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Yang-Mills connections over manifolds with Grassmann structure

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Abstract

Let M be a manifold with Grassmann structure, i.e. with an isomorphism of the cotangent bundle $T^*M \cong E \otimes H$ with the tensor product of two vector bundles E and H. We define the notion of a half-flat connection ∇^W in a vector bundle $W \to M$ as a connection whose curvature $F \in S^2 E \otimes \wedge^2 H \otimes \mathrm{End}\ W \subset \wedge^2 T^*M \otimes \mathrm{End}\ W$. Under appropriate assumptions, for example, when the Grassmann structure is associated with a quaternionic Kähler structure on M, half-flatness implies the Yang-Mills equations. Inspired by the harmonic space approach, we develop a local construction of (holomorphic) half-flat connections ∇^W over a complex manifold with (holomorphic) Grassmann structure equipped with a suitable linear connection. Any such connection ∇^W can be obtained from a prepotential by solving a system of linear first order ODEs. The construction can be applied, for instance, to the complexification of hyper-Kähler manifolds or more generally to hyper-Kähler manifolds with admissible torsion and to their higher-spin analogues. It yields solutions of the Yang-Mills equations.

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Contents

1 Introduction

The Yang-Mills self-duality equations have played an important role in field theory and in differential geometry. They are the main source of examples of solutions of the Yang-Mills equations on four-dimensional manifolds [AHS]. The self-duality equations $*F^\nabla = F^\nabla$ mean that the curvature F^∇ of a connection ∇ over a Riemannian 4-fold M is an eigenvector of the Hodge star operator, associated with the volume 4-form, which acts on two-forms. This apparently four-dimensional construction has an analogue in Riemannian manifolds M of arbitrary dimensions. Any 4-form Ω on M defines an endomorphism B_Ω of the space of 2-forms and one can define (Ω, λ)-self-duality as the condition, $B_\Omega F^\nabla = \lambda F^\nabla$, that the curvature is an eigenvector of B_Ω with eigenvalue $\lambda = \mathrm{const} \neq 0$. Under appropriate assumptions on Ω (for example, if it is co-closed) this implies the Yang-Mills equations, just as in four dimensions. For instance, this works for a constant Ω in flat space [CDFN, W] and for a parallel 4-form on a Riemannian manifold with special holonomy (some examples are discussed in [CS, N, BKS, DT, T]). If Ω is, for example, the canonical parallel 4-form associated to a quaternionic Kähler manifold M of dimension $4m$, then the eigenspaces of B_Ω are the irreducible $\mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$–submodules of the space of two-forms. In terms of the associated locally defined Grassmann structure $T^{*\mathbb{C}}M = E \otimes H$, i.e. the identification of the complexified cotangent bundle $T^{*\mathbb{C}}M$ with a tensor product of two vector bundles E and H of rank $2m$ and 2 respectively, the eigenspace decomposition is given by

$$\wedge^2 T^{*\mathbb{C}}M = S^2E \otimes \wedge^2 H \;\oplus\; \wedge_0^2 E \otimes S^2 H \;\oplus\; \omega_E \otimes S^2 H \;,$$

with corresponding B_Ω–eigenvalues $\lambda_1{=}1$, $\lambda_2{=} -1/3$, $\lambda_3{=} -(2m{+}1)/3$ [W, CGK]. Here ω_E and ω_H are two-forms on E^* and H^* such that the complex metric on $T^{\mathbb{C}}M$ is given by $\omega_E \otimes \omega_H$ and $\wedge_0^2 E$ denotes the traceless part of $\wedge^2 E$ with respect to ω_E. The eigenspaces of B_Ω can thus be described in terms of the Grassmann structure, which is a natural generalisation of the well-known spinor decomposition of a vector in four dimensions. A 2-form on any manifold with Grassmann structure is called half-flat if it belongs to the eigenspace $S^2E \otimes \wedge^2 H$ and a connection ∇ with half-flat curvature is called half-flat. If the Grassmann structure is associated with the quaternionic Kähler structure, then a half-flat connection is the same as an (Ω, λ_1)-self-dual connection and hence satisfies the Yang-Mills equations. Inspired by the harmonic space approach [GIOS], we develop a construction of locally-defined holomorphic half-flat connections on a manifold M with holomorphic admissible half-flat Grassmann structure, namely, a holomorphic Grassmann structure $T^*M = E \otimes H$ with holomorphic connections ∇^E and ∇^H in the bundles E and H respectively, such that ∇^H is flat and the torsion of the linear connection $\nabla = \nabla^E \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^H$ has no component in $S^3H \otimes E^* \otimes \wedge^2 E$. The construction associates to a holomorphic prepotential a half-flat connection through the solution of a system of linear first order ODEs. The construction can be applied, for instance, to the complexification of hyper-Kähler manifolds or, more generally, to hyper-Kähler manifolds with admissible torsion. Our construction of gauge fields on such curved backgrounds extends that of [GIOS], where flat torsionfree backgrounds were considered.

We note that using analytic continuation any real analytic connection ∇ over a real analytic Grassmann manifold allows extension to a holomorphic connection $\nabla^{\mathbb{C}}$ over a holomorphic Grassmann manifold and ∇ can be reconstructed from $\nabla^{\mathbb{C}}$ in terms of some antiholomorphic involution.

The main idea of our construction is to pull-back a half-flat connection ∇ in a holomorphic vector bundle $\nu : W \to M$ to the harmonic space S_H. The latter is the space of all symplectic frames $h = (h_+, h_-)$ in the vector bundle H^*. The group $\mathrm{Sp}(1, \mathbb{C})$ acts freely on S_H, with the orbit space $S_H/\mathrm{Sp}(1, \mathbb{C}) = M$. Hence, the projection $\pi : S_H \to M$ is an $\mathrm{Sp}(1, \mathbb{C})$-principal bundle. Choosing a (local) trivialisation, $M \ni x \mapsto (h_1(x), h_2(x)) \in S_H$, of π we can make the identification $S_H = \mathrm{Sp}(1, \mathbb{C}) \times M$. There exists a canonical decomposition,

$$TS_H = T^v S_H \oplus \mathcal{D}_+ \oplus \mathcal{D}_- \ ,$$

of the (holomorphic) tangent bundle into the vertical sub-bundle $T^v S_H$ and two (holomorphic) distributions $\mathcal{D}_+$ and $\mathcal{D}_-$ spanned respectively by vector fields X_+^e and X_-^e canonically associated with sections e of the bundle E^*. If the Grassmann structure is admissible and half-flat, the distributions $\mathcal{D}_+$ and $\mathcal{D}_-$ are integrable. The vertical distribution $T^v S_H$ is spanned by vector fields $\partial_0, \partial_{++}, \partial_{--}$, which correspond to the standard generators of the Lie algebra $\mathfrak{sp}(1, \mathbb{C})$. A half-flat connection ∇ in the bundle $\nu : W \to M$ induces the pull-back connection $\pi^*\nabla$ in the pull-back bundle $\pi^*\nu : \pi^*W \to S_H$. Since ∇ is half-flat, the curvature F of $\pi^*\nabla$ satisfies certain equations (see Definitions 6 and 7). A connection in $\pi^*\nu$ satisfying these equations is called a half-flat connection over S_H and is gauge equivalent to the pull-back of a half-flat connection over M. Any half-flat connection over S_H is flat along the leaves of the integrable distribution $\langle \mathcal{D}_+, \partial_0 \rangle$ spanned by ∂_0 and $\mathcal{D}_+$. We can therefore choose a frame of the vector bundle $\pi^*\nu$ which is parallel along its leaves. Such a frame is called an analytic frame. With respect to such a frame a half-flat connection has no potentials in the directions of the distribution $\langle \mathcal{D}_+, \partial_0 \rangle$. Starting from a matrix-valued function (prepotential) A_{++} on S_H, which is constant along the leaves of the distribution $\mathcal{D}_+$ and satisfies the homogeneity condition $\partial_0 A_{++} = 2A_{++}$, we construct a connection which satisfies almost all the conditions of half-flatness. We call such a connection an almost half-flat connection. It is half-flat if and only if its curvature satisfies the equation $F(\partial_{--}, \mathcal{D}_-) = 0$. The construction of an almost half-flat connection reduces to the solution of first order linear ODEs. Assuming that the almost half-flat connection ∇ is defined globally along the fibres (over $\pi^{-1}U$, where $U \subset M$ is a domain in M) we can modify ∇ to a half-flat connection over S_H which is the pull-back of a half-flat connection over M. In order to do this, we re-write ∇ with respect to a "central frame", namely, a frame parallel along the fibres of π. The transformation from the analytic to the central frame reduces to the solution of the system of equations

$$\partial_{++}\Phi = -A_{++}\Phi \quad , \quad \partial_0\Phi = 0 \ .$$

With respect to the central frame the potential $C(X_+^e)$ of the connection ∇ in the direction of the vector field $X_+^e \in \mathcal{D}_+$ has the form $C(X_+^e) = u_+^\alpha C_\alpha^e$, where C_α^e are matrix-valued functions on $M = M \times \{\mathrm{Id}\} \subset M \times \mathrm{Sp}(1, \mathbb{C})$ and $u_\pm^\alpha$, $\alpha = 1, 2$, are matrix coefficients of $\mathrm{Sp}(1, \mathbb{C})$. The matrix-valued functions C_1^e, C_2^e define the desired half-flat connection on M given by

$$\nabla^M_{e\otimes h_1} = e \otimes h_1 + C_1^e \quad , \quad \nabla^M_{e\otimes h_2} = e \otimes h_2 + C_2^e \ .$$

Moreover, any half-flat connection may be obtained in this way.

The above construction allows generalisation to manifolds with spin $\frac{m}{2}$ Grassmann structure. This means that the cotangent bundle is identified as $T^*M = E \otimes F = E \otimes S^m H$, where E and H are (holomorphic) vector bundles of rank p and 2 respectively. If a connection ∇^E on E and a flat connection ∇^H on H are given, then the Grassmann structure is called half-flat. The connection ∇^H defines a flat connection ∇^F on $F = S^m H$ and the linear connection $\nabla = \nabla^E \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^F$. The associated harmonic space $\pi : S_H \to M$ is defined as above, as the space of all symplectic frames $h = (h_+, h_-)$ in H^*. Its tangent space has decomposition

$$TS_H = T^v S_H \oplus \bigoplus_{k=0}^{m} \mathcal{D}_{k+} \oplus \bigoplus_{k=1}^{m} \mathcal{D}_{k-} \ .$$

Under certain conditions on the torsion of ∇ the distribution $\mathcal{D}^k_{(+)} := \bigoplus_{i=0}^k \mathcal{D}_{(m-2i)+}$, $k \le m/2$, is integrable. Such a half-flat Grassmann structure is called k-admissible. Generalising the notion of a half-flat connection, we may define a k-partially flat connection ∇ over a manifold with half-flat spin $\frac{m}{2}$ Grassmann structure such that the pull-back connection $\pi^*\nabla$ has no curvature in the directions of $\mathcal{D}^k_{(+)}$. The harmonic space method can be applied to construct k-partially flat connections over k-admissible half-flat spin $\frac{m}{2}$ Grassmann manifolds. In the final section we consider the case of $m = 3$ and sketch the construction of 0– and 1–partially flat connections. The latter are Yang-Mills connections.

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2 Generalised self-duality for manifolds of dimension greater than four

2.1 Yang-Mills data

Let $\nu : W \to M$ be a real vector bundle over M and ∇ a connection in ν, that is a bilinear map

$$\begin{array}{rcl} \nabla : \mathfrak{X}(M) \times \Gamma(\nu) & \to & \Gamma(\nu) \\ (X, \sigma) & \mapsto & \nabla_X \sigma \end{array}$$

which is $C^\infty(M)$-linear in the vector field $X \in \mathfrak{X}(M)$ and satisfies the Leibniz rule $\nabla_X(f\sigma) = (Xf)\sigma + f\nabla_X\sigma$, for any function $f \in C^\infty(M)$ and any section, $\sigma \in \Gamma(\nu)$, of ν. The map ∇ can be extended to a complex bilinear map,

$$\begin{array}{rcl} \nabla : \mathfrak{X}^{\mathbb{C}}(M) \times \Gamma(W^{\mathbb{C}} \to M) & \to & \Gamma(W^{\mathbb{C}} \to M) \\ (X, \sigma) & \mapsto & \nabla_X \sigma\ , \end{array} \tag{2.1}$$

where $\mathfrak{X}^{\mathbb{C}}(M)$ is the space of complex vector fields $X+iY\,;\, X, Y \in \mathfrak{X}(M)$ and $W^{\mathbb{C}} \to M$ is the complexification of the vector bundle ν. Note that ∇ satisfies the reality condition

$$\nabla_{\overline{X}}\overline{\sigma} = \overline{\nabla_X \sigma}\ ,\ X \in \mathfrak{X}^{\mathbb{C}}(M)\,,\, \sigma \in \Gamma(W^{\mathbb{C}} \to M)\,, \tag{2.2}$$

where the bar denotes complex conjugation. Conversely, any $\mathbb{C}$-bilinear map (2.1) which is $\mathfrak{X}^{\mathbb{C}}(M)$-linear and satisfies the Leibniz rule and the reality condition (2.2), defines a connection ∇ in the real vector bundle ν. If the reality condition (2.2) is dropped, then (2.1) defines a connection in the complex vector bundle $W^{\mathbb{C}} \to M$.

Let $\varphi = (\varphi_1, \ldots, \varphi_r)$ denote a local frame of ν such that for any section $\sigma \in \Gamma(\nu)$, $\sigma = \sum s^i \varphi_i =: \varphi \cdot s$, where s^i are the coordinates of σ with respect to the frame φ and $s = (s^1, \ldots, s^r)^t$. Then the connection ∇ in ν has local expression

$$\nabla_X \sigma = \nabla_X(s^i\varphi_i) = \varphi \cdot \nabla_X s := (Xs^i + \sum_j A^i_j(X)s^j)\varphi_i\,,$$

where $A^i_j(X) = (\nabla_X \varphi_j, \varphi^i)$ and $\varphi^* = (\varphi^1, \ldots, \varphi^r)$ denotes the dual frame. The locally defined matrix-valued 1-form $A = (A^i_j) : M \to \mathfrak{gl}(r, \mathbb{R})$ is called the Yang-Mills potential with respect to the frame φ. If the vector bundle ν has structure group G, i.e. if it is a bundle associated with a principal G-bundle $P \to M$ and a representation $\rho : G \to GL(r, \mathbb{R})$, such that $W = P \times_G \mathbb{R}^r$, then we may always choose a frame φ for which the potential takes values in the Lie algebra $\mathfrak{g} = \mathrm{Lie}\ \rho(G) \subset \mathfrak{gl}(r, \mathbb{R})$. We will symbolically write $\nabla_X = X + A\ ,\ A = A^\varphi$. A change of frame (gauge transformation) $\varphi' = \varphi U$ induces changes $s' = U^{-1}s$ and $\varphi'(X + A'(X))s' = \varphi'\nabla_X s' = \varphi\nabla_X s = \varphi(X + A(X))s = \varphi' U^{-1}(X + A(X))Us'$, yielding the transformation rule for the potential,

$$A' = U^{-1}(XU) + U^{-1}A(X)U = U^{-1}\nabla_X U\,. \tag{2.3}$$

The curvature of the connection ∇, $F = F^{\nabla} \in \Omega^2_M(\text{End } W) = \Gamma(\wedge^2 T^*M \otimes \text{End } W)$, is given by

$$\begin{aligned} F(X,Y) &= [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \\ &= XA(Y) - YA(X) + [A(X), A(Y)] - A([X,Y])\ . \end{aligned}$$

The Jacobi identity for ∇_X is equivalent to the Bianchi identity, $d^{\nabla}F^{\nabla} = 0$. Here the covariant derivative $d^{\nabla} : \Omega^p(\text{End } W) \to \Omega^{p+1}(\text{End } W)$ is defined by

$$d^{\nabla}(\omega \otimes C) = d\omega \otimes C \ + \ (-1)^p \omega \wedge \nabla C\ ,$$

where ω is a p-form and C is a section of End W. (The connection ∇ on W induces a connection on End W denoted by the same symbol.)

On any n-dimensional oriented pseudo-Riemannian (or complex Riemannian) manifold, (M, g) using the canonical volume form $\text{vol}^g \in \wedge^n T^*M$, we define the Hodge $*$ operator which interchanges forms of complementary degree, $* : \wedge^p T^*M \to \wedge^{n-p}T^*M$, by the relation: $\langle \alpha, \beta \rangle \text{vol}^g = \alpha \wedge *\beta$, where $\alpha, \beta \in \wedge^p T^*M$ and $\langle . , . \rangle$ is the natural scalar product on $\wedge^p T^*M$ induced by the metric g. We define $* : \wedge^p T^*M \otimes \text{End } W \to \wedge^{n-p}T^*M \otimes \text{End } W$ by $*(\omega \otimes C) := (*\omega \otimes C)$.

Definition 1 *Let $\nu : W \to M$ be a real vector bundle over a pseudo-Riemannian manifold (M, g). A* **YM connection** *∇ in ν is one which satisfies the Yang-Mills equation*

$$d^{\nabla} * F^{\nabla} = 0\ .$$

On a closed manifold this is the Euler-Lagrange equation for the YM functional

$$||F^{\nabla}||^2 = \int_M \left|F^{\nabla}\right|^2\ \text{vol}^g, \tag{2.4}$$

where the norm on $\wedge^2 T^*M \otimes \text{End } W$ is induced by the pseudo-Riemannian metric on M and the natural metric on End W.

2.2 Self-duality conditions

On a Riemannian four-manifold, the $*$ operator maps 2-forms to 2-forms and has eigenvalues ± 1. The curvature tensor therefore has decomposition into the eigenspaces of the $*$ operator,

$$F^{\nabla} = F^{\nabla}_{+1} \oplus F^{\nabla}_{-1} \in \Omega^+_M(\text{End } W) \oplus \Omega^-_M(\text{End } W).$$

This splitting corresponds to the decomposition of the SO(4)-module $\wedge^2\mathbb{R}^4 = \wedge^2_+ \oplus \wedge^2_- \cong \mathfrak{so}(4) = \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ into its irreducible submodules. We call ∇ and F^∇ self-dual or anti-self-dual if $F^\nabla_{-1} := \frac{1}{2}(F^\nabla - *F^\nabla) = 0$ or $F^\nabla_{+1} := \frac{1}{2}(F^\nabla + *F^\nabla) = 0$, respectively. For (anti-) self-dual connections, the YM equation, $d^\nabla * F^\nabla = 0$, is an immediate consequence of the Bianchi identity, $d^\nabla F^\nabla = 0$. On closed manifolds (anti-) self-dual connections in fact minimise the YM functional (2.4), since the inequality

$$||F^\nabla||^2 = ||F^\nabla_{+1}||^2 + ||F^\nabla_{-1}||^2 \geq \left|||F^\nabla_{+1}||^2 - ||F^\nabla_{-1}||^2\right| = 8\pi^2 \left|c_2(W)[M]\right|$$

is saturated. Here $c_2(W)[M] = \frac{1}{8\pi^2}\int_M \mathrm{tr}\quad F^\nabla \wedge F^\nabla$ is the evaluation of the second Chern class of the bundle W on the fundamental cycle.

The apparently four-dimensional notion of self-duality, has an analogue in higher dimensions. The construction originally given in [CDFN] for flat spaces, extends to arbitrary manifolds (M, g), of dimension greater than four, as follows.

For $\Omega \in \Omega^4(M)$ we define a symmetric tracefree endomorphism field $B_\Omega : \wedge^2 T^*M \to \wedge^2 T^*M$ by

$$B_\Omega\omega := *(*\Omega \wedge \omega)\ , \tag{2.5}$$

where $\omega \in \wedge^2 T^*M$. This endomorphism is zero if and only if the 4-form Ω is zero. Moreover, we have:

Lemma 1 *Let*

$$\Omega = \sum \Omega_{ijkl} e^i \wedge e^j \wedge e^k \wedge e^l, \quad \omega = \sum \omega_{ij} e^i \wedge e^j$$

be the expressions for Ω and ω with respect to a frame e^i of T^*M. Then B_Ω is given as the contraction

$$B_\Omega\omega = 12 \sum g^{ii'} g^{jj'}\ \Omega_{ijkl}\ \omega_{i'j'}\ e^k \wedge e^l\ .$$

Proof: It is sufficient to check the above formula for decomposable forms $\Omega = e^i \wedge e^j \wedge e^k \wedge e^l$ and $\omega = e^m \wedge e^n$, where the e^i form an orthonormal basis of T^*M. □

Definition 2 *A 4-form $\Omega \in \Omega^4(M)$ on a pseudo-Riemannian manifold M is called* **appropriate** *if there exists a nonzero real constant eigenvalue λ of the endomorphism field B_Ω.*

We note that on a Riemannian manifold the eigenvalues of B_Ω are real for any 4-form Ω. A generalisation of the four dimensional notion of self-duality may now be defined:

Definition 3 *Let Ω be an appropriate 4-form on a pseudo-Riemannian manifold (M,g) and $\lambda \neq 0 \in \mathbb{R}$. A connection ∇ in a vector bundle $\nu : W \to M$ is* **(Ω,λ)-self-dual** *if its curvature F^∇ satisfies the linear algebraic system*

$$B_\Omega F^\nabla = \lambda F^\nabla \tag{2.6}$$

$$(d * \Omega) \wedge F^\nabla = 0\ . \tag{2.7}$$

Theorem 1 *Let (M,g) be a pseudo-Riemannian manifold with an appropriate 4-form Ω. Then any (Ω,λ)-self-dual connection ∇ is a YM connection.*

Proof: Using (2.6) and (2.5) we obtain

$$\begin{aligned} d^\nabla * F^\nabla &= \tfrac{1}{\lambda} d^\nabla * B_\Omega F^\nabla = \pm\tfrac{1}{\lambda} d^\nabla(*\Omega \wedge F^\nabla) \\ &= \pm\tfrac{1}{\lambda}\left((d*\Omega)\wedge F^\nabla + *\Omega \wedge d^\nabla F^\nabla\right) \\ &= 0\,, \end{aligned}$$

in virtue of (2.7) and the Bianchi identity $d^\nabla F^\nabla{=}0$. □

Examples of manifolds admitting appropriate 4-forms are easily obtained. Let V be a pseudo-Euclidean vector space and $G \subset \mathrm{SO}(V)$ be a linear group preserving a nonzero element $\Omega_0 \in \wedge^4 V$. Denote by Ω_{ijkl} the components of Ω_0 with respect to an orthonormal basis of V. Given a manifold M with a G-structure, $\pi : P \to M$, i.e. a principal G-subbundle of the bundle of frames on M, we can define a 4-form $\Omega := \sum \Omega_{ijkl} e^i \wedge e^j \wedge e^k \wedge e^l$, where $(e^1,\ldots,e^n)$ is a coframe dual to a G-frame $p = (e_1,\ldots,e_n) \in P$. Since $G \subset \mathrm{SO}(V)$, M has the structure of an oriented pseudo-Riemannian manifold and we can define the operator B_Ω. The matrix components of $B_\Omega = \sum B_{ij}^{kl} e^i \wedge e^j \otimes e_k \wedge e_l$ are constant for any G-frame and so are its eigenvalues. Hence Ω is appropriate if the endomorphism $B_{\Omega_0} \in \wedge^4 V$ has a nonzero real eigenvalue λ. This is automatic in the Riemannian case.

There exist many examples of subgroups $G \subset \mathrm{SO}(V)$ admitting nonzero G-invariant 4-forms, as shown by the following construction. Let $G \subset \mathrm{SO}(V)$ be a closed subgroup of the pseudo-orthogonal group $\mathrm{SO}(V)$ and $\mathfrak{g}{\subset}\mathfrak{so}(V){\cong} \wedge^2 V^*$ its Lie algebra. Assume that $\mathfrak{g}$ admits a G-invariant symmetric nondegenerate bilinear form $B \in S^2(\mathfrak{g}^*)^G$, where W^G denotes the space of G-invariant elements of a G-module W. We can then identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ via B and consider B as an element of $(S^2(\mathfrak{g}))^G{\subset}(S^2 \wedge^2 V^*)^G$. A G-invariant 4-form is then defined by $\Omega_0^G{:=}\mathrm{alt}\, B \in (\wedge^4 V^*)^G$, where $\mathrm{alt}: S^2 \wedge^2 V^* {\to} \wedge^4 V^*$ denotes alternation. We denote the corresponding 4-form on a manifold with G-structure by Ω^G. The following variant of a theorem by Kostant [K] provides a wealth of examples of nonzero Ω_0^G's.

Theorem 2 *Let $G \subset \mathrm{SO}(V)$ be a closed subgroup whose Lie algebra $\mathfrak{g}$ admits a non-degenerate G-invariant bilinear form $B \in (S^2\mathfrak{g})^G$. If the G-module V is not equivalent to the isotropy module of a pseudo-Riemannian symmetric space, then the four-form $\Omega_0^G := alt\, B \in (\wedge^4 V)^G$ is nonzero.*

Proof: Recall that the $\mathrm{SO}(V)$-module $S^2 \wedge^2 V$ decomposes according to $S^2 \wedge^2 V = \mathcal{R}(\mathfrak{so}(V)) + \wedge^4 V$, where $\mathcal{R}(\mathfrak{so}(V))$ denotes the space of curvature tensors of type $\mathfrak{so}(V)$, i.e. the space of two-forms fulfilling the first Bianchi identity or the kernel of the map $\mathrm{alt} : S^2 \wedge^2 V \to \wedge^4 V$. If $\Omega_0^G = \mathrm{alt}\, B = 0$, then B is a nonzero element of $\mathcal{R}(\mathfrak{so}(V)) \cap S^2(\mathfrak{g})^G = \mathcal{R}(\mathfrak{g})^G$. Since B is a G-invariant two-form on V with values in $\mathfrak{g}$ it can be used to define a Lie bracket $[\cdot, \cdot]$ on the vector space $\mathfrak{l} = \mathfrak{g} \oplus V$ thus:
(i) $\mathfrak{g}$ is a subalgebra of $\mathfrak{l}$,
(ii) V is a $\mathfrak{g}$-submodule with action defined by the inclusion $\mathfrak{g} \subset \mathfrak{so}(V)$,
(iii) $[u, v] := B(u, v) \in \mathfrak{g}$ if $u, v \in V$.
The Jacobi identity follows from the Bianchi identity and the G-invariance. Let L be the simply connected Lie group with Lie algebra $\mathfrak{l}$. Then L/G_0 is a Riemannian symmetric space with V as its isotropy module, where $G_0 \subset L$ is the connected Lie subgroup with $\mathrm{Lie}\, G_0 = \mathfrak{g}$. □

Clearly, (2.7) is automatic if the 4-form Ω is co-closed, $d * \Omega = 0$. This is the case, for example, if Ω is parallel. In the Riemannian case the Berger list of irreducible holonomy groups [B] and a theorem of Kostant [K] yield the following result.

Theorem 3 *Let M be a complete simply connected irreducible Riemannian manifold of dimension $n \geq 4$ with holonomy group $\mathrm{Hol} \subset \mathrm{SO}(n)$, $\mathrm{Hol} \neq \mathrm{SO}(n)$. Then M admits a nontrivial parallel 4-form if one of the following holds: i) M is not a symmetric space or ii) M is a symmetric space and has a non-simple holonomy or, equivalently, isotropy group.*

Proof: By Berger's theorem on Riemannian irreducible holonomy groups [B], we have:
a) M is not a symmetric space and its holonomy group is one of: $\mathrm{U}(\frac{n}{2})$, $\mathrm{SU}(\frac{n}{2})$, $\mathrm{Sp}(\frac{n}{4})\mathrm{Sp}(1)$, $\mathrm{Sp}(\frac{n}{4})$, G_2, $\mathrm{Spin}(7)$, or
b) M is a symmetric space.
All the groups in a) admit invariant 4-forms. These are given below. A theorem of Kostant [K] states that a simply-connected irreducible Riemannian symmetric space G/K has no nonzero parallel 4-form if and only if the isotropy group K is simple. □

In the following examples we explicitly describe parallel (hence appropriate) 4-forms Ω on Riemannian n-manifolds with holonomy groups $\mathrm{Hol} \neq \mathrm{SO}(n)$ from Berger's list.

1. Kähler manifolds, Hol $\subset \mathrm{U}(m) \subset \mathrm{SO}(2m)$, $n = 2m$: $\Omega = \omega \wedge \omega$, where ω is the Kähler form. One can check that this is proportional to $\Omega^{\mathrm{SU}(m)}$ and that any parallel 4-form is proportional to $\omega\wedge\omega$ if the holonomy group is $\mathrm{SU}(m)$ or $\mathrm{U}(m)$. If Hol $\subset \mathrm{Sp}(k) \subset \mathrm{SU}(2k) \subset \mathrm{SO}(4k)$, $n = 4k > 4$, i.e. if the manifold is hyper-Kähler, there exist three skewsymmetric parallel complex structures $J_\alpha, \alpha = 1, 2, 3$. Then there exist six independent parallel 4-forms $\omega_\alpha \wedge \omega_\beta$, $\alpha, \beta = 1, 2, 3$, where ω_α is the Kähler form associated to J_α. For low dimensional examples, eigenvalues and eigenspaces of B_Ω are given in [CDFN].

2. Quaternionic Kähler manifolds, Hol $\subset \mathrm{Sp}(m)\mathrm{Sp}(1) \subset \mathrm{SO}(4m)$, $n = 4m$. In this case there exist 3 locally defined almost complex structures J_α, with corresponding Kähler forms ω_α, such that the 4-form $\Omega := \sum_\alpha \omega_\alpha \wedge \omega_\alpha$ is globally defined and parallel. This will be discussed in more detail in section 2.3.

3. Hol $\subset G_2 \subset \mathrm{SO}(7)$. Let $V = \mathbb{O} = \mathbb{R}1 + \mathrm{Im}\mathbb{O} = \mathbb{R} \oplus \mathbb{R}^7 = \mathbb{R}^8$ be the algebra of octonions. Recall that $G_2 = \mathrm{Aut}(\mathbb{O})$ is the group of automorphisms of the octonions. We can decompose the product of two octonions a, b into its real and imaginary parts as follows:

$$ab = a \cdot b = \langle a, b\rangle 1 + \tfrac{1}{2}[a, b] \,,$$

where $\langle a, b\rangle$ is the scalar product and $[a, b] = ab - ba$ is the commutator. We define a 3-form φ and a 4-form ψ on $\mathrm{Im}\,\mathbb{O} = \mathbb{R}^7$ by the formulas

$$\begin{aligned} \varphi(x, y, z) &:= \langle x \cdot y, z\rangle = \tfrac{1}{2}\langle [x, y], z\rangle \\ \psi(x, y, z, w) &:= \langle [x, y, z], w\rangle \,, \end{aligned}$$

where $[x, y, z] = (xy)z - x(yz)$ is the associator. It is known that $\psi = *\varphi$. Notice that G_2 is the group of isometries of $\mathbb{O} = \mathbb{R}^8$ which fix the identity element 1 and preserve the 3-form φ (or equivalently the 4-form ψ) on $\mathrm{Im}\,\mathbb{O}$. The 4-form ψ defines a parallel 4-form on any Riemannian 7-fold with holonomy $G_2 \subset \mathrm{SO}(7)$. It is known [OV] that $\wedge^4\mathbb{R}^7 = \mathbb{R}\psi \oplus V^7(\pi_1) \oplus V^{27}(2\pi_1)$, where $V^d(\pi)$ is the d-dimensional real irreducible representation of G_2 with highest weight π and π_i denotes the i-th fundamental weight of G_2. From this it follows that the 4-form $\Omega_0^{G_2}$ coincides with ψ up to scaling. The corresponding endomorphism B_ψ of $\wedge^2\mathbb{R}^7 = \mathfrak{g}_2 \oplus \mathbb{R}^7$ has two distinct eigenvalues which correspond to the two irreducible G_2-submodules $\mathfrak{g}_2$ and $\mathbb{R}^7 \subset \wedge^2\mathbb{R}^7$ [CDFN].

4. Hol $\subset \mathrm{Spin}(7) \subset \mathrm{SO}(8)$. Using the 3- and 4-forms φ and ψ on $\mathbb{R}^7$ introduced in the G_2-case, we construct the 4-form

$$\Omega = dt \wedge \varphi + \psi \,,$$

where t is the first coordinate on $\mathbb{R}^8 = \mathbb{R}1 + \mathbb{R}^7$. In particular,

$$\Omega(1, x, y, z) = \varphi(x, y, z) \quad , \quad \Omega(x, y, z, w) = \psi(x, y, z, w) \,, \quad x, y, z, w \in \mathbb{R}^7 \,.$$

This 4-form Ω defines a parallel 4-form on any Riemannian 8-fold with holonomy $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$. It is known that $\wedge^4\mathbb{R}^8 = \mathbb{R}\Omega \oplus V^7(\pi_1) \oplus V^{27}(2\pi_1) \oplus \wedge^4\mathbb{R}^7$. From this it follows that the 4-form $\Omega_0^{\mathrm{Spin}(7)}$ coincides with Ω up to scaling. The corresponding endomorphism B_Ω of $\wedge^2\mathbb{R}^8 = \mathfrak{spin}_7 \oplus \mathbb{R}^7$ has two distinct eigenvalues which correspond to the two irreducible Spin(7)-submodules $\mathfrak{spin}_7$ and $\mathbb{R}^7 \subset \wedge^2\mathbb{R}^8$ [CDFN].

2.3 Quaternionic Kähler case

Now we discuss in more detail the case of quaternionic Kähler manifolds (example 2 above). Riemannian manifolds (M, g) with holonomy group $\mathrm{Hol} \subset \mathrm{Sp}(m)\mathrm{Sp}(1)$ are called quaternionic Kähler manifolds. A quaternionic Kähler manifold with holonomy group $\mathrm{Hol} \subset \mathrm{Sp}(m)$ is called hyper-Kähler. On any quaternionic Kähler manifold M, there exists a rank 3 vector subbundle $Q \subset \mathrm{End}\ TM$, invariant under parallel transport, which is locally spanned by three almost complex structures $(J_\alpha) = (J_1, J_2, J_3{=}J_1J_2{=}-J_2J_1)$. The latter are in general only locally defined. The (globally defined) vector bundle Q is called the **quaternionic structure** of M. A local frame (J_α) as above is called a **standard frame** for Q. Similarly, a **standard basis** of Q at $m \in M$, is a triple $I, J, K = IJ = -JI \in Q_m$ of complex structures on T_mM. A quaternionic Kähler manifold is hyper-Kähler if and only if there exists a globally defined parallel standard frame $(J_\alpha) = (J_1, J_2, J_3{=}J_1J_2{=}-J_2J_1)$.

Given a standard frame, we may locally define three nondegenerate 2-forms $\omega_\alpha := g(J_\alpha\cdot, \cdot)$. The 4-form

$$\Omega := \sum_\alpha \omega_\alpha \wedge \omega_\alpha$$

is independent of the choice of standard frame and defines a global parallel 4-form.

To describe the eigenspace decomposition of Ω it is convenient to use the Grassmann structure (i.e. generalised spinor decomposition) of a quaternionic Kähler manifold. Recall that a **Grassmann structure** on a (real) manifold M is defined as an isomorphism $T^{*\mathbb{C}}M \cong E \otimes H$ of the complexified cotangent bundle with the tensor product of two complex vector bundles E and H over M. Any quaternionic Kähler manifold admits a (locally defined) Grassmann structure $T^{*\mathbb{C}}M = E \otimes H$, where H has rank 2, such that the holonomy group $\mathrm{Hol} \subset \mathrm{Sp}(E) \otimes \mathrm{Sp}(H)$. This follows from the fact that any complex irreducible representation of the group $\mathrm{Sp}(m) \times \mathrm{Sp}(1)$ is a tensor product of irreducible representations of its factors.

The complex extension $g^\mathbb{C}$ of the Riemannian metric defines a complex bilinear metric on $T^\mathbb{C}M$, which locally factorises as $g^\mathbb{C} = \omega_E \otimes \omega_H$, where ω_E and ω_H are sections of $\wedge^2 E$ and $\wedge^2 H$, defining complex symplectic forms on the fibres of E^* and H^*, respectively. We call ω_E and ω_H the symplectic forms of the symplectic vector bundles E^* and H^*.

In terms of the Grassmann structure the eigenspaces V_λ of the endomorphism B_Ω on $\wedge^2 T^{*\mathbb{C}}M$ are given by [W, CGK]:

$$V_{\lambda_1} = S^2E\otimes\omega_H \quad , \quad V_{\lambda_2} = \wedge^2_0 E\otimes S^2H \quad , \quad V_{\lambda_3} = \omega_E\otimes S^2H \, ,$$

where $\wedge^2_0 E$ is the space of ω_E-traceless 2-forms and the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1/3$ and $\lambda_3 = -(2m+1)/3$. In particular the λ_1-self-duality condition takes the form

$$F^\nabla \in S^2E\otimes\omega_H\otimes \mathrm{End}\ W \ . \tag{2.8}$$

Note that since Ω is parallel it is appropriate and co-closed and hence the (Ω,λ)-self-duality equations (Definition 3) reduce to (2.6), which implies the Yang-Mills equation. It is known (see Theorem 1 of [CS]) that λ_1- and λ_3-self-dual connections correspond to absolute minima of the Yang-Mills functional on compact quaternionic Kähler manifolds.

2.4 Self-duality as half-flatness

The λ_1-self-duality equation (2.8) in fact depends only the existence of the factorisation $T^{*\mathbb{C}}M \cong E\otimes H$ and the symplectic structure in H^*. A connection ∇ in a vector bundle W over a manifold M with a Grassmann structure is called **half-flat** if its curvature satisfies the condition

$$F^\nabla \in S^2E\otimes\wedge^2H\otimes \mathrm{End}\ W \ . \tag{2.9}$$

In general such half-flat connections are *not* YM connections (with respect to some metric), but it is possible to impose further conditions on F^∇ in order to enforce the YM equation. In fact it is the half-flatness of the connection, rather than the YM property, which is crucial for our construction of solutions.

Proposition 1 *A connection ∇ in a vector bundle $W \to M$ over a quaternionic Kähler manifold is half-flat if and only if it is λ_1-self-dual. Hence any such connection is a Yang-Mills connection.*

Proof: The result follows from (2.8) and (2.9) since $\wedge^2H$ is the line bundle generated by ω_H. □

The Levi-Civita connection on a hyper-Kähler manifold is an example of a half-flat linear connection. Its complexification gives an example of what we call an admissible half-flat Grassmann structure in the next section.

3 Manifolds with half-flat holomorphic Grassmann structure

Our goal is to give a construction of half-flat connections in a vector bundle $\nu : W \to M$ over a manifold M. If all objects are real analytic, using analytic continuation we may obtain corresponding complex analytic objects. Specifically, assume that the manifold M and the bundle ν are real analytic. Then M is defined by an atlas of charts with analytic transition functions. Extending these functions to complex holomorphic functions, we may extend M to a complex manifold $M^{\mathbb{C}}$ with antiholomorphic involution τ such that $M = (M^{\mathbb{C}})^{\tau}$, the fixed point set of τ. Similarly, a real analytic vector bundle $\nu : W \to M$ can be extended to a holomorphic vector bundle $\nu^{\mathbb{C}} : W^{\mathbb{C}} \to M^{\mathbb{C}}$. Moreover, an analytic connection ∇ in ν can be extended to a holomorphic connection ∇ in $\nu^{\mathbb{C}}$. A holomorphic extension of a Yang-Mills connection is also a Yang-Mills connection. In the rest of this paper, we shall assume that all objects (manifolds, bundles and connections) are holomorphic. In section 4 we shall give a construction of half-flat connections in a holomorphic bundle $W \to M$ over a complex manifold M with holomorphic Grassmann structure. Now we describe the required geometrical notions.

3.1 Grassmann structure

Let M be a complex manifold with holomorphic Grassmann structure $T^*M = E \otimes H$, the isomorphism of the holomorphic cotangent bundle over M with the tensor product of holomorphic vector bundles E and H over M of rank p and q respectively. Then $TM = E^* \otimes H^*$. A holomorphic linear connection ∇ on M is called a **holomorphic Grassmann connection** if it preserves the holomorphic Grassmann structure. This means that for any vector field X on M and local sections $e \in \Gamma(E)$ and $h \in \Gamma(H)$,

$$\nabla_X(e \otimes h) = \nabla^E_X e \otimes h + e \otimes \nabla^H_X h$$

where ∇^E, ∇^H are connections in the bundles E, H respectively.

Definition 4 *A holomorphic Grassmann structure, $T^*M = E \otimes H$, on a complex manifold M with a holomorphic Grassmann connection $\nabla = \nabla^E \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^H$ is called* **half-flat** *if the connection ∇^H in the holomorphic vector bundle $H \to M$ is flat. A manifold with such a half-flat holomorphic Grassmann structure is called a* **half-flat Grassmann manifold**.

Assumption. In this section we assume that M is a manifold with a half-flat holomorphic Grassmann structure ($T^*M = E \otimes H$, $\nabla = \nabla^E \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^H$), such that H has rank 2 and that a ∇^H-parallel nondegenerate fibre-wise 2-form $\omega_H \in \Gamma(\wedge^2 H)$ in the bundle H^* is fixed. If, in addition, a ∇^E-parallel nondegenerate two-form $\omega_E \in \Gamma(\wedge^2 E)$ is fixed, then we can define a ∇-parallel complex Riemannian metric $g = \omega_E \otimes \omega_H$ on M. We do not assume, in general, that the linear connection ∇ is torsion-free.

The torsion of a linear connection belongs to $TM \otimes \wedge^2 T^*M$. Since $T^*M = E \otimes H$, we have the decomposition,

$$\begin{aligned} TM \otimes \wedge^2 T^*M &= TM \otimes \left(\wedge^2 E \otimes S^2 H \;\oplus\; S^2 E \otimes \wedge^2 H\right) \\ &= E^* H^* \left(\wedge^2 E S^2 H \;\oplus\; S^2 E \omega_H\right) \\ &\cong E^* \wedge^2 E \left(S^3 H \;\oplus\; \omega_H H\right) \;\oplus\; E^* S^2 E \omega_H H \;, \end{aligned} \tag{3.1}$$

where we omit the $\otimes$'s and we identify H^* with H using ω_H.

Definition 5 *A half-flat connection is called* **admissible** *if its torsion tensor has no component in $E^* \otimes \wedge^2 E \otimes S^3 H$. A half-flat Grassmann manifold (M, ∇) is called* **admissible** *if ∇ is* **admissible**.

We remark that if the torsion of a half-flat connection is E-symmetric, i.e. if it belongs to $TM \otimes S^2 E \otimes \wedge^2 H = TM \otimes S^2 E \otimes \omega_H$, then the connection is admissible. It follows from the above decomposition that the torsion tensor of any admissible connection can be written as

$$T(e \otimes h, e' \otimes h') = T_1(e, e') \otimes \omega_H(h, h') h_1 \;+\; T_2(e, e') \otimes \omega_H(h, h_2) h' \;+\; T_2(e, e') \otimes \omega_H(h', h_2) h$$

where e, e' are sections of E^*, h_1, h_2 are fixed sections of $H \cong H^*$, $T_1 \in \Gamma(E^* \otimes S^2 E)$ and $T_2 \in \Gamma(E^* \otimes \wedge^2 E)$. This shows that admissibility of the connection means that the torsion can be represented as the sum of two tensors linear in ω_H.

3.2 Harmonic space

Let M be a half-flat Grassmann manifold. We denote by S_H the $\mathrm{Sp}(1, \mathbb{C})$-principal holomorphic bundle over M consisting of symplectic bases of $H_m^* \cong H_m \cong \mathbb{C}^2$, $m \in M$,

$$S_H = \{ s = (h_+, h_-) \mid \omega_H(h_+, h_-) = 1 \} .$$

The bundle $S_H \to M$ is called **harmonic space** [GIOS]. A parallel (local) section

$$m \mapsto s_m = (h_1(m), h_2(m)) \in S_H$$

defines a trivialisation

$$M \times \mathrm{Sp}(1, \mathbb{C}) \cong S_H \;,$$

given by

$$(m,\mathfrak{U}) \mapsto s_m\mathfrak{U} = \left(h_+ = \sum_{\alpha=1}^{2} h_\alpha u_+^\alpha \ , \ h_- = \sum_{\alpha=1}^{2} h_\alpha u_-^\alpha\right) , \quad \mathfrak{U} = \begin{pmatrix} u_+^1 & u_-^1 \\ u_+^2 & u_-^2 \end{pmatrix} \ ; \ \det\ \mathfrak{U} = 1 \, .$$

We denote by $\partial_{++}, \partial_{--}, \partial_0$ the fundamental vector fields on S_H generated by the standard generators of $\mathrm{Sp}(1,\mathbb{C})$,

$$\partial_{++} \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad \partial_{--} \sim \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \quad \partial_0 \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

They satisfy the relations

$$[\partial_{++} \, , \, \partial_{--}] = \partial_0 \ , \quad [\partial_0 \, , \, \partial_{++}] = 2\partial_{++} \ , \quad [\partial_0 \, , \, \partial_{--}] = -2\partial_{--} \ .$$

Consider $\mathrm{Mat}(2,\mathbb{C})$, the vector space of two by two matrices. The matrix coefficients $u_\pm^\alpha$ are coordinates on this vector space. One can easily check that the vector fields $u_+^\alpha \frac{\partial}{\partial u_-^\alpha}$, $u_-^\alpha \frac{\partial}{\partial u_+^\alpha}$ and $u_+^\alpha \frac{\partial}{\partial u_+^\alpha} - u_-^\alpha \frac{\partial}{\partial u_-^\alpha}$ annihilate the function $\det\ \mathfrak{U} = \epsilon_{\beta\gamma} u_+^\beta u_-^\gamma$, where $\epsilon_{\beta\gamma}$ are the matrix coefficients of the standard symplectic form of $\mathbb{C}^2$. Therefore these vector fields are tangent to the submanifold $\mathrm{Sp}(1,\mathbb{C}) = \{\det\ \mathfrak{U} = 1\} \subset \mathrm{Mat}(2,\mathbb{C})$. One can easily prove:

Lemma 2 *In terms of the identification, $S_H \cong M \times \mathrm{Sp}(1,\mathbb{C})$, the fundamental vector fields on S_H generated by the standard generators of* $\mathrm{Sp}(1,\mathbb{C})$ *may be written:*

$$\partial_{++} = u_+^\alpha \frac{\partial}{\partial u_-^\alpha} \ , \quad \partial_{--} = u_-^\alpha \frac{\partial}{\partial u_+^\alpha} \ , \quad \partial_0 = u_+^\alpha \frac{\partial}{\partial u_+^\alpha} - u_-^\alpha \frac{\partial}{\partial u_-^\alpha} \ .$$

We say that a function f on S_H has **charge** c if $\partial_0 f = cf$. The charge measures the difference in the degrees of homogeneity in u_+ and u_-.

Note that any frame $(h_+, h_-) \in S_H$ defines an isomorphism $\mathbb{C}^2 \xrightarrow{\sim} H_m^*$ given by $(z^1, z^2) \mapsto z^1 h_+ + z^2 h_-$. This induces an isomorphism

$$\mathfrak{sp}(1,\mathbb{C}) = \mathfrak{sp}(\mathbb{C}^2) \cong S^2\mathbb{C}^2 \xrightarrow{\sim} S^2 H_m^* = \mathrm{span}_{\mathbb{C}}\{h_+^2 \, , \, h_-^2 \, , \, h_+ \vee h_-\} \ ,$$

where we have identified $\mathfrak{sp}(\mathbb{C}^2)$ with $S^2\mathbb{C}^2$ using the symplectic form of $\mathbb{C}^2$. The generators of $\mathfrak{sp}(1,\mathbb{C})$ corresponding to h_+^2 , $-h_-^2$, $-h_+ \vee h_-$ under this identification are precisely ∂_{++} , ∂_{--} , ∂_0 respectively.

3.3 Canonical distributions on harmonic space

Let $S_H = \{(h_+ \, , \, h_-) | h_\pm = u_\pm^\alpha h_\alpha \ , \ (u_\pm^\alpha) \in \mathrm{Sp}(1,\mathbb{C})\}$ be the harmonic space associated to a half-flat Grassmann manifold M. Here we have fixed a parallel symplectic frame (h_1, h_2) of H^* which defines the trivialisation $S_H = M \times \mathrm{Sp}(1,\mathbb{C})$ of the holomorphic bundle S_H.

In particular, the matrix coefficients $u^\alpha_\pm$ of $\mathrm{Sp}(1,\mathbb{C})$ will be considered as holomorphic functions on S_H. Together with local coordinates (x^i) of M, we obtain a system $(x^i, u^\alpha_\pm)$ of local (nonhomogeneous–homogeneous) coordinates on S_H.

For any section $e \in \Gamma(E^*)$ we define vector fields $X^e_\pm \in \mathfrak{X}(S_H)$ by the formula

$$X^e_\pm|_{(h_+,h_-)} = \widetilde{e \otimes h_\pm} \ ,$$

where $\widetilde{Y}$ stands for the horizontal lift of a tangent vector Y on M with respect to the connection ∇^H. Since the frame h_α is parallel, this horizontal lift coincides with the horizontal lift with respect to the splitting $S_H = M \times \mathrm{Sp}(1,\mathbb{C})$. This shows that the vector fields $X^e_\pm$ are tangent to $M \times \{(h_+,h_-)\}$ and hence annihilate $u^\alpha_\pm$. If $h_\pm = u^\alpha_\pm h_\alpha$ then $X^e_\pm = u^\alpha_\pm \widetilde{X^e_\alpha}$, where $X^e_\alpha := e \otimes h_\alpha$.

There exists a canonical decomposition

$$TS_H = T^v S_H \oplus \mathcal{D}_+ \oplus \mathcal{D}_-$$

of the (holomorphic) tangent bundle into the vertical sub-bundle $T^v S_H$ and two (holomorphic) distributions $\mathcal{D}_+$ and $\mathcal{D}_-$ spanned respectively by vector fields X^e_+ and X^e_- associated with sections e of the bundle E^*. The vertical distribution $T^v S_H$ is spanned by the vector fields $\partial_0, \partial_{++}, \partial_{--}$, which correspond to the standard generators of the Lie algebra $\mathfrak{sp}(1,\mathbb{C})$.

Lemma 3 *The vector fields $X^e_\pm \in \mathfrak{X}(S_H)$ satisfy the following commutation relations:*

$$\begin{aligned}
&\big[\partial_0, X^e_\pm\big] = \pm X^e_\pm \ , \quad \big[\partial_{\pm\pm}, X^e_\pm\big] = 0 \ , \quad \big[\partial_{\pm\pm}, X^e_\mp\big] = X^e_\pm \\
&\big[X^e_+, X^{e'}_-\big] = X_-^{\nabla_{\pi_* X^e_+} e'} - X_+^{\nabla_{\pi_* X^{e'}_-} e} - \widetilde{T}(\pi_* X^e_+, \pi_* X^{e'}_-) \\
&\big[X^e_\pm, X^{e'}_\pm\big] = X_\pm^{\nabla_{\pi_* X^e_\pm} e'} - X_\pm^{\nabla_{\pi_* X^{e'}_\pm} e} - \widetilde{T}(\pi_* X^e_\pm, \pi_* X^{e'}_\pm) \ ,
\end{aligned} \tag{3.2}$$

where T is the torsion of the Grassmann connection, $\widetilde{T}(X,Y) := \widetilde{T(X,Y)}$ denotes the horizontal lift of the vector $T(X,Y)$ and we have used the abbreviation $\nabla_X e := \nabla^E_X e$.

Proof: The first three equations follow from $X^e_\pm = u^\alpha_\pm \widetilde{e \otimes h_\alpha}$ and the expression for the fundamental vector fields given in Lemma 2. To prove the last equation we first compute the Lie bracket of two vector fields $X = e \otimes h$ and $X' = e' \otimes h$ on M, where h is parallel:

$$[X, X'] \ = \ \nabla_X X' - \nabla_{X'} X - T(X,X') \ = \ (\nabla_X e' - \nabla_{X'} e) \otimes h - T(X,X') \, . \tag{3.3}$$

Using this we calculate the commutator

$$\begin{aligned}
[X^e_\pm\ ,\ X^{e'}_\pm] &= u^\alpha_\pm u^\beta_\pm \left(\widetilde{\nabla_{X^e_\alpha} X^{e'}_\beta} - \widetilde{\nabla_{X^{e'}_\beta} X^e_\alpha} - \widetilde{T}(X^e_\alpha, X^{e'}_\beta)\right) \\
&= (\nabla_{\pi_* X^e_\pm} e' \otimes h_\pm)^\sim - (\nabla_{\pi_* X^{e'}_\pm} e \otimes h_\pm)^\sim - u^\alpha_\pm u^\beta_\pm \widetilde{T}(X^e_\alpha, X^{e'}_\beta) \\
&= X_\pm^{\nabla_{\pi_* X^e_\pm} e' - \nabla_{\pi_* X^{e'}_\pm} e} - \widetilde{T}(\pi_* X^e_\pm, \pi_* X^{e'}_\pm)\ .
\end{aligned}$$

The expression for $[X^e_+, X^{e'}_-]$ follows similarly. □

We shall use the abbreviation $T(X^e_\pm, X^{e'}_\pm) := \widetilde{T}(\pi_* X^e_\pm, \pi_* X^{e'}_\pm)$.

Proposition 2 *The following conditions are equivalent:*

i) For any parallel section $h \in \Gamma(H^*)$ the distribution $E^* \otimes h$ on M is integrable.

ii) The distribution $\mathcal{D}_+$ (associated to any parallel frame (h_1, h_2)) on S_H is integrable.

iii) The distribution $\mathcal{D}_-$ on S_H is integrable.

iv) The holomorphic Grassmann structure is admissible, i.e. it has admissible connection.

Proof: The formula (3.3), where h is parallel shows that the distribution $E^* \otimes h$ is integrable if and only if

$$T(E^* \otimes h, E^* \otimes h) \subset E^* \otimes h\ . \tag{3.4}$$

Using the decomposition (3.1), one can check that this condition is satisfied for all parallel sections h if and only if the connection is admissible. This proves the equivalence of i) and iv). Since $\pi_* \left(X^e_+|_{(h_+, h_-)}\right) = e \otimes h_+$, the last equation in (3.2) shows that the distribution $\mathcal{D}_+$ is integrable if and only if (3.4) holds for all h. Thus i) is equivalent to ii). The equivalence of i) and iii) is proved similarly. □

4 Construction of half-flat connections over half-flat Grassmann manifolds

4.1 Half-flat connections over half-flat Grassmann manifolds

In this section we describe the *harmonic space method* [GIOS] for constructing half-flat connections ∇ (Def. 6 below) in a holomorphic vector bundle $\nu : W \to M$ over a complex manifold M with admissible half-flat holomorphic Grassmann structure. The basic ingredient of the construction is the lift of geometric data from M to S_H via $\pi : S_H \to M$. Let ∇ be a holomorphic connection in a holomorphic vector bundle $\nu : W \to M$. Its curvature

$$F(e \otimes h_\alpha\, ,\, e' \otimes h_\beta) = \omega_H(h_\alpha, h_\beta) F^{(ee')} + F^{[ee']}_{\alpha\beta}\, , \tag{4.1}$$

where (h_1, h_2) is the fixed parallel local frame of H^* and e, e' are local sections of E^*. The curvature component $F^{(ee')}$ is symmetric in e, e' and $F^{[ee']}_{\alpha\beta}$ is skew in e, e' and symmetric in α, β. Lifting (4.1) to S_H we obtain the curvature of the pull-back connection $\pi^*\nabla$ in $\pi^*\nu : \pi^*W \to S_H$ with components, $F(v\,,\,\cdot\,) = 0\ ,\ \forall v \in T^v S_H$, together with

$$\begin{aligned} F(X^e_\pm, X^{e'}_\pm) &= F^{[ee']}_{\pm\pm} := u^\alpha_\pm u^\beta_\pm F^{[ee']}_{\alpha\beta} \\ F(X^e_+, X^{e'}_-) &= F^{(ee')} + F^{[ee']}_{+-} := F^{(ee')} + u^\alpha_+ u^\beta_- F^{[ee']}_{\alpha\beta}\ . \end{aligned}$$

Definition 6 *A holomorphic connection ∇ in a holomorphic vector bundle $\nu : W \to M$ over a complex manifold M with holomorphic Grassmann structure, $T^*M = E \otimes H$, is called* **half-flat** *if its curvature F satisfies the equation*

$$F(e \otimes h_\alpha\,,\, e' \otimes h_\beta) = \omega_H(h_\alpha, h_\beta) F^{(ee')}\,, \tag{4.2}$$

where (h_1, h_2) is a parallel local frame of H^* and $F^{(ee')}$ is symmetric in the local sections e, e' of E^*.

Note that (4.2) is equivalent to (2.9). From this definition it follows that for any $h \in H^*$ we have $F(e \otimes h\,,\, e' \otimes h) = 0$.

Definition 7 *A connection in a holomorphic vector bundle $W \to S_H$ over harmonic space S_H is called* **half-flat** *if its curvature F satisfies the equations*

$$\begin{aligned} F(X^e_+, X^{e'}_+) &= 0 \\ F(X^e_+, X^{e'}_-) &= F^{(ee')} \\ F(X^e_-, X^{e'}_-) &= 0 \\ F(v\,,\,\cdot\,) &= 0 \quad , \quad \forall v \in T^v S_H\ , \end{aligned} \tag{4.3}$$

where $F^{(ee')}$ is symmetric in the local sections e, e' of E^*.

Definition 8 *Let $\nu : W \to M$ be a holomorphic vector bundle and ∇ a connection in $\pi^*\nu : \pi^*W \to S_H$, where $\pi : S_H \to M$. A local frame of $\pi^*\nu$ defined on $\pi^{-1}(U)$, where U is an open subset of M, is called a* **central frame** *with respect to ∇ if it is parallel along the fibres of the bundle $\pi : S_H \to M$.*

Remark: If $\chi = (\chi_1, \ldots \chi_r)$ is a local frame of ν then $\pi^*\chi$ will be a central frame with respect to the pull-back $\pi^*\nabla$ of any connection ∇ in ν. The connection 1-form A of $\pi^*\nabla$ with respect to the frame $\pi^*\chi$ satisfies $A(v) = 0\ ,\ A(X^e_\pm) = u^\alpha_\pm A^e_\alpha\,$, where v is any vertical vector and $A^e_\alpha = A(\widetilde{X^e_\alpha})$ is a matrix-valued function on M. Conversely, any connection satisfying these conditions is the pull-back of the connection over M with potential $A(X^e_\alpha) = A^e_\alpha$.

Proposition 3 *Let $\pi : S \to M$ be any fibre bundle with simply connected fibres over a simply connected manifold M. There is a natural one-to-one correspondence between gauge equivalence classes of connections ∇^M in the trivial bundle $\mathbb{C}^r \times M$ and gauge equivalence classes of connections ∇^S in $\mathbb{C}^r \times S$ satisfying the curvature constraint $F(v, \cdot) = 0$ for all vertical vectors v.*

Proof: It is clear that the pull-back $\nabla^S = \pi^* \nabla^M$ to S of a connection ∇^M defined over M satisfies the curvature constraint. To prove the converse, we will apply the following elementary lemma to the connection 1-form A of a connection ∇ over $N = S$.

Lemma 4 *Let $\pi : N \to M$ be a submersion with connected fibres and α a p-form on N. Then α is the pull-back $\pi^*\beta$ of a p-form β on M if and only if the inner products $\iota_v \alpha = \iota_v d\alpha = 0$ for all vertical tangent vectors v.*

Since the connection ∇^S is flat along the (simply connected) fibres of π there exists a central frame $\psi = (\psi_1, \ldots, \psi_r)$ for ∇^S. Let A be the connection 1-form of ∇^S with respect to this central frame. We then have $A(v) = 0$ for any vertical vector v and the curvature condition $F(v, \cdot) = 0$ implies $dA(v, \cdot) = 0$. Now the above lemma shows that A is the pull-back of a 1-form B on M, which defines a connection ∇^M in the trivial bundle $\mathbb{C}^r \times M$. Since any two central frames differ by a gauge transformation which is a matrix-valued function on M the connection ∇^M is well defined up to a gauge transformation. The pullback $\pi^*\nabla^M$ is gauge equivalent to ∇^S since it has the same expression with respect to the standard frame of $\mathbb{C}^r \times S$ (which is the pull-back of the standard frame of $\mathbb{C}^r \times M$) as ∇^S with respect to the central frame ψ. It is clear that the pull-backs of gauge equivalent connections over M are gauge equivalent connections over S. Applying a gauge transformation to a connection ∇^S which has connection 1-form A with respect to a central frame ψ we get a new connection $(\nabla^S)'$, which has the same connection form A with respect to the transformed frame ψ'. The frame ψ' is therefore central with respect to $(\nabla^S)'$ and the two connections ∇^S and $(\nabla^S)'$ define the same gauge equivalence class of connections over M. $\square$

Proposition 4 *Let $\nu : W = \mathbb{C}^r \times M \to M$ be a trivial vector bundle over a complex manifold M with admissible half-flat holomorphic Grassmann structure and $\pi^*\nu : \pi^*W = \mathbb{C}^r \times S_H \to S_H$ its pull-back to S_H. Then any half-flat connection over S_H is gauge equivalent to the pull-back of a half-flat connection over M.*

Proof: It is clear that the pull-back of a half-flat connection is half-flat. To prove the converse, we apply Proposition 3, by which a half-flat connection ∇^S over S_H is gauge equivalent to a pull-back connection $\pi^*\nabla^M$, which is necessarily half-flat. This implies that ∇^M is half-flat. In fact, if the connection ∇^M were not half-flat then it would have a nontrivial curvature component $F^{[ee']}_{\alpha\beta}$ which would imply that its pull-back $\pi^*\nabla^M$ has for instance a nonzero curvature component $F^{[ee']}_{++}$. But this is impossible since $\pi^*\nabla^M$ is half-flat. $\square$

Corollary 1 *The connection 1-form A of a half-flat connection over S_H with respect to a central frame ψ has the form*

$$A(v) = 0 \quad , \quad A(X^e_\pm) = u^\alpha_\pm A^e_\alpha \ ,$$

where v is any vertical vector and $A^e_\alpha = A(\widetilde{X^e_\alpha})$ is a matrix-valued function on M.

Remark: This shows that the half-flat connection is completely determined by the potential in the $\mathcal{D}_+$–direction, $A(X^e_+) = u^\alpha_+ A^e_\alpha$, with respect to a central frame.

Proof: This follows from Proposition 4 and the remark following Definition 8. $\square$

4.2 The construction

In this section we construct half-flat connections in a bundle $\nu : W \to M$ over a manifold M with a half-flat admissible Grassmann structure. First we define the weaker notion of an almost half-flat connection over S_H and show how to construct all such connections from appropriate prepotentials. Then we show that any almost half-flat connection over S_H may be used to construct a half-flat connection on M. Since our construction is local in M, we shall assume that the bundles π, ν and $\pi^*\nu$ are trivial, i.e. $\pi : M \times \mathrm{Sp}(1,\mathbb{C}) \to M$, $\nu : M \times \mathbb{C}^r \to M$ and $\pi^*\nu : S_H \times \mathbb{C}^r \to S_H$.

4.2.1 Construction of almost half-flat connections

The restriction of a half-flat connection to a leaf of the integrable distribution $\langle \mathcal{D}_+, \partial_0 \rangle$ is clearly flat.

Definition 9 *A frame $\varphi_1, ..., \varphi_r$ in the holomorphic vector bundle $\pi^*\nu : \mathbb{C}^r \times S_H$ which is parallel along leaves of the integrable distribution $\langle \mathcal{D}_+, \partial_0 \rangle$ is called an* **analytic frame***.*

With respect to an analytic frame a connection in the vector bundle $\pi^*\nu$ has components

$$\begin{aligned}
\nabla^S_{\partial_0} &= \partial_0 \\
\nabla^S_{X^e_+} &= X^e_+ \\
\nabla^S_{\partial_{++}} &= \partial_{++} + A_{++} := \partial_{++} + A(\partial_{++}) \\
\nabla^S_{\partial_{--}} &= \partial_{--} + A_{--} := \partial_{--} + A(\partial_{--}) \\
\nabla^S_{X^e_-} &= X^e_- + A(X^e_-) \ .
\end{aligned}$$

Definition 10 *A connection ∇^S over S_H is called* **almost half-flat** *if its curvature satisfies the following equations:*

$$F(X^e_+\,,\,X^{e'}_+) = F(X^e_+\,,\,v) = 0 \quad , \quad \forall v \in T^v S_H$$
$$F(\partial_{++}\,,\,\cdot\,) = F(\partial_0\,,\,\cdot\,) = 0\,. \tag{4.4}$$

In fact, these equations are not independent; for instance the Bianchi identity with arguments $(X_+, \partial_{++}, \partial_{--})$ together with $F(\partial_{++}, \partial_{--}) = F(\partial_{\pm\pm}, X^e_+) = 0$ implies the equation $F(\partial_{++}, X^e_-) = 0$.

Proposition 5 *Any almost half-flat connection satisfies the following equation:*

$$F(X^e_+, X^{e'}_-) = F(X^{e'}_+, X^e_-)\,.$$

Proof: Using the integrability of $\mathcal{D}_+$ and $F(X^e_+, X^{e'}_+) = F(\partial_{--}, X^e_+) = 0$, we obtain

$$\begin{aligned} 0 &= \left[\nabla^S_{\partial_{--}}\,,\,F(X^e_+, X^{e'}_+)\right] \\ &= \left[\nabla^S_{\partial_{--}}\,,\,\left[\nabla^S_{X^e_+}\,,\,\nabla^S_{X^{e'}_+}\right]\right] - \left[\nabla^S_{\partial_{--}}\,,\,\nabla^S_{[X^e_+\,,X^{e'}_+]}\right] \\ &= \left[\nabla^S_{X^e_-}\,,\,\nabla^S_{X^{e'}_+}\right] + \left[\nabla^S_{X^e_+}\,,\,\nabla^S_{X^{e'}_-}\right] - \nabla^S_{[X^e_-\,,X^{e'}_+]} - \nabla^S_{[X^e_+\,,X^{e'}_-]} \\ &= F(X^e_-, X^{e'}_+) - F(X^{e'}_-, X^e_+)\;. \end{aligned}$$
□

It follows that an almost half-flat connection is a generalisation of a half-flat connection, satisfying only those equations in (4.3), which involve curvatures with $\partial_0, \partial_{++}$ or X^e_+ in one of the arguments.

Proposition 6 *An almost half-flat connection is half-flat if and only if it satisfies $F(\partial_{--}, X^e_-) = 0$.*

Proof: By Proposition 5 an almost half-flat connection is required to satisfy all the half-flatness equations (4.3) with the exception of

$$F(\partial_{--}, X^e_-) = 0 \quad \text{and} \quad F(X^e_-, X^{e'}_-) = 0. \tag{4.5}$$

The second equation here follows from the first in virtue of the Bianchi identity with arguments $(X^e_+, X^{e'}_-, \partial_{--})$. □

The following proposition shows that an almost half-flat connection is completely determined by the potentials A_{++} and A_{--} with respect to an analytic frame.

Proposition 7 *Let ∇^S be an almost half-flat connection in the vector bundle $\pi^*\nu : \mathbb{C}^r \times S_H \to S_H$ with potentials A_{++}, A_{--} and $A(X^e_-)$ in an analytic frame. Then:*

(i) The potential A_{++} is analytic and has charge $+2$, i.e.

$$X^e_+ A_{++} = 0 \quad , \quad \partial_0 A_{++} = 2A_{++} \ . \tag{4.6}$$

(ii) The potential A_{--} satisfies

$$\partial_{++}A_{--} - \partial_{--}A_{++} + [A_{++}, A_{--}] = 0 \quad , \quad \partial_0 A_{--} = -2A_{--} \ . \tag{4.7}$$

(iii) The potential $A(X^e_-)$ is determined by A_{--} and has charge -1:

$$A(X^e_-) = -X^e_+ A_{--} \quad , \quad \partial_0 A(X^e_-) = -A(X^e_-) \ . \tag{4.8}$$

Conversely, any matrix-valued potentials A_{++}, A_{--} and $A(X^e_-)$ satisfying (4.6), (4.7) *and* (4.8) *define an almost half-flat connection.*

Proof: (i) The curvature constraints $F(X^e_+ , \partial_{++}) = 0$, $F(\partial_0 , \partial_{++}) = 0$, in an analytic frame, take the form (4.6).
(ii) The further almost half-flatness conditions, $F(\partial_{++} , \partial_{--}) = F(\partial_0 , \partial_{--}) = 0$ give equations (4.7) for the potential A_{--} .
(iii) Having obtained A_{--}, we can find $A(X^e_-)$ from the equations $F(X^e_+ , \partial_{--}) = F(\partial_0 , X^e_-) = 0$, which take the form

$$X^e_+ A_{--} = A([X^e_+, \partial_{--}]) = -A(X^e_-) \quad , \quad \partial_0 A(X^e_-) = -A(X^e_-) \ . \tag{4.9}$$

The second equation follows from the first. □

We can now write an algorithm for the construction of all almost half-flat connections:

Theorem 4 *Let A_{++} be an analytic prepotential, i.e. a matrix-valued function on a domain $U = \pi^{-1}(V) \subset S_H$, where $V \subset M$ is a simply connected domain, satisfying* (4.6)*. Let Φ be an invertible matrix-valued function on U which satisfies the equations*

$$\partial_{++}\Phi = -A_{++}\Phi \, , \quad \partial_0 \Phi = 0 \, . \tag{4.10}$$

It always exists. The pair (A_{++}, Φ) determines an almost half-flat connection $\nabla^S = \nabla^{(A_{++},\Phi)}$. Its potentials with respect to an analytic frame are given by A_{++}, $A_{--} = -(\partial_{--}\Phi)\Phi^{-1}$ and $A(X^e_-) = -X^e_+ A_{--}$. Conversely, any almost half-flat connection is of this form.

Proof: We consider the connection defined by A_{++} and $A(\partial_0) = 0$ along an orbit sB of the Borel subgroup of $\mathrm{SL}(2, \mathbb{C})$,

$$B = \left\{ \begin{pmatrix} t_0 & t_1 \\ 0 & t_0^{-1} \end{pmatrix} \middle| \ t_0 \in \mathbb{C}^*, \, t_1 \in \mathbb{C} \right\} \cong \mathbb{C}^* \times \mathbb{C} \quad \text{(diffeomorphic)} \ .$$

It is flat since the second equation of (4.6) is equivalent to $F(\partial_0\,,\,\partial_{++})$=0 (vanishing of the curvature along sB). Moreover it has trivial holonomy since the fundamental group of $B\cong\mathbb{C}^*\times\mathbb{C}$ coincides with the fundamental group of the $\mathbb{C}^*$-factor and the potential is zero in direction of ∂_0 which is tangent to $\mathbb{C}^*$. An invertible solution to the system (4.10) exits and defines a parallel frame Φ with respect to the flat connection with trivial holonomy defined along each orbit of the Borel group. Since the space of Borel orbits in U is diffeomorphic to $V\times\mathbb{C}P^1$ and is therefore simply connected, a solution Φ exists on the domain U. Now, given any such solution of (4.10), we define $A_{--} := -(\partial_{--}\Phi)\Phi^{-1}$. This solves (4.7), since $F(\partial_{\pm\pm},\partial_0) = F(\partial_{++},\partial_{--}) = 0$ is the integrability condition for the system $\partial_{\pm\pm}\Phi = -A_{\pm\pm}\Phi$, $\partial_0\Phi = 0$. Finally, we define $A(X^e_-) := -X^e_+ A_{--}$, obtaining an almost half-flat connection by Proposition 7. Now the converse statement follows also from Proposition 7. □

4.2.2 Transformation to the central frame

Since an almost half-flat connection $\nabla = \nabla^S$ is flat in vertical directions, it admits a central frame ψ. The following lemma shows that the solution Φ of equation (4.10) gives a gauge transformation from an analytic frame φ to a central frame $\psi = \varphi\Phi$ for the almost half-flat connection $\nabla^{(A_{++},\Phi)}$.

Lemma 5 *Let $\nabla = \nabla^{(A_{++},\Phi)}$ be the almost half-flat connection associated to the analytic prepotential A_{++} with respect to the analytic frame φ and an invertible solution Φ of (4.10). Then the frame $\psi := \varphi\Phi$ is a central frame for the connection ∇, i.e. the potentials $C(\partial_{\pm\pm})$ and $C(\partial_0)$ with respect to that frame vanish.*

Proof: The result follows from the pure gauge form of $A(\partial_{\pm\pm})$ and $A(\partial_0)$ and the transformation law (2.3) for potentials. □

With respect to the central frame ψ, the almost half flat connection constructed above takes the form:

$$\begin{cases} \nabla^S_{X^e_+} = X^e_+ + C(X^e_+) = X^e_+ + \Phi^{-1}\,X^e_+\,\Phi \\ \nabla^S_{X^e_-} = X^e_- + C(X^e_-) = X^e_- + \Phi^{-1}X^e_-\Phi + \Phi^{-1}X^e_+\,(\partial_{--}\Phi\Phi^{-1})\,\Phi \\ \nabla^S_{\partial_{++}} = \partial_{++} \quad , \quad \nabla^S_{\partial_{--}} = \partial_{--} \quad , \quad \nabla^S_{\partial_0} = \partial_0\ . \end{cases}$$

Moreover, the equations $F(\partial_{++}\,,\,X^e_+) = F(\partial_0\,,\,X^e_+) = 0$ imply that the potential $C(X^e_+)$ satisfies the equations

$$\partial_{++}C(X^e_+) = 0\ , \quad \partial_0 C(X^e_+) = C(X^e_+)\ . \tag{4.11}$$

4.2.3 Construction of half-flat connections

We assume now that the analytic prepotential A_{++} is defined globally along the fibres of $\pi : S_H \to M$. Then, restricting M to an appropriate domain, we may assume that A_{++} is defined globally on S_H. The previous construction then provides an almost half-flat connection over S_H. Using this connection, we may construct a half-flat connection on M. The crucial point is the following:

Proposition 8 *The potential $C(X^e_+)$ of an almost half-flat connection ∇ with respect to a central frame is linear in u^α_+, namely,*

$$C(X^e_+) = u^\alpha_+ C(\widetilde{X^e_\alpha}) =: u^\alpha_+ C^e_\alpha \ , \tag{4.12}$$

where $(x^i, u^\alpha_\pm)$ are the local coordinates associated with the trivialisation $S_H = M \times \mathrm{Sp}(1, \mathbb{C})$ and $C^e_\alpha = C^e_\alpha(x^i)$ is a matrix-valued function on M.

Proof: Due to equations (4.11), the result follows from:

Lemma 6 *i) If a holomorphic function f_+, defined on some domain*

$$U \subset \{u^2_+ \neq 0\} \subset \mathrm{Sp}(1, \mathbb{C}) = \left\{ \mathcal{U} = \begin{pmatrix} u^1_+ & u^1_- \\ u^2_+ & u^2_- \end{pmatrix} \ ; \ \det \mathcal{U} = 1 \right\}$$

satisfies

$$\partial_{++} f_+ = 0 \ , \quad \partial_0 f_+ = f_+ \ , \tag{4.13}$$

then $f_+ = u^\alpha_+ f_\alpha(\frac{u^1_+}{u^2_+})$. Here $f_\alpha(\frac{u^1_+}{u^2_+})$ are holomorphic functions on U invariant under the right action of the Lie algebra of upper-triangular matrices.

ii) Moreover, if the function f_+ is globally defined, then it is linear in u^α_+, i.e. $f_+ = u^\alpha_+ f_\alpha$, $f_\alpha =$const.

Proof: i) One can immediately check that $f_+ = u^\alpha_+ f_\alpha(\frac{u^1_+}{u^2_+})$ is a solution of (4.13). We note that the quotient of any two solutions of (4.13) is a solution of the corresponding homogeneous system,

$$\partial_{++} f = 0 \ , \quad \partial_0 f = 0 \ . \tag{4.14}$$

It is sufficient to check that any solution of (4.14) is a function of u^1_+/u^2_+. To prove this we use the local factorisation of $\mathrm{Sp}(1, \mathbb{C})$ into the product of a Borel subgroup $\mathcal{B}$ and a nilpotent subgroup as follows:

$$\begin{pmatrix} u^1_+ & u^1_- \\ u^2_+ & u^2_- \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \quad , \quad \mathcal{B} = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right\} \ .$$

Then $c = u_+^1/u_+^2$ and $\partial_0, \partial_{++}$ are generators of the right action of $\mathcal{B}$. This implies that the solutions of (4.14) are precisely the local functions on $\mathrm{Sp}(1,\mathbb{C})$ invariant under the right action of $\mathcal{B}$. In terms of the local coordinate system (a,b,c) on $\mathrm{Sp}(1,\mathbb{C})$ such functions are functions of $c = u_+^1/u_+^2$ alone.

ii) The restriction $V|_{\mathrm{Sp}(1)}$ to $\mathrm{Sp}(1)$ of any irreducible $\mathrm{Sp}(1,\mathbb{C})$-module V of holomorphic functions is a (finite dimensional) irreducible $\mathrm{Sp}(1)$-module of smooth functions on $\mathrm{Sp}(1)$. The condition (4.13) shows that f_+ is a highest weight vector with weight $+1$. Hence f_+ generates a 2-dimensional submodule $\langle f_+\rangle = \mathrm{span}\{f_+, f_- := \partial_{--} f_+\}$ of holomorphic functions. It remains to show that any two dimensional module of holomorphic functions on $\mathrm{Sp}(1,\mathbb{C})$ is spanned by linear functions. We know two such modules, generated by the highest weight vectors u_+^1 and u_+^2 respectively. On the other hand, by the Peter-Weyl Theorem the multiplicity of the two-dimensional irreducible representation of $\mathrm{Sp}(1)$ in $L^2(\mathrm{Sp}(1))$ is 2. $\square$

Using Proposition 8, with respect to a central frame, we can write $\nabla_{X_+^e} = X_+^e + u_+^\alpha C_\alpha^e$ where the coefficients $C_\alpha^e = C_\alpha^e(x^i)$ are matrix valued functions of coordinates x^i on M. Using them we define a new connection in the trivial bundle $\mathbb{C}^r \times S_H$ over S_H by

$$\begin{cases} \widehat{\nabla}_{X_\pm^e} &= X_\pm^e + u_\pm^\alpha C_\alpha^e \\ \widehat{\nabla}_{\partial_{\pm\pm}} &= \partial_{\pm\pm} \quad , \quad \widehat{\nabla}_{\partial_0} = \partial_0 \ . \end{cases}$$

Our main result now follows:

Theorem 5 *Let M be a complex manifold with a half-flat admissible Grassmann structure. Let A_{++} be an analytic prepotential, i.e. a solution of (4.6), and Φ an invertible solution of (4.10). Then the connection $\widehat{\nabla} = \widehat{\nabla}^{(A_{++},\Phi)}$ constructed from the data (A_{++},Φ) is a half-flat connection in the trivial vector bundle $\mathbb{C}^r \times S_H \to S_H$ and it is the pull-back of the following half-flat connection ∇^M in the bundle $\mathbb{C}^r \times M \to M$:*

$$\nabla^M_{X_\alpha^e} = X_\alpha^e + C_\alpha^e \ . \tag{4.15}$$

Conversely, any half-flat connection over S (or M) is gauge equivalent to one obtained from the above construction.

Proof: The remark after Definition 8 shows that the connection $\widehat{\nabla}$ is the pull-back of the connection ∇^M. It suffices now to show that ∇^M is half-flat. Note that the connections ∇ and $\widehat{\nabla}$ coincide in the direction of X_+^e. Hence, using $C_+^e := u_+^\alpha C_\alpha^e$, we have

$$\begin{aligned} 0 &= F^\nabla(X_+^e, X_+^{e'}) = F^{\widehat{\nabla}}(X_+^e, X_+^{e'}) \\ &= X_+^e C_+^{e'} - X_+^{e'} C_+^e + [C_+^e , C_+^{e'}] - C([X_+^e , X_+^{e'}]) \\ &= u_+^\alpha u_+^\beta \left(X_\alpha^e C_\beta^{e'} - X_\beta^{e'} C_\alpha^e + [C_\alpha^e , C_\beta^{e'}] - C([X_\alpha^e , X_\beta^{e'}]) \right) \\ &= u_+^\alpha u_+^\beta F^{\nabla^M}(X_\alpha^e , X_\beta^{e'}) \ , \end{aligned}$$

since $X^e_+ u^\beta_+ = 0$. This shows that the curvature $F^{\nabla^M}(X^e_\alpha, X^{e'}_\beta)$ is skewsymmetric in α, β, i.e. it belongs to $\wedge^2 H \otimes S^2 E \otimes \mathrm{End}\ W$. In other words, the connection ∇^M is half-flat.

Conversely, let ∇^S be a half-flat connection over S_H. By Proposition 4 we may assume that it is a pull-back of a half-flat connection ∇^M over M. Since the restriction of ∇^S to the leaves of $\langle \mathcal{D}_+, \partial_0 \rangle$ is flat, there exists an analytic frame (i.e. a frame such that $A(X^e_+) = A(\partial_0) = 0$, in which the potential $A(\partial_{++})$ satisfies the equations (4.6)). Since ∇^S is flat along the (simply-connected) fibres, there exists an invertible solution Φ to the system

$$\partial_{++}\Phi + A_{++}\Phi \;=\; \partial_{--}\Phi + A_{--}\Phi \;=\; \partial_0 \Phi \;=\; 0\,.$$

This shows that ∇^S is gauge-equivalent to the almost half-flat connection $\nabla^{(A_{++},\Phi)} = \widehat{\nabla}^{(A_{++},\Phi)}$. □

4.3 Application to hyper-Kähler manifolds with admissible torsion

The above construction can be applied to the complexification of hyper-Kähler manifolds. Recall that any hyper-Kähler manifold admits a (locally defined) Grassmann structure $T^{*\mathbb{C}}M = E \otimes H$, such that the Levi-Civita connection on the cotangent bundle $\nabla = \nabla^E \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^H$ is half-flat, i.e. the connection ∇^H is flat. Since the hyper-Kähler metric is Ricci flat, hence analytic, we may, using analytic continuation, extend the manifold M to a complex manifold $M^\mathbb{C}$ with holomorphic extension of the hyper-Kähler structure, in particular, we have a holomorphic Ricci flat metric on $M^\mathbb{C}$ with holonomy in $\mathrm{Sp}(n,\mathbb{C})$ and half-flat Grassmann structure. This Grassmann structure is admissible since the Levi-Civita connection on $M^\mathbb{C}$ has no torsion. Hence we can apply the harmonic space method to construct half-flat connections on holomorphic vector bundles $W \to M^\mathbb{C}$. The complex version of Proposition 1 shows that such connections are Yang-Mills connections. More generally, the method of construction of half-flat connections extends to real analytic (possibly indefinite) hyper-Kähler manifolds with admissible torsion, i.e. with torsion which has zero component in $S^3H \otimes E^* \otimes \wedge^2 E$. A **hyper-Kähler manifold with admissible torsion** is defined as a pseudo-Riemannian manifold (M, g) with a linear metric connection ∇ with holonomy in $\mathrm{Sp}(k,l)$ which has admissible torsion. As in the (torsionfree) hyper-Kähler case there exists a parallel 4-form given by $\Omega = \sum_\alpha \omega_\alpha \wedge \omega_\alpha$, $\omega_\alpha := g(J_\alpha \cdot, \cdot)$, and half-flat connections are characterised as connections with curvature in $V_{\lambda_1} \otimes \mathrm{End}\ W$, where V_{λ_1} is the λ_1-eigenspace of the endomorphism B_Ω associated to Ω. If the form Ω is co-closed then any half-flat connection will be (Ω, λ_1)-self-dual and thus a Yang-Mills connection. We remark that co-closedness of Ω is equivalent to a linear $\mathrm{Sp}(k,l)$-invariant condition on the torsion.

5 Generalisation to higher-spin Grassmann manifolds

5.1 Higher-spin Grassmann structures

The construction discussed in the previous section is in fact the m=1 specialisation of a more general construction of connections on *spin* $\frac{m}{2}$ *Grassmann manifolds*, which we discuss in this section. These manifolds were considered in [AG].

Definition 11 *A* **spin $\frac{m}{2}$ Grassmann structure** *on a (complex) manifold M is a holomorphic Grassmann structure of the form $T^*M \cong E \otimes F = E \otimes S^m H$, with a holomorphic Grassmann connection $\nabla = \nabla^E \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^F$, where H is a rank 2 holomorphic vector bundle over M with holomorphic symplectic connection ∇^H and symplectic form $\omega_H \in \Gamma(\wedge^2 H)$, and ∇^F is the connection in $F = S^m H$ induced by ∇^H. M is called* **half-flat** *if the connection ∇^F is flat.*

The bundle $S^m H$ is associated with the spin $\frac{m}{2}$ representation of the group $\mathrm{Sp}(1, \mathbb{C})$. Any frame (h_1, h_2) for H^* defines a frame for $S^m H^*$, $(h_A := h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_m})$, where the multi-index $A := \alpha_1 \alpha_2 \ldots \alpha_m$, $\alpha_i = 1, 2$. The ∇^H-parallel symplectic form ω_H on H^* induces a bilinear form ω_H^m on $F^* = S^m H^*$ given by

$$\omega_H^m(h_A, h_B) := \underset{A}{\mathfrak{S}}\,\underset{B}{\mathfrak{S}}\, \omega_H(h_{\alpha_1}, h_{\beta_1}) \omega_H(h_{\alpha_2}, h_{\beta_2}) \cdots \omega_H(h_{\alpha_m}, h_{\beta_m}) \ ,$$

where $\mathfrak{S}_A$ denotes the sum over all permutations of the α's. This form is skew-symmetric if m is odd and symmetric if m is even. To any section $e \in \Gamma(E^*)$ and multi-index A we associate the vector field $X_A^e := e \otimes h_A$ on M.

The construction of half-flat connections described in section 4.2 may be adapted to obtain certain 'partially flat' connections in vector bundles $W \to M$, provided that the torsion of ∇ obeys certain admissibility conditions.

Definition 12 *Let (M, ∇) be a half-flat spin $\frac{m}{2}$ Grassmann manifold. For any section $e \in \Gamma(E^*)$ we define vector fields*

$$\begin{aligned} X^e_{(m-2i)+} &:= u_-^{\alpha_1} \cdots u_-^{\alpha_i} u_+^{\alpha_{i+1}} \cdots u_+^{\alpha_m} X_A^e \quad \text{if} \quad m - 2i \geq 0 \\ X^e_{(2i-m)-} &:= u_-^{\alpha_1} \cdots u_-^{\alpha_i} u_+^{\alpha_{i+1}} \cdots u_+^{\alpha_m} X_A^e \quad \text{if} \quad m - 2i < 0 \end{aligned}$$

on the principal bundle S_H of symplectic frames in H. The distribution spanned by these vector fields is denoted by $\mathcal{D}_{k+} := \langle X^E_{k+} \rangle$ for $k \geq 0 \, , k \equiv m \mod 2$ and $\mathcal{D}_{k-} := \langle X^E_{k-} \rangle$ for $k > 0 \, , k \equiv m \mod 2$. We define also

$$\mathcal{D}^k_{(\pm)} := \bigoplus_{i=0}^{k} \mathcal{D}_{(m-2i)\pm} \ .$$

The Grassmann connection ∇ *is called* **k–admissible** *if it preserves the distribution* $\mathcal{D}^k_{(\pm)}$, *i.e.*

$$T\left(\mathcal{D}^k_{(\pm)}, \mathcal{D}^k_{(\pm)}\right) \subset \mathcal{D}^k_{(\pm)} \ . \tag{5.1}$$

The Grassmann manifold (M, ∇) *is called* **k–admissible** *if the connection* ∇ *is* k*–admissible.*

For small m we shall write X^e_0, X^e_+, X^e_-, X^e_{++} etc. instead of X^e_{0+}, X^e_{1+}, X^e_{1-}, X^e_{2+} etc.

Proposition 9 *Let* (M, ∇) *be a half-flat spin* $\frac{m}{2}$ *Grassmann manifold. Then the distribution* $\mathcal{D}^k_{(\pm)}$ *is integrable if and only if the torsion of the Grassmann connection* ∇ *satisfies equation* (5.1).

The proof is similar to that of Proposition 2.

5.2 Partially flat connections over higher-spin Grassmann manifolds

Let (M, ∇) be a half-flat spin $\frac{m}{2}$ Grassmann manifold and $\nu : W \to M$ a holomorphic vector bundle. Since our constructions are local we will assume that W is trivial. In the higher spin ($m > 1$) case, there exists, as a natural generalisation of the notion of a half-flat connection, the more refined notion of a k-partially flat connection in ν. The space of two-forms $\wedge^2 T^*M$ has the following decomposition into $GL(E) \otimes \mathrm{Sp}(1, \mathbb{C})$-submodules:

$$\wedge^2 T^*M = \wedge^2\big(E \otimes S^m H\big) = \wedge^2 E \otimes S^2 S^m H \ \oplus \ S^2 E \otimes \wedge^2 S^m H \,,$$

where

$$\begin{aligned} S^2 S^m H &= S^{2m} H \ \oplus \ \omega_H^2 S^{2m-4} H \ \oplus \ \cdots \oplus \omega_H^{2[\frac{m}{2}]} S^{2m-4[\frac{m}{2}]} H \\ \wedge^2 S^m H &= \omega_H S^{2m-2} H \ \oplus \ \omega_H^3 S^{2m-6} H \oplus \cdots \oplus \omega_H^{2[\frac{m}{2}]+1} S^{2m-4[\frac{m}{2}]-2} H \,. \end{aligned}$$

Here we use the convention that $S^l H = 0$ if $l < 0$.

Let ∇ be a connection in the vector bundle $W \to M$. Its curvature has the following decomposition, corresponding to the above decomposition of $\wedge^2 T^*M$ into irreducible $GL(E){\cdot}\mathrm{Sp}(1, \mathbb{C})$-submodules:

$$\begin{aligned} F(X^e_A, X^{e'}_B) &= \underset{A}{\mathfrak{S}}\,\underset{B}{\mathfrak{S}} \sum_{k=0}^{[m/2]} \Big(\omega_H(h_{\alpha_1}, h_{\beta_1}) \cdots \omega_H(h_{\alpha_{2k}}, h_{\beta_{2k}}) \overset{(2k)}{F}{}^{[ee']}_{\alpha_{2k+1}\ldots\alpha_m\beta_{2k+1}\ldots\beta_m} \\ &\quad + \omega_H(h_{\alpha_1}, h_{\beta_1}) \cdots \omega_H(h_{\alpha_{2k+1}}, h_{\beta_{2k+1}}) \overset{(2k+1)}{F}{}^{(ee')}_{\alpha_{2k+2}\ldots\alpha_m\beta_{2k+2}\ldots\beta_m} \Big), \end{aligned} \tag{5.2}$$

where the tensors $\overset{(2k)}{F} \in \Gamma(\wedge^2 E \otimes S^{2m-4k} H)$ and $\overset{(2k+1)}{F} \in \Gamma(S^2 E \otimes S^{2m-4k-2} H)$.

We note that half-flat connections are those which satisfy the conditions

$$\overset{(2i)}{F} = 0 \ , \quad \text{for all } i \in \mathbb{N}. \tag{5.3}$$

For $m>1$ these conditions are not suitable for application of the harmonic space method. However, the following more refined restrictions on the curvature are amenable to the method (c.f. [DN1]).

Definition 13 *A connection ∇ in the vector bundle $\nu : W \to M$ is called* **k-partially flat** *if $\overset{(i)}{F} = 0$ for all $i \leq 2k$. Here $0 \leq k \leq [\frac{m+2}{2}]$.*

Clearly, $[\frac{m+2}{2}]$-partially flat connections are simply flat connections. We note that for $m=1$, 0-partially flat connections are precisely half-flat connections. For general odd $m=2p+1$, 0-partially flat connections in a vector bundle ν over flat spaces with spin $\frac{m}{2}$ Grassmann structure were considered by Ward [W]. He chose E to be a rank 2 flat bundle and showed that 0-partially flat connections, for $m>1$, do not correspond to Yang-Mills connections. Therefore, in our more general setting, we clearly cannot expect 0-partially flat connections to satisfy the Yang-Mills equations for $m>1$. On the other hand, the penultimate case, $k = [\frac{m}{2}]$, is particularly interesting for odd m:

Theorem 6 *Let M be a half-flat spin $\frac{m}{2}$ Grassmann manifold M. If m is odd and the vector bundle $E^* \to M$ admits a ∇^E-parallel symplectic form ω_E, then M has canonical $\mathrm{Sp}(E){\cdot}\mathrm{Sp}(H)$-invariant metric $g = \omega_E \otimes \omega_H^m$ and 4-form $\Omega \neq 0$. If Ω is co-closed with respect to the metric g, then any $\frac{m-1}{2}$-partially flat connection ∇ in a vector bundle W over M is (Ω, λ)-self-dual and hence it is a Yang-Mills connection.*

Proof: To describe Ω we use the following notation: e_a is a basis of E^*, h_α is a basis of H^*, h_A is the corresponding basis of $S^m H^*$ and $X_{aA} := e_a \otimes h_A$ the corresponding basis of $TM = E^* \otimes S^m H^*$. With respect to these bases, the skew symmetric forms ω_E, ω_H and ω_H^m are represented by the matrices ω_{ab}, $\omega_{\alpha\beta}$ and ω_{AB} respectively. We define Ω by

$$\Omega := \sum \omega_{ab}\, \omega_{cd}\, \omega_{AC}\, \omega_{BD}\, X^{aA} \wedge X^{bB} \wedge X^{cC} \wedge X^{dD} \,,$$

where X^{aA} is the basis dual to X_{aA}. This form is obviously $\mathrm{Sp}(E){\cdot}\mathrm{Sp}(H)$-invariant since we used only ω_E and ω_H in the definition. One can easily check that $\Omega \neq 0$. The connection ∇ is $\frac{m-1}{2}$-partially flat if and only if its curvature F belongs to the space

$$S^2 E \otimes \omega_H^m \otimes \mathrm{End}\ W \ \subset\ S^2 E \otimes \wedge^2 S^m H \otimes \mathrm{End}\ W \ \subset\ \wedge^2 (E \otimes S^m H) \otimes \mathrm{End}\ W \,.$$

Here we use the decomposition

$$\wedge^2 S^m H = \omega_H S^{2m-2} H \ \oplus\ \omega_H^3 S^{2m-6} H \ \oplus\ \ldots\ \oplus\ \mathbb{C}\omega_H^m \,.$$

The $\mathrm{Sp}(E)\cdot\mathrm{Sp}(H)$-submodule $S^2E\otimes\omega_H^m \subset \wedge^2 T^*M$ is irreducible. Therefore it is contained in an eigenspace V_λ of the $\mathrm{Sp}(E)\cdot\mathrm{Sp}(H)$-invariant operator $B_\Omega : \wedge^2T^*M \to \wedge^2T^*M$. It remains to check that $\lambda \neq 0$. By Lemma 1 it suffices to compute the contraction $K = K_{cCdD}X^{cC}X^{dD}$ of a tensor $S = S_{ab}\omega_{AB}e^a e^b \otimes h^A h^B$ in $S^2E\otimes\omega_H^m$ with Ω:

$$\begin{aligned}
-K_{cCdD} &= S^{ab}\omega^{AB}(\omega_{ab}\omega_{cd}\omega_{AC}\omega_{BD} + \omega_{ac}\omega_{db}\omega_{AD}\omega_{CB} + \omega_{ad}\omega_{bc}\omega_{AB}\omega_{DC} - \omega_{ba}\omega_{cd}\omega_{BC}\omega_{AD} \\
&\quad -\omega_{bc}\omega_{da}\omega_{BD}\omega_{CA} - \omega_{bd}\omega_{ac}\omega_{BA}\omega_{DC} - \omega_{ca}\omega_{bd}\omega_{CB}\omega_{AD} - \omega_{cb}\omega_{da}\omega_{CD}\omega_{BA} \\
&\quad -\omega_{cd}\omega_{ab}\omega_{CA}\omega_{DB} + \omega_{da}\omega_{bc}\omega_{DB}\omega_{AC} + \omega_{db}\omega_{ca}\omega_{DC}\omega_{BA} + \omega_{dc}\omega_{ab}\omega_{DA}\omega_{CB}) \\
&= 4(m+1)S_{cd}\omega_{CD}\,.
\end{aligned}$$

Here ω^{ab} and ω^{AB} denote the inverses of ω_{ab} and ω_{AB} and $S^{ab} = \omega^{aa'}\omega^{bb'}S_{a'b'}$. We have used that $S^{ab}\omega^{AB}$ is skew symmetric under interchange of aA with bB and that $\omega^{AB}\omega_{AB} = -(m+1)$. The above calculation shows that $\lambda = -4(m+1) \neq 0$ and hence any $\frac{m-1}{2}$-partially flat connection is (Ω,λ)-self-dual and is a Yang-Mills connection by Theorem 1. □

The analogous result does not hold if m is even.

Proposition 10 *If m is even and the vector bundle $E^*\to M$ admits a ∇^E-parallel metric γ_E, then M has canonical $\mathrm{SO}(E)\cdot\mathrm{Sp}(H)$-invariant metric $g = \gamma_E\otimes\omega_H^m$ and 4-form $\Omega \neq 0$.*

Proof: Analogously to the case of m odd, we can define Ω by

$$\Omega := \sum \gamma_{ab}\,\gamma_{cd}\,\omega_{AC}\,\omega_{BD}\,X^{aA}\wedge X^{bB}\wedge X^{cC}\wedge X^{dD}\,.$$

Here $\gamma_{ab} = \gamma_E(e_a, e_b)$ and we recall that for even m the bilinear form ω_H^m is symmetric: $\omega_{AB} = \omega_{BA}$. □

For even m, a connection ∇ in a vector bundle W over M is $\frac{m}{2}$-partially flat if and only if its curvature F belongs to the space

$$\begin{aligned}
&\left(\wedge^2E\otimes\omega_H^m \ \oplus\ S^2E\otimes S^2H\otimes\omega_H^{m-1}\right)\otimes \mathrm{End}\ W \\
&\subset \left(\wedge^2E\otimes S^2S^mH \ \oplus\ S^2E\otimes\wedge^2 S^mH\right)\otimes\mathrm{End}\ W = \wedge^2(E\otimes S^mH)\otimes\mathrm{End}\ W.
\end{aligned}$$

The $\mathrm{SO}(E)\cdot\mathrm{Sp}(H)$-submodule $\wedge^2E\otimes\omega_H^m\oplus S^2E\otimes S^2H\otimes\omega_H^{m-1} \subset \wedge^2T^*M$ is not irreducible, so unlike the odd m case we cannot conclude that it is contained in an eigenspace V_λ of the $\mathrm{SO}(E)\cdot\mathrm{Sp}(H)$-invariant operator $B_\Omega : \wedge^2T^*M \to \wedge^2T^*M$. In fact, examples are known (see Appendix B of [DN2]) where B_Ω has different eigenvalues on each irreducible summand of $\wedge^2T^*M$. Therefore in the case of even m we cannot expect that $\frac{m}{2}$-partial flatness implies the Yang-Mills equations.

5.3 Construction of partially flat connections over higher spin Grassmann manifolds

Now we generalise the construction of half-flat connections over admissible half-flat Grassmann manifolds to the case of k–partially flat connections over k–admissible higher spin Grassmann manifolds M. The natural extension of the harmonic construction given in section 4.2 yields k-partially flat connections in the vector bundle ν over the k–admissible spin $\frac{m}{2}$ Grassmann manifold M. Again, we lift the geometric data from M to S_H via the projection $\pi : S_H \to M$. The pull back $\pi^*\nabla$ of a k–partially flat connection ∇ in the trivial vector bundle $\nu : W = \mathbb{C}^r \times M \to M$ is a connection in the vector bundle $\pi^*\nu : \pi^*W \to S_H$ which satisfies equations defining the notion of a k–partially flat gauge connection on S_H. One can also define the weaker notion of an almost k–partially flat connection in $\pi^*\nu : \pi^*W \to S_H$. The latter may be constructed from a prepotential and it affords the construction of a k-partially flat connection in the bundle $W \to M$. To simplify our exposition we explain the construction in the m=3 case. Here the decomposition (5.2) of the curvature tensor takes the form

$$\begin{aligned} F(X^e_{\alpha_1\alpha_2\alpha_3}, X^{e'}_{\beta_1\beta_2\beta_3}) \quad = \quad \underset{A}{\mathfrak{S}}\,\underset{B}{\mathfrak{S}} \Big(& \overset{(0)}{F}{}^{[ee']}_{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3} + \omega_H(h_{\alpha_1}, h_{\beta_1}) \overset{(1)}{F}{}^{(ee')}_{\alpha_2\alpha_3\beta_2\beta_3} \\ & + \omega_H(h_{\alpha_1}, h_{\beta_1})\omega_H(h_{\alpha_2}, h_{\beta_2}) \overset{(2)}{F}{}^{[ee']}_{\alpha_3\beta_3} \\ & + \omega_H(h_{\alpha_1}, h_{\beta_1})\omega_H(h_{\alpha_2}, h_{\beta_2})\omega_H(h_{\alpha_3}, h_{\beta_3}) \overset{(3)}{F}{}^{(ee')} \Big) . \end{aligned} \tag{5.4}$$

In this case we have two nontrivial notions of partial flatness:

$$\text{0–partial flatness :} \qquad \overset{(0)}{F} = 0 \tag{5.5}$$

$$\text{1–partial flatness :} \qquad \overset{(0)}{F} = \overset{(1)}{F} = \overset{(2)}{F} = 0 \ . \tag{5.6}$$

Clearly, 2-partial flatness is tantamount to flatness. By Theorem 6, a 1-partially flat connection is a Yang-Mills connection.

5.3.1 Construction of 0-partially flat connections

Let M be a 0–admissible spin $\frac{3}{2}$ Grassmann manifold with a 0-partially flat connection ∇ (satisfying (5.5)) in a holomorphic vector bundle $W \to M$. The pull-back of such a connection ∇ to a connection in the bundle $\pi^*W \to S_H$, where $\pi : S_H \to M$, has curvature

F with components given by:

$$
\begin{aligned}
F(X^e_{\pm\pm\pm}, X^{e'}_{\pm\pm\pm}) &= 0\\
F(X^{e'}_{\pm\pm\pm}, X^e_{\pm}) &= u^{\alpha_1}_{\pm} u^{\alpha_2}_{\pm} u^{\alpha_3}_{\pm} u^{\beta_1}_{\pm} u^{\beta_2}_{\pm} u^{\beta_3}_{\mp} F(X^e_{\alpha_1\alpha_2\alpha_3}, X^{e'}_{\beta_1\beta_2\beta_3})\\
&= \pm 12 \overset{(1)}{F}{}^{(ee')}_{\alpha_1\alpha_2\beta_1\beta_2} u^{\alpha_1}_{\pm} u^{\alpha_2}_{\pm} u^{\beta_1}_{\pm} u^{\beta_2}_{\pm} =: \pm 12 \overset{(1)}{F}{}^{(ee')}_{\pm\pm\pm\pm}\\
F(X^e_{\pm}, X^{e'}_{\pm}) &= u^{\alpha_1}_{\pm} u^{\alpha_2}_{\pm} u^{\alpha_3}_{\mp} u^{\beta_1}_{\pm} u^{\beta_2}_{\pm} u^{\beta_3}_{\mp} F(X^e_{\alpha_1\alpha_2\alpha_3}, X^{e'}_{\beta_1\beta_2\beta_3})\\
&= -8 \overset{(2)}{F}{}^{[ee']}_{\alpha_1\beta_1} u^{\alpha_1}_{\pm} u^{\beta_1}_{\pm} =: -8 \overset{(2)}{F}{}^{[ee']}_{\pm\pm}\\
F(X^e_{+++}, X^{e'}_{---}) &= u^{\alpha_1}_{+} u^{\alpha_2}_{+} u^{\alpha_3}_{+} u^{\beta_1}_{-} u^{\beta_2}_{-} u^{\beta_3}_{-} F(X^e_{\alpha_1\alpha_2\alpha_3}, X^{e'}_{\beta_1\beta_2\beta_3})\\
&= 36 \left(\overset{(1)}{F}{}^{(ee')}_{\alpha_1\alpha_2\beta_1\beta_2} u^{\alpha_1}_{+} u^{\alpha_2}_{+} u^{\beta_1}_{-} u^{\beta_2}_{-} + \overset{(2)}{F}{}^{[ee']}_{\alpha_1\beta_1} u^{\alpha_1}_{+} u^{\beta_1}_{-} + \overset{(3)}{F}{}^{(ee')} \right)\\
&=: 36 \left(\overset{(1)}{F}{}^{(ee')}_{0} + \overset{(2)}{F}{}^{[ee']}_{0} + \overset{(3)}{F}{}^{(ee')}_{0} \right)\\
F(X^e_{\pm\pm\pm}, X^{e'}_{\mp}) &= u^{\alpha_1}_{\pm} u^{\alpha_2}_{\pm} u^{\alpha_3}_{\pm} u^{\beta_1}_{\pm} u^{\beta_2}_{\mp} u^{\beta_3}_{\mp} F(X^e_{\alpha_1\alpha_2\alpha_3}, X^{e'}_{\beta_1\beta_2\beta_3})\\
&= \pm 24 \overset{(1)}{F}{}^{(ee')}_{\pm\pm} + 12 \overset{(2)}{F}{}^{[ee']}_{\pm\pm}\\
F(X^e_{+}, X^{e'}_{-}) &= u^{\alpha_1}_{+} u^{\alpha_2}_{+} u^{\alpha_3}_{-} u^{\beta_1}_{+} u^{\beta_2}_{-} u^{\beta_3}_{-} F(X^e_{\alpha_1\alpha_2\alpha_3}, X^{e'}_{\beta_1\beta_2\beta_3})\\
&= 12 \overset{(1)}{F}{}^{(ee')}_{0} - 4 \overset{(2)}{F}{}^{[ee']}_{0} - 12 \overset{(3)}{F}{}^{(ee')}_{0}\\
F(v\,,\,\cdot\,) &= 0\,, \qquad (5.7)
\end{aligned}
$$

where v, v' are vertical vector fields on S_H. The form of these components lead us to:

Definition 14 *A connection in a holomorphic vector bundle $W \to S_H$ is* **0-partially flat** *if its curvature satisfies the equations,*

$$
\begin{aligned}
F(X^e_{\pm\pm\pm}, X^{e'}_{\pm\pm\pm}) &= 0\\
F(X^e_{\pm\pm\pm}, X^{e'}_{\pm}) &= F(X^{e'}_{\pm\pm\pm}, X^e_{\pm})\\
F(X^e_{\pm}, X^{e'}_{\pm}) &= -F(X^{e'}_{\pm}, X^e_{\pm})\\
F(v\,,\,\cdot\,) &= 0 \quad , \quad \forall v \in T^v S_H\,.
\end{aligned}
$$

The restriction of a 0-partially flat connection to a leaf of the integrable distribution $\langle \mathcal{D}_{3+}, \partial_0 \rangle$ is clearly flat. In this case, an **analytic frame** in the holomorphic vector bundle $\pi^*\nu : \mathbb{C}^r \times S_H \to S_H$ is a frame which is parallel along leaves of this integrable distribution.

With respect to such a frame, a connection in the vector bundle $\pi^*\nu$ can be written as

$$\begin{cases} \nabla^S_{\partial_0} & = \partial_0 \\ \nabla^S_{X^e_{+++}} & = X^e_{+++} \\ \nabla^S_{X^e_{\pm}} & = X^e_{\pm} + A(X^e_{\pm}) \\ \nabla^S_{\partial_{\pm\pm}} & = \partial_{\pm\pm} + A(\partial_{\pm\pm}) \\ \nabla^S_{X^e_{---}} & = X^e_{---} + A(X^e_{---}) \ . \end{cases}$$

Definition 15 *A connection ∇^S over S_H is called* **almost 0-partially flat** *if its curvature satisfies the following equations:*

$$F(X^e_{+++}, X^{e'}_{+++}) = F(X^e_{+++}, v) = F(X^e_{\pm}, v) = 0 \quad , \quad \forall v \in T^v S_H \ ,$$
$$F(\partial_{++}, \cdot\,) = F(\partial_0, \cdot\,) = 0 \, . \tag{5.8}$$

Following the construction of almost half-flat connections, we may construct almost 0-partially flat connections, which allow deformation to a 0-partially flat connection. As in the case of a half-flat connection (c.f. Proposition 7), an almost 0-partially flat connection is completely determined by the potentials $A_{\pm\pm} =: A(\partial_{\pm\pm})$ with respect to an analytic frame.

Proposition 11 *Let ∇ be an almost 0-partially flat connection in the vector bundle $\pi^*\nu : \mathbb{C}^r \times S_H \to S_H$ with potentials A_{++}, A_{--}, $A(X^e_{\pm})$ and $A(X^e_{---})$ in an analytic frame. Then:*

(i) The potential A_{++} is analytic and has charge 2, i.e.

$$X^e_{+++} A_{++} = 0 \quad , \quad \partial_0 A_{++} = 2A_{++} \ . \tag{5.9}$$

(ii) The potential A_{--} satisfies

$$\partial_{++} A_{--} - \partial_{--} A_{++} + [A_{++}, A_{--}] = 0 \quad , \quad \partial_0 A_{--} = -2A_{--} \ . \tag{5.10}$$

(iii) The potentials $A(X^e_{\pm})$ and $A(X^e_{---})$ are then recursively determined as follows:

$$\begin{aligned} A(X^e_{+}) &= -\tfrac{1}{3} X^e_{+++} A_{--} \\ A(X^e_{-}) &= \tfrac{1}{2}\left(\partial_{--} A(X^e_{+}) - X^e_{+} A_{--} + [A_{--}, A(X^e_{+})]\right) \\ A(X^e_{---}) &= \partial_{--} A(X^e_{-}) - X^e_{-} A_{--} + [A_{--}, A(X^e_{-})] \end{aligned} \tag{5.11}$$

and they have charges $+1, -1$ and -3 respectively, i.e.

$$\partial_0 A(X^e_{\pm}) = \pm A(X^e_{\pm}) \quad , \quad \partial_0 A(X^e_{---}) = -3A(X^e_{---}) \ . \tag{5.12}$$

Conversely, any set of matrix-valued potentials A_{++}, A_{--}, $A(X^e_{\pm})$ and $A(X^e_{---})$ satisfying (5.9)–(5.12) define an almost 0-partially flat connection.

Proof: (i) The curvature constraints $F(X^e_+ \,,\, \partial_{++}) = 0$, $F(\partial_0 \,,\, \partial_{++}) = 0$, in an analytic frame, take the form (5.9).
(ii) The further almost 0-partial-flatness conditions, $F(\partial_{++} \,,\, \partial_{--}) = F(\partial_0 \,,\, \partial_{--}) = 0$ give equations (5.10) for the potential A_{--} .
(iii) Having obtained A_{--} , we can find $A(X^e_\pm)$ and $A(X^e_{---})$ from the equations

$$F(\partial_{--}, X^e_{+++}) = 0 \;\Leftrightarrow\; -X^e_{+++}A_{--} = A([\partial_{--}, X^e_{+++}]) = 3A(X^e_+)$$

$$F(\partial_{--}, X^e_+) = 0 \;\Leftrightarrow\; \partial_{--}A(X^e_+) - X^e_+ A_{--} + [A_{--}, A(X^e_+)] = A([\partial_{--}, X^e_+]) = 2A(X^e_-)$$

$$F(\partial_{--}, X^e_-) = 0 \;\Leftrightarrow\; \partial_{--}A(X^e_-) - X^e_- A_{--} + [A_{--}, A(X^e_-)] = A([\partial_{--}, X^e_-]) = A(X^e_{---})\,.$$

Equations (5.12) follow from (5.11). □

Now, using this proposition, a modification of Theorem 4 gives an algorithm for the construction of all almost 0-partially flat connections.

Theorem 7 *Let A_{++} be an analytic prepotential, i.e. a matrix-valued function on a domain $U = \pi^{-1}(V) \subset S_H$, $V \subset M$ a simply connected domain, satisfying* (5.9)*, and Φ an invertible matrix-valued function on U which satisfies the equations*

$$\partial_{++}\Phi = -A_{++}\Phi\,, \quad \partial_0\Phi = 0\,. \tag{5.13}$$

Such a function Φ always exists. Then the pair (A_{++}, Φ) determines an almost 0-partially flat connection $\nabla^S = \nabla^{(A_{++},\Phi)}$. Its potentials with respect to an analytic frame are given by A_{++}, $A_{--} = -(\partial_{--}\Phi)\Phi^{-1}$ and (5.11)*. Conversely, any almost 0-partially flat connection is of this form.*

The proof follows that for Theorem 4 and uses Proposition 11.

To deform an almost 0-partially flat connection into a 0-partially flat connection, we need to find a transformation from the above analytic frame to a central frame. Analogously to Lemma 5 we may prove:

Lemma 7 *Let $\nabla = \nabla^{(A_{++},\Phi)}$ be the almost 0-partially flat connection associated to the analytic prepotential A_{++} with respect to the analytic frame φ and an invertible solution Φ of* (5.13)*. Then the frame $\psi := \varphi\Phi$ is a central frame for the connection ∇, i.e. the potentials $C(\partial_{\pm\pm})$ and $C(\partial_0)$ with respect to that frame vanish.*

With respect to the central frame ψ, the above almost 0-partially flat connection then takes the form:

$$\begin{cases} \nabla^S_{X^e_{+++}} &= X^e_{+++} + C(X^e_{+++}) \;=\; X^e_{+++} + \Phi\, X^e_{+++}\, \Phi^{-1} \\ \nabla^S_{X^e_{\pm}} &= X^e_{\pm} + C(X^e_{\pm}) \\ \nabla^S_{X^e_{---}} &= X^e_{---} + C(X^e_{---}) \\ \nabla^S_{\partial_{++}} &= \partial_{++} \quad , \quad \nabla^S_{\partial_{--}} \;=\; \partial_{--} \quad , \quad \nabla^S_{\partial_0} \;=\; \partial_0 \,. \end{cases}$$

where in terms of the analytic frame potentials $A(X)$, the central frame potentials $C(X)$ are given by $C(X) = \Phi^{-1}A(X)\Phi + \Phi^{-1}(X\Phi)$. Moreover, the equations $F(\partial_{++}\,,\, X^e_{+++}) = F(\partial_0\,,\, X^e_{+++}) = 0$ imply that the potential $C(X^e_{+++})$ satisfies the equations

$$\partial_{++}C(X^e_{+++}) = 0\;, \quad \partial_0 C(X^e_{+++}) = 3C(X^e_{+++})\,. \tag{5.14}$$

The following proposition is analogous to Proposition 8 in the half-flat case.

Proposition 12 *The potential $C(X^e_{+++})$ of an almost 0-partially flat connection ∇ with respect to a central frame is cubic in u^α_+,*

$$C(X^e_{+++}) = u^\alpha_+ u^\beta_+ u^\gamma_+ C(\widetilde{X^e_{\alpha\beta\gamma}}) = u^\alpha_+ u^\beta_+ u^\gamma_+ C^e_{\alpha\beta\gamma}\;, \tag{5.15}$$

where the coefficients $C^e_{\alpha\beta\gamma} = C^e_{\alpha\beta\gamma}(x^i)$, symmetric in α, β, γ, are matrix valued functions of coordinates x^i on M and $(x^i, u^\alpha_\pm)$ are the local coordinates associated with the trivialisation $S_H = M \times \mathrm{Sp}(1, \mathbb{C})$.

With respect to a central frame, we can therefore write $\nabla_{X^e_{+++}} = X^e_{+++} + u^\alpha_+ u^\beta_+ u^\gamma_+ C^e_{\alpha\beta\gamma}$. Using $C^e_{\alpha\beta\gamma}$, we now define a new connection in $\pi^*\nu$ over S_H by

$$\begin{cases} \widehat{\nabla}_{X^e_{+++}} &= X^e_{+++} + u^\alpha_+ u^\beta_+ u^\gamma_+ C^e_{\alpha\beta\gamma} \\ \widehat{\nabla}_{X^e_{+}} &= X^e_{+} + u^\alpha_+ u^\beta_+ u^\gamma_- C^e_{\alpha\beta\gamma} \\ \widehat{\nabla}_{X^e_{-}} &= X^e_{-} + u^\alpha_- u^\beta_- u^\gamma_+ C^e_{\alpha\beta\gamma} \\ \widehat{\nabla}_{X^e_{---}} &= X^e_{---} + u^\alpha_- u^\beta_- u^\gamma_- C^e_{\alpha\beta\gamma} \\ \widehat{\nabla}_{\partial_{++}} &= \partial_{++} \quad , \quad \widehat{\nabla}_{\partial_{--}} \;=\; \partial_{--} \quad , \quad \widehat{\nabla}_{\partial_0} \;=\; \partial_0 \,. \end{cases}$$

The following theorem is the analogue of Theorem 5 in the half-flat case.

Theorem 8 *The constructed connection $\widehat{\nabla}$ is a 0-partially flat connection in $\pi^*\nu$ over S_H and it is the pull-back of the following 0-partially flat connection ∇^M in ν over M:*

$$\nabla^M_{X^e_{\alpha\beta\gamma}} = X^e_{\alpha\beta\gamma} + C^e_{\alpha\beta\gamma}\;.$$

Proof: As in Lemma 1 we may show that the connection $\widehat{\nabla}$ is the pull-back of the connection ∇^M. It then suffices to show that ∇^M is 0-partially flat. The connections ∇ and $\widehat{\nabla}$ coincide in the direction of X^e_{+++}. Hence, using $C^e_{+++} := u_+^{\alpha_1} u_+^{\alpha_2} u_+^{\alpha_3} C^e_{\alpha_1\alpha_2\alpha_3}$, we have

$$
\begin{aligned}
0 &= F^{\nabla}(X^e_{+++}, X^{e'}_{+++}) \\
&= F^{\widehat{\nabla}}(X^e_{+++}, X^{e'}_{+++}) \\
&= X^e_{+++} C^{e'}_{+++} - X^{e'}_{+++} C^e_{+++} + \left[C^e_{+++}\,,\, C^{e'}_{+++}\right] - C\big(\left[X^e_{+++}\,,\, X^{e'}_{+++}\right]\big) \\
&= u_+^{\alpha_1} u_+^{\alpha_2} u_+^{\alpha_3} u_+^{\beta_1} u_+^{\beta_2} u_+^{\beta_3} \Big(X^e_{\alpha_1\alpha_2\alpha_3} C^{e'}_{\beta_1\beta_2\beta_3} - X^{e'}_{\beta_1\beta_2\beta_3} C^e_{\alpha_1\alpha_2\alpha_3} + \left[C^e_{\alpha_1\alpha_2\alpha_3}\,,\, C^{e'}_{\beta_1\beta_2\beta_3}\right] \\
&\qquad\qquad - C\big(\left[X^e_{\alpha_1\alpha_2\alpha_3}\,,\, X^{e'}_{\beta_1\beta_2\beta_3}\right]\big)\Big) \\
&= u_+^{\alpha_1} u_+^{\alpha_2} u_+^{\alpha_3} u_+^{\beta_1} u_+^{\beta_2} u_+^{\beta_3} \; F^{\nabla^M}\big(X^e_{\alpha_1\alpha_2\alpha_3}\,,\, X^{e'}_{\beta_1\beta_2\beta_3}\big) \;,
\end{aligned}
$$

since $X^e_{+++} u_+^\beta = 0$. This shows that the component $\overset{(0)}{F} = 0$ in the decomposition (5.4), i.e. the connection ∇^M is 0-partially flat. □

5.3.2 Construction of 1-partially flat connections

Let M be a 1–admissible spin $\frac{3}{2}$ Grassmann manifold with a 1-partially flat connection ∇ (satisfying (5.6)) in a holomorphic vector bundle $W \to M$. The pull-back of such a connection to a connection in the bundle $\pi^* W \to S_H$, where $\pi : S_H \to M$, has curvature F with components given by:

$$
\begin{aligned}
F(X^e_{\pm\pm\pm}, X^{e'}_{\pm\pm\pm}) &= 0 \\
F(X^{e'}_{\pm\pm\pm}, X^e_{\pm}) &= 0 \\
F(X^e_{\pm}, X^{e'}_{\pm}) &= 0 \\
F(X^e_{+++}, X^{e'}_{---}) &= 36 \overset{(3)}{F}{}^{(ee')} \\
F(X^e_{\pm\pm\pm}, X^{e'}_{\mp}) &= 0 \\
F(X^e_{+}, X^{e'}_{-}) &= -12 \overset{(3)}{F}{}^{(ee')} \\
F(v\,,\,\cdot) &= 0\,,
\end{aligned}
\qquad (5.16)
$$

where v is any vertical vector field on S_H. A connection in a holomorphic vector bundle $W \to S_H$ is **1-partially flat** if its curvature satisfies the above equations. The restriction of a 1-partially flat connection to a leaf of the integrable distribution $\langle \mathcal{D}_{3+}, \mathcal{D}_+, \partial_0 \rangle$ is clearly flat. In this case, an **analytic frame** in the holomorphic vector bundle $\pi^*\nu : \mathbb{C}^r \times S_H \to S_H$

is a frame which is parallel along leaves of this distribution. With respect to such a frame, a connection in the vector bundle $\pi^*\nu$ can be written as

$$\begin{cases} \nabla^S_{\partial_0} &= \partial_0 \\ \nabla^S_{X^e_{+++}} &= X^e_{+++} \\ \nabla^S_{X^e_{+}} &= X^e_{+} \\ \nabla^S_{\partial_{++}} &= \partial_{++} + A_{++} \\ \nabla^S_{\partial_{--}} &= \partial_{--} + A_{--} \\ \nabla^S_{X^e_{---}} &= X^e_{---} + A(X^e_{---}) \\ \nabla^S_{X^e_{-}} &= X^e_{-} + A(X^e_{-}) \ , \end{cases}$$

with potentials $A(X^e_{+++}) = A(X^e_{+}) = A(\partial_0) = 0$. We look for solutions of the system (5.16) in this analytic gauge.

Definition 16 *A connection ∇^S over S_H is called* **almost 1-partially flat** *if its curvature satisfies the equations:*

$$F(X^e_{+++}, X^{e'}_{+++}) = F(X^e_{+++}, X^{e'}_{+}) = F(X^e_{+}, X^{e'}_{+}) = 0$$
$$F(X^e_{+++}, v) = F(X^e_{\pm}, v) = F(\partial_{++}, \cdot\,) = F(\partial_0, \cdot\,) = 0 \quad , \quad \forall v \in T^v S_H \ . \quad (5.17)$$

In virtue of these equations, the potentials $A_{\pm\pm} =: A(\partial_{\pm\pm})$ determine all other potentials:

Proposition 13 *Let ∇ be an almost 1-partially flat connection in the vector bundle $\pi^*\nu : \mathbb{C}^r \times S_H \to S_H$ with potentials A_{++}, A_{--}, $A(X^e_{-})$ and $A(X^e_{---})$ in an analytic frame. Then:*
(i) The potential A_{++} is analytic and has charge 2, i.e.

$$X^e_{+++} A_{++} = 0 \quad , \quad X^e_{+} A_{++} = 0 \quad , \quad \partial_0 A_{++} = 2A_{++} \ . \quad (5.18)$$

(ii) The potential A_{--} satisfies

$$\partial_{++} A_{--} - \partial_{--} A_{++} + [A_{++}, A_{--}] = 0 \quad , \quad \partial_0 A_{--} = -2A_{--} \ . \quad (5.19)$$

(iii) The potentials $A(X^e_{-})$ and $A(X^e_{---})$ are then recursively determined as follows:

$$\begin{aligned} A(X^e_{-}) &= -\tfrac{1}{2} X^e_{+} A_{--} \\ A(X^e_{---}) &= \partial_{--} A(X^e_{-}) - X^e_{-} A_{--} + [A_{--}, A(X^e_{-})] \end{aligned} \quad (5.20)$$

and they have charges -1 and -3 respectively; i.e.

$$\partial_0 A(X^e_{-}) = \pm A(X^e_{-}) \quad , \quad \partial_0 A(X^e_{---}) = -3A(X^e_{---}) \ . \quad (5.21)$$

Conversely, any set of matrix-valued potentials A_{++}, A_{--}, $A(X^e_{-})$ and $A(X^e_{---})$ satisfying (5.18)–(5.21) *define an almost 1-partially flat connection.*

Proof: (i) Equations (5.18) are equivalent to $F(\partial_{++}, X^e_{+++}) = F(\partial_{++}, X^e_{+}) = F(\partial_0, \partial_{++})=0$.

(ii) The further almost 1-partial-flatness conditions, $F(\partial_{++}\,,\, \partial_{--}) = F(\partial_0\,,\, \partial_{--}) = 0$ give equations (5.19).

(iii) Having obtained A_{--}, we can find $A(X^e_{-})$ and $A(X^e_{---})$ from the equations

$$F(\partial_{--}, X^e_{+}) = 0 \quad \Leftrightarrow \quad -X^e_{+} A_{--} = A([\partial_{--}, X^e_{+}]) = 2A(X^e_{-})$$

$$F(\partial_{--}, X^e_{-}) = 0 \quad \Leftrightarrow \quad \partial_{--}A(X^e_{-}) - X^e_{-} A_{--} + [A_{--}, A(X^e_{-})] = A([\partial_{--}, X^e_{-}]) = A(X^e_{---})\,.$$

The equations (5.21) follow from (5.20). □

Now, starting from a prepotential A_{++}, which solves (5.18), we may construct an almost 1-partially flat connection. The potential $A_{--} = -(\partial_{--}\Phi)\Phi^{-1}$ is determined, as before, from a solution Φ of (5.13). Then, with the remain potentials in an analytic frame being given by (5.20) and satisfying (5.21), all the other equations in (5.17) follow. This shows that a almost 1-partially flat connection is determined by an arbitrary analytic prepotential A_{++} and an invertible solution Φ of (5.13). As before, Φ is a transition function from an analytic frame to a central frame, in which the above almost 1-partially flat connection takes the form:

$$\begin{cases} \nabla^S_{X^e_{+++}} & = \; X^e_{+++} + C(X^e_{+++}) \; = \; X^e_{+++} + \Phi\, X^e_{+++}\, \Phi^{-1} \\ \nabla^S_{X^e_{+}} & = \; X^e_{+} + C(X^e_{+}) \; = \; X^e_{+} + \Phi\, X^e_{+}\, \Phi^{-1} \\ \nabla^S_{X^e_{-}} & = \; X^e_{-} + C(X^e_{-}) \\ \nabla^S_{X^e_{---}} & = \; X^e_{---} + C(X^e_{---}) \\ \nabla^S_{\partial_{++}} & = \; \partial_{++} \quad , \quad \nabla^S_{\partial_{--}} = \partial_{--} \quad , \quad \nabla^S_{\partial_0} = \partial_0\ . \end{cases}$$

Moreover, the equations $F(\partial_{++}\,,\, X^e_{+++}) = F(\partial_0\,,\, X^e_{+++}) = 0$ imply that the potential $C(X^e_{+++})$ satisfies the equations

$$\partial_{++}C(X^e_{+++}) = 0\ , \quad \partial_0 C(X^e_{+++}) = 3C(X^e_{+++})\ .$$

Proposition 14 *The potentials $C(X^e_{+++})$ and $C(X^e_{+})$ of an almost 1-partially flat connection ∇ with respect to a central frame have the form,*

$$C(X^e_{+++}) = u^\alpha_+ u^\beta_+ u^\gamma_+ C^e_{\alpha\beta\gamma} \quad , \quad C(X^e_{+}) = u^\alpha_+ u^\beta_+ u^\gamma_- C^e_{\alpha\beta\gamma}\,,$$

where $C^e_{\alpha\beta\gamma}$ is a function on M, symmetric in α, β, γ.

With respect to a central frame, we can therefore write

$$\nabla_{X^e_{+++}} = X^e_{+++} + u^\alpha_+ u^\beta_+ u^\gamma_+ C^e_{\alpha\beta\gamma} \quad , \quad \nabla_{X^e_{+}} = X^e_{+} + u^\alpha_+ u^\beta_+ u^\gamma_- C^e_{\alpha\beta\gamma}\,.$$

We define a modified connection in the bundle $\pi^*\nu$ over S_H by

$$\left\{\begin{array}{lcl} \widehat{\nabla}_{X^e_{+++}} & = & X^e_{+++} + u^\alpha_+ u^\beta_+ u^\gamma_+ C^e_{\alpha\beta\gamma} \\ \widehat{\nabla}_{X^e_{+}} & = & X^e_{+} + u^\alpha_+ u^\beta_+ u^\gamma_- C^e_{\alpha\beta\gamma} \\ \widehat{\nabla}_{X^e_{-}} & = & X^e_{-} + u^\alpha_- u^\beta_- u^\gamma_+ C^e_{\alpha\beta\gamma} \\ \widehat{\nabla}_{X^e_{---}} & = & X^e_{---} + u^\alpha_- u^\beta_- u^\gamma_- C^e_{\alpha\beta\gamma} \\ \widehat{\nabla}_{\partial_{++}} & = & \partial_{++} \quad , \quad \widehat{\nabla}_{\partial_{--}} \;=\; \partial_{--} \quad , \quad \widehat{\nabla}_{\partial_0} \;=\; \partial_0 \,. \end{array}\right. \tag{5.22}$$

As in the 0-partially flat case, we have:

Theorem 9 *The constructed connection $\widehat{\nabla}$ is a 1-partially flat connection in $\pi^*\nu$ over S_H and it is the pull-back of the following 1-partially flat connection ∇^M in ν over M:*

$$\nabla^M_{X^e_{\alpha\beta\gamma}} = X^e_{\alpha\beta\gamma} + C^e_{\alpha\beta\gamma}\,. \tag{5.23}$$

Proof: As before one shows that the connection $\widehat{\nabla}$ is the pull-back of the connection ∇^M. It remains to show that ∇^M is 1-partially flat. Since any almost 1-partially flat connection is almost 0-partially flat we have $\overset{(0)}{F}= 0$ by Theorem 8. Next we show that $\overset{(1)}{F}= 0$. The connections ∇ and $\widehat{\nabla}$ coincide in the direction of X^e_{+++} and X^e_{+}. Hence, using equation (5.7), which holds for 0-partially flat connections, we have

$$\begin{array}{rcl} 0 & = & F^{\nabla}(X^e_{+++}, X^{e'}_{+}) \\ & = & F^{\widehat{\nabla}}(X^e_{+++}, X^{e'}_{+}) \\ & = & 12 \, \overset{(1)}{F}{}^{(ee')}_{\alpha_1\alpha_2\beta_1\beta_2} u^{\alpha_1}_+ u^{\alpha_2}_+ u^{\beta_1}_+ u^{\beta_2}_+ \,. \end{array}$$

This shows that the component $\overset{(1)}{F}$ in the decomposition (5.4) vanishes. Similarly,

$$\begin{array}{rcl} 0 & = & F^{\nabla}(X^e_{+}, X^{e'}_{+}) \\ & = & F^{\widehat{\nabla}}(X^e_{+}, X^{e'}_{+}) \\ & = & -8 \, \overset{(2)}{F}{}^{[ee']}_{\alpha_1\beta_1} u^{\alpha_1}_+ u^{\beta_1}_+ \,, \end{array}$$

implies $\overset{(2)}{F}{}^{[ee']} = 0$, and hence that $\widehat{\nabla}$ is 1-partially flat. □

By Theorem 6, the 1-partially flat connection $\nabla^M_{X^e_{\alpha\beta\gamma}}$ in (5.23) is a Yang-Mills connection.

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