NEW INVARIANTS FOR SURFACES

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Abstract. We define a new invariant of surfaces, stable on connected components of moduli space. The new invariant comes from the polycyclic structure of the fundamental group of the complement of a branch curve. We compute this invariant for a few examples. Moreover, we indicate the possibility of using braid monodromy factorizations of branch curves as an invariant of a surface. (Recall that braid monodromy factorizations related to curves is a first step in computing the fundamental group of the complement of the curve.)

0. Introduction

After the remarkable discovery of the Donaldson invariants and the Sieberg-Witten invariants [SW], there was hope that using them one could distinguish between different connected components of moduli space of surfaces. In our earlier work we indicated that we believe that these invariants are not fine enough for this differentiation and a more geometrical approach is needed. Indeed, in 1997, M. Manetti produced examples of surfaces which are diffeomorphic but are not a deformation of each other; and thus it is clear that a more direct geometric approach is needed. We went on to suggest the following distinguishing invariant:

Let X be a complex algebraic surface of general type embedded in $\mathbb{C}P^n$. Take a generic projection of X to $\mathbb{C}P^2$ and let S_X be its branch curve. Clearly $\pi_1(\mathbb{C}P^2 - S_X)$ is stable on a connected component of moduli space of surfaces. We believe

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that these groups can distinguish between different components. We base our belief on the structure of such groups which already have been computed.

1. History of computations of fundamental groups of complements of branch curves

Let X be a complex algebraic surface of general type embedded in $\mathbb{C}P^N$. Let $f : X \to \mathbb{C}P^2$ be a generic projection and let $S_X \in \mathbb{C}P^2$ be its branch curve. Let $\mathbb{C}^2$ be a big affine piece of $\mathbb{C}P^2$ s.t. S_X is transversal to the line at infinity. We denote:

$$\bar{G}_X = \pi_1(\mathbb{C}P^2 - S_x), \quad G = \pi_1(\mathbb{C}^2 - S_X).$$

The first computations of $\bar{G}_X$ were done by Zariski [Z]. He computed $\bar{G}_X$ for X a cubic surface in $\mathbb{C}P^2$, to be $\mathbb{Z}_2 * \mathbb{Z}_3$ (free products of 2 finite cyclic groups). The topic was renewed by Moishezon in the late 1970's, when he generalized Zariski's result to X_n, a $\deg n$ surface in $\mathbb{C}P^3$, and proved that in this case G_{X_n} is the braid group B_n and $\bar{G}_n$ is the braid group over its centers, B_n/center. In fact, Moishezon's result [Mo1] for $n = 3$ coincides with Zariski's result since $B_3/\text{center} \simeq \mathbb{Z}_2 * \mathbb{Z}_3$. ($B_3 = \langle x, y \mid xyx = yxy \rangle$, center $B_3 = \langle (xy)^3 \rangle$. Thus in B_3/center, $x = (xy)^3 x = xy \cdot yx \cdot xyx$ and $y = y(xy)^3 = yxy \cdot (xy)^2 = xyx \cdot (xy)^2$. Thus B_3/center is generated by xy and xyx while $(xy)^3 = 1$ and $(xyx)^2 = xyx \cdot yxy = (xy)^3 = 1$).

The next example was V_2 Veronese of order 2 [MoTe3]. In all the above examples, G contained a free noncommutative subgroup of 2 elements (and in other related examples as in [DOZ]). We call such group "big". This gave the basis to the feeling that this will always be the case. Since 1991 new examples were computed and these examples were not "big".

2. The new examples: almost solvable groups

To our great surprise the invariants that were computed after 1991 were not "big". It turns out that in all of the new examples, $\bar{G}_X = P\psi_1(\mathbb{C}P^2 - S_X)$ and $G_X = \pi_1(\mathbb{C}^2 - S_X)$ satisfy the following conditions:

1) There exists a quotient of the braid group namely $\tilde{B}_n$ s.t. $\tilde{B}_n$ acts on G_X and $\bar{G}_X$.
2) G_X and $\bar{G}_X$ are not "big".
3) G_X and $\bar{G}_X$ are "small": They are almost solvable (or virtually solvable in another terminology), i.e., they contain an almost solvable subgroup of finite index.
4) Moreover, G_X and $\bar{G}_X$ are extensions of a solvable group by a symmetric group.
5) $\bar{G}_X = G/\text{central element}$.

Moreover, in all the new examples we have the following series:

$$\{1\} \triangleleft A_0 \triangleleft A_1 \triangleleft A_2 \triangleleft A_3 \triangleleft G_x \tag{1}$$

where $G/A_3 \simeq S_n$ and A_i/A_{i-1} is a direct sum of $\mathbb{Z}$ and one or two finite cyclic groups, to some power $(\mathbb{Z} + \mathbb{Z}_p + \mathbb{Z}_q)^t$. For example we have the following computations:

2.1. V_p Veronese of order $p, p \geq 3$. There exists a series

$$1 = A_0 \triangleleft A_1 \triangleleft A_2 \triangleleft A_3 \triangleleft G_{V_p} \quad \text{s.t.}$$

$G/A_3 \simeq S_{p^2}$, symmetric groups on p^2 elements.

$$A_3/A_2 \simeq \mathbb{Z}$$

$$A_2/A_1 \simeq \begin{cases} (\mathbb{Z} \oplus \mathbb{Z}_3)^{p^2-1} & p \equiv (\operatorname{mod} 3) \\ \mathbb{Z}^{p^2-1} & p \not\equiv 0 (\operatorname{mod} 3) \end{cases}$$

$$A_1/A_0 \simeq \begin{cases} \mathbb{Z}_2 & p \quad \text{odd} \\ 0 & p \quad \text{even} \end{cases}$$

Moreover, we know that A_0 is the commutant subgroup of A_3, and $A_0 \subset \text{Center}(G)$.

There exists a series

$$1 \bar{A}_0 \lhd \bar{A}_1 \lhd \bar{A}_2 \lhd \bar{A}_3 \lhd \bar{G}_p \quad \text{s.t.}$$

$$\bar{G}_p/\bar{A}_3 \simeq S_{p^2}$$

$$\bar{A}_3/\bar{A}_2 \simeq \mathbb{Z}_q \quad q = \frac{3p(p-1)}{2}$$

$$\bar{A}_2/\bar{A}_1 \simeq \begin{cases} (\mathbb{Z} \oplus \mathbb{Z}_3)^{p^2-1} & p \equiv 0 (\operatorname{mod} 3) \\ \mathbb{Z}^{p^2-1} & p \not\equiv (\operatorname{mod} 3) \end{cases}$$

$$\bar{A}_1/\bar{A}_0 = \begin{cases} \mathbb{Z}_2 & p \quad \text{odd} \\ 0 & p \quad \text{even} \end{cases}$$

2.2. X_a projective embedding of $\mathbb{C}P' \times \mathbb{C}P'$ w.r.t. to $\mid al_1 + bl_2 \mid$. There exists a series

$$1 \lhd A_0 \lhd A_1 \lhd A_2 \lhd A_3 \subset G_{X_{ab}} \quad s.t.$$

for $n = 2ab = \deg X_{ab}$ we have:

$$G_{X_{ab}}/A_3 \simeq S_n$$

$$A_3/A_2 \cong \mathbb{Z}$$

$$A_2/A_1 \cong \begin{cases} (\mathbb{Z}_2)_0^{n-1} (\mathbb{Z}_{a-b})^{n-1} & a, b \quad \text{even} \\ (\mathbb{Z}_{2(a-b)}^{n-1}) & \text{otherwise} \end{cases}$$

$$A_1/A_0 \simeq \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & a,b \quad \text{even} \\ \mathbb{Z}_2 & \text{otherwise} \end{cases}$$

There exists a series

$$1 \triangleleft \bar{A}_= \triangleleft \bar{A}_1 \triangleleft \bar{A}_2 \triangleleft \bar{A}_3 \triangleleft \bar{G}_{X_{ab}}$$

s.t.

$$\bar{G}_{X_{ab}}/\bar{A}_3 \simeq S_n$$
$$\bar{A}_3/\bar{A}_2 \simeq Z_{m_1} \qquad m_1 = 3ab - a - b$$
$$\bar{A}_2/\bar{A}_1 \simeq A_2/A_1$$
$$\bar{A}_1/\bar{A}_0 \simeq A_1/A_0$$

2.3. Complete intersection of $\deg n$ **(not a hypersurface).** There exists a series

$$1 A_0 \triangleleft A_1 \triangleleft A_2 \triangleleft A_3 \triangleleft G$$

s.t.

$$G/A_3 \simeq S_n$$
$$A_3/A_2 \simeq \mathbb{Z}$$
$$A_2/A_1 \simeq \mathbb{Z}^{n-1}$$
$$A_1/A_0 \simeq \mathbb{Z}_2$$

In this case G is $\tilde{B}_n$ itself.

There are other computations in progress on Hirzebruch surfaces (following [MoRoTe] and [FRoTe]), on K3-surfaces (following [CiMT]) and on toric.

The proofs are based on our braid monodromy techniques as presented in detail in [MoTe4], [MoTe6], [MoTe7], [MoTe8], [MoTe9], [MoTe10], [Te2], [Ro], [FRoTe], and on the Van Kampen Theorem [VK].

In fact, all these groups are polycyclic with $A_1 \subseteq \text{Center}(G)$. $A_1 = A_3 = A_2$. Group theoretic classification of these groups might answer questions such as the following: Does every almost polycyclic $\tilde{B}_n$-group appear as the fundamental group of complements of branch curves? How many non-isomorphic groups of that type appear? (We transfer the question to a classification problem in group theory.)

The interest in fundamental groups is growing in general, see for example, [ToCa], [Si], [BoKa], [L1], [L2], [RoTe], [To].

3. The mysterious quotient of B_n that acts on G_x and $\bar{G}_X$

In this section we bring the definition of the braid group and we distinguish certain elements, called half-twists. Using half-twists we present Artin's Structure Theorem for the braid group and the natural homomorphism to the symmetric group. We also define transversal half-twists and the quotient of B_n called $\tilde{B}_n$, and quote the almost solvability theorem for this group.

Definition 3.1. Braid group $B_n = B_n[D, K]$

Let D be a closed disc in $\mathbb{R}^2, K \subset D, K$ finite. Let B be the group of all diffeomorphisms β of D such that $\beta(K) = K, \beta \mid_{\partial D} = Id_{\partial D}$. For $\beta_1, \beta_2 \in B$, we say that β_1 is equivalent to β_2 if β_1 and β_2 induce the same automorphism of $\pi_1(D - K, u)$. The quotient of B by this equivalence relation is called the braid group $B_n[D, K] (n = \sharp K)$. The elements of $B_n[D, K]$ are called braids.

Definition 3.2. $H(\sigma)$ half-twist defined by σ

Let D, K be as above. Let $a, b \in K, K_{a,b} = K - a - b$ and σ be a simple path in $D - \partial D$ connected a with b s.t. $\sigma \cap K = \{a, b\}$. Choose a small regular neighborhood U of σ and an orientation preserving diffeomorphism $f : \mathbb{R}^2 \to \mathbb{C}^1$ ($\mathbb{C}^1$ is taken with the usual "complex" orientation) such that $f(\sigma) = [-1,1], f(U) = \{z \in \mathbb{C}^1 \parallel z \mid < 2\}$. Let $\alpha(r), r \geq 0$, be a real smooth monotone function such that $\alpha(r) = 1$ for $r \in [0, \frac{3}{2}]$ and $\alpha(r) = 0$ for $r \geq 2$.

Define a diffeomorphism $h : \mathbb{C}^1 \to \mathbb{C}^1$ as follows. For $z \in \mathbb{C}^1, z = re^{i\varphi}$, let $h(z) = re^{i(\varphi + \alpha(r))}$. It is clear that on $\{z \in \mathbb{C}^1 \parallel z \mid \leq \frac{3}{2}\}, h(z)$ is the positive rotation by $180°$ and that $h(z) = \text{Identity}$ on $\{z \in \mathbb{C}^1 \parallel z \mid \geq 2\}$, in particular, on $\mathbb{C}^1 - f(U)$. Considering $(f \circ h \circ f^{-1}) \mid_D$ (we always take composition from left to right), we get a diffeomorphism of D which interchanges a and b and is the identity on $D - U$. Thus it defines an element of $B_n[D,K]$, called the half-twist defined by σ and denoted $H(\sigma)$.

Using half-twists we build a set of generators for B_n.

Definition 3.3. Frame of $B_n[D,K]$

Let D be a disc in $\mathbb{R}^2$. Let $K = \{a_1, \dots, a_n\}$, $K \subset D$. Let $\sigma_1, \dots, \sigma_{n-1}$ be a system of simple paths in $D - \partial D$ such that each σ_i connects a_i with a_{i+1} and for

$$i,j \in \{1, \dots, n-1\}, i < j, \quad \sigma_i \cap \sigma_j = \begin{cases} \emptyset & \text{if} \quad \mid i-j \mid \geq 2 \\ a_{i+1} & \text{if} \quad j = i+1. \end{cases}.$$

Let $H_i = H(\sigma_i)$. We call the ordered system of half-twists $(H_1, \dots, H_{n-1})$ a frame of $B_n[D,K]$ defined by $(\sigma_1, \dots, \sigma_{n-1})$, or a frame of $B_n[D,K]$ for short.

Notation 3.4.

$$[A,B] = ABA^{-1}B^{-1}$$

$$\langle A,B \rangle = ABAB^{-1}A^{-1}B^{-1}$$

$$(A)_B = B^{-1}AB.$$

Theorem 3.5. (E. Artin's braid group presentation). *Let $\{H_i\}$ be a frame of B_n. Then B_n is generated by the half-twists H_i and all the relations between $H_1, \dots, H_{n-1}$ follow from*

$$[H_i, H_j] = 1 \quad \textit{if} \quad \mid i-j \mid > 1,$$

$$\langle H_i, H_j \rangle = 1 \quad \textit{if} \quad \mid i-j \mid = 1,$$

$$1 \leq i, j \leq n - 1.$$

Proof. [A] (or [MoTe4], Chapter 5).

Theorem 3.6. *Let $\{H_i\}$ be a frame of B_n. Then*

(i) *for $n \geq 2$,* Center B_n *is isomorphic to Z with a generator*

$$\Delta_n^2 = (H_1 \cdot \ldots \cdot H_{n-1})^n.$$

(ii) *$B_2 \simeq \mathbb{Z}$ with a generator H_i.*

Proof. [MoTe4], Corollary V.2.3.

Proposition 3.7. *There is a natural homomorphism $B_n \to S_n$ (symmetric group on n elements) defined by $H_i \to (ii+1)(i\,iH)$.*

Proof. Since the transportations $\alpha_i = (ii+1)$ satisfy the relations from Artin's theorem (2.5), the above is well defined.

Definition 3.8. $\underline{P_n}$

The kernel of the above homomorphism is denoted by P_n.

Remark 3.9. The transpositions α_i satisfy a relation that H_i do not satisfy, which is $\alpha_i^2 = 1$. In fact it is true for any transposition. Under the above homomorphism the image of any half-twist is a transposition and thus any square of a half-twist belongs to $\ker(B_n \to S_n)$ which is P_n.

Definition 3.10. <u>Transversal half-twist, adjacent half-twist, disjoint half-twist</u>

Let σ_1 and σ_2 be 2 paths in D with endpoints in K which do not intersect K otherwise (like in 2.2). The half-twists $H(\sigma_1)$ and $H(\sigma_2)$ will be called *transversal* if σ_1 and σ_2 intersect transversally in one point which is not an end point of either of the σ_i's.

The half-twists $H(\sigma_1)$ and $H(\sigma_2)$ will be called *adjacent* if σ_1 and σ_2 have one endpoint in common.

The half-twists $H(\sigma_1)$ and $H(\sigma_2)$ will be called *disjoint* if σ_1 and σ_2 do not intersect.

Claim 3.11. *Disjoint half-twists commute and adjacent half-twists satisfy the triple relation* $ABA = BAB$.

Proof of 3.11. By Proposition 2.7 and the fact that every 2 half-twists are conjugated to each other.

Definition 3.12. $\underline{\tilde{B}_n}$

Let Q_n be the subgroup of B_n normally generated by $[X, Y]$ for X, Y transversal half-twists. $\tilde{B}_n$ is the quotient of B_n modulo Q_n.

A basic property of $\tilde{B}_n$ is the following:

Theorem 3.13. $\tilde{B}_n$ *is an almost solvable group; there exist a series*

$$1 \lhd \tilde{P}_n' \lhd \tilde{P}_{n,0} \lhd \tilde{P}_n \lhd \tilde{B}_n \quad s.t.$$

$$\tilde{B}_n/\tilde{P}_n \simeq S_n$$

$$\tilde{P}_n/\tilde{P}_{n,0} \simeq \mathbb{Z}$$

$$\tilde{P}_{n,0}/\tilde{P}_n' \simeq$$

$$\tilde{P}_n' \simeq \mathbb{Z}_2 (\subseteq \text{Center}\, \tilde{b}_n).$$

Proof. [Te1].

4. Applications of Kulikov's proof of Chisini conjecture

V. Kulikov proved in 1988 [Ku] that the so-called Chisini conjecture is true in many (and in fact for the most important) families of surfaces. Basically he proved that if a curve S in $\mathbb{C}P^2$ is a branch curve of a generic projection to $\mathbb{C}P^2$, then it is the branch curve of only one generic projection. Thus if we can distinguish different branch curves by fundamental groups of their complements, we will also be able to distinguish between the corresponding surfaces.

5. The new invariant

We conjecture that for many classes of embedded (in $\mathbb{C}P^N$) surfaces of general type the fundamental group of the complement of the branch curve of a generic projection is almost solvable. Moreover, we believe that like in all the new examples there exist:

$$1 = A_0 \lhd \ldots \lhd A_{m-1} \lhd A_m \lhd G$$

$$1 = \bar{A}_0 \lhd \ldots \lhd \bar{A}_{m-1} \lhd \bar{A}_m \lhd G$$

$$\text{s.t.} \quad G/A_m \simeq \bar{G}/\bar{A}_m \simeq S_n \quad n = \deg X$$

$$A_i/A_{i-1} \simeq (\mathbb{Z}_{t_i} \oplus \mathbb{Z}_{s_i} + \mathbb{Z})^{q_i}$$

$$\bar{A}_0/\bar{A}_{i-1} \simeq (\mathbb{Z}_{t'_{t_i}} \oplus \mathbb{Z}_{a'_i} \oplus \mathbb{Z}^{r'_1})^{q'_i}$$

Thus we attach to X the invariants

$$n(X) = (t_1, s_1, v_1, q_1, \ldots, t_m, s_m, v_m, q_m, n)$$

$$\bar{n}(X) = (t'_1, s'_1, y'_1, q'_1, \ldots, t'_m, s'_m, v'_m, q'_m, n).$$

Clearly, if X and Y are in the same connected component, these invariants coincide for them and if they do not coincide, they are not in the same component.

Examples 5.1.

$$n(V_p) = (2, 4, 0, 1, 3, 1, 1, p^2 - 1, 1, 1, 1, 1, p^2) \qquad p \text{ odd}, \ p = 0(3)$$

$$= (2, 1, 0, 1, 1, 1, 1, p^2 - 1, 1, 1, 1, 1, p^2) \qquad p \text{ odd}, \ p \not\equiv 0(3)$$

$$= (1, 1, 0, 0, 3, 1, 1, p^2 - 1, 1, 1, 1, 1, p^2) \qquad p \text{ even}, \ p = 0(3)$$

$$= (1, 1, 0, 0, 1, 1, 1, p^2 - 1, 1, 1, 1, 1, p^2) \qquad p \text{ even}, \ q \neq 0(3)$$

$$\bar{n}(V_p) = (2,4,0,1,3,1,1,p^2-1,\frac{3p(p-1)}{2},1,0,1,p^2) \qquad p \quad \text{odd} \quad p = 0(3)$$

$$= (2,1,0,1,1,1,1,p^2-1,\frac{3p(p-1)}{2},1,0,1,p^2) \qquad p \quad \text{odd} \quad p \not\equiv 0(3)$$

$$= (1,1,0,0,3,1,1,p^2-1,\frac{3p(p-1)}{2},1,0,1,p^2) \qquad p \quad \text{even} \quad p = 0(3)$$

$$= (1,1,0,0,1,1,1,p^2-1,\frac{3p(p-1)}{2},1,0,1,p^2) \qquad p \quad \text{even} \quad q \neq 0(3)$$

$$n(X_{ab}) = (2,2,0,1,2,a-b,0,2ab-1,1,1,1,1,2ab) \qquad a,b \quad \text{even}$$

$$\text{otherwise}$$

$$= (2,1,0,1,2(a-b),1,0,2ab-1,1,1,1,1,2ab)$$

$$\bar{n}(X_{ab}) = (2,2,0,1,2,a-b,0,2ab-1,3ab-a-b,1,0,1,2ab) \qquad a,b \quad \text{even}$$

$$\text{otherwise}$$

$$= (2,1,0,1,2(a-b),1,0,2ab-1,3ab-a-b,1,0,1,2ab)$$

$$n(CI) = (2,1,0,1,1,1,1,n-1,1,1,1,1,n)$$

On the other hand, $\bar{n}(CI)$ is not completely determined by n.

6. Example: precise computation of G

In this section we give the precise statement of G_X for X the Veronese of order 3. ¿From this structure it will be understood how the almost solvability phenomenon came up.

Let $G = \pi_1(\mathbb{C}^2 - S)$ for S the branch curve of $V_3 \to \mathbb{CP}^2$.

In order to formulate the theorem we need a few definitions.

Definitions 6.1.

$\underline{G_0(9)}$

$G_0(9)$ is a $\mathbb{Z}_2$ extension of a free group on 8 elements. We take the following model for $G_0(9)$:

Let $G_0(9)$ be generated by $\{g_i\}_{i=1\ i\neq 4}^{9}$ s.t.

$$[g_i, g_j] = \begin{cases} 1 & T_i, T_j \text{ are disjoint} \\ \tau & \text{otherwise} \end{cases}$$

where $\tau^2 = 1$, $\tau \in \operatorname{Center} G_0(9)$.

We take the following action of $\tilde{B}_9$ on $G_0(9)$

$$(g_i)_{\tilde{T}_k} = \begin{cases} g_i^{-1}\tau & k = i \\ g_i & T_i, T_k \text{ are disjoint} \\ g_i g_k^{-1} & T_i, T_k \text{ are not orderly adjacent} \\ g_k g_i & \text{otherwise} \end{cases}$$

$\underline{\tilde{B}_n \ltimes G_0(9)}$

Consider the semidirect product $\tilde{B}_9 \ltimes G_0(9)$ w.r.t. the chosen action.

$\underline{N_9}$

Let $c = [\tilde{T}_1^2, \tilde{T}_2^2]$.

Let $\xi_1 = (\tilde{T}_2\tilde{T}_1\tilde{T}_2^{-1})^2\tilde{T}_2^{-2}$.

Let $N_9 \triangleleft \tilde{B}_9 \ltimes G_0(9)$ be normally generated by $c\tau^{-1}$ and $(g_1\xi_1^{-1})^3$.

$\underline{G_9}$

Let $G_9 = \dfrac{\tilde{B}_9 \ltimes G_0(9)}{N_9}$.

$\underline{\hat{\psi}_9}$

Let $\tilde{\psi}_9$ be the homomorphism $\tilde{B}_9 \to S_9$ induced from the standard homomorphism $B_9 \to S_9$ (see 2.7). $\tilde{\psi}_9$ exists since $[X, Y] \to 1$ under the standard homomorphism. Let $\hat{\psi}_9 : G_9 \to S_9$ be defined by the first coordinate $\hat{\psi}_9(\alpha, \beta) = \tilde{\psi}_9(\alpha)$.

$\underline{\psi}$

The projection $V_3 \to \mathbb{C}^2$, of degree 9, induces a standard monodromy homomorphism $\pi_1(\mathbb{C}^2 - S, *) \to S_9$ which we denote by ψ.

Theorem 6.2. *$G \simeq \tilde{B}_9 \ltimes G_0(9)/N_9$ s.t. ψ is compatible with ψ_9.*

Proof. [MoTe9], [MoTe10], [Te2].

We also know about G that:

$\underline{H_9, H_{9,0}, H_9', H_{9,0}'}$

Let $Ab : B_9 \to \mathbb{Z}$ be the abelianization of B_9 and B_9 over its commutator subgroup.
Let $\widetilde{A_b} : \tilde{B}_9 \to \mathbb{Z}$ be a homomorphism induced from Ab (which exists since $Ab([X, Y]) = 1$).
Let $\widehat{Ab} : G_9 \to \mathbb{Z}$ be defined by the first coordinate $\widehat{Ab}(\alpha, \beta) = \widetilde{Ab}(\alpha)$.
Let $H_9 = \ker \hat{\psi}_9$.
Let $H_{9,0} = \ker \hat{\psi}_9 \cap \ker \widehat{Ab}$.
Let $H_9', H_{9,0}'$ be the commutant subgroup of H_9 and $H_{9,0}$, respectively.

Proposition 6.3. *We have $1 \triangleleft H_{9,0}' \triangleleft H_{9,0} \triangleleft H_9 \triangleleft G_9$, where $G_9/H_9 \simeq S_9$, $H_9/H_{9,0} \simeq \mathbb{Z}$, $H_{9,0}/H_{9,0}' \simeq (\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z})^8$, $H_{9,0}' = H_9' \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. [MoTe10], Proposition 2.4.

7. A more basic invariant: braid monodromy factorizations related to branch curves

The first step in computing fundamental groups of complements of a curve is computing their braid monodromy. In fact in order to really understand the group, one has to compute braid monodromy factorizations of Δ^2 (the central element of braid groups related to the curve).

There are many interesting questions which are still open. They include:

Is the data in a braid monodromy factorization of Δ^2, related to a branch curve, enough in order to distinguish between different connected components of moduli space? In recent research of V. Kulikov and the author ([KuTe]), it was proved that if 2 curves have equivalent braid monodromy factorizations, the pairs (P_i^2, B_i) are homomorphic and the surfaces (in case of branch curves) are diffeomorphic.

Will equivalent braid monodromy indicate deformation type within the algebraic surfaces category or outside of it? Is π_2 over π_1 needed for this purpose?

How does one distinguish between two type braid monodromy factorizations: those induced from algebraic curves and those induced from other curves (see [Mo2])?

How does one determine if 2 braid monodromy factorizations are equivalent?

Can one use braid monodromy factorizations to find simplectic invariants (see [CKTe])?

References on diffeomorphism types can be found in [Ma] and [C].

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