Cuspidal modular symbols are transportable

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Abstract

Modular symbols of weight 2 for a congruence subgroup Γ satisfy the identity $\{\alpha, \gamma(\alpha)\} = \{\beta, \gamma(\beta)\}$ for all α, β in the upper half plane and $\gamma \in \Gamma$. The analogue of this identity is false for modular symbols of weight greater than 2. In this paper we define transportable modular symbols, which are symbols for which the above identity holds, and prove that every cuspidal symbol can be written as a transportable symbol. As a corollary, we obtain an algorithm for computing periods of cuspforms.

Introduction

It is well known that modular symbols of weight 2 for a congruence subgroup Γ satisfy the identity $\{\alpha, \gamma(\alpha)\} = \{\beta, \gamma(\beta)\}$ for all α, β in the upper half plane and $\gamma \in \Gamma$. The analogue of this identity is, in general, false for modular symbols of weight greater than 2. To investigate further, we define transportable modular symbols, which are symbols that can be expressed in such a way that the above identity holds. We then prove that every cuspidal symbol is transportable. As a corollary we obtain an algorithm for computing periods of cuspforms.

In Section 1 we review the definition of modular symbols. In Section 2 we define transportable modular symbols and prove our main theorem. Section 3 contains an application of our transportability result to the computation of periods of modular forms. Finally, Section 4 contains two examples in which we verify the assertion of Theorem 2.5 and apply the period computation algorithm.

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1 Modular symbols

In Section 1.1 we recall the definition of modular symbols given in [4], then in Section 1.2 we introduce a slight generalization of the definition. Let N and k be positive integers with $k \geq 2$, and let $\varepsilon : \mathbf{Z}/N\mathbf{Z} \to \mathbf{C}$ be a Dirichlet character modulo N.

1.1 Definition

Let $\boldsymbol{\mathcal{M}}$ be the abelian group generated by all symbols $\{\alpha, \beta\}$ with $\alpha, \beta \in \mathbf{P}^1(\mathbf{Q})$, modulo the relations $\{\alpha, \beta\} + \{\beta, \gamma\} + \{\gamma, \alpha\} = 0$, and modulo any torsion. Let

V_{k-2} denote the group of homogeneous polynomials in $\mathbf{Z}[X,Y]$ of degree $k-2$. Then each element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbf{Z})$ acts on the left on V_{k-2} by

$$\gamma(P(X,Y)) = P(dX - bY, -cX + aY),$$

and on $\boldsymbol{\mathcal{M}}_k = V_{k-2} \otimes \boldsymbol{\mathcal{M}}$ by

$$\gamma(P(X,Y) \otimes \{\alpha,\beta\}) = P(dX - bY, -cX + aY) \otimes \{\gamma(\alpha),\gamma(\beta)\}.$$

Fix a Dirichlet character $\varepsilon : \mathbf{Z}/N\mathbf{Z} \to \mathbf{C}$, and denote by $\mathbf{Z}[\varepsilon]$ the ring generated by the image of ε. We also view ε as a homomorphism $\Gamma_0(N) \to \mathbf{C}^*$ by setting $\varepsilon\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \varepsilon(d)$.

The space $\boldsymbol{\mathcal{M}}_k(N,\varepsilon)$ of *modular symbols* of level N and character ε is the quotient of the $\mathbf{Z}[\varepsilon]$-module $\boldsymbol{\mathcal{M}}_k \otimes \mathbf{Z}[\varepsilon]$ by the $\mathbf{Z}[\varepsilon]$-submodule generated by $\gamma(x) - \varepsilon(\gamma)x$ for all $x \in \boldsymbol{\mathcal{M}}_k$, all $\gamma \in \Gamma_0(N)$, and by any torsion. Denote by $P\{\alpha,\beta\}$ the image of $P \otimes \{\alpha,\beta\}$ in $\boldsymbol{\mathcal{M}}_k(N,\varepsilon)$. The $\mathbf{Q}[\varepsilon]$-vector space

$$\boldsymbol{\mathcal{M}}_k(N,\varepsilon;\mathbf{Q}) = \boldsymbol{\mathcal{M}}_k(N,\varepsilon) \otimes_{\mathbf{Z}} \mathbf{Q}$$

contains $\boldsymbol{\mathcal{M}}_k(N,\varepsilon)$.

Let $\boldsymbol{\mathcal{B}}$ be the free abelian group generated by all symbols $\{\alpha\}$, for $\alpha \in \mathbf{P}^1(\mathbf{Q})$. Define a left action of $\mathrm{SL}(2,\mathbf{Z})$ on $\boldsymbol{\mathcal{B}}_k = V_{k-2} \otimes \boldsymbol{\mathcal{B}}$ by

$$\gamma(P \otimes \{\alpha\}) = \gamma(P) \otimes \{\gamma\alpha\}.$$

The space $\boldsymbol{\mathcal{B}}_k(N,\varepsilon)$ of *boundary symbols* is the quotient of $\boldsymbol{\mathcal{B}}_k \otimes \mathbf{Z}[\varepsilon]$ by the $\mathbf{Z}[\varepsilon]$-submodule generated by $\gamma(x) - \varepsilon(\gamma)x$ for all $x \in \boldsymbol{\mathcal{B}}_k$, all $\gamma \in \Gamma_0(N)$, and by any torsion. The subspace $\boldsymbol{\mathcal{S}}_k(N,\varepsilon)$ of *cuspidal symbols* is the kernel of the map $\delta : \boldsymbol{\mathcal{M}}_k(N,\varepsilon) \to \boldsymbol{\mathcal{B}}_k(N,\varepsilon)$ given by $\delta(P\{\alpha,\beta\}) = P\{\beta\} - P\{\alpha\}$.

When $\varepsilon = 1$ is the trivial character, we will also write $\boldsymbol{\mathcal{M}}_k(\Gamma_0(N))$ for $\boldsymbol{\mathcal{M}}_k(N,1)$, and similarly for $\boldsymbol{\mathcal{S}}_k$ and $\boldsymbol{\mathcal{B}}_k$.

1.2 Extended modular symbols

It is useful to extend the notion of modular symbols to allows symbols of the form $P\{z,w\}$ where z and w are arbitrary elements of $\mathfrak{h}^* = \mathfrak{h} \cup \mathbf{P}^1(\mathbf{Q})$.

Definition 1.1 (Extended modular symbols). The group $\overline{\boldsymbol{\mathcal{M}}}_k$ of *extended modular symbols* is the free abelian group with basis the set of all symbols $P\{z,w\}$ with $z,w \in \mathfrak{h}^*$.

Note that $\overline{\boldsymbol{\mathcal{M}}}_k$ is of uncountable rank over $\mathbf{Z}$. It is equipped with an action of $\Gamma_0(N)$; we let $\overline{\boldsymbol{\mathcal{M}}}_k(N,\varepsilon)$ be the largest torsion-free quotient of $\overline{\boldsymbol{\mathcal{M}}}_k$ by the relations $\gamma x = \varepsilon(\gamma)x$ for $\gamma \in \Gamma_0(N)$.

2 Transportable modular symbols

In Section 2.1 we define transportable modular symbols, and we prove an elementary proposition that motivates the definition. Section 2.2, which is the heart of this paper, contains a proof that characterizes transportable modular symbols.

2.1 Definition

Definition 2.1 (Transportable). A modular symbol is *transportable* if it can be written in the form

$$\sum_{i=1}^{m} P_i\{\infty, \gamma_i(\infty)\},$$

for $\gamma_i \in \Gamma_0(N)$ and $P_i \in V_{k-2}$ with

$$\sum_{i=1}^{m} P_i\{\infty, \gamma_i(\infty)\} = \sum_{i=1}^{m} P_i\{\alpha, \gamma_i(\alpha)\}$$

for all $\alpha \in \mathfrak{h}^*$.

When $k = 2$, the identity $\{\infty, \gamma(\infty)\} = \{\alpha, \gamma(\alpha)\}$ holds for any $\alpha \in \mathfrak{h}^*$, so in weight 2 there is a plentiful supply of transportable modular symbols. This identity sometimes fails when $k > 2$, as indicated below.

Proposition 2.2. *For any $\gamma \in \Gamma_0(N)$, $P \in V_{k-2}$ and $\alpha \in \mathfrak{h}^*$,*

$$\begin{aligned} P\{\infty, \gamma(\infty)\} &= P\{\alpha, \gamma(\alpha)\} + (P - \varepsilon(\gamma)\gamma^{-1}P)\{\infty, \alpha\} \\ &= \varepsilon(\gamma)(\gamma^{-1}P)\{\alpha, \infty\} - P\{\gamma(\alpha), \infty\}. \end{aligned} \tag{1}$$

In particular,

$$P\{\infty, \gamma(\infty)\} = P\{\alpha, \gamma(\alpha)\} \Leftrightarrow P = \varepsilon(\gamma)\gamma^{-1}P. \tag{2}$$

Proof. If $x \in \boldsymbol{\mathcal{M}}_k(N, \varepsilon)$ is a modular symbol and $\gamma \in \Gamma_0(N)$ then $\gamma x = \varepsilon(\gamma)x$, where, as usual, ε is viewed as a homomorphism $\Gamma_0(N) \to \mathbf{C}^*$ via $\varepsilon(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = \varepsilon(d)$. In particular, $\varepsilon(\gamma)\gamma^{-1}x = x$, so

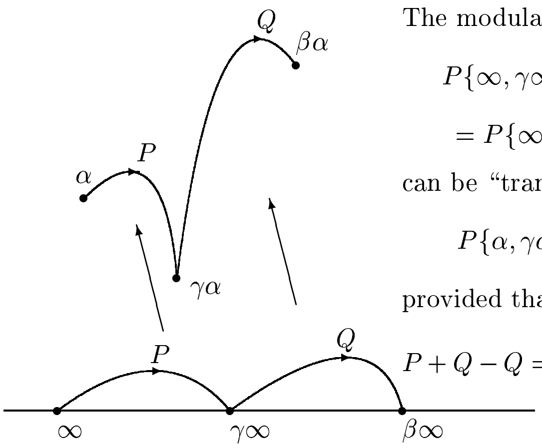
$$\begin{aligned} P\{\infty, \gamma(\infty)\} &= P\{\infty, \alpha\} + P\{\alpha, \gamma(\alpha)\} + P\{\gamma(\alpha), \gamma(\infty)\} \\ &= P\{\infty, \alpha\} + P\{\alpha, \gamma(\alpha)\} + \varepsilon(\gamma)\gamma^{-1}(P\{\gamma(\alpha), \gamma(\infty)\}) \\ &= P\{\infty, \alpha\} + P\{\alpha, \gamma(\alpha)\} + \varepsilon(\gamma)(\gamma^{-1}P)\{\alpha, \infty\} \\ &= P\{\alpha, \gamma(\alpha)\} + P\{\infty, \alpha\} - \varepsilon(\gamma)(\gamma^{-1}P)\{\infty, \alpha\} \\ &= P\{\alpha, \gamma(\alpha)\} + (P - \varepsilon(\gamma)\gamma^{-1}P)\{\infty, \alpha\}. \end{aligned}$$

The remaining statements of the proposition now follow easily. □

Example 2.3. In some cases it is easy to construct examples of symbols that are obviously transportable. Suppose $k \geq 2$ is an even integer. If P is a polynomial such that $\gamma(P) = P$ for some $\gamma \in \Gamma_0(N)$, then $P\{\infty, \gamma^{-1}(\infty)\}$ is transportable. Given $\gamma \in \Gamma_0(N)$, an example of such a P is

$$P(X, Y) = (cX^2 + (d - a)XY - bY^2)^{\frac{k-2}{2}}.$$

We found this polynomial by viewing V_{k-2} as the $(k-2)$th symmetric product of the 2-dimensional space on which $\Gamma_0(N)$ acts naturally. If γ, which has determinant 1, has eigenvalues α and α^{-1}, then the eigenvalues of the $k-2$ fold symmetric product of γ are given by α^{k-2-2j} for $0 \leq j \leq k-2$.



The modular symbol

$$P\{\infty, \gamma\infty\} + Q\{\gamma\infty, \beta\infty\}$$
$$= P\{\infty, \gamma\infty\} + Q\{\infty, \beta\infty\} - Q\{\infty, \gamma\infty\}$$

can be "transported" to

$$P\{\alpha, \gamma\alpha\} + Q\{\gamma\alpha, \beta\alpha\},$$

provided that

$$P + Q - Q = \gamma^{-1}P + \beta^{-1}Q - \gamma^{-1}Q.$$

Figure 1: "Transporting" a transportable modular symbol.

2.2 Characterization of transportable modular symbols

Lemma 2.4. *A modular symbol in* $\boldsymbol{\mathcal{M}}_k(N, \varepsilon; \mathbf{Q})$ *is transportable if and only if it can be written in the form* $\sum_{i=1}^{m} P_i\{\infty, \gamma_i(\infty)\}$ *with*

$$\sum P_i = \sum \varepsilon(\gamma_i)\gamma_i^{-1} P_i.$$

Proof. This follows from Proposition 2.2. □

Figure 1 illustrates Lemma 2.4 with a trivial-character example.

Theorem 2.5. *A modular symbol is transportable if and only if it is cuspidal.*

Proof. By Lemma 2.4 every transportable modular symbols is cuspidal, so we must prove that every cuspidal symbol is transportable.

Let $I = I_{N,\varepsilon}$ be the ideal in the group ring of $\Gamma_0(N)$ generated by all elements of the form $\varepsilon(\gamma) - \gamma$ for $\gamma \in \Gamma_0(N)$. Suppose $v \in \boldsymbol{\mathcal{S}}_k(N, \varepsilon)$. Use the relation $\{\alpha, \beta\} = \{\infty, \beta\} - \{\infty, \alpha\} \in \boldsymbol{\mathcal{M}}$ to see that any v is the image of an element $\tilde{v} \in \boldsymbol{\mathcal{M}}_k$ of the form

$$\tilde{v} = \sum_{\beta \in \mathbf{Q}} P_\beta \otimes \{\infty, \beta\} \in \boldsymbol{\mathcal{M}}_k$$

with only finitely many P_β nonzero. For later convenience we set $P_\infty = 0$, and take sums over all $\beta \in P^1(\mathbf{Q})$. The boundary map δ lifts in a natural way to $\boldsymbol{\mathcal{M}}_k = V_{k-2} \otimes \boldsymbol{\mathcal{M}}$, as illustrated.

$$\begin{tikzcd}
 & I(V_{k-2} \otimes \boldsymbol{\mathcal{M}}) \arrow[r] \arrow[d] & I(V_{k-2} \otimes \boldsymbol{\mathcal{B}}) \arrow[d] \\
 & V_{k-2} \otimes \boldsymbol{\mathcal{M}} \arrow[r, "\tilde{\delta}"] \arrow[d] & V_{k-2} \otimes \boldsymbol{\mathcal{B}} \arrow[d] \\
\boldsymbol{\mathcal{S}}_k(N, \varepsilon) \arrow[r, hook] & \boldsymbol{\mathcal{M}}_k(N, \varepsilon) \arrow[r, "\delta"] & \boldsymbol{\mathcal{B}}_k(N, \varepsilon)
\end{tikzcd}$$

Bearing in mind torsion, our assumption that $\delta(v) = 0$ implies that for some nonzero $M \in \mathbf{Z}$ we have $M\tilde{\delta}(\tilde{v}) \in I(V_{k-2} \otimes \boldsymbol{\mathcal{B}})$. So there are $Q_{\gamma,\beta} \in V_{k-2}$, for $\gamma \in \Gamma_0(N)$ and $\beta \in \mathbf{P}^1(\mathbf{Q})$, only finitely β many nonzero, such that

$$M\tilde{\delta}(\tilde{v}) = \sum_{\gamma,\beta} (\varepsilon(\gamma) - \gamma)(Q_{\gamma,\beta} \otimes \{\beta\}).$$

We now use a summation trick.

$$\begin{aligned} M\tilde{\delta}(\tilde{v}) &= M\sum_{\beta}\Big(P_\beta \otimes \{\beta\} - P_\beta \otimes \{\infty\}\Big) \\ &= \sum_{\gamma,\beta}\Big(\varepsilon(\gamma)Q_{\gamma,\beta} \otimes \{\beta\} - (\gamma Q_{\gamma,\beta}) \otimes \{\gamma\beta\}\Big) \\ &= \sum_{\gamma,\beta} \varepsilon(\gamma)Q_{\gamma,\beta} \otimes \{\beta\} - (\gamma Q_{\gamma,\gamma^{-1}\beta}) \otimes \{\beta\} \\ &= \sum_{\gamma,\beta}\Big(\varepsilon(\gamma)Q_{\gamma,\beta} - \gamma Q_{\gamma,\gamma^{-1}\beta}\Big) \otimes \{\beta\}. \end{aligned}$$

This shows that

$$M\sum_{\beta}\Big(P_\beta \otimes \{\beta\} - P_\beta \otimes \{\infty\}\Big) = \sum_{\gamma,\beta}\Big(\varepsilon(\gamma)Q_{\gamma,\beta} - \gamma Q_{\gamma,\gamma^{-1}\beta}\Big) \otimes \{\beta\}. \tag{3}$$

Equating terms we deduce that for $\beta \neq \infty$,

$$MP_\beta = \sum_{\gamma}\Big(\varepsilon(\gamma)Q_{\gamma,\beta} - \gamma Q_{\gamma,\gamma^{-1}\beta}\Big). \tag{4}$$

Using this expression for P_β and that $\varepsilon(\gamma)\gamma^{-1}$ acts trivially on $\boldsymbol{\mathcal{M}}_k(N,\varepsilon)$, we find that

$$\begin{aligned} Mv = M\sum_{\beta} P_\beta\{\infty,\beta\} &= \sum_{\gamma,\beta}\Big(\varepsilon(\gamma)Q_{\gamma,\beta} - \gamma Q_{\gamma,\gamma^{-1}\beta}\Big)\{\infty,\beta\} \\ &= \sum_{\gamma,\beta} \varepsilon(\gamma)Q_{\gamma,\beta} - \varepsilon(\gamma)\gamma^{-1}\Big((\gamma Q_{\gamma,\gamma^{-1}\beta})\{\infty,\beta\}\Big) \\ &= \sum_{\gamma,\beta} \varepsilon(\gamma)Q_{\gamma,\beta}\{\infty,\beta\} - \varepsilon(\gamma)Q_{\gamma,\gamma^{-1}\beta}\{\gamma^{-1}\infty,\gamma^{-1}\beta\} \\ &= \sum_{\gamma,\beta} \varepsilon(\gamma)Q_{\gamma,\beta}\{\infty,\beta\} - \varepsilon(\gamma)Q_{\gamma,\beta}\{\gamma^{-1}\infty,\beta\} \\ &= \sum_{\gamma,\beta} \varepsilon(\gamma)Q_{\gamma,\beta}\{\infty,\gamma^{-1}\infty\}. \end{aligned} \tag{5}$$

Equating coefficients of $\{\infty\}$ in Equation 3, we have

$$-M\sum_{\beta} P_\beta = \sum_{\gamma}\Big(\varepsilon(\gamma)Q_{\gamma,\infty} - \gamma Q_{\gamma,\gamma^{-1}\infty}\Big),$$

which, combining with Equation 4, and recalling that $P_\infty = 0$, means that

$$-\sum_{\gamma,\beta\neq\infty}\Big(\varepsilon(\gamma)Q_{\gamma,\beta} - \gamma Q_{\gamma,\gamma^{-1}\beta}\Big) = \sum_{\gamma}\Big(\varepsilon(\gamma)Q_{\gamma,\infty} - \gamma Q_{\gamma,\gamma^{-1}\infty}\Big),$$

and hence

$$\sum_{\gamma,\beta}\Big(\varepsilon(\gamma)Q_{\gamma,\beta} - \gamma Q_{\gamma,\beta}\Big) = 0.$$

Using the expression $v = -\frac{1}{M}\sum_{\beta,\gamma}\varepsilon(\gamma)Q_{\gamma,\beta}\{\infty,\gamma^{-1}\infty\}$ obtained from Equation 5, we see this is the condition for v to be transportable. $\square$

Corollary 2.6. *Any element of $\boldsymbol{\mathcal{S}}_k(N,\varepsilon)$ is a sum of modular symbols of the form $P\{\infty,\gamma\infty\}$.*

Proof. Every element of $\boldsymbol{\mathcal{S}}_k(N,\varepsilon)$ is transportable and transportable modular symbols are sums of symbols of the form $P\{\infty,\gamma\infty\}$. $\square$

When $k=2$, the corollary follows from [3, §12.1], which asserts that the group homomorphism $\Gamma_0(N)\to\boldsymbol{\mathcal{S}}_2(\Gamma_0(N)) = H_1(X_0(N),\mathbf{Z})$ sending γ to $\{\infty,\gamma(\infty)\}$ is surjective. When $k>2$, we have not found a similar group-theoretic statement.

2.3 When do the symbols $P\{\infty,\gamma(\infty)\}$ span $\boldsymbol{\mathcal{M}}_k(\Gamma_0(N);\mathbf{Q})$?

Fix an even integer $k\geq 2$ and a positive integer N. Let W denote the subspace of $\boldsymbol{\mathcal{M}}_k(\Gamma_0(N);\mathbf{Q})$ spanned by symbols of the form $P\{\infty,\gamma(\infty)\}$, for $P\in V_{k-2}$ and $\gamma\in\Gamma_0(N)$. In this section we prove that the $W=\boldsymbol{\mathcal{M}}_k(\Gamma_0(N);\mathbf{Q})$ if and only if $N=1$.

Lemma 2.7. $\dim W \leq \dim\boldsymbol{\mathcal{S}}_k(\Gamma_0(N);\mathbf{Q})+1$.

Proof. In light of Corollary 2.6, W contains the kernel $\boldsymbol{\mathcal{S}}_k(\Gamma_0(N)$ of the boundary map δ. It thus suffices to show that $\dim\delta(W)\leq 1$. For $\gamma\in\Gamma_0(N)$,

$$\delta(P\{\infty,\gamma(\infty)\}) = P\{\gamma(\infty)\} - P\{\infty\} = (\gamma^{-1}P - P)\{\infty\}\in V_{k-2}\{\infty\}.$$

It remains to show that $\dim V_{k-2}\{\infty\}\leq 1$ by using the relations that define $\boldsymbol{\mathcal{B}}_k(\Gamma_0(N);\mathbf{Q})$. Since $t=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\in\Gamma_0(N)$, we have $P\{\infty\} = t(P\{\infty\}) = (tP)\{t(\infty)\} = (tP)\{\infty\}$. Explicitly, the action of t on polynomials is as follows:

$$\begin{aligned} t(X^iY^{k-2-i}) &= X^i(X+Y)^{k-2-i}\\ &= X^iY^{k-2-i} + (k-2-i)X^{i+1}Y^{k-2-i-1} + X^{i+2}(\cdots). \end{aligned}$$

Thus the matrix $[t]$ of the action of t on the basis $X^{k-2}, X^{k-1}Y,\ldots,Y^{k-2}$ is upper triangular with diagonal entries all equal to 1, and the entries directly above the main diagonal of $[t]$ are nonzero. It follows that the dimension of $\ker([t]-1)$ is 1, so the quotient of V_{k-2} by the relations $tP=P$ has dimension 1. This implies that $V_{k-2}\{\infty\}$ has dimension at most 1. $\square$

Proposition 2.8. $W=\boldsymbol{\mathcal{M}}_k(\Gamma_0(N);\mathbf{Q})$ *if and only if* $N=1$.

Proof. First suppose that $N=1$. Then γ can be any element of $SL_2(\mathbf{Z})$, so the symbols $P\{\infty,\gamma(\infty)\}$ run through all generating modular symbols. Thus they must span $\boldsymbol{\mathcal{M}}_k(\Gamma_0(1);\mathbf{Q})$.

Conversely, suppose that $W=\boldsymbol{\mathcal{M}}_k(\Gamma_0(N);\mathbf{Q})$. If $k=2$ then by [3, §12.1] $W=\boldsymbol{\mathcal{S}}_k(\Gamma_0(N);\mathbf{Q})$, so $N=1$. Thus we may assume that $k>2$. Lemma 2.7 implies that $\dim\boldsymbol{\mathcal{M}}_k(\Gamma_0(N);\mathbf{Q}) - \dim\boldsymbol{\mathcal{S}}_k(\Gamma_0(N);\mathbf{Q})\leq 1$. Since $k>2$, there is a nonzero modular symbol $E_k\in\boldsymbol{\mathcal{M}}_k(\mathrm{SL}_2(\mathbf{Z});\mathbf{Q})$ that is linear disjoint from

$\boldsymbol{\mathcal{S}}_k(\mathrm{SL}_2(\mathbf{Z});\mathbf{Q})$. If $N>1$, then there are at least two linearly independent images of E_k under the degeneracy maps (see [4, §2.6]), and these images span a two-dimensional space that is linearly disjoint from $\boldsymbol{\mathcal{S}}_k(\Gamma_0(N);\mathbf{Q})$. Thus $\dim \boldsymbol{\mathcal{M}}_k(\Gamma_0(N);\mathbf{Q}) - \dim \boldsymbol{\mathcal{S}}_k(\Gamma_0(N);\mathbf{Q}) \geq 2$, which is a contradiction, so $N=1$, as claimed. □

3 Application to computing periods of newforms

The authors were led to introduce transportable modular symbols in order to study the error term $(P-\varepsilon(\gamma)\gamma^{-1}P)\{\infty,\alpha\}$ of Equation 1 of Proposition 2.2 in the context of computing periods of newforms. There are many ways to compute periods of newforms, but we hope that the method given below will be of value in some contexts.

Section 3.1 contains an algorithm for computing periods that relies on Theorem 2.5. We present a potentially more efficient method in Section 3.2.

3.1 An algorithm for computing periods

Let $f=\sum a_n q^n \in S_k(N,\varepsilon)$ be a cuspform and let $x\in \boldsymbol{\mathcal{M}}_k(N,\varepsilon)$ be a modular symbol. Then $\langle f,x\rangle$ (see [4, §1.5]) is a linear combination of integrals of the form

$$\langle f, X^m Y^{k-2-m}\{\alpha,\infty\}\rangle = 2\pi i \int_\alpha^{i\infty} f(z) z^m dz, \qquad (6)$$

where $\alpha\in\mathfrak{h}^*$ and the integer m satisfies $0\leq m\leq k-2$. If $\alpha\in\mathfrak{h}$ then the imaginary part of α is positive, so

$$2\pi i \int_\alpha^{i\infty} f(z) z^m dz = \sum_{n\geq 1} a_n c_n,$$

where

$$c_n = 2\pi i \int_\alpha^{i\infty} z^m e^{2\pi i n z} dz.$$

The reversal of summation and integration is justified because the sum converges absolutely. We compute the c_n using the following formula, which we obtain using repeated integration by parts.

Lemma 3.1.

$$\int_\alpha^{i\infty} e^{2\pi i n z} z^m dz = e^{2\pi i n\alpha} \sum_{s=0}^{m} \left\{ \frac{(-1)^s \alpha^{m-s}}{(2\pi i n)^{s+1}} \cdot \prod_{j=(m+1)-s}^{m} j \right\}.$$

If α has large imaginary part, the c_n will rapidly converge to 0 as $n\to\infty$. However, the reversal of summation and integration above need not be valid when α is a real number, so for computational purposes we are led to express periods in terms of integrals with end points that are in $\mathfrak{h}$. When $k=2$ this is easy because of the identity $\{\infty,\gamma(\infty)\}=\{\alpha,\gamma(\alpha)\}$, which is valid for any $\alpha\in\mathfrak{h}^*$. However, this identity can fail when $k>2$; the failure is made precise in Proposition 2.2.

Since we can take the real part of α to be greater than 0, each of the terms on the right hand side of Equation 1 can be computed using the sum given by Lemma 3.1.

We showed in Section 2 that every cuspidal modular symbols can be expressed as a sum of symbols of the form $P\{\infty, \gamma(\infty)\}$. Periods of modular symbol of this form can then be computed using the following algorithm.

Algorithm 3.2. *Given a triple $\gamma \in \Gamma_0(N)$, $P \in V_{k-2}$ and $g \in S_k(N, \varepsilon)$ this algorithm computes the period integral $\langle g, P\{\infty, \gamma(\infty)\}\rangle$.* Express γ as $\left(\begin{smallmatrix} a & b \\ cN & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and set $\alpha = \frac{-d+i}{cN}$ in Proposition 2.2. Replacing γ by $-\gamma$ if necessary, we find that the imaginary parts of α and $\gamma(\alpha) = \frac{a+i}{cN}$ are both equal to the positive number $1/cN$. Equation 6 and Lemma 3.1 can now be used to compute the period integrals provided by Proposition 2.2.

3.2 The W_N-trick

In this section, in order to obtain a potentially more efficient way of computing periods than Algorithm 3.2, we generalize the method of Cremona [2] to even integer weight $k \geq 2$. In Algorithm 3.2, with $\gamma = \left(\begin{smallmatrix} a & b \\ cN & d \end{smallmatrix}\right)$, the endpoints of the corresponding integrals have imaginary part $\frac{1}{cN}$. However, using the following trick one can increase the imaginary part of all endpoints involved to at least $\frac{1}{d\sqrt{N}}$, which is sometimes a significant improvement.

Recall that the Atkin-Lehner involution $W = W_N$ is induced by the matrix $\left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$; it acts on modular forms by sending a cuspform $f \in S_k(N, \varepsilon)$ to the form

$$f|_W(z) = N^{-k/2} z^{-k} f(-1/(Nz)) \in S_k(N, \varepsilon^{-1}).$$

If f is an eigenvector for W, then necessarily $\varepsilon = \varepsilon^{-1}$. For the rest of this section we assume that $\varepsilon^2 = 1$. Then W acts on $\boldsymbol{\mathcal{M}}_k(N, \varepsilon)$ by

$$W\Big(P(X,Y)\{\alpha, \beta\}\Big) = \frac{P(Y, -NX)}{N^{k/2-1}} \left\{\frac{1}{N\alpha}, -\frac{1}{N\beta}\right\},$$

and this action is compatible with the integration pairing.

Proposition 3.3. *Let $g \in S_k(N, \varepsilon)$ be a cuspform that is an eigenform for the Atkin-Lehner involution W having eigenvalue w. Then for any transportable modular symbol $\sum_{j=1}^m P_j\{\infty, \gamma_j(\infty)\}$ with $\gamma_j \in \Gamma_0(N)$ and $P_j \in V_{k-2}$, we have for any $\alpha \in \mathfrak{h}$ the following formula:*

$$\begin{aligned} \langle g, \sum_{j=1}^m P_j\{\infty, \gamma_j(\infty)\}\rangle \;=\; & \Bigg\langle g,\ \sum_{j=1}^m w \frac{P_j(Y, -NX)}{N^{k/2-1}} \{W(\alpha), \infty\} \\ & + \sum_{j=1}^m \left(P_j - w\frac{P_j(Y,-NX)}{N^{k/2-1}}\right) \left\{\frac{i}{\sqrt{N}}, \infty\right\} \\ & - \sum_{j=1}^m P_j \{\gamma_j(\alpha), \infty\} \Bigg\rangle. \end{aligned}$$

Here $W(\alpha) = -1/(N\alpha)$.

If $\gamma_j = \left(\begin{smallmatrix} a_j & b_j \\ c & d \end{smallmatrix}\right)$, *where* c *and* d *are fixed integers that do not depend on* j, *then*

$$\begin{aligned}\langle g, \sum_{j=1}^{m} P_j\{\infty, \gamma_j(\infty)\}\rangle &= \Bigg\langle g, \sum_{j=1}^{m} w\frac{P_j(Y,-NX)}{N^{k/2-1}}\left\{\frac{c}{d}+\frac{i}{d\sqrt{N}},\infty\right\} \\ &\quad + \sum_{j=1}^{m}\left(P_j - w\frac{P_j(Y,-NX)}{N^{k/2-1}}\right)\left\{\frac{i}{\sqrt{N}},\infty\right\} \\ &\quad - \sum_{j=1}^{m} P_j\left\{\frac{b_j}{d}+\frac{i}{d\sqrt{N}},\infty\right\}\Bigg\rangle.\end{aligned}$$

Proof. By Proposition 2.2 our condition of transportability implies that we have $\sum_{j=1}^{m} P_j\{\infty,\gamma_j(\infty)\} = \sum_{j=1}^{m} P_j\{\alpha,\gamma_j(\alpha)\}$. The steps of the following computation are described below.

$$\begin{aligned}&\langle g, P_j\{\alpha,\gamma_j(\alpha)\}\rangle \\ &= \langle g, P_j\{\alpha, \frac{i}{\sqrt{N}}\} + P_j\{\frac{i}{\sqrt{N}}, W(\alpha)\} + P_j\{W(\alpha),\gamma_j(\alpha)\}\rangle \\ &= \langle g, w\frac{W(P_j)}{N^{k/2-1}}\{W(\alpha),\frac{i}{\sqrt{N}}\} + P_j\{\frac{i}{\sqrt{N}}, W(\alpha)\} + P_j\{W(\alpha),\gamma_j(\alpha)\}\rangle \\ &= \langle g, \left(w\frac{W(P_j)}{N^{k/2-1}} - P_j\right)\{W(\alpha),\frac{i}{\sqrt{N}}\} + P_j\{W(\alpha),\infty\} - P_j\{\gamma_j(\alpha),\infty\}\rangle \\ &= \langle g, w\frac{W(P_j)}{N^{k/2-1}}\{W(\alpha),\infty\} + \left(P_j - w\frac{W(P_j)}{N^{k/2-1}}\right)\{\frac{i}{\sqrt{N}},\infty\} - P_j\{\gamma_j(\alpha),\infty\}\rangle.\end{aligned}$$

In the first step, we break the path from α to $\gamma_j(\alpha)$ into three paths. In the second step, we apply the W-involution to the first term, and use that the action of W is compatible with the pairing $\langle\,,\,\rangle$. The third step involves combining the first two terms and breaking up the third. In the final step, we replace $\{W(\alpha), i/\sqrt{N}\}$ by $\{W(\alpha),\infty\} + \{\infty, i/\sqrt{N}\}$ and regroup. Taking the sum of both sides of the expression over j from 1 to m gives the first result of the proposition.

Now, following Cremona [1, §2.10.8], in order to simultaneously maximize the imaginary parts of $\gamma_j(\alpha)$ and $W(\alpha)$, we take $\alpha = \gamma_1^{-1}\left(\frac{b_1}{d} + \frac{i}{d\sqrt{N}}\right)$. In this case we have $W(\alpha) = \frac{c}{d} + \frac{i}{d\sqrt{N}}$, and $\gamma_j(\alpha) = \frac{b_j}{d} + \frac{i}{d\sqrt{N}}$. The second formula then follows. □

Remark 3.4. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Since the imaginary parts of the terms $i/\sqrt{N}$, $\gamma_j(\alpha)$ and $W(\alpha)$ in the second part of the proposition are all relatively large, the sums appearing in Equation 6 converge relatively quickly if d is small. However, we emphasize that *it is extremely important to choose* γ_j *in Proposition 3.3 with* d *small; otherwise, the series will converge very slowly.*

4 Examples

The example of Section 4.1 illustrates some of the results of this paper for the weight-12 modular form Δ, and Section 4.2 concerns a nonrational form of level

11 and weight 4. The computations below were done using the first author's implementation of the algorithms of [5] in MAGMA.

4.1 The weight-12 form Δ

Let $f = \Delta = q \cdot \prod(1 - q^n)^{24}$ be the unique normalized eignform in $S_{12}(1)$. The space $\boldsymbol{\mathcal{M}}_{12}(1; \mathbf{Q})$ of modular symbols has dimension 3 and is spanned by $a_1 = X^{10}\{0, \infty\}$, $a_2 = X^8Y^2\{0, \infty\}$, and $a_3 = X^9Y\{0, \infty\}$, and the cuspidal subspace $\boldsymbol{\mathcal{S}}_{12}(1; \mathbf{Q})$ has dimension 2 and is spanned by a_2 and a_3.

As explained in Example 2.3, there is a transportable modular symbol associated to each nonidentity element $\gamma \in \mathrm{SL}_2(\mathbf{Z})$. The transportable symbol $(2X^2 + 2XY - Y^2)^5\{\infty, \frac{1}{2}\} = -300X^9Y\{0, \infty\}$ is attached to $\left(\begin{smallmatrix} 1 & 1 \\ 2 & 3 \end{smallmatrix}\right)$, and $-4665600X^8Y^2\{0, \infty\} - 87300X^9Y\{0, \infty\}$ is attached to $\left(\begin{smallmatrix} -8 & 5 \\ 19 & -12 \end{smallmatrix}\right)$. Together these two transportable symbols span $\boldsymbol{\mathcal{S}}_{12}(1; \mathbf{Q})$.

The period map Φ_f sends $X^iY^{10-i}\{0, \infty\}$ to $2\pi i \int_0^\infty z^i f(z)\,\mathrm{d}z$. These integrals are, up to scalars, special values of $L(f, s)$ at critical integers, so they could be computed using any of the standard methods. In any case, we obtain an approximation for the period map: $\Phi_f(a_1) \sim 0.0374412812$, $\Phi_f(a_2) \sim -0.0159703242$, $\Phi_f(a_3) \sim -0.0232962319i$. The period lattice Λ of f is spanned by $\Phi_f(\frac{1}{14}a_2)$ and $\Phi_f(\frac{1}{48}a_3)$. (The fractions appear because $\boldsymbol{\mathcal{S}}_{12}(1; \mathbf{Z})$ has basis $\frac{1}{14}a_2$ and $\frac{1}{48}a_3$.) Since $\mathbf{C}/\Lambda$ is a one-dimensional torus, it makes sense to consider the corresponding elliptic curve over $\mathbf{C}$. This is the elliptic curve $y^2 = x^3 + c_4x + c_6$, where $c_4 \sim 28091951348793344.58$ and $c_6 \sim 0$. The j-invariant of this curve is approximately 2592849.394270. Is j a transcendental number?

4.2 Level 11, weight 4

The unique normalized eigenform in $S_4(\Gamma_0(11))$ is

$$f = q + \alpha q^2 + (-4\alpha + 3)q^3 + (2\alpha - 6)q^4 + (8\alpha - 7)q^5 + \cdots.$$

The space $\boldsymbol{\mathcal{M}}_4(\Gamma_0(11); \mathbf{Q})$ has basis $a_1 = X^2\{0, \infty\}$, $a_2 = (64X^2 + 16XY + Y^2)\{-\frac{1}{8}, 0\}$, $a_3 = (49X^2+14XY+Y^2)\{-\frac{1}{7}, 0\}$, $a_4 = (25X^2+10XY+Y^2)\{-\frac{1}{5}, 0\}$, $a_5 = (100X^2+20XY+Y^2)\{-\frac{1}{10}, 0\}$, $a_6 = Y^2\{\infty, 0\}$. The subspace $\boldsymbol{\mathcal{S}}_4(\Gamma_0(11); \mathbf{Q})$ has basis $b_1 = a_2 - a_6$, $b_2 = a_3 - a_6$, $b_3 = a_4 - a_6$, $b_4 = a_5 - a_6$.

As explained in Example 2.3, there is a transportable modular symbol associated to each nonidentity element $\gamma \in \Gamma_0(11)$. For example the transportable symbol $(11X^2 - 11XY + Y^2)\{\infty, \frac{10}{11}\} = 11(a_5 - a_6)$ is associated to $\gamma = \left(\begin{smallmatrix} 10 & -1 \\ 11 & -1 \end{smallmatrix}\right)$. The symbol $-\frac{5}{4}b_1 + \frac{5}{4}b_2 - \frac{1}{4}b_3 + \frac{1}{4}b_4$ is the transportable symbol associated to $\left(\begin{smallmatrix} 5 & -1 \\ 11 & -2 \end{smallmatrix}\right)$. The symbol $-\frac{9}{8}b_1 - \frac{19}{8}b_2 + \frac{19}{8}b_3 + \frac{99}{8}b_4$ is associated to $\left(\begin{smallmatrix} 4 & 1 \\ 11 & 3 \end{smallmatrix}\right)$, and $-\frac{27}{8}b_1 + \frac{11}{8}b_2 + \frac{9}{8}b_3 + \frac{49}{8}b_4$ is associated to $\left(\begin{smallmatrix} 3 & -2 \\ 11 & -7 \end{smallmatrix}\right)$. Together these four transportable symbols span $\boldsymbol{\mathcal{S}}_4(\Gamma_0(11); \mathbf{Q})$.

In order to illustrate Section 2.3, we remark that symbols of the form $P\{\infty, \gamma(\infty)\}$ do not span all of $\boldsymbol{\mathcal{M}}_4(\Gamma_0(11); \mathbf{Q})$, but they do span a space bigger than $\boldsymbol{\mathcal{S}}_4(\Gamma_0(11); \mathbf{Q})$. Corollary 2.6 implies that their span contains $\boldsymbol{\mathcal{S}}_4(\Gamma_0(11); \mathbf{Q})$; however, the symbol $Y^2\{\infty, \frac{1}{11}\}$ does not lie in $\boldsymbol{\mathcal{S}}_4(\Gamma_0(11); \mathbf{Q})$.

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