Phi(\mathcal{G}_F) \to \mathcal{G}_F \to E \to 1$$

where $E \cong (\dot{F}/\dot{F}^2)^*$, the Pontrjagin dual of the mod 2 vector space $\dot{F}/\dot{F}^2$. The starting point for our work is a field-theoretic characterization of a condition on groups (known as the 2C property, see [1]) which has substantial cohomological implications:

Theorem. (2.2) *If $|\mathcal{G}_F| > 2$, then $-1 \in F$ is a sum of squares (F is not formally real) if and only if every element of order 2 in $\mathcal{G}_F$ is central.*

In terms of the defining extension this implies that $\Phi(\mathcal{G}_F)$ is the unique maximal elementary abelian subgroup of $\mathcal{G}_F$; in the formally real case we show that $\Phi(\mathcal{G}_F)$ is an index two subgroup of any maximal elementary abelian subgroup.

Given this simple expression for $\mathcal{G}_F$ as a central extension, the next logical step is to identify it using cohomological and field-theoretic data. Recall that if

$$l\colon \dot{F}/\dot{F}^2 \to K_1F/2K_1F$$

is the canonical isomorphism between $\dot{F}/\dot{F}^2$ written multiplicatively and additively respectively, then Milnor K-theory mod 2 (see [33]) can be expressed as $\mathbb{F}_2[x_i \mid i \in \Omega]/I_F$, where $\{x_i \mid i \in \Omega\}$ are one dimensional polynomial generators which constitute a basis for $K_1F/2K_1F$ and I_F is the ideal generated by the quadratic polynomials corresponding to $l(a)l(1-a)$, for $a \in \dot{F}/\dot{F}^2$, $a \neq 1$. Let $\mathcal{B}_F$ denote the subspace of $H^2(E, \mathbb{F}_2)$ spanned by these polynomials. From the five term exact sequence associated to the defining extension for $\mathcal{G}_F$, we obtain an injective map $\delta\colon H^1(\Phi(\mathcal{G}_F), \mathbb{F}_2) \to H^2(E, \mathbb{F}_2)$.

Theorem. (3.7) *The image of δ is the subspace $\mathcal{B}_F \subset H^2(E, \mathbb{F}_2)$, and $\mathcal{G}_F$ is the uniquely determined central extension associated to this subspace.*

One can think of $\mathcal{G}_F$ as the "minimal" extension where the relations in K-theory are satisfied. As a consequence of this and the main result in [38] we obtain

Corollary. (3.9) *If F and L are fields as above then $\mathcal{G}_F \cong \mathcal{G}_L$ if and only if $K_1F/2K_1F \cong K_1L/2K_1L$. Hence if $WF \cong WL$, $K_1F/2K_1F \cong K_1L/2K_1F$, and the converse is true provided F and L have the same level if $\langle 1,1\rangle_F$ is universal.*

Next we use the Milnor Conjecture to prove

Theorem. (3.14) *Let $\mathcal{R} \subset H^*(\mathcal{G}_F, \mathbb{F}_2)$ denote the subring generated by one dimensional classes, then $\mathcal{R} \cong H^*(F, \mathbb{F}_2)$, the mod 2 Galois Cohomology of F.*

An important point to make is that $H^*(\mathcal{G}_F, \mathbb{F}_2)$ can contain substantially more cohomology than $H^*(F, \mathbb{F}_2)$ alone, and in fact its qualitative structure will reflect properties of F. This is best illustrated when $|\dot{F}/\dot{F}^2| < \infty$, an assumption which we make for the rest of this introduction. If $|\dot{F}/\dot{F}^2| = 2^n$, then E and $\Phi(\mathcal{G}_F)$ are finite elementary abelian groups of ranks n and r respectively, where $r = \binom{n+1}{2} - \dim\, H^2(\mathrm{Gal}(F_q/F), \mathbb{F}_2)$. Using methods from the cohomology of finite groups, we prove

Theorem. (3.10, 3.12) *There exist polynomial classes*

$$\zeta_1, \dots, \zeta_r \in H^2(\mathcal{G}_F, \mathbb{F}_2)$$

which form a regular sequence. If the field F is not formally real, then $H^*(\mathcal{G}_F, \mathbb{F}_2)$ is free and finitely generated over the polynomial subalgebra $\mathbb{F}_2[\zeta_1, \dots, \zeta_r]$ (in particular Cohen-Macaulay). In the formally real case, $H^*(\mathcal{G}_F, \mathbb{F}_2)$ has depth equal to r or $r+1$ and Krull dimension equal to $r+1$.

Using the results in [1], we show

Theorem. (3.16) *Let F be a field which is not formally real. Then there exist non-zero classes $x \in H^*(\mathcal{G}_F, \mathbb{F}_2)$ which restrict trivially on all proper subgroups of $\mathcal{G}_F$. Any such class must be exterior, i.e. $x^2 = 0$.*

The undetectable classes are also called *essential* cohomology classes; in fact they constitute an ideal $\mathcal{E} \subset H^*(\mathcal{G}_F, \mathbb{F}_2)$ such that any two elements in it multiply trivially. Note that the result above indicates that the usual detection methods for computing cohomology will not work for these W-groups. In contrast, for formally real fields detection can occur and in fact the element $[-1] \in H^1(\mathcal{G}_F, \mathbb{F}_2)$ plays a key role. Recall that a field is said to be pythagorean if $F^2 + F^2 = F^2$. Our main result for formally real fields is the following:

Theorem. (3.15) *Let F be a formally real field, then the following conditions are equivalent:*

1. *$[-1] \in H^1(\mathcal{G}_F, \mathbb{F}_2)$ is not a zero divisor.*
2. *F is pythagorean and $H^*(\mathcal{G}_F, \mathbb{F}_2)$ is Cohen-Macaulay.*
3. *F is pythagorean and $H^*(\mathcal{G}_F, \mathbb{F}_2)$ is detected on its elementary abelian subgroups.*

To calculate the quotient algebra $H^*(\mathcal{G}_F,\mathbb{F}_2)/(\zeta_1,\dots,\zeta_r)$, we construct what we call a *topological model*, its main properties are summarized in

Theorem. (4.6) *Given a field F with $|\dot{F}/\dot{F}^2| = 2^n$, there exists an action of $E_n \cong (\mathbb{Z}/2)^n$ on $X_F \cong (\mathbb{S}^1)^r$ with the following properties:*

1. *E_n only has cyclic isotropy subgroups,*
2. *the action is free if and only if F is not formally real,*
3. *$H^*(\mathcal{G}_F,\mathbb{F}_2)/(\zeta_1,\dots,\zeta_r) \cong H^*_{E_n}(X_F;\mathbb{F}_2)$.*

The term appearing in (3) is the *equivariant cohomology*, i.e. the mod 2 cohomology of the *Borel construction* $X_F \times_{E_n} EE_n$. In case (2) we of course obtain the cohomology of X_F/E_n, a compact r-dimensional Euclidean space form such that the subring generated by one dimensional classes is isomorphic to $H^*(F,\mathbb{F}_2)$. The fact that the ζ_i form a regular sequence implies [17, 10.3.4] that if $p_F(t)$ is the Poincaré series for the cohomology of $\mathcal{G}_F$, and $q_F(t)$ the one for the equivariant cohomology $H^*_{E_n}(X_F,\mathbb{F}_2)$, then $p_F(t) = q_F(t)/(1-t^2)^r$. If F is not formally real, $q_F(t)$ will describe a basis for $H^*(\mathcal{G}_F,\mathbb{F}_2)$ as a module over $\mathbb{F}_2[\zeta_1,\dots,\zeta_r]$. Also we should point out that the geometry of the action reflects the field theory in other ways besides condition (2). For example, if F is pythagorean, one can find a hyperplane $H \subset E_n$ which acts freely on X_F; this will correspond to the index two subgroup $\mathcal{G}_{F(\sqrt{-1})} \subset \mathcal{G}_F$.

We apply the results above to examples of interest in field theory. For example, if $F = \mathbb{Q}_2$ is the field of 2-adic numbers, we obtain a compact 5-dimensional manifold with Poincaré series equal to $1+3t+6t^2+6t^3+3t^4+t^5$. In the case of superpythagorean fields with $|\dot{F}/\dot{F}^2| < \infty$, (such as $F_n = \mathbb{R}((t_1))((t_2))\dots((t_{n-1}))$, the field of iterated power series over $\mathbb{R}$), we obtain a complete description of the cohomology of $\mathcal{G}_{F_n}$, and from there the Galois cohomology of any superpythagorean field.

Given a W-group $\mathcal{G}_F$ with $|\dot{F}/\dot{F}^2| = 2^n$, it can be expressed as a quotient of a unique "universal" W-group $W(n)$ generated by n elements. In terms of the extension data this is the group corresponding to the entire vector space $H^2(E,\mathbb{F}_2)$. Given their defining properties, these groups will have the most interesting and complicated cohomology. In particular $W(n)$ has the 2C property, hence its cohomology is "undetectable". To compute $H^*(W(n),\mathbb{F}_2)$, we replace its given topological model (an orbit space which is interesting in its own right) by one which has much more plentiful *rational* cohomology. Using an Eilenberg-Moore spectral sequence associated to this model, we obtain the following substantial combinatorial computation:

Theorem. (6.12) *If $\mathcal{G}_F = W(n)$ and $q_F(t) = 1 + a_1 t + \cdots + a_r t^r$, then*

$$a_i \geq \sum_{p+q=i} \sum_{Y_\lambda} \prod_{(s,t)\in Y_\lambda} \frac{n+t-s}{h(s,t)}$$

where Y_λ ranges over all symmetric, $p+2q$-box, p-hook Young diagrams, and $h(s,t)$ denotes the hooklength of the box (s,t).

We can verify that this theorem gives an equality for $i \leq 3$, whence we obtain

$$a_1 = n, \quad a_2 = \frac{n(n+1)(n-1)}{3}, \quad a_3 = \frac{n(n^2-1)(3n-4)(n+3)}{60}.$$

It is worthwhile to note that our method uses rational techniques to produce classes in the cohomology of a finite 2-group, which seems to be a somewhat novel approach. Determining whether or not we have an equality for all coefficients $a_1, \dots, a_r$ is an interesting problem, equivalent to showing that the homology of an integral Koszul complex is 2-torsion free (see §6 for details).

The subsequent sections of this paper are organized as follows: in §2 we provide the background on W-groups involving Galois theory; in §3 we discuss the basic cohomological structure of the $\mathcal{G}_F$; in §4 we introduce our topological models together with examples, including a discussion of the possible low-dimensional euclidean space forms which can occur; in §5 we analyze the situation for formally real fields, discussing at length the general pythagorean and superpythagorean case; in §6 we provide the computation for universal W-groups; and finally in §7 we discuss a plausible general calculation for the cohomology of W-groups.

Throughout this paper coefficents will be assumed in the field $\mathbb{F}_2$ with two elements unless stated otherwise, hence they are suppressed from now on. The results in this paper have been announced in [2]. We are grateful to J. Carlson, F. Cohen, D. Kotschick and J.-P. Serre for their useful comments on aspects of this work.

2. Preliminaries on W-groups

In this section we will provide preliminary information on W-groups, our basic reference is [38]. Assume that F is a field of characteristic different from 2. Let F_q denote the *quadratic closure* of F, and denote by G_F the Galois group of this extension over F. We will begin by defining the W-group associated to F.

Definition 2.1. Let $G_F^4[G_F^2, G_F] \subset G_F$ denote the closure of the subgroup generated by fourth powers and commutators of squares with arbitrary elements. The W-group of F is the quotient group

$$\mathcal{G}_F = G_F / G_F^4[G_F^2, G_F].$$

The group $\mathcal{G}_F$ is a pro-2-group, and it is finite if and only if $|\dot{F}/\dot{F}^2| < \infty$. Let $F^{(2)} = F(\sqrt{a} \mid a \in \dot{F})$, $\mathcal{E} = \{y \in F^{(2)} \mid F^{(2)}(\sqrt{y})/F \quad \text{is Galois}\}$, and $F^{(3)} = F^{(2)}(\sqrt{y} \mid y \in \mathcal{E})$. Then we have a sequence of field extensions $F \subset F^{(2)} \subset F^{(3)} \subset F_q$ where $\mathrm{Gal}(F^{(3)}/F) = \mathcal{G}_F$, $\mathrm{Gal}(F^{(2)}/F) \cong G_F/G_F^2 \cong E$, an elementary abelian 2-group such that $E \cong (\dot{F}/\dot{F}^2)^*$, the Pontrjagin dual, and G_F^2 is the Frattini subgroup of G_F. Now if we let $\Phi(\mathcal{G}_F) \subset \mathcal{G}_F$ denote its Frattini subgroup, then it can be identified with the subgroup $\mathrm{Gal}(F^{(3)}/F^{(2)})$, and we have a diagram of extensions

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
 & G_F^4[G_F^2, G_F] \arrow[r, equal] \arrow[d] & G_F^4[G_F^2, G_F] \arrow[d] & & \\
1 \arrow[r] & G_F^2 \arrow[r] \arrow[d] & G_F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \Phi(\mathcal{G}_F) \arrow[r] \arrow[d] & \mathcal{G}_F \arrow[r] \arrow[d] & E \arrow[r] & 1 \\
 & 1 & 1 & &
\end{tikzcd}$$

where the subgroup $\Phi(\mathcal{G}_F)$ is elementary abelian as well as central.

Let $\mathcal{C}$ denote the class of finite 2-groups H such that $H^4[H^2, H] = \{1\}$. If $\{e_i \mid i \in I\}$ is a basis for the vector space $\dot{F}/\dot{F}^2$, then it is not hard to show that $\mathcal{G}_F$ is a pro-$\mathcal{C}$-group with a minimal set of generators of cardinality $|I|$. The *universal* W-group on I, $W(I)$, can be defined as the unique pro-$\mathcal{C}$-group satisfying the following condition: for any pro-$\mathcal{C}$-group H and any (set) map $f\colon I \to H$, there is a unique extension of f to a pro-$\mathcal{C}$-group-homomorphism $\tilde{f}\colon W(I) \to H$. ¿From [38], we know that every element of order 2 in $W(I)$ is central (i.e. $W(I)$ satisfies the 2C condition, see [1]) and that its Frattini subgroup $\Phi(W(I))$ is a maximal elementary abelian subgroup. Now if $\mathcal{G}_F$ is any W-group, then it fits into an extension

$$1 \to V \to W(I) \to \mathcal{G}_F \to 1$$

where I is a set in one to one correspondence with a minimal set of generators for $\dot{F}/\dot{F}^2$ and $V \subseteq \Phi(W(I))$. In the special case when $|\dot{F}/\dot{F}^2| = 2^n$, $\Phi(\mathcal{G}_F)$ is an elementary abelian group of rank equal to $r = n + \binom{n}{2} - \dim H^2(G_F)$

Recall that F is said to be *formally real* if -1 is not a sum of squares in F. The following theorem provides a characterization of W-groups for fields which are not formally real.

Theorem 2.2. *Suppose that $|\mathcal{G}_F| > 2$; then the field F is not formally real if and only if each element of order 2 in $\mathcal{G}_F$ is central.*

Proof. Assume first that F is not formally real. Let σ be any element in $\mathcal{G}_F$ of order 2. If σ belongs to $\Phi(\mathcal{G}_F)$, then it is central, as all the elements in this subgroup are central. Therefore we may assume that $\sigma \notin \Phi(\mathcal{G}_F)$. However, in [39] it was shown that

$$P_\sigma = \{f \in F \mid \sqrt{f}^{\sigma} = \sqrt{f}\}$$

is an ordering in the field F. Hence F is a formally real field, a contradiction. Therefore if F is not formally real, condition 2C must hold.

Assume now that F is formally real. In [39] it was shown that the correspondence $\sigma \leftrightarrow P_\sigma$, $\sigma \in \mathcal{G}_F - \Phi(\mathcal{G}_F)$ where $\sigma \in \mathcal{G}_F$ is an involution, is a one-to-one correspondence between classes of involutions not contained in $\Phi(\mathcal{G}_F)$ and orderings in F. Hence we may assume the existence of an involution $\sigma \in \mathcal{G}_F - \Phi(\mathcal{G}_F)$. Due to the fact that $|\mathcal{G}_F| \geq 4$, there exists an element $\tau \in \mathcal{G}_F - \{1, \sigma\}$. Since σ is an involution, it cannot generate the whole group $\mathcal{G}_F$. Therefore τ can be chosen in $\mathcal{G}_F - \Phi(\mathcal{G}_F)$ and moreover σ and τ are linearly independent in the vector space $\mathcal{G}_F/\Phi(\mathcal{G}_F)$.

As F is formally real, -1 is not a square in $\dot{F}$. Moreover, in [39, p.521], it is proved that $(\sqrt{-1})^\sigma = -\sqrt{-1}$. As σ, τ are linearly independent modulo $\Phi(\mathcal{G}_F)$, we see that there exists an element $a \in \dot{F}$ such that $(\sqrt{a})^\sigma = \sqrt{a}$ but $(\sqrt{a})^\tau = -\sqrt{a}$. Set $L = F(\sqrt[4]{a}, \sqrt{-1})$; then L/F is a Galois extension with Galois group D_4, the dihedral group of order eight. ¿From the information we have it is easy to check that the images of τ and σ generate D_4 and in particular they *do not* commute. This implies that σ is not central and we conclude that the 2C condition does not hold. □

Remark 2.3. In the paper [34] similar dihedral tricks are used to exclude certain subgroups of $\mathcal{G}_F$. For later use we define the *level* of a field F as $s(F)$, the minimum m such that there exist $f_1, \ldots, f_m \in F$ with $-1 = f_1^2 + \cdots + f_m^2$. Then F is not formally real if and only if $s(F) < \infty$ and Pfister has proved that $s(F) \in \{1, 2, 4, 8, 16, \ldots, \infty\}$.

As a consequence of the theorem above, we see that if F is not formally real the Frattini subgroup $\Phi(\mathcal{G}_F)$ is the unique maximal elementary abelian subgroup, which in particular is central. In the formally real case we have

Theorem 2.4. *If F is formally real then $\Phi(\mathcal{G}_F) \subset \mathcal{G}_F$ is a central elementary abelian subgroup of index two in any maximal elementary abelian subgroup in $\mathcal{G}_F$.*

Proof. We see from the proof of 2.2 that any pair of involutions in $\mathcal{G}_F - \Phi(\mathcal{G}_F)$ which are linearly independent in $\mathcal{G}_F/\Phi(\mathcal{G}_F)$ cannot commute. On the other hand as the 2C condition cannot hold we conclude that the Frattini subgroup must be a central, index two subgroup of any maximal elementary abelian subgroup of $\mathcal{G}_F$. □

As we shall see in the next section, the group theoretic results in this section have important cohomological consequences. In particular we will make use of the recent cohomological characterization of finite 2-groups with the 2C property provided in [1].

3. Group Cohomology and Extensions

We will require some basic facts from the cohomology of finite groups, all of which can be found in [3] or [17]. First we have

Lemma 3.1. *Let G denote a finite 2-group satisfying the 2C condition with $E \subseteq G$ the elementary abelian subgroup of maximal rank k. Let $\mathcal{P} \subset H^*(G)$ be a polynomial subalgebra such that $H^*(E)$ is a finitely generated module over $res^G_E(\mathcal{P})$. Then $H^*(G)$ is a free and finitely generated module over $\mathcal{P}$.*

Proof. Let $\mathcal{P} = \mathbb{F}_2[\zeta_1, \ldots, \zeta_n]$. By a theorem of Duflot [15], we know that $H^*(G)$ is Cohen-Macaulay and by a standard result in commutative algebra we know that under that condition the cohomology will be a free module over any polynomial subring over which it is finitely generated. (For a proof of this result see [43, p.IV-20, Thm.2]; for a historical discussion see [16].)

Hence we only need to prove that $H^*(G)$ is a finitely generated $\mathcal{P}$-module. To prove this we will use the more geometric language of cohomological varieties (see [17] for background).

Let $V_G(\zeta_i)$ denote the homogeneous hypersurface in V_G, (the maximal ideal spectrum for $H^*(G)$) defined by ζ_i. Then $H^*(G)$ will be finitely generated over $\mathcal{P}$ if and only if $V_G(\zeta_1) \cap \cdots \cap V_G(\zeta_n) = \{0\}$. If we represent the class ζ_i by an epimorphism $\Omega^{n_i}(\mathbb{F}_2) \to \mathbb{F}_2$ with kernel L_{ζ_i}, then we know that $V_G(\zeta_i) = V_G(L_{\zeta_i})$, the variety associated to the annihilator of $Ext^*_{\mathbb{F}_2 G}(L_{\zeta_i}, L_{\zeta_i})$. Moreover using basic properties of these varieties, we have that $V_G(\zeta_1) \cap \cdots \cap V_G(\zeta_n) = V_G(L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_i})$. Now the cohomological variety of a module will be 0 if and only if the module is projective, hence what we need to prove is that the module $L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_n}$ is projective. However by Chouinard's Theorem (see [17]) we know that it is enough to check this by restricting to maximal elementary abelian subgroups; in this case E is the only such group and projectivity follows from our hypothesis, as $\mathrm{res}^G_E(\mathcal{P}) \subseteq H^*(E)$ is a polynomial subalgebra over which it is finitely generated (note that by Quillen's detection theorem, the kernel of res^G_E is nilpotent, hence $\mathcal{P}$ embeds in $H^*(E)$ under this map). Hence we conclude that $\zeta_1, \ldots, \zeta_n$ form a "homogeneous system of parameters" and so $H^*(G)$ is free and finitely generated as a module over $\mathcal{P}$. $\square$

Next we give a slight generalization of this lemma which is also well-known to the experts (the idea is in [15]), but which is not available in the literature in precisely the form we want. Since the proof of the generalization is not so conceptual as that of the lemma above, we confine ourselves to indicating how an existing proof may be modified to yield the result we want.

Lemma 3.2. *Let $E \subset G$ be an central elementary abelian p-subgroup and let $\zeta_1, \ldots, \zeta_n$ be a sequence of elements in $H^*(G)$ which restrict to a regular sequence in $H^*(E)$. Then $\zeta_1, \ldots, \zeta_n$ is a regular sequence in $H^*(G)$.*

Proof. We will not need the case $p \neq 2$, so we ignore it. In [9, 1.1], this theorem is proved in the special case where there exists a basis $u_1, \ldots, u_n$ for $H^1(E)$ and numbers l_i such that $\mathrm{res}_E(\zeta_i) = u_i^{l_i}$. A brief examination of the proof shows that the only property of these elements $u_i^{l_i}$ used is that they form a regular sequence in $H^*(E)$. (The reader who is not inclined to examine the proof in [9] may be excused from doing so on the grounds that we will actually be able to take our restrictions to have this special form.) $\square$

Our next lemma is about group extensions.

Lemma 3.3. *Let*

$$1 \to V \to G \to W \to 1$$

be a central extension of elementary abelian 2-groups where $V = \Phi(G)$, the Frattini subgroup. Then, the differential $\delta\colon H^1(V) \to H^2(W)$ in the five term exact sequence for the extension above is a monomorphism, and if G is finite, the isomorphism class of the extension is determined by the subspace $\delta(H^1(V)) \subset H^2(W)$.

Proof. The fact that δ is injective follows directly from the five-term exact sequence associated to the extension. Now a well-known fact in group cohomology is that the isomorphism class of a finite extension as above is determined by an *extension class* in $H^2(W,V)$. We will show that determining this isomorphism class is equivalent to identifying the subspace mentioned above. To obtain the extension class we consider the five term exact sequence for the extension with coefficients in V; we have:

$$0 \to H^1(W,V) \to H^1(G,V) \to H^1(V,V) \xrightarrow{\delta} H^2(W,V) \to H^2(G,V).$$

Our extension class is $\delta(\mathrm{Id})$ (see [23, p. 207]); we can decompose it by using a basis for $H^1(V)$. This gives rise to a basis for the subspace $\operatorname{Im}\delta$ and hence determines it uniquely. Conversely up to a change of basis, the subspace $\operatorname{Im}\delta \subset H^2(W)$ determines the map $\delta\colon H^1(V) \to H^2(W)$ and hence $\delta(Id)$. □

Following standard usage, a basis for *im* δ will be called a collection of defining k-invariants for the group extension.

Remark 3.4. This result easily extends to pro-finite groups such as our W-groups (see [41], page 100). Hence $\mathcal{G}_F$ is uniquely determined by the extension data.

Remark 3.5. Consider the universal group on n generators, $W(n)$, described as a central extension

$$1 \to \Phi(W(n)) \to W(n) \to E_n \to 1$$

where $E_n \cong (\mathbb{Z}/2)^n$, and $\Phi(W(n)) \cong (\mathbb{Z}/2)^{n+\binom{n}{2}}$. Then $W(n)$ is the central extension associated to the *entire* vector space $H^2(E_n)$.

Remark 3.6. The mod 2 cohomology of a finite group is known to be Noetherian (see [17]). On the other hand it is elementary to verify that if G is a pro-2-group such that $G/\Phi(G)$ is infinite, then $H^*(G)$ cannot be Noetherian; hence $H^*(\mathcal{G}_F)$ is Noetherian if and only if $|\dot{F}/\dot{F}^2| < \infty$.

Our objective will be to use 3.3 to identify the group extension

$$1 \to \Phi(\mathcal{G}_F) \to \mathcal{G}_F \to E \to 1$$

in terms of a subspace $I_F \subset H^2(E)$. To do this we require some basic notions from K-theory (see [33]).

Let

$$l\colon \dot{F}/\dot{F}^2 \to k_1F$$

denote the canonical isomorphism between $\dot{F}/\dot{F}^2$ written multiplicatively and additively (which is k_1F by definition). Let $\mathbb{F}_2[k_1F] \cong H^*(E)$ denote the polynomial algebra generated by the vector space k_1F, where the generators are assumed to be one-dimensional. Then, if $a \in \dot{F}/\dot{F}^2$, $a \neq 1$, the element $l(a)l(1-a)$ can be thought of as a quadratic polynomial in this ring. We define $I_F \subset H^2(E)$ as the subspace generated by these classes. If $\{e_i \mid i \in I\}$ is a basis for $\dot{F}/\dot{F}^2$, then by definition we have that Milnor K-theory mod 2 is given by

$$k_*F = \mathbb{F}_2[e_i \mid i \in I]/I_F.$$

We will now show that our description of $\mathcal{G}_F$ as a central extension of elementary abelian groups can be made explicit in terms of I_F. We use the notation from 3.3.

Theorem 3.7. *For the central extension*

$$1 \to \Phi(\mathcal{G}_F) \to \mathcal{G}_F \to E \to 1$$

$\delta(H^1(\Phi(\mathcal{G}_F))) = I_F$ *and hence the* W*-group* $\mathcal{G}_F$ *is uniquely determined by the subspace* $I_F \subset H^2(E)$*. In particular there exists a basis* $\{e_i \mid i \in I\}$ *for* $H^1(\Phi(\mathcal{G}_F))$ *such that all the* k*-invariants* $\delta(e_i)$ *are of the form* $u_i v_i$ *for* $u_i, v_i \in H^1(E)$*.*

Proof. We make use of the diagram of extensions described in section 2. First we recall a theorem due to Merkurjev [32], namely that the map $H^2(E) \to H^2(G_F)$ is surjective, where the kernel can be described as the subspace I_F generated by the defining relations in Milnor K-theory. We claim that there is also an isomorphism $H^1(\Phi(\mathcal{G}_F)) \cong H^1(G_F^2)^E$. Indeed, let $u\colon G_F^2 \to \mathbb{Z}/2$ and assume it is E and hence G_F-invariant. This means that for all $g \in G_F$, $r \in G_F^2$, we have $u(g^{-1}r^{-1}gr) = 1$, hence $u(G_F^4[G_F^2, G_F]) = 1$ and so it defines a unique element in $H^1(\Phi(\mathcal{G}_F)) = H^1(G_F^2/G_F^4[G_F^2, G_F])$. ¿From the above we obtain a diagram of exact sequences, where the kernels and middle terms are mapped isomorphically:

$$\begin{tikzcd}
0 \arrow[r] & H^1(\Phi(\mathcal{G}_F)) \arrow[r, "\delta"] \arrow[d] & H^2(E) \arrow[r] \arrow[d] & H^2(\mathcal{G}_F) \cap \mathcal{R} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^1(G_F^2)^E \arrow[r, "\delta'"] & H^2(E) \arrow[r] & H^2(G_F) \arrow[r] & 0.
\end{tikzcd}$$

Hence we conclude that by choosing the right basis, we can assume that the k-invariants for $\mathcal{G}_F$ are precisely the generators for the relations in Milnor K-theory. However we have already mentioned that the relations there are of the form $l(a)l(1-a)$ for $a \in \dot{F}/\dot{F}^2$, $a \neq 1$, where $l\colon \dot{F}/\dot{F}^2 \to k_1F$ is the canonical isomorphism. Making the identification with the quotient of the cohomology of $H^*(E)$ we see that the k-invariants must indeed be products of linear forms, completing the proof. $\square$

Remark 3.8. Actually this result only requires the injectivity part of Merkurjev's theorem (the proof of the Milnor conjecture in dimension 2). Observe that for any W-group $\mathcal{G}_F$ we obtain all non-trivial k-invariants of the form $l(a)l(1-a)$, $a \neq 1$, because they correspond to $\mathbb{Z}/4$ and D_4 quotients of $\mathcal{G}_F$. Hence we see that all K-theoretic relations are present in $im\ \delta$, and from the injectivity of Merkurjev's theorem and the diagram used before, we see that these provide all the defining k-invariants for the W-group.

Note that both $\mathcal{G}_F$ and k_*F are unambiguously determined by the same ideal I_F. From this and the main result in [38] we obtain

Corollary 3.9. $\mathcal{G}_F \cong \mathcal{G}_L$ *if and only if* $k_*F \cong k_*L$*. Moreover if* $WF \cong WL$ *then* $k_*F \cong k_*L$*, and the converse holds provided* F *and* L *have the same level if* $\langle 1,1\rangle_F$ *is universal.*

The following basic fact follows from our description of the k-invariants (the key idea is due to W. Gao and is used in his thesis [19], pg. 51):

Theorem 3.10. *Let F be a field which is not formally real and suppose that $|\dot{F}/\dot{F}^2| = 2^n$. Then there exists a collection of 2-dimensional classes $\zeta_1, \dots, \zeta_r$ in $H^*(\mathcal{G}_F)$ where $r = n + \binom{n}{2} - \dim H^2(G_F)$, such that $H^*(\mathcal{G}_F)$ is free and finitely generated as a module over the subalgebra $\mathbb{F}_2[\zeta_1, \dots, \zeta_r]$.*

Proof. We consider the Lyndon-Hochschild-Serre spectral sequence for the extension

$$1 \to \Phi(\mathcal{G}_F) \to \mathcal{G}_F \to E \to 1.$$

If $H^*(E) \cong \mathbb{F}_2[x_1, \dots, x_n]$ and $H^*(\Phi(\mathcal{G}_F)) \cong \mathbb{F}_2[e_1, \dots, e_r]$ then we have already remarked that we can assume $d_2(e_i) = u_i v_i$ where the u_i, v_i are 1-dimensional linear forms. We claim that the squares e_i^2 are permanent cocycles in the spectral sequence. For this it suffices to note that transgressions commute with Steenrod squares, and hence

$$d_3(e_i^2) = d_3(Sq^1(e_i)) = Sq^1 d_2(e_i) = Sq^1(u_i v_i) = u_i^2 v_i + u_i v_i^2 = 0$$

in $E_3^{3,0}$, as we obtain an element in the ideal generated by the image of d_2.

Hence the elements e_i^2 are in the image of the edge homomorphism, which in this case simply means that they are in the image of res^G_Φ. We can therefore find polynomial classes $\zeta_i \in H^2(G)$, $i = 1, \dots, r$, such that $\mathrm{res}^G_\Phi(\zeta_i) = e_i^2$ and invoking lemma 3.1 we conclude that $H^*(G)$ is free and finitely generated over the polynomial subring which they generate. $\square$

In [5] it was shown that the Poincaré series for a Cohen-Macaulay cohomology ring has a very special form. Namely, if $H^*(G)$ is free and finitely generated over a polynomial subring $\mathbb{F}_2[u_1, \dots, u_k]$ where $u_i \in H^{n_i}(G)$, then it is of the form $q(t)/(1 - t^{n_1}) \dots (1 - t^{n_k})$, where $q(t)$ is a "palindromic polynomial" (i.e. if $q(t) = 1 + a_1 t + \dots + a_d t^d$, then $a_i = a_{d-i}$), with integral coefficients and of degree $n_1 + \dots + n_k - k$. Applying this we obtain

Corollary 3.11. *If F is not formally real, the Poincaré series for $H^*(\mathcal{G}_F)$ is of the form*

$$p_F(t) = \frac{q_F(t)}{(1 - t^2)^r}$$

where $q_F(t)$ is a palindromic polynomial of degree $r = n + \binom{n}{2} - \dim H^2(G_F)$ in $\mathbb{Z}[t]$.

The proof above can be combined with 2.4 to establish a modified version of the above for formally real fields. The appeal to lemma 3.1 can be replaced by an appeal to lemma 3.2.

Theorem 3.12. *If F is a formally real field with $|\dot{F}/\dot{F}^2| = 2^n$, then there exists a regular sequence of 2-dimensional elements in $H^*(\mathcal{G}_F)$ of length $r = n + \binom{n}{2} - \dim H^2(\mathrm{Gal}(F_q/F))$, i.e. $H^*(\mathcal{G}_F)$ has depth at least equal to r; this is one less than its Krull dimension.*

The point here is that we have a sequence of elements restricting nontrivially to the maximal 2-torus in the center. Hence we conclude that $H^*(\mathcal{G}_F)$ has depth at least one less than the Krull dimension. We cannot conclude that it is Cohen-Macaulay, although that may occur, as we shall see in §5.

Remark 3.13. There is a factorization for the Poincaré series in the formally real case; one factor is the expected $(1-t^2)^{-r}$, but the other factor need not be a polynomial. The existence of this factorization follows from the fact [17, 10.3.4] that if $\zeta_1, \dots, \zeta_r$ is a regular sequence in $H^*(\mathcal{G}_F)$, then $H^*(\mathcal{G}_F)$ is a free module over $\mathbb{F}_2[\zeta_1, \dots, \zeta_r]$. The other factor to which we have referred will be given a very concrete form later, as it is the equivariant cohomology of our "topological model" (see theorem 4.6).

We will now show that the mod 2 cohomology of a W-group contains at least as much information as the Galois Cohomology of F. Let $\bar{F}$ denote a separable closure of F, and let $\mathbb{G} = \mathrm{Gal}(\bar{F}/F)$ denote the Galois group of this extension. Then by definition, the *Galois cohomology* of F, denoted $H^*(F)$, is the cohomology of the group $\mathbb{G}$, where, as remarked earlier, we will assume mod 2 coefficients. Let $E = \mathbb{G}/\Phi_2(\mathbb{G})$ where $\Phi_2(\mathbb{G})$ is the 2-Frattini subgroup of $\mathbb{G}$. Note that $H^1(F) \cong \mathrm{Hom}(E, \mathbb{Z}/2)$. The quotient map above factors through the W-group associated to F and so we have a commutative diagram

$$\begin{tikzcd} \mathbb{G} \arrow[r] \arrow[dr] & \mathcal{G}_F \arrow[d] \\ & E \end{tikzcd}$$

where by construction the group E can also be identified with the quotient $\mathcal{G}_F/\Phi(\mathcal{G}_F)$. Hence we have isomorphisms $H^1(F) \cong H^1(\mathcal{G}_F) \cong H^1(E)$. Let $\mathcal{R} \subset H^*(\mathcal{G}_F)$ denote the subring generated by one dimensional classes. We have the following fundamental result

Theorem 3.14. *The map $\phi\colon \mathbb{G} \to \mathcal{G}_F$ induces an isomorphism $\mathcal{R} \cong H^*(F)$.*

Proof. The group E is an elementary abelian group such that $H^1(E) \cong \dot{F}/\dot{F}^2$. In cohomology the surjection $\mathbb{G} \to E$ gives rise to a ring map $H^*(E) \to H^*(F)$. On the other hand we have seen that Milnor K-theory k_*F can be identified with the quotient of the polynomial ring $H^*(E) = \mathbb{F}_2[k_1F]$ by the ideal I_F generated by 2-dimensional relations, of the form $l(a)l(1-a)$, where $l\colon \dot{F}/\dot{F}^2 \to k_1F$ is the canonical isomorphism (see [33, p. 319]). In the finite case, the K-theory is simply expressible as a ring $\mathbb{F}_2[x_1, \dots, x_n]/(\mu_1, \dots, \mu_s)$, where $\mu_1, \dots, \mu_s$ are an irredundant collection of quadratic polynomials in the variables $x_1, \dots, x_n$. By Voevodsky's theorem (the Milnor conjecture [46]), we can identify k_*F with $H^*(\mathbb{G})$ using the map $H^*(E) \to H^*(\mathbb{G})$ i.e. we can identify its kernel with the ideal I_F.

We want to fit the W-group into this picture. Note that as a consequence of our previous remarks, $\mathcal{R}$ surjects onto $H^*(F)$. Now we have already established in 3.7 that if the group $\mathcal{G}_F$ is expressed as a central extension

$$1 \to \Phi(\mathcal{G}_F) \to \mathcal{G}_F \to E \to 1$$

defined by k-invariants $\rho_j \in H^2(E)$, $j \in \Omega$, then we can assume that the ideal J generated by the ρ_j is precisely the ideal I_F generated by the basic Steinberg relations in K-theory. Hence we can express $\mathcal{R}$ as a quotient of $H^*(E)/I_F$. Putting this together we deduce that we have a factorization of the isomorphism in the Milnor conjecture into a composition of two epimorphisms:

$$k_*F \longrightarrow \mathcal{R} \longrightarrow H^*(F)$$

from which the result follows. □

One can think of the W-group $\mathcal{G}_F$ as the central extension of an elementary abelian group by an elementary abelian group such that its associated k-invariants are precisely the generating relations for K-theory. Note that it is a consequence of the Milnor Conjecture that $\mathcal{R}$ is precisely the quotient $H^*(E)/I_F$. Computing $H^*(\mathcal{G}_F)$ using the Lyndon-Hochschild-Serre spectral sequence associated to the extension above, this means that

$$E_\infty^{*,0} = E_3^{*,0} \cong H^*(F) \cong \mathcal{R}$$

i.e. no differentials beyond d_2 can hit the cohomology of the base. We will see later that a general collapse of these spectral sequences at the E_3 term for all W-groups is not altogether unlikely.

We can use the above to deduce the following

Corollary 3.15. *If F is a field of characteristic different from 2 such that $\mathcal{G}_F \neq \mathbb{Z}/2$, then the following four conditions are equivalent:*

1. *every $a \in H^1(\mathcal{G}_F)$ is nilpotent*
2. *every positive dimensional element in $H^*(F)$ is nilpotent,*
3. *F is not formally real,*
4. *$\mathcal{G}_F$ satisfies the 2C condition.*

Proof. The equivalence of (1) and (2) follows from theorem 3.14. The equivalence of (2) and (3) is theorem 1.4 in [33]. The equivalence of (3) and (4) is proposition 2.2. □

In [1] it was shown that if G is a finite 2-group satisfying the 2C condition, then it contains non-zero cohomology classes $x \in H^*(G, \mathbb{F}_2)$ such that $\mathrm{res}_H^G(x) = 0$ for every proper subgroup $H \subset G$ (these are called *essential* cohomology classes). We conclude

Proposition 3.16. *If F is not formally real and $|\dot{F}/\dot{F}^2| < \infty$, and $|\mathcal{G}_F| > 2$, then $H^*(\mathcal{G}_F)$ contains essential cohomology classes. Furthermore if x is an essential class then it is exterior, i.e. $x^2 = 0$.*

Proof. Only the second part needs justification. Given our hypotheses we have a non-trivial relation in K-theory, i.e. $I_F \neq 0$. Hence we see that there exist non-zero classes $x_1, x_2 \in H^1(\mathcal{G}_F)$ such that $x_1x_2 = 0$. On the other hand due to the fact that x is essential, it must be divisible by any one dimensional class. Choose u_1, u_2 such that $x = x_1u_1 = x_2u_2$; then obviously $x^2 = x_1u_1x_2u_2 = x_1x_2u_1u_2 = 0$. □

Remark 3.17. Observe that the same proof shows that all products of essential classes are zero. In other words, the ideal of essential classes $\mathcal{E} \subset H^*(\mathcal{G}_F)$ is a ring with trivial products. An interesting problem would be to obtain a description of $\mathcal{E}$. For example, one can ask for the minimal degree of a homogeneous element $x \in \mathcal{E}$. As an example consider the case when F is a local field (a finite extension of some $\mathbb{Q}_p$) then we know (see [44], 4.5) that the cup product

$$H^1(\mathcal{G}_F) \times H^1(\mathcal{G}_F) \to \mathbb{Z}/2 \subset H^2(\mathcal{G}_F)$$

is a non–degenerate bilinear pairing (here we use the fact that the Galois cohomology is a 2-dimensional Poincaré duality algebra). Hence there is a fixed element $a \in H^2(\mathcal{G}_F)$ such that if $x, y \in H^1(\mathcal{G}_F)$, then $xy = 0$ or $xy = a$. Clearly $a \in \mathcal{E}$, and so the essential ideal has elements of degree two.

It remains to consider the case when F is formally real. As we have seen, the subgroup $\Phi(\mathcal{G}_F) \subset \mathcal{G}_F$ is a central elementary abelian subgroup of index 2 in any maximal elementary abelian subgroup in $\mathcal{G}_F$. Consequently there exists a non-nilpotent class $\alpha \in H^1(\mathcal{G}_F)$. We can make explicit the identification between $H^1(\mathcal{G}_F)$ and $\dot{F}/\dot{F}^2$ as follows. If $a \in \dot{F}$, then we define a homomorphism $\mathcal{G}_F \to \mathbb{Z}/2$ by the formula

$$x_a(\sigma) = (\sqrt{a})^\sigma/\sqrt{a},\ \sigma \in \mathcal{G}_F.$$

In the sequel this class will often be denoted by $[a]$. In terms of this we can now specify a non-nilpotent class when the field is formally real.

Theorem 3.18. *Let F denote a field of characteristic different from two and such that $|\dot{F}/\dot{F}^2| < \infty$. Then F is formally real if and only if $x_{-1} \in H^1(\mathcal{G}_F)$ is non-nilpotent.*

Proof. Assume that F is formally real. As we saw in the proof of 2.2, we can choose an involution $\sigma \in \mathcal{G}_F - \Phi(\mathcal{G}_F)$ such that $(\sqrt{-1})^\sigma = -\sqrt{-1}$; hence we deduce that $x_{-1}(\sigma) \neq 1$. This means of course that x_{-1} restricts non trivially to $H^1(\langle\sigma\rangle)$, and hence x_{-1} is non-nilpotent. The converse has already been established. □

Remark 3.19. Actually the condition $|\dot{F}/\dot{F}^2| < \infty$ can be removed if one is willing to apply the other half of the Milnor Conjecture, namely the isomorphism

$$\phi_n : H^n(F) \to I^n/I^{n+1}$$

given via

$$\phi_n([a_1] \cup \cdots \cup [a_n]) = [\langle\langle -a_1, \dots, -a_n\rangle\rangle]$$

where I^n is the n-th power of the fundamental ideal in WF and

$$[\langle\langle -a_1, \dots, -a_n\rangle\rangle] \in I^n/I^{n+1}$$

is the class of the Pfister form (see [46], [40] and [26]). In particular we have that $\phi_n([-1]^n) = [\langle\langle 1, \dots, 1\rangle\rangle]$; however from the Arason-Pfister Theorem (see [26], Ch.10, Section 3, Th. 3.1) we see that $[-1]^n = 0$ in $H^*(F)$ if and only if $\langle\langle 1, \dots, 1\rangle\rangle$ is isotropic. Since this form is *never* isotropic over a formally real field, we see that $[-1]$ is non–nilpotent in $H^*(F)$ and consequently in $H^*(\mathcal{G}_F)$.

In the case of a field F which is not formally real, we can specify exactly the height of the nilpotent class $[-1]$.

Proposition 3.20. *Let F be a field that is not formally real, and let n denote the smallest positive integer such that $[-1]^n = 0$ in $H^*(\mathcal{G}_F)$. Then, if $s(F)$ denotes the level of the field F, $s(F) = 2^{n-1}$.*

Proof. Clearly the height of $[-1]$ in $H^*(F)$ is the same as its height in $H^*(\mathcal{G}_F)$. However from the previous remark we see that $[-1]^n = 0$ if and only if $\langle\langle 1, \dots, 1\rangle\rangle$ is isotropic, and therefore hyperbolic (see [26], Chapter 10). Therefore the Pfister form $\langle\langle 1, \dots, 1\rangle\rangle$ ($n-1$ times) represents -1, and this means that -1 is expressible as a sum of 2^{n-1} squares, whence our proposition follows. □

This type of argument can also be used to find an upper bound on the highest degree in which $\mathcal{R} \subset H^*(\mathcal{G}_F)$ has non–zero classes (we denote this by $c(F)$).

Theorem 3.21. *Let F be a field that is not formally real and such that $|\dot{F}/\dot{F}^2| = 2^n$; then $c(F) \leq n$.*

Proof. ¿From Kneser's Theorem (see [26], Chapter 11, Th.4.4), we see that each Pfister form $\langle\langle -a_1, \dots, -a_{n+1}\rangle\rangle$ with $a_1, \dots, a_{n+1} \in \dot{F}$ is hyperbolic; hence each product $[a_1] \cup \dots \cup [a_{n+1}] \in H^{n+1}(\mathcal{G}_F)$ is zero. □

4. Topological Models

As a consequence of the cohomological analysis carried out in the previous section, it is fairly evident that the key step in computing the cohomology of a W-group is calculating the quotient algebra $H^*(\mathcal{G}_F)/(\zeta_1, \dots, \zeta_r)$. In this section we will construct a topological space X_F together with a very explicit action of $E_n = (\mathbb{Z}/2)^n$ on it, such that the quotient above can be identified with the equivariant cohomology $H^*_{E_n}(X_F)$. This *topological model* has geometric properties which reflect the field theory in a very nice way, and as we shall see, computing the equivariant cohomology rings is an interesting problem in its own right.

Before proceeding with the description of our construction and proving its main properties, we briefly recall some basic facts about equivariant cohomology that will be used in the rest of the paper. All of these facts can be found in standard texts, e.g. [3, Ch. V], [4], [7].

Let X be a topological space with the action of a finite group G on it. Let EG denote a universal G–space i.e. a contractible space with a free action of G. The *Borel construction* is defined as the orbit space $EG \times_G X = (EG \times X)/G$, where G acts diagonally on $EG \times X$. The cohomology of this construction is denoted by $H^*_G(X) = H^*(EG \times_G X, \mathbb{F}_2)$, and is referred to as the mod 2 *equivariant cohomology* of X. For our applications we will always assume that X is a compact space, in most cases a closed manifold, and that G is a finite 2-group. As the action on EG is free, the projection map $EG \times X \to EG$ gives rise to a fibration $EG \times_G X \to BG$, where $BG = EG/G$ is the classifying space of G, and the fiber is X. Similarly if the G-action on X is free, we get a fibration $EG \times_G X \to X/G$ with contractible fiber EG, hence we have a homotopy equivalence $EG \times_G X \simeq X/G$.

The basic structural result concerning equivariant cohomology is that $H^*_G(X)$ has Krull dimension equal to the largest rank of a 2-elementary abelian isotropy subgroup. We will use two simple consequences of this:

Lemma 4.1. *If X is a finite dimensional G-space, then $H^*_G(X)$ is finite dimensional if and only if G acts freely on X, in which case $H^*_G(X) \cong H^*(X/G)$.*

Lemma 4.2. *If the isotropy groups G_x are cyclic (or trivial) for all $x \in X$, then $H^*_G(X)$ is eventually periodic.*

The *singular set* $\mathcal{S}_G(X)$ is the set of points of X which have non-trivial isotropy groups, or equivalently, the set of points of X which are not permuted freely by G. In sufficiently large dimensions $H^*_G(X)$ is determined by the equivariant cohomology of the singular set. More precisely, we have

Lemma 4.3. *If $i > \dim X$, then the inclusion $\mathcal{S}_G(X) \subset X$ induces an isomorphism $H^i_G(X) \cong H^i_G(\mathcal{S}_G(X))$.*

We will also use the following simple fact:

Lemma 4.4. *If $H \subset G$ is a subgroup, $H^*_G(G/H \times Y) \cong H^*_H(Y)$.*

The inclusion $X \subset EG \times_G X$ gives a map $H^*_G(X) \to H^*(X)$, and if this map is surjective, we say that the cohomology of X is *totally nonhomologous to zero* in the cohomology of the *Borel construction* $EG \times_G X$. Writing the Poincaré series for a graded ring R as $P(R)$ we have:

Lemma 4.5. *If $H^*(X)$ is totally nonhomologous to zero in $H^*_G(X)$, then $P(H^*_G(X)) = P(H^*(X)) \cdot P(H^*(G))$.*

We now state and prove the main result in this section.

Theorem 4.6. *Let F denote a field of characteristic different from 2 such that $|\dot{F}/\dot{F}^2| = 2^n$. Let r denote the rank of $\Phi(\mathcal{G}_F)$. There exists a homomorphism $\rho\colon (\mathbb{Z}/2)^n \to O(2)^r$ which defines an action of $E_n = (\mathbb{Z}/2)^n$ on $X_F \cong (\mathbb{S}^1)^r$ with the following properties:*

1. *E_n acts freely on X_F if and only if F is not formally real.*
2. *E_n acts with non-trivial cyclic isotropy subgroups if and only if F is formally real.*
3. *There exists a regular sequence $\{\mu_F\} \subset H^2(\mathcal{G}_F)$ of length*
$$r = n + \binom{n}{2} - \dim H^2(\mathrm{Gal}(F_q/F))$$
such that
$$H^*(\mathcal{G}_F)/(\mu_F) \cong H^*_{E_n}(X_F).$$

Proof. To prove this we will make use of the fact that $\mathcal{G}_F$ is defined as a central extension
$$1 \to \Phi(\mathcal{G}_F) \to \mathcal{G}_F \to E_n \to 1$$
where we can assume that the r k-invariants are of the form
$$u_1v_1, \ldots, u_rv_r,$$
for elements $u_s, v_t \in H^1(E_n)$. Given any element $\beta \in H^1(E_n)$, there is a representation $\rho_\beta\colon E_n \to O(1)$ such that it has Stiefel-Whitney class $w_1(\rho_\beta) = \beta$. Hence we may construct a direct sum representation $\rho_{u_t} \oplus \rho_{v_t} = \rho_{u_tv_t}\colon E_n \to O(2)$ such that it has second Stiefel-Whitney class $w_2(\rho_{u_tv_t}) = w_1(\rho_{u_t})w_1(\rho_{v_t}) = u_tv_t$. Taking a product we obtain a homomorphism $\rho\colon E_n \to O(2)^r$ such that on each factor the corresponding w_2 is mapped to the appropriate k-invariant.

Consider the action of $O(2)$ on $\mathbb{S}^1$ induced by matrix multiplication on $\mathbb{R}^2$. Restricted to the subgroup $O(1) \times O(1) \subset O(2)$ we obtain the following action: let x_1, x_2 denote generators of $G = O(1) \times O(1)$, then if $z \in \mathbb{S}^1$, $x_1(z) = \bar{z}$, $x_2(z) = -\bar{z}$. Here we have used complex conjugation to simplify the notation. Note that the subgroups $\langle x_1\rangle$, $\langle x_2\rangle$ are isotropy subgroups, while x_1x_2 acts freely via multiplication by -1. We can use the universal G space EG to obtain the usual fibration

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r] & EG \times_G \mathbb{S}^1 \arrow[d] \\ & BG \end{tikzcd}$$

If $e \in H^1(\mathbb{S}^1)$ is a generator and $H^*(G) \cong \mathbb{F}_2[x_1, x_2]$ (a slight abuse of notation), then the transgression can be computed as $d_2(e) = x_1x_2$. This follows from the fact that $d_2(e)$ restricts trivially to the cohomology of both of the isotropy subgroups (indeed both restricted fibrations have sections) and non-trivially to the cohomology of $\langle x_1x_2\rangle$. In fact $\pi_1(EG \times_G \mathbb{S}^1) \cong \mathbb{Z}/2 * \mathbb{Z}/2$, the infinite dihedral group. Hence in the fibration

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r] & EO(2) \times_{O(2)} \mathbb{S}^1 \arrow[d] \\ & BO(2) \end{tikzcd}$$

the cohomology generator on the fiber will transgress to the Stiefel-Whitney class $w_2 = x_1x_2$.

Now let X_F denote the space $(\mathbb{S}^1)^r$ with the action of E_n through the homomorphism $\rho\colon E_n \to O(2)^r$ described previously. Using the naturality of the corresponding fibration for the Borel construction $X_F \times_{E_n} EE_n$, one can verify that the cohomology generators on the fiber will transgress to the images of the w_2 classes under the map induced by ρ, which by construction are precisely the desired k-invariants $u_1v_1, \ldots, u_rv_r$. Note that X_F is a product of r E_n-spaces, as the action on each coordinate is independent of the others. Let $Y_F = X_F \times_{E_n} EE_n$, then we have the usual fibration

$$\begin{tikzcd} X_F \arrow[r] & Y_F \arrow[d] \\ & BE_n \end{tikzcd}$$

which corresponds to a group extension

$$1 \to L \to \Upsilon_F \to (\mathbb{Z}/2)^n \to 1$$

where $\Upsilon_F = \pi_1(Y_F)$ and L is a free abelian group of rank r. Note that if the E_n action is free, then Y_F is an aspherical, compact r-dimensional manifold, with necessarily torsion-free fundamental group.

We now consider the mod 2 spectral sequence for this extension (or equivalently the fibration). Let $H^*(E_n) \cong \mathbb{F}_2[x_1, \ldots, x_n]$ and $H^*(X_F) = \Lambda(e_1, \ldots, e_r)$. We have seen that *by construction* the fibration will have k-invariants $d_2(e_i) = u_iv_i$, for $i = 1, \ldots, r$.

The subgroup $2L \subset \Upsilon_F$ is a normal subgroup, and hence fits into a commutative diagram of extensions:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
 & 2L \arrow[r, equal] \arrow[d] & 2L \arrow[d] & & \\
1 \arrow[r] & L \arrow[r] \arrow[d] & \Upsilon_F \arrow[r] \arrow[d] & (\mathbb{Z}/2)^n \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & (\mathbb{Z}/2)^r \arrow[r] \arrow[d] & \mathcal{G}'_F \arrow[r] \arrow[d] & (\mathbb{Z}/2)^n \arrow[r] & 1 \\
 & 1 & 1 & &
\end{tikzcd}$$

Now observe that in the spectral sequence for the bottom row, the transgressions are precisely $\{u_i v_i \mid i = 1, \ldots, r\}$. As the original action on L was a sum of rank one sign-twists, we conclude that the quotient group $\mathcal{G}'_F$ is expressed as a central extension of $(\mathbb{Z}/2)^n$ by $(\mathbb{Z}/2)^r$ with precisely the same extension class as $\mathcal{G}_F$. We conclude from 3.3 that $\mathcal{G}'_F \cong \mathcal{G}_F$. We now make use of the 2 columns; comparing the spectral sequences for them we see that we may choose $\tilde{e}_i$ generating $H^1(2L)$ such that the $d_2(\tilde{e}_i)$ restrict to the squares of the polynomial generators in $H^*((\mathbb{Z}/2)^r)$. These elements, which we denote by ζ_i, $i = 1, \ldots, r$, form a regular sequence in $H^*(\mathcal{G}_F)$ by lemma 3.2. As a consequence of this, the spectral sequence collapses at E_3 and

$$H^*(Y_F) \cong H^*(\Upsilon_F) \cong H^*(\mathcal{G}_F)/(\zeta_i).$$

We now consider the case when F is not formally real. As we have seen, this is equivalent to having $\mathcal{G}_F$ satisfy the 2C condition. In other words, $\Phi(\mathcal{G}_F)$ is the unique elementary abelian subgroup of maximal rank, and it is central. This means that no cyclic subgroup in the quotient E_n can split off; cohomologically this means that if we consider the restricted extension

$$1 \to \Phi(\mathcal{G}_F) \to H_C \to C \to 1$$

where $C \subset E_n$ is any cyclic subgroup, then the restricted k-invariants must generate $H^2(C) = \mathbb{F}_2$. Looking at the associated fibration for the C-action on X_F; we see that the k-invariants must also necessarily generate $H^2(C)$; hence C acts freely on X_F and we conclude that the entire group E_n must be acting freely.

It remains to see what happens in the formally real case. Here we know that $\Phi(\mathcal{G}_F)$ is a central elementary abelian subgroup, of index 2 in any maximal elementary abelian subgroup. Moreover, if s_1, s_2 are commuting involutions in $\mathcal{G}_F - \Phi(\mathcal{G}_F)$, then $s_1 s_2 \in \Phi(\mathcal{G}_F)$. If there were a subgroup $A \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ in E_n which fixed a point in X_F, then the associated splitting would imply the existence of an elementary abelian subgroup of rank equal to two more than the rank of $\Phi(\mathcal{G}_F)$, a contradiction. On the other hand we do know that any cyclic summand in E_n which forms a maximal elementary abelian subgroup together with $\Phi(\mathcal{G}_F)$ must split, hence it splits cohomologically and must have a fixed point. □

If F is not formally real, we have shown that E_n acts freely on X_F. Hence we have a homotopy equivalence

$$Y_F \simeq X_F/E_n$$

and so $H^*_{E_n}(X_F) \cong H^*(X_F/E_n)$. Now X_F/E_n is a compact r-dimensional manifold, such that its mod 2 Poincaré series is precisely the polynomial $q_F(t)$ defined in 3.11. In the formally real case the the equivariant cohomology is infinite dimensional, although eventually periodic, as the isotropy is cyclic (4.2). An interesting alternative here would be to compute the equivariant Tate cohomology $\widehat{H}^*_{E_n}(X_F)$; this invariant vanishes for free actions and more generally agrees with the ordinary equivariant cohomology in sufficiently high dimensions. For the case of fields which are formally real, the entire Tate Cohomology is periodic and it would seem possible to find an expression for it in field-theoretic terms, but we will not explore this any further here.

The result above shows that the calculation of $H^*(\mathcal{G}_F)$ can be reduced to computing the cohomology of a crystallographic group. When the field is not formally real then in fact it reduces to computing the cohomology of a Bieberbach group. It is worthwhile to compare the spectral sequences arising from the two extensions

$$1 \to M_F \to \Upsilon_F \to E_n \to 1$$

$$1 \to \Phi(\mathcal{G}_F) \to \mathcal{G}_F \to E_n \to 1$$

using the morphism of extensions which we defined above. The E_2 terms are mapped via

$$\mathbb{F}_2[x_1, \dots, x_n] \otimes \mathbb{F}_2[e_1, \dots, e_r] \longrightarrow \mathbb{F}_2[x_1, \dots, x_n] \otimes \Lambda(e_1, \dots, e_r)$$

where e_i^2 is sent to zero. Using this and the fact that the e_i^2 are permanent cocycles (due to the form of the k-invariants), we conclude that at any stage the map above induces an isomorphism of spectral sequences

$$E_r^{*,*}(\mathcal{G}_F)/(e_1^2, \dots, e_r^2) \cong E_r^{*,*}(\Upsilon_F).$$

We can deduce a few facts from this. First of all, the spectral sequences will both collapse at the same stage, in particular $E_3(\mathcal{G}_F) = E_\infty(\mathcal{G}_F)$ if and only if $E_3(\Upsilon_F) = E_\infty(\Upsilon_F)$. Finding conditions that imply the collapse at E_3 for either type of extension is a basic question which also relates to the collapse of an associated Eilenberg-Moore spectral sequence, as we shall discuss later on. ¿From the above we also can conclude that $E_3^{*,0}(\mathcal{G}_F) = E_\infty^{*,0}(\mathcal{G}_F) \cong E_\infty^{*,0}(\Upsilon_F)$, hence we see that

$$H^*(F) \cong \mathcal{R} \subset H^*_{E_n}(X_F)$$

where we can now identify $\mathcal{R}$ with the subring in $H^*_{E_n}(X_F)$ generated by one dimensional elements. For the case of fields which are not formally real this means that the Galois cohomology $H^*(F)$ can be computed as the subring generated by one dimensional elements in the cohomology of a compact manifold.

A consequence of the proof of 4.6 is a description of the isotropy subgroups in terms of group theoretic data for $\mathcal{G}_F$. The elements in E_n which can fix a point in X_F are exactly those which are the homomorphic image of a (noncentral) involution under the natural projection $\mathcal{G}_F \to E_n$. Moreover, given any two elements which fix points in X_F, their product will act freely on it. Another observation for the case of fields which are not formally real is that the fact that E_n acts freely on X_F inducing a trivial action in mod 2 cohomology implies (see [12]) that $n \leq r$. This can be stated in a purely algebraic way as follows: if F is a field which is not formally real, then $\dim H^2(\mathrm{Gal}(F_q/F)) \leq \binom{n}{2}$. In the formally real case (using a modified result involving the co-rank of an isotropy subgroup of maximal rank) the bound is $\binom{n}{2}+1$.

One would like to comprehend in a precise way how unique our construction is. For this we must keep in mind the data which was used. To begin we fixed a choice of the k-invariants (or, equivalently, a basis for $H^1(\Phi(\mathcal{G}_F))$) such that each of the r k-invariants was a product. Then, given a k-invariant of the form $\alpha = uv$ we constructed a map $E_n \to O(2)$ and from there an action on $\mathbb{S}^1$. Now E_n will act on $H^1(\mathbb{S}^1, \mathbb{Z}) \cong \mathbb{Z}$; there will be a subgroup $H_\alpha \subset E_n$ which acts trivially on this module, and which has index at most 2 in E_n. This subgroup corresponds to the one dimensional cohomology class $u+v$; note that $H_\alpha = E_n$ if and only if $u = v$. If the action is non-trivial, this module, denoted M_{H_α}, fits into a short exact sequence of $\mathbb{Z}E_n$-modules

$$0 \to M_{H_\alpha} \to \mathbb{Z}[E_n/H_\alpha] \to \mathbb{Z} \to 0.$$

Applying group cohomology to the sequence above, it follows that

$$H^2(E_n, M_{H_\alpha}) \cong H^2(H_\alpha, \mathbb{Z}),$$

and furthermore as it is 2-torsion, the mod 2 reduction map $M_{H_\alpha} \to \mathbb{F}_2$ induces a monomorphism $0 \to H^2(E_n, M_{H_\alpha}) \to H^2(E_n, \mathbb{F}_2)$. Our construction can be interpreted as showing the existence of an element $\bar{\alpha} \in H^2(E_n, M_{H_\alpha})$ which maps to α under this map, note that it is automatically unique once the representation M_{H_α} has been fixed (this also works if $H_\alpha = E_n$). Geometrically this corresponds to taking the extension $\pi_1(\mathbb{S}^1 \times_{E_n} EE_n)$, with the action inducing M_{H_α}. We do this for each k-invariant and then take a direct sum, to obtain the unique class $\alpha_F \in H^2(E_n, M_F)$ which under mod 2 reduction maps to the sum of all of the k-invariants (here $M_F = \bigoplus M_{H_{\alpha_i}}$). Again, this lifting will be unique for a prescribed module M_F. However, there are many possible choices for the module.

As an example, consider the case of the universal W-group $W(2)$. The k-invariants are $\{x_1^2, x_2^2, x_1x_2\}$ and our construction in this case provides $X_F = (\mathbb{S}^1)^3$ with a $\mathbb{Z}/2 \times \mathbb{Z}/2$ action defined by $x_1(z_1, z_2, z_3) = (-z_1, z_2, \bar{z}_3)$ and $x_2(z_1, z_2, z_3) = (z_1, -z_2, -\bar{z}_3)$. Now we can modify the k-invariants without changing the quotient group to obtain $\{x_1^2 + x_1x_2, x_2^2 + x_1x_2, x_1x_2\}$. In this case our construction provides $X'_F \cong (\mathbb{S}^1)^3$ with an action given by $x_1(z_1, z_2, z_3) = (-z_1, \bar{z}_2, -\bar{z}_3)$ and $x_2(z_1, z_2, z_3) = (\bar{z}_1, -z_2, \bar{z}_3)$. Note that the first Betti number for X_F/E_2 is 2 whereas the first betti number for X'_F/E_2 is zero. Nevertheless the two spaces have the same mod 2 cohomology groups.

Thus, to summarize, while the W-group $\mathcal{G}_F$ depends only on the subspace of $H^2(E_n)$ generated by the k-invariants (lemma 3.3), our topological model X_F is constructed from, and depends on, a particular choice of basis for that subspace. It follows from remark 3.13 that the cohomology groups $H^*(Y_F)$ are determined by $\mathcal{G}_F$. Although different models may yield different ring structures for $H^*(Y_F)$, they will all contain the Milnor K-theory of F (theorem 3.14) and the ring structure of $H^*(\mathcal{G}_F)$ can in principle be recovered from any one of them by solving an extension problem.

The case of fields which are not formally real is particularly interesting, as we are simply dealing with certain quotients of $(\mathbb{S}^1)^r$ by the free action of an elementary abelian 2-group. By our construction, the subgroup $H_F = \cap_{\alpha_F} H_\alpha$ is acting freely and homologically trivially on the product of circles. In fact the elements in this group are acting on each circle either as the identity or as multiplication by -1. From this it follows that $X_F/H_F \cong (\mathbb{S}^1)^r$. This space, denoted $\bar{X}_F$, now has a free action of the quotient group E_n/H_F which induces a faithful representation on $H^1(\bar{X}_F, \mathbb{Z})$. This is the standard model for a Bieberbach group (see [13]). In the language of extensions, what we have done is take a *maximal* abelian subgroup $\pi_1(\bar{X}_F) \subset \Upsilon_F$, with quotient the *holonomy group* E_n/H_F. Going back to the examples above for $W(2)$, we see that in the first model the subgroup generated by x_1x_2 acts via diagonal multiplication by -1, hence trivially in homology. Dividing out by this action we obtain $\bar{X}_F \cong (\mathbb{S}^1)^3$ with holonomy $\mathbb{Z}/2 = E_2/\langle x_1x_2\rangle$. We can locate this space in the list of all compact connected flat riemannian 3-manifolds (which have been classified, see [47], page 122). From the fact that X_F/E_2 is non-orientable, with $\beta_1 = 2$ we deduce that this must be the manifold of type $\mathcal{B}_2$. On the other hand, the other model does have the full E_2 as holonomy group and X'_F/E_2 is orientable, with $\beta_1 = 0$. Looking at the classification, we conclude that this must be the celebrated *Hantzsche-Wendt* manifold (of type $\mathcal{G}_6$ in Wolf's notation).

¿From the discussion above we see that the classification of W-groups for fields which are not formally real will involve information about the classification of compact connected, flat Riemannian manifolds with elementary 2-abelian holonomy. In dimension 3 the possibilities are fairly limited: 3 orientable ones $(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_6)$ and 4 non-orientable ones $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4)$. Of course we are most interested in a classification "up to mod 2 cohomology". This provides an additional simplification: there are only 2 distinct mod 2 Poincaré series which occur among the 7 manifolds above: the ones with Poincaré series $1 + 2t + 2t^2 + t^3$ $(\mathcal{G}_6, \mathcal{B}_2, \mathcal{B}_4)$ and those with Poincaré series $1 + 3t + 3t^2 + t^3$ $(\mathcal{G}_1, \mathcal{G}_2, \mathcal{B}_1, \mathcal{B}_3)$. The first type occurs for $W(2)$ whereas the second kind occurs for $\mathcal{G}_F = (\mathbb{Z}/4)^3$. This completely describes the possibilities for the mod 2 cohomology of W-groups of formally real fields F such that $\dim \Phi(\mathcal{G}_F) = 3$. Of course we can subdivide this into the two cases

1. $|\dot{F}/\dot{F}^2| = 4$, $\dim H^2(G_F) = 0$ and $q_F(t) = 1 + 2t + 2t^2 + t^3$
2. $|\dot{F}/\dot{F}^2| = 8$, $\dim H^2(G_F) = 3$ and $q_F(t) = 1 + 3t + 3t^2 + t^3$.

Making use of this type of classification in higher dimensions would be much more complicated, but we can still make complete statements for dimension four. Even though there are 74 equivalence classes of compact

4-dimensional euclidean space forms (see [47, p. 126] and [22]), the possibilities for us are severely restricted by the fact that the holonomy can be at most $(\mathbb{Z}/2)^4$ and the k-invariants must be linearly independent. Hence $|\dot{F}/\dot{F}^2|$ can only have either 8 or 16 elements. As before we obtain only two possible cases:

1. $|\dot{F}/\dot{F}^2| = 8$, $\dim H^2(G_F) = 2$, and X_F/E_3 with Poincaré series equal to $q_F(t) = 1 + 3t + 4t^2 + 3t^3 + t^4$
2. $|\dot{F}/\dot{F}^2| = 16$, $\dim H^2(G_F) = 6$, and X_F/E_4 with Poincaré series equal to $q_F(t) = 1 + 4t + 6t^2 + 4t^3 + t^4$.

To make this discussion complete, we list fields which give rise to the two possible Poincaré series. Let K denote a field such that $\mathcal{G}_K = W(2)$ (this is discussed in §6). If we let $F = K((t))$, then the cohomology of $\mathcal{G}_F$ will give rise to the Poincaré series in (1).

If we take $F = \mathbb{F}_p((t_1))((t_2))((t_3))$ where p is a prime congruent to 3 mod 4, then $\mathcal{G}_F$ gives rise to the Poincaré series in (2). An analysis of the possible cohomology *rings* is of course a more delicate issue.

We will now look at further explicit examples of interest in field theory.

Example 4.7. $F = \mathbb{Q}_2$, the field of 2-adic numbers. In this case, the vectors $[-1], [2], [5]$ form a basis for $\dot{F}/\dot{F}^2$. Let us denote by x_{-1}, x_2, x_5 the elements in $H^1(\mathcal{G}_{\mathbb{Q}_2})$ which correspond to this basis. Then the k-invariants for this W-group are

$$\{x_2^2, x_5^2, x_{-1}x_2, x_{-1}x_5, x_{-1}^2 + x_2x_5\}$$

(see [38]). To apply 4.6, we must find equivalent k-invariants expressed as products of one dimensional classes. By a simple change of basis, we can modify the list above to get

$$\{x_2^2, x_5^2, x_{-1}x_2, x_{-1}x_5, (x_{-1} + x_2)(x_{-1} + x_5)\}$$

without altering the group (in fact this simply corresponds to making the right choice of generators for $H^1(\Phi(\mathcal{G}_F))$ before applying the transgression map).

Let $E_3 = \langle e_1, e_2, e_3\rangle$, with the duals of the e_i corresponding to x_{-1}, x_2, x_5 respectively. We now define the action on $X_{\mathbb{Q}_2}$, using complex coordinates:

$$\begin{aligned} e_1(z_1, z_2, z_3, z_4, z_5) &= (z_1, z_2, \bar{z}_3, \bar{z}_4, -z_5) \\ e_2(z_1, z_2, z_3, z_4, z_5) &= (-z_1, z_2, -\bar{z}_3, z_4, -\bar{z}_5) \\ e_3(z_1, z_2, z_3, z_4, z_5) &= (z_1, -z_2, z_3, -\bar{z}_4, \bar{z}_5). \end{aligned}$$

One notices immediately that the action is free. Moreover if we project onto the first, second and fifth coordinates, we obtain a fibration

$$\begin{tikzcd} \mathbb{S}^1 \times \mathbb{S}^1 \arrow[r] & X_{\mathbb{Q}_2}/E_3 \arrow[d] \\ & Y \end{tikzcd}$$

where Y is a 3-dimensional manifold with

$$H^*(Y) \cong \mathbb{F}_2[e_1, e_2, e_3]/((e_1 + e_2)(e_1 + e_3), e_2^2, e_3^2).$$

The fibre generators transgress to the products e_1e_2 and e_1e_3. The cohomology of the orbit space $X_{\mathbb{Q}_2}/E_3$ can be computed iterating a Gysin sequence argument, and it has Poincaré series equal to

$$1 + 3t + 6t^2 + 6t^3 + 3t^4 + t^5.$$

The example indicates a general fact which we now state.

Theorem 4.8. *Let X_F be the free E_n space associated to a field F which is not formally real, where $|\dot{F}/\dot{F}^2| = 2^n$ and $r = n + \binom{n}{2} - \dim H^2(\mathrm{Gal}(F_q/F))$. There is a free action of E_n on a direct factor $(\mathbb{S}^1)^n \subset X_F$, and hence we have a fibration*

$$\begin{tikzcd} (\mathbb{S}^1)^{r-n} \arrow[r] & X_F/E_n \arrow[d] \\ & (\mathbb{S}^1)^n/E_n \end{tikzcd}$$

Proof. Let $e_1, \ldots, e_r \in H^1(X_F)$ denote the 1-dimensional cohomology generators. In the Cartan-Leray spectral sequence associated to the action of E_n, these classes transgress to 2-dimensional polynomials $\kappa_1, \ldots, \kappa_r \in H^*(E_n)$. By theorem 3.14,

$$H^*(F) \cong \mathcal{R} \cong H^*(E_n)/(\kappa_1, \ldots, \kappa_r) \subset H^*(X_F/E_n)$$

hence in particular no higher differentials can hit the bottom edge of the spectral sequence. Notice that $H^*(X_F/E_n)$ is finite, hence the κ_j form a possibly redundant homogeneous system of parameters in $H^*(E_n)$. We can therefore choose a minimal subset of parameters in $\{\kappa_1, \ldots, \kappa_r\}$; this subset must necessarily have n elements in it. By relabelling if necessary, we can assume that the first n of them form such a subset; note that they will constitute a regular sequence in $H^*(E_n)$. Now take the direct factor Z consisting of the first n-circles in X_F; recalling that the action we constructed is a product of actions on each of the circles we see that the factorization of X_F is compatible with respect to the action of E_n. Now consider the action on this subcomplex; the transgressions form a regular sequence, hence $H^*_{E_n}(Z) \cong H^*(E_n)/(\kappa_1, \ldots, \kappa_n)$, an algebra of finite total dimension. Hence the action of E_n is free, and we obtain the desired conclusions. $\square$

5. Formally Real Fields

In this section we specialize to fields which are formally real. As we have seen, the model X_F is endowed with an action of an elementary abelian group with non-trivial cyclic isotropy subgroups. We begin our analysis by looking at a very interesting collection of examples.

Consider the 2-group defined as an extension

$$1 \to (\mathbb{Z}/2)^{\binom{n}{2}} \to T(n) \to (\mathbb{Z}/2)^n \to 1$$

which can be regarded as the quotient of $W(n)$ obtained by making the squares of the generators equal to zero. $T(n)$ can also be thought of as the extension defined by taking n generators of order 2 such that their commutators are central and also of order 2. More concretely, we note that $T(2) \cong D_8$, the dihedral group of order 8. The results of [39] can be combined to yield the following information about fields with W-group $T(n)$:

Proposition 5.1. *If $\mathcal{G}_F \cong T(n)$, F is a formally real pythagorean field with $|\dot{F}/\dot{F}^2| = 2^n$. Conversely, the W-group of such a field is generated by n involutions, and is therefore a quotient of $T(n)$.*

Proof. By [39, 2.11], F is pythagorean if and only if $\mathcal{G}_F$ is generated by involutions, and $T(n)$ is, by definition, generated by involutions. By [39, 2.7.2], F is formally real if there are involutions of $\mathcal{G}_F$ which do not lie in the Frattini subgroup. Again, for $T(n)$ this condition obviously holds. The condition on $|\dot{F}/\dot{F}^2|$ follows immediately from the fact that $\mathcal{G}_F/\Phi(\mathcal{G}_F)$ is dual to $|\dot{F}/\dot{F}^2|$. The partial converse follows from the fact that *any* W-group generated by n involutions is a quotient of $T(n)$ and [39, 2.11]. ☐

This statement can be strengthened to a characterization of fields with $T(n)$ as W-group. The proof of this characterization involves some field theory which may not be totally familiar, but most of the results which we use can be found in [27]. For the reader's convenience we recall briefly the relevant definitions. A subset $P \neq F$ of a field F is said to be an *ordering* if P is closed under addition and multiplication and $P \cup -P = F$. Elements of P are said to be *positive* with respect to the ordering. The motivating example is $F = \mathbb{R}$ and $P = \{x \geq 0\}$; by the Artin-Schreier theorem [27, 1.5], an ordering of F exists if and only if F is formally real. A *preordering* [27, p. 2] weakens the requirement $P \cup -P = F$ to simply $P \supset F^2$; it follows that every preordering contains ΣF^2, the set of all sums of squares of elements of F, and this set is called the *weak preordering* of F. Roughly speaking, a preordering T of F has the *strong approximation property* (abbreviated SAP) if, given disjoint subsets of the set of orderings containing T, it is possible to choose an element of F positive with respect to the first set of orderings and negative with respect to the second. F itself is said to be *SAP* if the weak preordering of F has the SAP [27, p. 126]. It turns out that a formally real pythagorean field with $|\dot{F}/\dot{F}^2| = 2^n$ is SAP if and only if F admits exactly n orderings [27, 17.4]. This last is a key fact we will use in proving the characterization below:

Theorem 5.2. *$\mathcal{G}_F \cong T(n)$ if and only if F is a formally real pythagorean SAP field with $|\dot{F}/\dot{F}^2| = 2^n$. Moreover, for each $n \in \mathbb{N}$ there exists such a field.*

Proof. The existence of formally real pythagorean SAP fields with $|\dot{F}/\dot{F}^2| = 2^n$ for each $n \in \mathbb{N}$ was proved in [8], Satz (4). We therefore turn to the proof of the "if and only if" part.

Necessity. Assume that $\mathcal{G}_F \cong T(n)$. By proposition 5.1, we need only show that F is SAP, i.e. that F has exactly n orderings. By [39, 2.10], there is a bijection between the set of all orderings of F and the non-trivial cosets $\sigma\Phi(\mathcal{G}_F)$, where σ is an involution. (By "non-trivial" we mean $\sigma \notin \Phi(\mathcal{G}_F)$.) Since $\mathcal{G}_F \cong T(n)$, let us write the involutions generating $\mathcal{G}_F$ as $\sigma_1, \ldots, \sigma_n$. We claim that the only non-trivial cosets $\sigma\Phi(\mathcal{G}_F)$ with σ an involution are those with $\sigma = \sigma_i$ for some i, or equivalently, that the maximal elementary abelian subgroups of $\mathcal{G}_F$ are the subgroups $\langle \sigma_i, \Phi(\mathcal{G}_F)\rangle$. To see this, write the general element $\sigma \in \mathcal{G}_F$ as a product:

$$\sigma = \sigma_1^{\epsilon_1}\sigma_2^{\epsilon_2}\cdots\sigma_n^{\epsilon_n}\prod[\sigma_i,\sigma_j]^{\epsilon_{ij}}$$

where $\epsilon_i, \epsilon_{ij} \in \{0,1\}$, and note that if $\sigma \notin \mathcal{G}_F$, $\epsilon_i = 1$ for at least one i, while if $\epsilon_i = 1$ for more than one i, $\sigma^2 \neq 1$.

Sufficiency. The converse is more interesting. Let F be a formally real pythagorean SAP field, with $|\dot{F}/\dot{F}^2| = 2^n$ and suppose that $\mathcal{G}_F \neq T(n)$. Then we want to show that there exist more than n orderings of F. This will require some field theory; to start choose $a_1, \ldots, a_n \in \dot{F}$ such that $\sigma_i(\sqrt{a_j})/\sqrt{a_j} = -1$ if $i = j$ and is equal to 1 if $i \neq j$. Then $a_i \in \cap_{i\neq j} P_j - P_i$, where $P_i = \{f \mid \sqrt{f}^{\sigma_i} = \sqrt{f}\}$ are orderings corresponding to the σ_i (see [39, p. 522]). (The a_i exist since F is assumed to be SAP.) It follows that the cosets $[a_1], \ldots, [a_n]$ form a basis for the vector space $\dot{F}/\dot{F}^2$. If $\mathcal{G}_F \neq T(n)$, then $\mathcal{G}_F$ is a proper quotient of $T(n)$ (5.1), and there exists a nontrivial relation between commutators $[\sigma_i, \sigma_j]$, $1 \leq i < j \leq n$. Without loss of generality we may assume that our relation $\gamma = [\sigma_1, \sigma_2]\gamma_1 \in T(n)$ is a nontrivial product of commutators of the form $[\sigma_i, \sigma_j]$, $i < j$, and that $[\sigma_1, \sigma_2]$ does not enter the expression for γ_1. Since γ is a relation, the image $\bar{\gamma}$ of γ in $\mathcal{G}_F$ is 1. From [38, 2.21] we can conclude that the quaternion algebra

$$A = \langle \alpha_1, \alpha_2 \mid \alpha_1^2 = a_1, \alpha_2^2 = a_2, \alpha_1\alpha_2 = -\alpha_2\alpha_1\rangle$$

does not split. Applying the standard criterion for the splitting of quaternion algebras over F (see [26]), we see that $a_2 \notin D\langle 1, -a_1\rangle$, where by $D\langle 1, -a_1\rangle$ we mean the set of values of the quadratic form $x^2 - a_1y^2$ over F. However, using [27, 1.6], we see that $D\langle 1, -a_1\rangle$ is the intersection of all orderings P of F such that $-a_1 \in P$. If the set of all orderings were just $P_1, \ldots, P_n$ (the ones determined by the cosets $\sigma_i\Phi(\mathcal{G}_F)$), then we would have that this intersection is equal to P_1, but $a_2 \in P_1$, a contradiction. Therefore we see that there exists some ordering P of F such that $P \neq P_1, \ldots, P_n$.

This finishes the proof of the fact that $\mathcal{G}_F \cong T(n)$ if and only if F is a formally real pythagorean SAP field with $|\dot{F}/\dot{F}^2| = 2^n$. □

Remark 5.3. The theorem above also follows from [14, prop. 4.1], where the results in [36] were applied, but we have chosen to provide a direct proof.

We now study the cohomology of the groups $T(n)$ using the topological models previously described in 4.6. If $H^*((\mathbb{Z}/2)^n) = \mathbb{F}_2[x_1, \ldots, x_n]$, then the k-invariants for the defining extension for $T(n)$ are precisely x_ix_j, with $i < j$. We consider an action of $E_n = (\mathbb{Z}/2)^n$ on $Z_n = (\mathbb{S}^1)^{\binom{n}{2}}$ defined as follows: if $\langle e_1, \ldots, e_n\rangle$ is a basis for E_n, then we define

$$e_k(z_{ij}) = \begin{cases} \bar{z}_{ij}, & \text{if } k = i; \\ -\bar{z}_{ij}, & \text{if } k = j; \\ z_{ij}, & \text{otherwise.} \end{cases}$$

In the notation of 4.6, if F is a field with $\mathcal{G}_F \cong T(n)$, then we can assume that $X_F = Z_n$, with the above action of $(\mathbb{Z}/2)^n$. We now analyze the equivariant cohomology ring $H^*_{E_n}(X_F)$. To do this we need to understand the "singular set" of the action, i.e. the subspace of elements in Z_n which are not freely permuted by the group E_n. In sufficiently high dimensions the equivariant cohomology is completely determined on this subspace (4.3).

Let $C_k = \langle e_k \rangle \subset E_n$; we examine its fixed-point set. Fixing k, there are $k-1$ coordinates of the form z_{ik} and $n-k$ coordinates of the form z_{kj} in Z_n. The group acts on each coordinate independently. Hence an $\binom{n}{2}$-tuple z_{ij} will be fixed by C_k if and only if $z_{ik} \in \{i, -i\}$ for $i < k$ and $z_{kj} \in \{1, -1\}$ for $j > k$ (note that the action on the other coordinates is trivial). Hence we obtain

$$Z_n^{C_k} \cong \{1, -1\}^{n-k} \times \{i, -i\}^{k-1} \times (\mathbb{S}^1)^{\binom{n-1}{2}}.$$

Now let $E_{n,k} = E_n/C_k$; then this group acts on $Z_n^{C_k}$, and we can express this space as a $E_{n,k}$ space as follows:

$$Z_n^{C_k} \cong E_{n,k} \times Z_{n-1}$$

where $E_{n,k} \cong E_{n-1}$ acts as before on Z_{n-1} (here we have all the coordinates z_{ij} which do not involve k; the basis we use for $E_{n,k}$ is simply the one obtained by omitting e_k). We need some understanding of the geometry of these fixed point sets. We have

Proposition 5.4. *$Z_n^{C_k} \cap Z_n^{C_l} = \varnothing$ if $k \neq l$ and any cyclic subgroup in E_n distinct from $C_1, \ldots, C_n$ acts freely on Z_n.*

Proof. Assume that $k < l$ and consider the coordinate z_{kl}; then the fixed point set for C_k has 1 or -1 in this coordinate and the fixed point set for C_l has i or $-i$ in this coordinate. Hence they cannot intersect. Now let $U \subset E_n$ be any cyclic subgroup distinct from the C_i. It will contain an element of the form $x_{i_1}x_{i_2}\ldots x_{i_r}$ where $r > 1$ and $i_1 < i_2 < \cdots < i_r$. This will act on the coordinate $z_{i_1 i_2}$ as multiplication by -1, hence it acts freely on all of Z_n. □

We have shown that the singular set $\mathcal{S}$ of the action is a disjoint union of precisely n subspaces, which, as E_n spaces, are of the form $E_{n,k} \times Z_{n-1}$. Since $H^*_{E_n}(E_{n,k} \times Z_{n-1}) \cong H^*_{C_k}(Z_{n-1})$ (4.4), we obtain

$$H^*_{E_n}(\mathcal{S}) \cong H^*_{C_1}(Z_{n-1}) \oplus \cdots \oplus H^*_{C_n}(Z_{n-1}).$$

Each of these factors has Poincaré series equal to $(1+t)^{\binom{n-1}{2}}/(1-t)$; hence we deduce that the Poincaré series for the equivariant cohomology of the singular set is $n(1+t)^{\binom{n-1}{2}}/(1-t)$. In dimensions larger than $\binom{n}{2}$ this will coincide with the equivariant cohomology (4.3). Using this we obtain that there is a polynomial $s_n(t)$ of degree $\binom{n}{2}$ such that the cohomology of $T(n)$ is given by the expression

$$(1) \qquad \frac{(1-t)s_n(t) + n2^{\binom{n-1}{2}} t^{\binom{n}{2}+1}}{(1-t^2)^{\binom{n}{2}}(1-t)}.$$

The polynomial $s_n(t)$ is simply the Poincaré series for the equivariant cohomology $H^*_{E_n}(Z_n)$ through dimension $\binom{n}{2}$. For example, if $n = 2$ this is

$1+2t$ and we obtain the series

$$\frac{(1-t)(1+2t)+2t^2}{(1-t^2)(1-t)} = \frac{1}{(1-t)^2}$$

which agrees with that for the cohomology of the dihedral group $D_8 \cong T(2)$.

It is worthwhile to observe that there is a one dimensional class, $\beta = x_1 + x_2 + \cdots + x_n \in H^1(T(n)) \cong H^1((\mathbb{Z}/2)^n)$ which restricts non trivially on $H^1_{C_i}(Z_{n-1})$ for all $i = 1, \dots, n$. On each of these summands this class restricts to a polynomial class coming from the cohomology of the cyclic isotropy subgroup. In sufficiently high dimensions, multiplication by this class induces a periodicity isomorphism. Hence if we take $\{\mu_{ij}\}$ and adjoin β, the quotient $H^*(T(n))/(\mu_{ij}, \beta)$ is finite dimensional. Hence we conclude that β together with the μ_{ij}, $i < j$ form a *homogeneous system of parameters* for $H^*(T(n))$. We also record the following simple consequence of our calculations:

Corollary 5.5. *Let F be a field such that $\mathcal{G}_F = T(n)$; then*

$$H^*(F) \cong \mathbb{F}_2[x_1, \dots, x_n]/(x_i x_j) \cong \mathbb{F}_2[x_1] \bar{\oplus} \cdots \bar{\oplus} \mathbb{F}_2[x_n],$$

where $\bar{\oplus}$ indicates that we identify the unit elements in the rings involved in the direct sum. In particular, multiplication by β is an isomorphism.

To obtain further calculations, one needs to understand the series $s_n(t)$. In some instances this can be obtained by direct geometric methods. Consider the case $n = 3$. Z_3 is a 3-torus with an action of E_3 such that the singular set is homeomorphic to the disjoint union of 12 circles. One can readily see that there is a CW-decomposition for Z_3 which is permuted by this action (this is the product of the natural cellular structure for each circle). There are 64 three-dimensional cells and 192 two-dimensional cells, all of which are freely permuted by E_3. In fact the cellular chain complex has the following form:

$$\mathbb{F}_2[E_3]^8 \to \mathbb{F}_2[E_3]^{24} \to [\oplus_{i=1}^3 \mathbb{F}_2[E_3/C_i]^2] \oplus \mathbb{F}_2[E_3]^{18} \to$$
$$\to [\oplus_{i=1}^3 \mathbb{F}_2[E_3/C_i]^2] \oplus \mathbb{F}_2[E_3]^2.$$

The singular set is of codimension 2, and the pair $(Z_3, \mathcal{S})$ is a relative 3-manifold; hence we deduce that $H^3(Z_3/E_3) \cong H^3_G(Z_3, \mathcal{S}) \cong \mathbb{F}_2$. Using this fact and the long exact sequence associated to the equivariant pair $(Z_3, \mathcal{S})$, one can show that $s_3(t) = 1 + 3t + 5t^2 + 6t^3$. Note that the coefficients arise from the isomorphisms $H^1_{E_3}(Z_3) \cong H^1(E_3)$, $H^2_{E_3}(\mathcal{S}) \cong H^2_{E_3}(Z_3) \oplus H^3(Z_3/E_3)$ and $H^3_{E_3}(Z_3) \cong H^3_{E_3}(\mathcal{S})$. Substituting in our formula above, we obtain that the Poincaré series for the cohomology of $T(3)$ is

$$\frac{(1-t)(1+3t+5t^2+6t^3)+6t^4}{(1-t)(1-t^2)^3} = \frac{1+2t+2t^2+t^3}{(1-t)(1-t^2)^3}.$$

The cohomology of this group has already been computed (see [10], group number 144) and it agrees with the above; it is detected on the 3 conjugacy classes of maximal elementary abelian subgroups.

For later use we point out that in proving 5.2 we established that $T(n)$ has precisely n maximal 2-tori, all of rank $\binom{n}{2}+1$. If $\sigma_1,\ldots,\sigma_n$ generate the group, then the tori are of the form

$$U_k = \langle [\sigma_i,\sigma_j],\sigma_k \mid 1\le i<j\le n\rangle.$$

Note that they all intersect along the central elementary abelian subgroup generated by the commutators, and they are non-conjugate. These groups are all self-centralizing, and their Weyl groups are isomorphic to $(\mathbb{Z}/2)^{n-1}$.

To have a better understanding of the situation we will describe a rather interesting relationship between the groups $T(n)$ and $W(n-1)$. If $T(n)$ is generated by $\sigma_1,\ldots,\sigma_n$, consider the subgroup $P(n)$ generated by the elements $\sigma_1\sigma_2,\ldots,\sigma_1\sigma_n$ and the $\binom{n}{2}$ central commutators of order two. This is an index two subgroup, which we claim is isomorphic to $W(n-1)$.

Proposition 5.6. *The group $T(n)$ is isomorphic to the semi-direct product $W(n-1)\times_T\mathbb{Z}/2$, where the element of order two acts by inverting the $n-1$ generators in $W(n-1)$.*

Proof. We will use the group $P(n)$ to establish this isomorphism. Note that $T(n)/P(n)$ is generated by the class represented by $\bar\sigma_1$; this of course splits as σ_1 is an element of order two in $T(n)$. Hence $T(n)$ is a semi-direct product, and it will suffice to show that $P(n)\cong W(n-1)$ and identify the action of σ_1.

Observe that $W(n-1)$ is the universal group in the category of all 2-groups with $n-1$ generators and satisfying the following two conditions: (1) all squares are central, (2) all elements have order ≤ 4. Given that $P(n)\subset T(n)$, it will automatically satisfy both of these conditions and hence is a homomorphic image of $W(n-1)$. However by comparing orders we see that this in fact they must be isomorphic. The action is determined by conjugation with σ_1. Note that $\sigma_1\sigma_1\sigma_i\sigma_1=\sigma_i\sigma_1=(\sigma_1\sigma_i)^{-1}$, whence the proof is complete. ☐

We would like to understand 5.6 from the point of view of field theory. Assume that F is a field such that $\mathcal{G}_F=T(n)$. We claim that $P(n)$ can be interpreted as $\mathrm{Gal}(F^{(3)}/F(\sqrt{-1}))$. We have already remarked that under these conditions F is a formally real, pythagorean SAP field. In particular $\sqrt{-1}\notin\dot F$. From [39, p. 521], we see that for each σ_i, $\sigma_i(\sqrt{-1})=-\sqrt{-1}$. Hence for each σ_i,σ_j we have $\sigma_i\sigma_j(\sqrt{-1})=\sqrt{-1}$, and therefore $P(n)\subset\mathrm{Gal}(F^{(3)}/F(\sqrt{-1}))$; as it has index two in $\mathcal{G}_F$ this must be an equality. More explicitly, we claim that $P(n)$ can be identified with the W-group associated to $F(\sqrt{-1})$. Indeed, from [26, thm. 3.4, p. 202] we have an exact sequence induced by the inclusion of F in $F(\sqrt{-1})$:

$$1\to\{\dot F^2,-\dot F^2\}\to\dot F/\dot F^2\xrightarrow{\epsilon}\dot F(\sqrt{-1})/\dot F(\sqrt{-1})^2\xrightarrow{N}\dot F/\dot F^2$$

where ϵ is the map induced by inclusion, and N is the homomorphism induced by the norm. In our case N is trivial, and ϵ is surjective. Note that $|\dot F(\sqrt{-1})/\dot F(\sqrt{-1})^2|=2^{n-1}$, hence $\mathcal{G}_{F(\sqrt{-1})}$ is a homomorphic image of $W(n-1)$. On the other hand we see that $F(\sqrt{-1})^{(2)}$ (the compositum of all quadratic extensions of $F(\sqrt{-1})$) is just $F^{(2)}(\sqrt{-1})$. Since $F^{(3)}$ is Galois over $F(\sqrt{-1})$, we see that $F^{(3)}\subset F(\sqrt{-1})^{(3)}$. Hence $P(n)=$

$\mathrm{Gal}(F^{(3)}/F(\sqrt{-1}))$ is a homomorphic image of $\mathcal{G}_{F(\sqrt{-1})}$. We conclude that $P(n) \cong \mathcal{G}_{F(\sqrt{-1})} \cong W(n-1)$. We have identified the class of the extension associated to $P(n) \subset T(n)$: it is the class $\alpha \in H^1(T(n))$ which corresponds to the element $[-1] \in \dot{F}/\dot{F}^2 \cong H^1(T(n))$. Geometrically we see that $\alpha = \beta$, a class which "lives" on all components of the singular set of the action on X_F.

Associated to the extension class α we have a Gysin sequence

$$\dots H^i(T(n)) \xrightarrow{\cup\alpha} H^{i+1}(T(n)) \xrightarrow{\mathrm{res}} H^{i+i}(P(n)) \xrightarrow{\mathrm{tr}} H^{i+1}(T(n)) \to \dots$$

If multiplication by α were injective, then this sequence would collapse, i.e. the cohomology of $P(n)$ would be totally nonhomologous to zero, and the Poincaré series for the cohomology of $T(n)$ would be of the form $p_{n-1}(t)(1-t)^{-1}$ where $p_{n-1}(t)$ is the Poincaré series for $W(n-1)$ (4.5). Moreover the homogeneous system of parameters formed by adjoining the class α, to the regular sequence $\{\mu_F\}$ would be regular, and hence $H^*(T(n))$ would be Cohen-Macaulay. Conversely if $H^*(T(n))$ is Cohen-Macaulay, any homogeneous system of parameters is a regular sequence, and this in turn implies that multiplication by α is injective. Moreover if $H^*(T(n))$ is Cohen-Macaulay, it must be detected on the centralizers of elementary abelian subgroups of maximal rank (a result appearing in [11]); in this case we obtain detection on elementary abelian subgroups. Conversely, note that the homogeneous system of parameters consisting of α and the μ_F restricts to regular sequences on all the maximal elementary abelian subgroups; hence if they detect the entire cohomology this must be a regular sequence.

¿From our preceding calculations we see that in the Gysin sequence for the extension

$$1 \to P(3) \to T(3) \to \mathbb{Z}/2 \to 1$$

the cohomology of $P(3)$ is totally non-homologous to zero, and hence it collapses to yield the Poincaré series described previously. We also have explicit detection on elementary abelian subgroups. We claim that this simple scheme breaks down for $T(4)$ and from there on. Assume that multiplication by α is injective; then the Poincaré series for $H^*(T(4))$ would be given by

$$\frac{1+3t+8t^2+12t^3+8t^4+3t^5+t^6}{(1-t)(1-t^2)^6}.$$

In particular if we divide out by the regular sequence μ_F, we see that in dimension 7 there must be precisely $1+3+8+12+8+3+1=36$ linearly independent elements. However, from our previous calculation of Poincaré series, there can only be 32. This contradiction means that multiplication by α cannot be injective, and so we deduce that $H^*(T(4))$ is not Cohen-Macaulay.

More generally, there is a natural inclusion of pairs $(T(n), P(n)) \subset (T(n+1), P(n+1))$ which gives rise to a commutative diagram of restriction maps:

$$\begin{array}{ccc} H^*(T(n+1)) & \longrightarrow & H^*(T(n)) \\ \downarrow & & \downarrow \\ H^*(P(n+1)) & \longrightarrow & H^*(P(n)) \end{array}$$

The two horizontal arrows are naturally split and hence are automatically surjective. Now injectivity of multiplication by α is equivalent to surjectivity of the restriction map $H^*(T(n)) \to H^*(P(n))$. Hence we deduce that if multiplication by α is injective for $T(n+1)$, then it must be true for $T(n)$ (e.g. $T(3)$ and $T(2)$). Hence as it fails for $T(4)$, it must fail for $T(n)$ for all $n \geq 4$. We therefore have

Proposition 5.7. *For $n \geq 4$, $H^*(T(n))$ has depth equal to one less than the rank of the group. Hence it is not Cohen-Macaulay and it is not detected on elementary abelian subgroups.*

Using the insight acquired from our analysis of $T(n)$, we now make explicit the role played by the class $[-1]$ in the cohomology of $\mathcal{G}_F$ for any formally real field.

Proposition 5.8. *Let F be formally real. Then the class $[-1] \in H^1(\mathcal{G}_F)$ restricts non-trivially on any cyclic subgroup of the form $\langle\sigma\rangle$, where $\sigma \in \mathcal{G}_F - \Phi(\mathcal{G}_F)$ is a non-central involution. Furthermore if $\alpha \in H^*(\mathcal{G}_F)$ satisfies $[-1] \cup \alpha = 0$, then α is nilpotent.*

Proof. Let $\sigma \in \mathcal{G}_F$ denote a non-central involution. Now let P be the ordering corresponding to σ, i.e. $P = \{p \in \dot{F} \mid \sqrt{p}^{\sigma} = \sqrt{p}\}$. Since $-1 \notin P$, we see that $\sqrt{-1}^{\sigma} = -\sqrt{-1}$, and therefore $[-1](\sigma) \neq 0$. This shows that $[-1]$ restricts non-trivially on any such cyclic subgroup $\langle\sigma\rangle$ (note that in particular this shows that $[-1]$ is not nilpotent). The maximal elementary abelian subgroups in $\mathcal{G}_F$ are of the form $E_\sigma = \langle\Phi(\mathcal{G}_F), \sigma\rangle$, where is σ is a non-central involution as before. Hence in particular we have shown that $[-1]$ restricts non-trivially to every maximal elementary abelian subgroup. Thus, if $[-1] \cup \alpha = 0$, we know that α must restrict trivially on all these subgroups, and by Quillen's theorem (see [3]) must be nilpotent. $\square$

We are interested in investigating under what conditions $[-1]$ is not a zero divisor. We have

Lemma 5.9. *Let F be a formally real field. Then, if $[-1]$ is not a zero divisor in $H^*(\mathcal{G}_F)$, F must be pythagorean.*

Proof. Suppose that F is a formally real, non-pythagorean field. Then there exists an element $a \in \dot{F}$ such that $a \notin \dot{F}^2$ but is a sum of two (non-zero) squares in F, say $a = x^2 + y^2$. If we rewrite this as $a - x^2 - y^2 = 0$, then we obtain the relation $[a][-x^2][-y^2] = 0$ in $H^*(\mathcal{G}_F)$. Here we are using a well-known identity in Milnor K-theory (see [33, p. 320]) and the fact that we have identified it with the subring $\mathcal{R} \subset H^*(\mathcal{G}_F)$ generated by 1-dimensional classes (see 3.14). Expanding this out by using the K-theoretic relations, we obtain

$$0 = [a][-x^2][-y^2] = [a]([-1] + [x^2])([-1] + [y^2]) = [a][-1]^2$$

from which we deduce that $[-1]$ is necessarily a zero divisor, a contradiction. Hence F must be pythagorean. $\square$

Now given any formally real field F we can construct an index 2 subgroup $\mathcal{H} \subset \mathcal{G}_F$ corresponding to $[-1] \in \dot{F}/\dot{F}^2$. We can use this subgroup to obtain a simple condition that implies the Cohen-Macaulay property.

Proposition 5.10. *If F is formally real and $[-1]$ is not a zero divisor, then $H^*(\mathcal{G}_F)$ is Cohen-Macaulay.*

Proof. Assume that $[-1]$ is not a zero divisor. Then using the Gysin sequence we can express

$$H^*(\mathcal{H}) \cong H^*(\mathcal{G}_F)/([-1]).$$

Now recall that the sequence $\{\mu_F\}$ in $H^*(\mathcal{G}_F)$ is regular; but more is true: in fact the classes μ_F restrict non-trivially to $\Phi(\mathcal{G}_F)$, the maximal central elementary abelian subgroup. The index 2 subgroup $\mathcal{H}$ contains this central elementary abelian subgroup, and hence the μ_F restrict to a regular sequence in its cohomology. This means that $\{\mu_F\}$ is a regular sequence in the quotient $H^*(\mathcal{G}_F)/([-1])$, and we conclude that $\{\mu_F, [-1]\}$ is a regular sequence in $H^*(\mathcal{G}_F)$, and so it is Cohen-Macaulay. $\square$

Next we observe that the centralizers of the maximal elementary abelian subgroups $E_\sigma \subset \mathcal{G}_F$ are simply the centralizers of the non-central involutions $\sigma \in \mathcal{G}_F - \Phi(\mathcal{G}_F)$. An immediate consequence of lemma 2.1 in [34] is that these centralizers are precisely the elementary abelian subgroups themselves i.e. they are self-centralizing. Applying the detection results in [11] we obtain a simple consequence of 5.10.

Proposition 5.11. *If F is formally real then $[-1]$ is not a zero divisor if and only if $H^*(\mathcal{G}_F)$ is detected on the cohomology of its elementary abelian subgroups.*

Proof. According to [11], if $H^*(\mathcal{G}_F)$ is Cohen-Macaulay, it is detected on centralizers of maximal 2-tori. On the other hand, if we assume detection on elementary abelian subgroups, then $[-1]$ cannot be a zero divisor. Indeed, we have shown that it restricts non-trivially on all the maximal elementary abelian subgroups, hence given any $x \in H^*(\mathcal{G}_F)$, there must exist an E_σ on which $[-1] \cup x$ restricts non-trivially, as this is true for x. $\square$

It remains to identify the subgroup $\mathcal{H}$ in terms of field-theoretic data. Following the arguments given after 5.6, one can show that $\mathcal{H}$ is a quotient of the W-group $\mathcal{G}_{F(\sqrt{-1})}$. Indeed the arguments provided there can be extended to provide the following entirely analogous characterization of fields where these two groups agree (see [37] for details).

Theorem 5.12. *If F is not formally real then the index two subgroup $\mathcal{H} \subset \mathcal{G}_F$ corresponding to $[-1]$ is isomorphic to $\mathcal{G}_{F(\sqrt{-1})}$ if and only if F is pythagorean. Furthermore, in that case the group $\mathcal{G}_F$ can be expressed as a semi-direct product*

$$\mathcal{G}_F \cong \mathcal{G}_{F(\sqrt{-1})} \times_T \mathbb{Z}/2$$

where the involution acts by inverting a suitable collection of minimal generators for $\mathcal{G}_F$.

Now let F denote a pythagorean field. Evidently $F(\sqrt{-1})$ is not formally real, hence its W-group satisfies the 2C property. This has an interesting geometric interpretation. Take the model X_F with an action of E_n. Then there exists a hyperplane $H_{-1} \subset E_n$ such that X_F with the action of H_{-1} is a model for $\mathcal{G}_{F(\sqrt{-1})}$ (indeed, we can identify this kernel with

$H^1(\mathcal{G}_{F(\sqrt{-1})}) \subset H^1(\mathcal{G}_F) \cong E_n)$. Note that $\Phi(\mathcal{G}_F) = \Phi(\mathcal{G}_{F(\sqrt{-1})})$. By our previous observation, H_{-1} must act freely on X_F. Let $\mathcal{I} = \{C_1, \ldots, C_t\}$ denote the set of all isotropy subgroups for the action of E_n on X_F. Then H_{-1} is a hyperplane which does not contain any of these subgroups.

Putting things together we obtain:

Theorem 5.13. *If F is a formally real field then the following conditions are equivalent*

1. *F is pythagorean and $H^*(\mathcal{G}_F)$ is Cohen-Macaulay.*
2. *F is pythagorean and $H^*(\mathcal{G}_F)$ is detected on its elementary abelian subgroups.*
3. *$[-1] \in H^1(\mathcal{G}_F)$ is not a zero divisor.*

Proof. By our previous results it suffices to show that (1) implies (3). If we have the Cohen-Macaulay condition, any non-redundant homogeneous system of parameters forms a regular sequence. We claim that this is true for $\{\mu_F, [-1]\}$. It suffices to show that multiplication by $[-1]$ on $H^*(\mathcal{G}_F)/(\mu_F)$ is an isomorphism in sufficiently large dimensions. Consider the action of E_n on X_F; we can identify the quotient above with the equivariant cohomology $H^*_{E_n}(X_F)$. Similarly we can identify $H^*(\mathcal{G}_{F(\sqrt{-1})}/(\mu_F)$ with $H^*(X_F/H_{-1})$. The Gysin sequence is recovered on equivariant cohomology as the following long exact sequence:

$$\cdots H^i(X_F/H_{-1}) \to H^i_{E_n}(X_F) \xrightarrow{\cup[-1]} H^{i+1}_{E_n}(X_F) \to H^{i+1}(X_F/H_{-1}) \cdots$$

We observe that $H^i(X_F/H_{-1}) = 0$ for $i > \dim X_F$, whence multiplication by $[-1]$ is eventually injective on the quotient, and hence we can conclude that $[-1]$ is not a zero-divisor. $\square$

We will now consider examples of fields where the above conditions are satisfied. We first recall from [27, p. 45] that a field F is said to be *superpythagorean* if it satisfies the following conditions. F is a formally real field with the property that for any set S containing $\dot{F}^2$ but such that $-1 \notin S$, if S is a subgroup of index 2 in $\dot{F}$, then S is an ordering on F. It is easy to see that superpythagorean fields are pythagorean (we refer the reader to [27], appendix to section 5, for additional information and references). Nice examples of such fields are given by $F = \mathbb{R}((t_1))((t_2))\ldots((t_n))$, the field of iterated power series over $\mathbb{R}$. In this case $|\dot{F}/\dot{F}^2| = 2^{n+1}$, with a basis given by the classes $[-1], [t_1], \ldots, [t_n]$.

We will compute the cohomology of any W-group arising from such a field. For this we make use of the following group-theoretic characterization described in [39]. If F is superpythagorean with $|\dot{F}/\dot{F}^2| = 2^n$, then

$$\mathcal{G}_F \cong (\mathbb{Z}/4)^{n-1} \times_T \mathbb{Z}/2$$

a semidirect product where the element of order two acts by inverting the $n-1$ standard generators in $(\mathbb{Z}/4)^{n-1}$. We shall denote this group by $S(n)$; then the Frattini subgroup is generated by the elements of order 2 in $(\mathbb{Z}/4)^{n-1}$ and we can express it as a central extension

$$1 \to (\mathbb{Z}/2)^{n-1} \to S(n) \to (\mathbb{Z}/2)^n \to 1.$$

If $x_1, \ldots, x_n$ generate the cohomology of the quotient group, then we can assume that the k-invariants of this extension are given by

$$x_2(x_2 + x_1), x_3(x_3 + x_1), \ldots, x_n(x_n + x_1).$$

Note that these elements form a regular sequence in $\mathbb{F}_2[x_1, \ldots, x_n]$. Now let X_F denote the space constructed as before; in this case $X_F \simeq (\mathbb{S}^1)^{n-1}$ with an action of $E_n = (\mathbb{Z}/2)^n$. Consider the spectral sequence for computing the equivariant cohomology of X_F; it has E_2 term

$$\mathbb{F}_2[x_1, \ldots, x_n] \otimes \Lambda(u_1, \ldots, u_{n-1})$$

where the exterior generators on the fiber transgress to the k-invariants above. As these form a regular sequence, the spectral sequence collapses at E_3, and furthermore the edge homomorphism induces an isomorphism of algebras

$$E_3^{*,0} \cong H^*_{E_n}(X_F)$$

(see [12] or [15]). In this case we simply obtain

$$H^*_{E_n}(X_F) \cong H^*(F) \cong \mathbb{F}_2[x_1, \ldots, x_n]/x_2(x_1 + x_2), \ldots, x_n(x_1 + x_n).$$

Note that in this example we can identify $[-1]$ with the class x_1, and $\mathcal{G}_{F(\sqrt{-1})} = (\mathbb{Z}/4)^{n-1}$. Clearly multiplication by $[-1]$ is injective and so the conclusions of 5.13 all apply. Indeed if $\zeta_1, \ldots, \zeta_{n-1}$ denote the regular sequence obtained from the Frattini subgroup, then a basis for $H^*(S(n))$ as a module over $\mathbb{F}_2[x_1, \zeta_1, \ldots, \zeta_{n-1}]$ is given by the monomials $x_{i_1} x_{i_2} \ldots x_{i_s}$ where $2 \leq i_1 < i_2 < \cdots < i_s \leq n$. In particular the Poincaré series for $H^*(S(n))$ is given by

$$p_F(t) = \frac{(1+t)^{n-1}}{(1-t)(1-t^2)^{n-1}} = \frac{1}{(1-t)^n}.$$

We have calculated the cohomology rings of superpythagorean fields with finite square class group. It is an easy matter to complete our results and compute $H^*(\mathcal{G}_F)$ for any superpythagorean field F. Let $\{[-1]\} \cup \{[a_i] \mid i \in I\}$ be any basis of $\dot{F}/\dot{F}^2$ which contains $[-1]$. Then a set $\{\sigma_{-1}\} \cup \{\sigma_i \mid i \in I\} \subset \mathcal{G}_F$ such that $\sigma_{-1}(\sqrt{-1}) = -\sqrt{-1}$, $\sigma_{-1}(\sqrt{a_i}) = \sqrt{a_i}$ for all $i \in I$, and $\sigma_i(\sqrt{a_j}) = (-1)^{\delta_{ij}}\sqrt{a_j}$ (where $\delta_{ij} = -1$ if $i = j$, $\delta_{ij} = 1$ otherwise) and finally $\sigma_i(\sqrt{-1}) = \sqrt{-1}$ is a minimal set of generators for $\mathcal{G}_F$. For any finite subset S of I consider the subgroup generated by σ_i, $i \in S$, and by σ_{-1}. Each G_S is isomorphic to $S(n+1)$, where $n = |S|$. Moreover each G_S is also a quotient of $\mathcal{G}_F$, and the (G_S) (as S ranges over finite subsets in I) form a projective system of finite subgroups in $\mathcal{G}_F$ such that $\mathcal{G}_F = \varprojlim G_S$. Therefore from [44, I.2.8] we see that $H^*(\mathcal{G}_F) \cong \varinjlim H^*(G_S)$ and hence the subring generated by one-dimensional classes will simply be $\mathcal{R} = \mathbb{F}_2[x_i \mid i \in I]/M$ where M is the ideal generated by the set $\{x_i(x_1 + x_i) \mid i \in I\}$.

6. Cohomology of Universal W-Groups

¿From the point of view of finite group cohomology, the universal groups $W(n)$ are perhaps the most interesting examples of W-groups. Given that $W(n)$ will surject onto any other W-group $\mathcal{G}_F$ for $|\dot{F}/\dot{F}^2| = 2^n$, $H^*(W(n))$

must necessarily be rather complicated. Given that the $W(n)$ satisfy the 2C condition, we know by [1] that their cohomology is not detectable on proper subgroups, and hence the usual methods for computing cohomology of finite groups will run into difficulties.

Before discussing general properties of these groups, it seems natural to describe how the universal groups arise as W-groups. A basic example is given by taking the field $F = \mathbb{C}(t)$, where the elements $\{[t-c] \mid c \in \mathbb{C}\}$ form a basis for $\dot{F}/\dot{F}^2$. The well-known theorem of Tsen-Lang (see [26, pp. 45, 46, 296]) implies that each quaternion algebra $(\frac{a,b}{F})$ with $a, b \in \dot{F}$ splits. From theorem 2.20 in [38], we see that the W-group of F is the universal W-group on the set of generators $\{\sigma_{(t-c)} \mid c \in \mathbb{C}\}$ One can also construct an infinite algebraic extension $F \subset K \subset F_q$ (taking $F = \mathbb{C}(t)$ again) such that $|\dot{K}/\dot{K}^2| = 2^n$ (n any prescribed natural number), and $\mathcal{G}_K \cong W(n)$. For details of this construction we refer the reader to [20, pp. 102–103].

We now describe some subgroups and quotient groups of $W(n)$. ¿From the description we gave for $W(n)$ in terms of generators and relations, it is apparent that we can construct a surjective map $\phi_i \colon W(n+1) \to W(n)$ simply by making $x_i = 1$ and leaving the other generators unchanged. This map clearly splits with the canonical injection and hence we have an embedding $H^*(W(n)) \hookrightarrow H^*(W(n+1))$. In fact these maps can be described in terms of extensions of the form

$$1 \to \mathbb{Z}/4 \times (\mathbb{Z}/2)^n \to W(n+1) \to W(n) \to 1,$$

where the $\mathbb{Z}/4$ corresponds to the subgroup generated by x_i and the $\mathbb{Z}/2$ factors correspond to the commutators $[x_i, x_j]$, where $i \neq j$.

On the other hand, we can also make n of the generators in $W(n+1)$ equal to 1, and hence obtain an extension of the form

$$1 \to W(n) \times (\mathbb{Z}/2)^n \to W(n+1) \to \mathbb{Z}/4 \to 1.$$

Recall that we have established in 3.10 that $H^*(W(n))$ is free and finitely generated over the polynomial subalgebra generated by $r = n + \binom{n}{2}$ 2-dimensional classes $\{\zeta_{ij}\}_{i \leq j}$. Furthermore, the Poincaré series for $H^*(W(n))$ is of the form

$$p_n(t) = \frac{q_n(t)}{(1-t^2)^{n+\binom{n}{2}}}$$

where $q_n(t)$ is a palindromic polynomial of degree $n + \binom{n}{2}$ in $\mathbb{Z}[t]$. Our goal will now be to study the polynomial $q_n(t)$.

Given E_n an elementary abelian 2-group with basis $\{x_1, \ldots, x_n\}$, we define an action on $X_n = (\mathbb{S}^1)^r$, where as before $r = n + \binom{n}{2}$.

Definition 6.1. Let (z_{ij}) denote the complex coordinates for the space X_n, ordered lexicographically. Then we define the action of E_n on X_n as follows:

$$x_l(z_{ij}) = \begin{cases} -z_{ij}, & \text{if } i = j = l; \\ \bar{z}_{ij}, & \text{if } i = l,\ j \neq l; \\ -\bar{z}_{ij}, & \text{if } j = l,\ i \neq l; \\ z_{ij}, & \text{otherwise.} \end{cases}$$

Following the notation from 4.6, if $\mathcal{G}_F = W(n)$, we can take $X_F = X_n$ with the E_n action described as above. The action is evidently free (indeed, $W(n)$ satisfies the 2C property) and so the orbit space $Y_n = X_n/E_n$ is a compact r dimensional manifold. We shall denote its fundamental group by $\Upsilon(n)$. We are now faced with a very explicit topological problem, namely

Problem 6.2. Calculate $H^*(X_n/E_n)$.

To begin we consider $W(2)$; it is a group of order 32 expressed as a central extension

$$1 \to (\mathbb{Z}/2)^3 \to W(2) \to (\mathbb{Z}/2)^2 \to 1.$$

Its cohomology has been computed by Rusin [42] but we will redo this calculation from our point of view. In this case the group $\Upsilon(2)$ is the fundamental group of a closed, non-orientable Seifert manifold, i.e. a circle fibering over a 2-dimensional surface (the torus). Hence it fits into an extension of the form

$$1 \to \mathbb{Z} \to \Upsilon(2) \to \mathbb{Z} \oplus \mathbb{Z} \to 1$$

with trivial twisting over $\mathbb{F}_2$. This yields a long exact sequence in mod 2 cohomology,

$$\begin{aligned} 0 \to H^1(\mathbb{Z} \oplus \mathbb{Z}) \cong H^1(\Upsilon(2)) \to \\ \to H^1(\mathbb{Z}) \to H^2(\mathbb{Z} \oplus \mathbb{Z}) \to H^2(\Upsilon(2)) \to H^1(\mathbb{Z} \oplus \mathbb{Z}, H^1(\mathbb{Z})) \to 0 \end{aligned}$$

as well as the evident isomorphism $H^3(\Upsilon(2)) \cong H^2(\mathbb{Z} \oplus \mathbb{Z}, H^1(\mathbb{Z})) \cong \mathbb{F}_2$. Hence we conclude that $q_2(t) = 1 + 2t + 2t^2 + t^3$. In terms of ring structure it is easy to verify (using mod 2 Poincaré Duality) that

$$H^*(\Upsilon(2)) \cong \Lambda(x_1, y_1, u_2, v_2)/\langle R \rangle$$

where x_1, y_1, u_2, v_2 are exterior classes with degree equal to their subscript and R is the set of relations

$$x_1 y_1 = u_2 v_2 = x_1 u_2 = y_1 v_2 = 0, \quad x_1 v_2 = y_1 u_2.$$

Hence the Poincaré series for $W(2)$ is

$$p_2(t) = \frac{1 + 2t + 2t^2 + t^3}{(1 - t^2)^3}.$$

Now we consider the case $n = 3$. Here we have $X_3 = (\mathbb{S}^1)^6$ with a free action of E_3. If we label these complex coordinates using z_{ij}, then we can project (equivariantly)

$$(z_{11}, z_{22}, z_{33}, z_{12}, z_{13}, z_{23}) \mapsto (z_{11}, z_{22}, z_{33}, z_{12}, z_{13})$$

where $Z = (\mathbb{S}^1)^5$ still has a free E_3 action due to the presence of the 3 free coordinates z_{ii}. Hence we obtain a circle bundle $X_3/E_3 \to Z/E_3$. Similarly, using the projections

$$(z_{11}, z_{22}, z_{33}, z_{12}, z_{13}) \mapsto (z_{11}, z_{22}, z_{33}, z_{12})$$

and

$$(z_{11}, z_{22}, z_{33}, z_{12}) \mapsto (z_{11}, z_{22}, z_{33})$$

we obtain circle bundles

$$Z/E_3 \to W/E_3 \quad \text{and} \quad W/E_3 \to U/E_3$$

where $W \cong (\mathbb{S}^1)^4$ and $U \cong (\mathbb{S}^1)^3$, all of which have a free E_3-action.

For our cohomology computation we start at the bottom, first we show

Proposition 6.3.

$$H^*(W/E_3) \cong \Lambda(u_1, v_1, w_1, b_2, c_2)/\langle R \rangle$$

where R is the set of relations

$$u_1 v_1 = u_1 b_2 = v_1 c_2 = 0, \quad u_1 c_2 = v_1 b_2$$

and the Poincaré series for W/E_3 is $1 + 3t + 4t^2 + 3t^3 + t^4$.

Proof. First we recall that $U/E_3 \cong (\mathbb{S}^1)^3$. Next we use the Gysin sequence for the circle bundle $W/E_3 \to U/E_3$. Let e represent the generator for the cohomology of the fiber, and u_1, v_1, w_1 generators for the cohomology of the base. The key differential is given by $d_2(e) = u_1 v_1$; the following is an explicit list of non-zero classes in $E_3^{*,*}$:

$$\{u_1, v_1, w_1, eu_1, ev_1, u_1 w_1, v_1 w_1, eu_1 v_1, eu_1 w_1, ev_1 w_1, eu_1 v_1 w_1\},$$

from which the calculation above readily follows. □

Our strategy will now be to iterate this method. We consider the circle bundle $Z/E_3 \to W/E_3$; the Gysin sequence simplifies to yield a sequence

$$0 \to \mathbb{F}_2^3 \to H^2(Z/E_3) \to \mathbb{F}_2^3 \xrightarrow{d_2} \mathbb{F}_2^3 \to H^3(Z/E_3) \to \mathbb{F}_2^3 \to 0$$

and the isomorphisms $H^1(Z/E_3) \cong \mathbb{F}_2^3$, $H^5(Z/E_3) \cong \mathbb{F}_2$. If e' generates the cohomology of the fiber, then $d_2(e') = u_1 w_1$, and hence $d_2(u_1 e') = u_1^2 w_1 = 0$, $d_2(v_1 e') = v_1 u_1 w_1 = 0$, $d_2(w_1 e') = u_1 w_1^2 = 0$ and so $d_2 \equiv 0$ in the sequence above. From this we deduce that $H^*(Z/E_3)$ has Poincaré series equal to

$$1 + 3t + 6t^2 + 6t^3 + 3t^4 + t^5$$

and is in fact generated by 3 one-dimensional classes, 5 two-dimensional classes and one three-dimensional class.

The final extension to X/E_3 can be done similarly, using the circle bundle $X/E_3 \to Z/E_3$. As before, the Gysin sequence is completely determined by a single differential, which is also identically zero. We obtain

Proposition 6.4. *The Poincaré series for $H^*(X_3/E_3)$ is*

$$q_3(t) = 1 + 3t + 8t^2 + 12t^3 + 8t^4 + 3t^5 + t^6$$

and hence the Poincaré series for $H^*(W(3))$ is

$$p_3(t) = \frac{1 + 3t + 8t^2 + 12t^3 + 8t^4 + 3t^5 + t^6}{(1 - t^2)^6}.$$

Unfortunately this process of iterating Gysin sequences becomes unmanageable after a few steps. This is related to interesting questions in differential homological algebra and homotopy theory which we will discuss further on.

We will now introduce another torsion-free group to study the cohomology of W-groups. Let $L(n)$ denote the universal central extension

$$(2) \qquad 1 \to \mathbb{Z}^{\binom{n}{2}} \to L(n) \to \mathbb{Z}^n \to 1$$

which is often called the free 2-step nilpotent group on n generators. Indeed the group $L(n)$ is generated by elements $\sigma_1, \sigma_2, \dots, \sigma_n$ of infinite order with the relations imposed by requiring that the commutators $[\sigma_i, \sigma_j]$ for $i < j$ be central (hence the extension above).

Recall from our description of $W(n)$ that it is generated by elements $x_1, \dots, x_n$ of order 4 such that their squares and commutators are central (and of order 2). Hence if we abelianize $W(n)$ we obtain $(\mathbb{Z}/4)^n$, and so a central extension

$$1 \to (\mathbb{Z}/2)^{\binom{n}{2}} \to W(n) \to (\mathbb{Z}/4)^n \to 1$$

where the generators of the kernel correspond to the commutators. Reducing mod 4 we obtain a map $L(n) \to (\mathbb{Z}/4)^n$ which factors through $W(n)$ and hence we obtain an extension

$$1 \to L_4(n) \to L(n) \to W(n) \to 1 \tag{3}$$

where $L_4(n)$ appears as some sort of "congruence subgroup". Note that it will be generated by the fourth powers $u_i = \sigma_i^4$ and the squares of the commutators $w_{ij} = [\sigma_i, \sigma_j]^2$.

We begin our cohomological analysis by computing the mod 2 cohomology ring of $L_4(n)$.

Theorem 6.5. *The mod 2 cohomology of $L_4(n)$ is an exterior algebra on $r = n + \binom{n}{2}$ one dimensional generators.*

Proof. ¿From the above it is clear that we can also realize $L_4(n)$ as a central extension of the subgroup generated by commutators,

$$1 \to \mathbb{Z}^{\binom{n}{2}} \to L_4(n) \to \mathbb{Z}^n \to 1.$$

To compute the mod 2 Lyndon-Hochschild-Serre spectral sequence for this extension we need to determine the commutators $[u_i, u_j]$ for $i < j$. A straightforward calculation shows that $\sigma_i^4 \sigma_j^4 \sigma_i^{-4} \sigma_j^{-4} = [\sigma_i, \sigma_j]^{16}$. Now if we consider the E_2 term of the aforementioned spectral sequence, it is of the form $\Lambda(u_1, u_2, \dots, u_n) \otimes \Lambda(v_1, \dots, v_{\binom{n}{2}})$ where by abuse of notation we have identified generators from the base with their duals. ¿From the extension data computed above, we see that the $v_1, \dots, v_{\binom{n}{2}}$ are permanent cocycles, as they transgress to elements divisible by 8 and hence 0 mod 2. The spectral sequence collapses at E_2, there are evidently no extension problems and the result follows. □

We will now use this to analyze the cohomology of $L(n)$.

Theorem 6.6. *In the mod 2 LHS spectral sequence associated to*

$$1 \to L_4(n) \to L(n) \to W(n) \to 1$$

the one dimensional generators in the cohomology of $L_4(n)$ transgress to a regular sequence in $H^2(W(n), \mathbb{F}_2)$, $E_3 = E_\infty$, and in particular if I is the ideal generated by these transgressions, then

$$H^*(W(n), \mathbb{F}_2)/I \cong H^*(L(n), \mathbb{F}_2).$$

Proof. Let $L_2(n)$ denote the subgroup of $L(n)$ generated by the commutators w_{ij}, $i < j$ and the squares σ_i^2; then this group fits into a commutative diagram of extensions:

$$\begin{tikzcd}
 & & 1 & 1 & \\
 & & (\mathbb{Z}/2)^n \arrow[u] \arrow[r, equal] & (\mathbb{Z}/2)^n \arrow[u] & \\
1 \arrow[r] & L_4(n) \arrow[r] & L(n) \arrow[r] \arrow[u] & W(n) \arrow[r] \arrow[u] & 1 \\
1 \arrow[r] & L_4(n) \arrow[r] \arrow[u, equal] & L_2(n) \arrow[r] \arrow[u] & (\mathbb{Z}/2)^r \arrow[r] \arrow[u] & 1 \\
 & & 1 \arrow[u] & 1 \arrow[u] &
\end{tikzcd}$$

Note that $H = (\mathbb{Z}/2)^r \subset W(n)$ is the maximal central elementary abelian subgroup. For reasons analogous to the arguments in 6.5, $H^*(L_2(n), \mathbb{F}_2)$ is also an exterior algebra on r one dimensional generators. Consider the mod 2 LHS spectral sequence for the bottom row. Let I' denote the ideal generated by the transgressions. Then evidently $H^*((\mathbb{Z}/2)^r)/I' \cong H^*(L_2(n))$, and so we conclude that the ideal is precisely the ideal generated by the squares of the one dimensional generators. Comparing spectral sequences, we see that I is generated by trangressions ζ_{ij}, $i < j$ such that they restrict to squares in $H^2(H, \mathbb{F}_2)$. As we saw in section 3 these form a regular sequence of maximal length in the cohomology of $W(n)$. Indeed, they generate a polynomial subalgebra over which it is free and finitely generated. From this it follows that $E_3 = E_\infty$ and hence that $H^*(L(n), \mathbb{F}_2) \cong H^*(W(n))/(\zeta_{ij})$. □

Remark 6.7. From the above we see that the mod 2 cohomology of $L(n)$ plays the same role as the cohomology of $\Upsilon(n)$. An important difference is that the latter group is virtually abelian, whereas $L(n)$ is not. The part played by the associated circle classes in cohomology is played by the cohomology classes in $H^1(L_4(n), \mathbb{F}_2)$. In other words we use a "mod 2 cohomology torus" instead. Note that the integral cohomology of $L_4(n)$ can be quite complicated.

The interesting feature of $L(n)$ is that it has substantial torsion-free cohomology. Furthermore, there are reasons to believe that all of it may be 2-torsion-free. In contrast, $H^*(\Upsilon(n), \mathbb{Q})$ is fairly small, even though it has the same mod 2 cohomology as $L(n)$. Later we shall provide a complete calculation for the rational cohomology $H^*(L(n), \mathbb{Q})$. From our constructions it is clear that these rational classes come from integral classes which in turn will appear as mod 2 classes in the cohomology of $W(n)$. This seems like a rather novel approach—using rational cohomology to produce mod 2 cohomology for a finite group.

We will now describe a well-known topological method for approaching the cohomology of $L(n)$. This is based on [21] and [28], although these methods were known even before these references appeared. To begin we observe that the central extension defining $L(n)$ gives rise to a classifying map $\phi\colon (\mathbb{S}^1)^n \longrightarrow (\mathbb{C}P^\infty)^{\binom{n}{2}}$. If we consider the pullback of the universal bundle over $(\mathbb{C}P^\infty)^{\binom{n}{2}}$, we obtain (up to homotopy) a diagram of fibrations

$$\begin{array}{ccc} BL(n) & \longrightarrow & U \\ \downarrow & & \downarrow \\ (\mathbb{S}^1)^n & \xrightarrow{\phi} & (\mathbb{C}P^\infty)^{\binom{n}{2}} \end{array}$$

where $BL(n)$ is the classifying space for $L(n)$ (and hence its cohomology is the group cohomology of $L(n)$) and the right hand column is the universal bundle (and so U is contractible). Note that the map ϕ is determined by the extension data, which in this case is the fact that the commutators are central. Note that if b_{ij} for $i<j$ form an integral basis for $H^2((\mathbb{C}P^\infty)^{\binom{n}{2}},\mathbb{Z})$, then we can assume $\phi^*(b_{ij}) = x_ix_j$, where $x_1,\dots,x_n$ is an exterior basis of one dimensional elements for the cohomology of $(\mathbb{S}^1)^n$.

Associated to any such pull-back diagram of spaces we have an *Eilenberg-Moore spectral sequence* converging to the cohomology of the upper left hand corner (in this case the cohomology of $L(n)$). In our situation the E_2 term can be computed homologically as

$$E_2^{*,*} = \mathrm{Tor}_{\mathbb{Z}[b_{ij}]}(\Lambda(x_1,\dots,x_n),\mathbb{Z}),$$

where we have written $\Lambda(x_1,\dots,x_n)$ for $H^*((\mathbb{S}^1)^n,\mathbb{Z})$.

Remark 6.8. This identification means that our convention for exterior algebras over $\mathbb{Z}$ requires the relation $e\wedge e = 0$ and not just $e\wedge e = -e\wedge e$. In particular, $\Lambda(x_1,\dots,x_n)$ is a free $\mathbb{Z}$-module.

We can obtain a great deal of information about this E_2 term using methods from commutative algebra. The map ϕ induces an algebra map

$$\phi^*\colon \mathbb{Z}[b_{ij}] \longrightarrow \Lambda(x_1,x_2,\dots,x_n)$$

with $\phi^*(b_{ij}) = x_ix_j$. To compute the Tor term in the spectral sequence, we make use of a particular free $\mathbb{Z}[b_{ij}]$ resolution for $\mathbb{Z}$, namely the Koszul complex (see [28] for complete details) which is of the form $\Lambda(u_{ij})\otimes\mathbb{Z}[b_{ij}]$. The E_2 term above can then be computed in the usual way, i.e. setting

$$K^{*,*} = \Lambda(u_{ij})\otimes\mathbb{Z}[b_{ij}]\otimes_{\mathbb{Z}[b_{ij}]}\Lambda(x_1,\dots,x_n), \tag{4}$$

we have $E_2^{*,*} = H(K^{*,*})$. In fact, more is true: the EMSS starts out with an E_1-term which can be indentified with our complex $K^{*,*}$.

Let us now record a few facts about the functoriality of this complex which will be used later. First, we have been working with integer coefficients in cohomology; if we liked we could tensor $K^{*,*}$ with $\mathbb{Q}$ or $\mathbb{F}_2$ to get the E_1-term for the rational or mod 2 version of the EMSS. Second, note that $K^{*,*}$ can be written in coordinate-free fashion as $\Lambda^*\Lambda^2V\otimes\Lambda^*V$, where V is a free $\mathbb{Z}$-module of rank n. In the bigrading associated with this description (which is *not* the same as the usual bigrading for the EMSS), the differential has bidegree $(2,-1)$ and a simple coordinate-free description: if $\theta\in\Lambda^rV$, and $\omega_1, \ldots, \omega_m \in \Lambda^2 V$, then

(5)

$$d_1(\theta \otimes \omega_1 \wedge \cdots \wedge \omega_m) = \sum_{i=1}^{m} (-1)^{i+1} (\theta \wedge \omega_i) \otimes (\omega_1 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega_m),$$

where the sum on the right hand side is an element of $\Lambda^{r+2} V \otimes \Lambda^{m-1} \Lambda^2 V$. These coordinate-free descriptions show that K is functorial in V; in particular $K \otimes \mathbb{Q}$ and its homology are representations of $GL(V \otimes \mathbb{Q})$.

At this stage we have two problems to address. First we have the purely algebraic problem of computing the homology of the Koszul complex. As we shall see shortly this has been done for rational coefficients, but for coefficients in $\mathbb{F}_2$ it is an open problem. Secondly, to make this an effective method of computation we need to determine the higher differentials or whether in fact $E_2 = E_\infty$. If the combinatorial determination of the homology of the Koszul complex and the collapse at E_2 can *both* be established then we will have a complete computation. We should also note that the E_3 term of the Lyndon-Hochschild-Serre spectral sequence for the central extension defining $L(n)$ will also be isomorphic to the cohomology of the Koszul complex. Hence its collapse at E_3 is equivalent to the collapse of the EMSS at E_2. There are however some advantages in dealing with higher differentials in the EMSS related to methods from homological perturbation theory (see [21]).

Over the rationals there is a very nice collapse theorem.

Proposition 6.9. *The Eilenberg-Moore spectral sequence for $R = \mathbb{Q}$ collapses at $E_2 = E_\infty$ without extension problems, and hence we have*

$$H^*(L(n), \mathbb{Q}) \cong H(K \otimes \mathbb{Q}).$$

Proof. This is proposition 4.3.1 in [28]. Without elaborating too much we shall only say that this works because we can use "rational de Rham complexes" which are commutative, from which it follows that the higher differentials are zero. □

Note that the isomorphism above is an isomorphism of algebras, i.e. we can endow the homology of the Koszul complex with a natural product induced from that on the original complex.

Combining 6.6 with 6.9 we can construct a large number of mod 2 cohomology classes for $W(n)$. Let T denote the ideal of torsion classes, then we have an inclusion

$$H^*(L(n), \mathbb{Z})/T \otimes \mathbb{F}_2 \hookrightarrow H^*(W(n))/(\zeta_{ij}).$$

Note that in particular we can determine the dimension of this subspace as the dimension of the *rational* cohomology of $L(n)$ or of the corresponding rational Koszul complex.

It is interesting to link the "algebraic 2-torsion" in the Koszul complex with 2-torsion in the cohomology of $L(n)$. Both of these are open questions as far as we know. They can be easily related via

Proposition 6.10. *The cohomology $H^*(L(n), \mathbb{Z})$ has no 2-torsion and the associated mod 2 Eilenberg-Moore spectral sequence collapses at E_2 if and only if the homology of the Koszul complex $K^{*,*}$ has no 2-torsion.*

Proof. Suppose that $H^*(L(n), \mathbb{Z})$ has no 2-torsion and that the mod 2 EMSS collapses at E_2. This means that this E_2 term accounts for all the mod 2 cohomology, which by our hypothesis must have the same dimension as the rational cohomology. In terms of the Koszul complex this simply means that the homology of the rational version must have the same total dimension as the mod 2 complex. Hence we know that the homology of the integral complex must be 2-torsion free. Conversely if this condition holds, the E_2 terms of both the rational and mod 2 EMSS for $L(n)$ must have the same dimensions, and as the rational one always collapses, the same must hold for the mod 2 one and in addition there cannot be any 2-torsion in $H^*(L(n), \mathbb{Z})$, as this would produce unaccountable mod 2 cohomology classes. $\square$

An explicit calculation due to Lambe [28] shows that for $n \leq 5$ the conditions in 6.10 hold. Hence we have complete calculations for the cohomology of the corresponding W-groups. In other words if as before $I \subset H^*(W(n), \mathbb{F}_2)$ is the ideal generated by the regular sequence ζ_{ij}, then we have an extension of algebras

$$0 \longrightarrow I \longrightarrow H^*(W(n), \mathbb{F}_2) \longrightarrow H(K \otimes \mathbb{F}_2) \longrightarrow 0.$$

Using the computations in [28] we can now record the Poincaré series for $W(4)$ and $W(5)$, following the format previously established. We have that $p_4(t) = q_4(t)/(1-t^2)^{10}$, where $q_4(t)$ is the polynomial

$$1 + 4t + 20t^2 + 56t^3 + 84t^4 + 90t^5 + 84t^6 + 56t^7 + 20t^8 + 4t^9 + t^{10}.$$

For $p_5(t)$ the denominator will be $(1-t^2)^{15}$ and the numerator will be $q_5(t)$, the polynomial

$$\begin{gathered} 1 + 5t + 40t^2 + 176t^3 + 440t^4 + 835t^5 + 1423t^6 + 1980t^7 \\ +1980t^8 + 1423t^9 + 835t^{10} + 440t^{11} + 176t^{12} + 40t^{13} + 5t^{14} + t^{15}. \end{gathered}$$

We will now describe the homology of the rational Koszul complex defined previously. The computation of $H(K \otimes \mathbb{Q})$ has appeared in many guises (see [24], also [6], [25], and [45]), so we do not repeat it here. The connections between the various contexts in which this computation occurs are not completely transparent, so we point out that the calculation is equivalent to computing the Lie Algebra cohomology of the graded Lie algebra associated to the group $L(n)$. To state the result, let us take V to be an n-dimensional rational vector space, identify $K \otimes \mathbb{Q}$ with $\Lambda^*\Lambda^2 V \otimes \Lambda^* V$, and use the associated bigrading. As noted earlier, this complex and its homology are representations of $GL(V)$. Irreducible representations of this group are parametrized by Young diagrams, which are a way of representing partitions of natural numbers; the reader unfamiliar with this theory may wish to consult [18].

Theorem 6.11. *The homology of $K \otimes \mathbb{Q}$ in the (p,q) position is the sum of all representations corresponding to symmetric $p + 2q$-box p-hook diagrams.*

The corresponding rational ranks are easily computable by a standard dimension formula [31]. We state this formula for the convenience of the reader. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition with trailing zeroes added to fill out its length to n if necessary. Recall that by convention, we take $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Associated to λ we have a diagram Y_λ with λ_1 boxes in the first column, λ_2 in the second, etc. where the columns all begin on the same horizontal line. Hence our partition gives a diagram with n columns of boxes, of length $\lambda_1, \ldots, \lambda_n$ respectively (some at the end may be zero). In total we will have $\lambda_1 + \cdots + \lambda_n$ boxes in Y_λ. We label these boxes by pairs (i,j) corresponding to rows and columns. Note that a diagram will be *symmetric* if it is invariant under the *transposition* of exchanging rows and columns. The number of "hooks" will be equal to the number of boxes in the diagram which are bisected by the diagonal; to each box we associate a *hook* by taking its union with all boxes below it and all boxes to the right of it. By definition, the *hooklength* of the box (i,j) is $h(i,j)$, the number of boxes below (i,j) plus the number of boxes to the right of (i,j) plus 1. Let V be a vector space of dimension n and let $S_\lambda V$ be the representation of $GL(V)$ corresponding to λ, then we have:

$$\dim S_\lambda V = \prod_{(i,j)\in Y_\lambda} \frac{n+j-i}{h(i,j)}$$

We can use this formula to find a lower bound on the coefficients of the Poincaré series $p_n(t)$. Indeed we have

Theorem 6.12. *Let $p_n(t) = q_n(t)/(1-t^2)^r$ denote the Poincaré series for $H^*(W(n))$, where $r = \binom{n+1}{2}$. If $q_n(t) = 1 + a_1 t + a_2 t^2 + \cdots + a_r t^r$, then*

$$a_k \geq \sum_{p+q=k} \sum_{Y_\lambda} \prod_{(i,j)\in Y_\lambda} \frac{n+j-i}{h(i,j)}$$

where Y_λ ranges over all symmetric $p+2q$-box, p-hook Young diagrams, and $h(i,j)$ denotes the hooklength of the box (i,j).

As we have noted, the rational cohomology $H^{p,q}(K \otimes \mathbb{Q})$ obtained from Young diagrams gives us mod 2 cohomology in $H^{p,q}(K \otimes \mathbb{F}_2)$. If we show that the *integral* cohomology $H^{p,q}(K)$ is 2-torsion free, we will be able to conclude that there is no mod 2 cohomology unaccounted for by our constructions, or in other words, that the inequality of theorem 6.12 is actually an equality. We will demonstrate by ad hoc methods that this is the case for certain small values of p and q:

Lemma 6.13. *$H^{p,q}(K)$ is 2-torsion free if $p+q \leq 4$ and $q \leq 3$.*

It then follows from the universal coefficient theorem that we have:

Theorem 6.14. *For all integers $n \geq 1$, we have*

$$a_2 = \frac{n(n+1)(n-1)}{3} \quad \text{and} \quad a_3 = \frac{n(n^2-1)(3n-4)(n+3)}{60}.$$

These results could certainly be extended, but as we have no method that would give all the coefficients a_k, we have elected to stop here.

This section has been rather technical, so for the weary reader who would like to see some easily-understandable consequences of the results we offer the following facts. ¿From the very definition of the groups $W(n)$ we know that their one dimensional cohomology classes multiply trivially, i.e. the subring in $\mathcal{R} \subset H^*(W(n))$ generated by $H^1(W(n))$ is an n dimensional space with no products. Hence there are precisely $n(n+1)(2n+1)/6$ 2-dimensional generators in $H^*(W(n))$. More generally, if n is sufficiently large, the k-th coefficient a_k of $q_n(t)$ will satisfy the inequality

$$a_k \geq \frac{(n+k-1)(n+k-2)\cdots(n-k+1)}{(2k-1)[(k-1)!]^2},$$

which can be derived from theorem 6.12.

We finish this section with the proof of our lemma about the 2-torsion freeness of $H^{p,q}(K)$.

Proof of lemma 6.13. Recall that the groups $K^{p,q} = \Lambda^p V \otimes \Lambda^q \Lambda^2 V$ are free abelian, and that the differential (5) has bidegree $(2,-1)$. If p is 0 or 1, then $H^{p,q}$ is a kernel and therefore also free abelian, in particular 2-torsion free. It is easy to show that $H^{p,q}$ is zero if $q = 0$ and $p > 1$, so we need only consider the cases $(p,q) = (2,2)$, $(2,1)$, and $(3,1)$.

$(p,q) = (2,1)$. First note that for any $\mathbb{Z}$-module W, there is an exact sequence $\Lambda^2 W \to W \otimes W \to S^2 W$, where the first map sends $u \wedge v$ to $u \otimes v - v \otimes u$. Applying this to $W = \Lambda^2 V$, we see that the cokernel of $\Lambda^2\Lambda^2 V \to \Lambda^2 V \otimes \Lambda^2 V$ is $S^2\Lambda^2 V$, so $H^{2,1}$ is a submodule of the free module $S^2\Lambda^2 V$, and therefore 2-torsion free.

$(p,q) = (3,1)$. We show that the sequence

$$V \otimes \Lambda^2\Lambda^2 V \to^{\phi} \Lambda^3 V \otimes \Lambda^2 V \to^{\mu} \Lambda^5 V$$

is exact at $\Lambda^3 V \otimes \Lambda^2 V$, so $H^{3,1}$ is zero. To see this, let $\{e_i\}$ be a basis for V, note that the kernel of μ is generated by elements of the form $e_i \wedge e_j \wedge e_k \otimes e_l \wedge e_m - e_i \wedge e_l \wedge e_m \otimes e_j \wedge e_k$, and that all of these elements are in the image of ϕ.

$(p,q) = (2,2)$. Notice that the natural map $\mu\colon \Lambda^2 V \otimes \Lambda^2\Lambda^2 V \to \Lambda^3\Lambda^2 V$ given by $\omega_1 \otimes \omega_2 \wedge \omega_3 \mapsto \omega_1 \wedge \omega_2 \wedge \omega_3$ has $\mu d_1 = 3$ (see equation 5). For purposes of 2-torsion, this means that the differential $\mathrm{d}_1\colon \Lambda^3\Lambda^2 V \to \Lambda^2 V \otimes \Lambda^2\Lambda^2 V$ is split, and that there is therefore no 2-torsion in $H^{2,2}$.

This completes the proof of lemma 6.13. □

7. Final Remarks

In the preceding sections we have described the basic cohomological structure of W-groups by making use of certain topological models. However one could equally well attempt to compute $H^*(\mathcal{G}_F)$ directly from the central extension

$$1 \to \Phi(\mathcal{G}_F) \to \mathcal{G}_F \to E \to 1.$$

The most meaningful situation occurs when $|\dot{F}/\dot{F}^2| = 2^n$, in which case $E = E_n \cong (\mathbb{Z}/2)^n$ and $\Phi(\mathcal{G}_F) \cong (\mathbb{Z}/2)^r$. There is an Eilenberg-Moore spectral sequence associated to this extension, with E_2-term given by the bigraded algebra

$$\mathrm{Tor}_{H^*(K(\Phi(\mathcal{G}_F),2))}(H^*(E_n), \mathbb{F}_2).$$

Now if V is an elementary abelian 2-group, $H^*(K(V,2))$ is a polynomial algebra on countably many generators. By using the explicit form of the

k-invariants as described in 3.7 this algebra can be simplified as follows. Let $\kappa_1, \ldots, \kappa_r \in H^2(E_n)$ denote the k-invariants of the extension. They can be used to define a map of polynomial algebras:

$$\mathbb{F}_2[b_1, \ldots, b_r] \to H^*(E_n)$$

where the $b_1, \ldots, b_r$ have degree 2 and $b_i \mapsto \kappa_i$. Then we can express the E_2 term above as an extension

$$0 \to (\zeta_1, \ldots, \zeta_r) \to E_2^{*,*} \to \mathrm{Tor}_{\mathbb{F}_2[b_1,\ldots,b_r]}(\mathbb{F}_2[\dot{F}/\dot{F}^2], \mathbb{F}_2) \to 0$$

where the polynomial classes $\zeta_1, \ldots, \zeta_r$ are permanent cocyles in bidegree $(-1, 3)$, and can be chosen to represent the regular sequence we have already obtained (see 3.10). We now make a conjecture which has been verified for all examples we know

Conjecture 7.1. Up to filtration, we have an isomorphism of algebras

$$H^*(\mathcal{G}_F)/(\zeta_1, \ldots, \zeta_r) \cong \mathrm{Tor}_{\mathbb{F}_2[b_1,\ldots,b_r]}(\mathbb{F}_2[\dot{F}/\dot{F}^2], \mathbb{F}_2)$$

The validity of this conjecture remains an interesting open question. It is equivalent to the collapse at E_3 for the Lyndon-Hochschild Serre spectral associated to the extension above. The advantage of the EMSS is the fact that in this kind of situation models with explicit differentials have been developed (see [21]) and hence substantial insight can be obtained; however a definitive proof (or counterexample) would seem to require additional ideas. It is interesting to note however, that the Galois cohomology occurs as an edge in the spectral sequence above, and the Tor algebra we describe seems to be the most natural extension of this to a global computation for W-groups. As we have mentioned previously, collapse on the edge is implied by the Milnor Conjecture. It would seem reasonable to expect that the intrinsic field theory input which determines $\mathcal{G}_F$ will play an important role here. The higher differentials can be determined using cup-1 products; one can only speculate that a sensible approach would be to get a hold on them by using the field theory context in which W-groups are defined.

References

[1] Adem, A. and Karagueuzian, D. *Essential Cohomology of Finite Groups*, Comm. Math. Helv. **72** (1997), pp. 101–109.

[2] Adem, A., Karagueuzian, D. and Minac, J. *Topological Models and the Cohomology of Galois Groups*, submitted research announcement, 1998.

[3] Adem, A. and R. J. Milgram, R. J. *Cohomology of Finite Groups* (Springer-Verlag Grundlehren 309), Springer-Verlag, Berlin, 1994.

[4] Allday, C. and Puppe, V. *Cohomological Methods in Transformation Groups*, Cambridge University Press, 1994.

[5] Benson, D. and Carlson, J. *Projective Resolutions and Poincaré Duality Complexes*, Transactions of the A.M.S. **342** (1994), pp. 447–488.

[6] Bouc, S. *Homologie de certains ensembles de 2-sous-groupes des groupes symétriques.* (French) [Homology of certain sets of 2-subgroups of symmetric groups] J. Algebra **150** (1992), no. 1, pp. 158–186.

[7] Bredon, G. *Introduction to compact transformation groups.* Pure and Applied Mathematics, Vol. **46**. Academic Press, New York-London, 1972.

[8] Bröcker, L. *Uber die Anzahl der Anordnungen Eines kommutativen Körpers*, Arch. Math. **29** (1977), pp. 149–163.

[9] Broto, C. and Henn, H.-W. *Some remarks on central elementary abelian p-subgroups and cohomology of classifying spaces*, Quart. J. Math. Oxford Ser. (2) **44** (1993), no. 174, pp. 155–163.
[10] Carlson, J. `http://www.math.uga.edu/~jfc/groups/Groups64`.
[11] Carlson, J. *Depth and Transfer Maps in the Cohomology of Groups*, Math. Zeit. **218** (1995), pp. 461–468.
[12] Carlsson, G. *On the Non–existence of Free Actions of Elementary Abelian Groups on Products of Spheres*, Am. J. Math. **102** (1980), pp. 1147–1157.
[13] Charlap, L. *Bieberbach Groups and Flat Manifolds*, (Springer-Verlag Universitext) Springer-Verlag, Berlin, 1986.
[14] Craven, T. and Smith, T. *Formally Real Fields from a Galois Theoretic Perspective*, to appear in Journal of Pure and Applied Algebra.
[15] Duflot, J. *Depth and Equivariant Cohomology*, Comm. Math. Helv. **56** (1981), pp. 627–637.
[16] Eisenbud, D., Green, M., and Harris, J. *Cayley-Bacharach Theorems and Conjectures*, Bull. Amer. Math. Soc. (N.S.) **33** (1996), no. 3, pp. 295–324.
[17] Evens, L. *Cohomology of Groups*, Oxford University Press, Oxford, 1991.
[18] Fulton, W. and Harris, J. *Representation Theory, a First Course* (GTM 129), Springer-Verlag, Berlin, 1991.
[19] Gao, W. Ph.D. Thesis, U. Western Ontario, 1996.
[20] Gao, W. and Minac, J. *Milnor's Conjecture and Galois Theory I*, Fields Institute Communications Vol. 16 (1997), pp. 95–110.
[21] Gugenheim, V.K.A.M. and May, J.P. *On the Theory and Application of Torsion Products*, Memoirs of the A.M.S. **142** (1974).
[22] Hillmann, J. *Flat 4–manifold Groups*, New Zealand J. Math. **24** (1995), pp. 29–40.
[23] Hilton, P. and Stammbach, U. *An Introduction to Homological Algebra* (GTM 4), Springer Verlag, Berlin, 1971.
[24] Jozefiak, T. and Weyman, J. *Representation–theoretic interpretation of a formula of D.E. Littlewood*, Math Proc. Cambridge Phil. Soc. **103** (1988), pp. 193–196.
[25] Karagueuzian, D. *Homology of Complexes of Degree One Graphs*, Ph.D. Thesis, Stanford University, 1994.
[26] Lam, T.. Y. *The Algebraic Theory of Quadratic Forms*, Benjamin/Addison–Wesley, Reading, Mass., 1973.
[27] Lam, T. Y. *Orderings, Valuations and Quadratic Forms*, C.B.M.S. Vol. 52, A.M.S. (1983).
[28] Lambe, L. *Cohomology of Principal G–Bundles over a Torus when $H^*(BG, R)$ is Polynomial*, Bull. Soc. Math. Belgique **38** (1986), pp. 247–264.
[29] Lang, S. *Algebra* (Third Edition), Addison-Wesley, Reading, Mass., 1993.
[30] Lazard, *Groupes Analytiques p-adiques*, Inst. Hautes Etudes Scientifiques, Publ. Math. **26**, pp. 389–603.
[31] Macdonald, I. G. *Symmetric functions and Hall polynomials.* Second edition. With contributions by A. Zelevinsky. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.
[32] Merkurjev, A. *On the Norm Residue Symbol of Degree Two*, Doklad. Akad. Nauk. SSR **261** (1981), pp. 542–547 (English Translation: Soviet Math. Doklady **24** (1981) pp. 546–551.)
[33] Milnor, J. *Algebraic K-theory and Quadratic Forms*, Inventiones Math. **9** (1970), pp. 318–344.
[34] Mahé, L., Minac, J., and Smith, T. *Additive Structure of Subgroups of $\dot{F}/\dot{F}^2$ and Galois Theory*, preprint 1997.
[35] Minac, J. *Elementary 2-Abelian Subgroups of W-groups*, preprint.
[36] Minac J. and Smith, T. *Decomposition of Witt Rings and Galois groups*, Can. J. Math. **47** (1995), pp. 1274–1289.
[37] Minac, J. and Smith, T. *W-groups of Quadratic Extensions*, in preparation.
[38] Minac, J. and Spira, M. *Witt Rings and Galois Groups*, Annals of Mathematics **144** (1996), pp. 35–60.

[39] Minac, J. and Spira, M. *Formally Real Fields, Pythagorean Fields, C-fields and W-groups*, Math. Zeit. **205** (1990), pp. 519–530.
[40] Orlov, D., Vishik, A. and Voevodsky V. *Motivic Cohomology of Pfister Quadrics*, in preparation.
[41] Ribes, L. *Introduction to Profinite Groups and Galois Cohomology*, Queen's Papers in Pure and Applied Mathematics No. 24, Queen's University, Kingston, Ontario Canada (1970).
[42] Rusin, D. *The Cohomology of the Groups of Order 32*, Math. Comp. **53** No. 187 (1989), pp. 359–385.
[43] Serre, J.-P. *Algèbre Locale—Multiplicités*, Springer Lecture Notes in Mathematics, **11**, Springer-Verlag, Berlin, 1965.
[44] Serre, J-P. *Galois Cohomology*, Springer-Verlag, Berlin, 1997.
[45] Sigg, S. *The Laplacian and the Homology of Free Two-step Nilpotent Lie Algebras*, J. Algebra **185** (1996), pp. 144–161.
[46] Voevodsky, V. *The Milnor Conjecture*, preprint 1997.
[47] Wolf, J. *Spaces of Constant Curvature* (5th edition), Publish or Perish, 1984.

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