General existence of minimal surfaces with prescribed flux

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Introduction

Let $x : \mathbf{C} \cup \{\infty\} \setminus \{q_1, \dots, q_n\} \to \mathbf{R}^3$ be a complete conformal minimal immersion. For each end q_j ($j = 1, \dots, n$) of x, the *flux vector* is defined by

$$\varphi_j := \int_{\gamma_j} \vec{n}\, ds,$$

where γ_j is a positively oriented curve surrounding q_j, and $\vec{n}$ the conormal such that $(\gamma', \vec{n})$ is positively oriented. It is well known that the flux vectors satisfy a "balancing" condition so called the *flux formula*

$$\sum_{j=1}^{n} \varphi_j = 0.$$

The minimal immersion x is called an *n-end catenoid* if each end q_j is of catenoid type. The catenoid and the Jorge-Meeks surfaces [JM] are typical ones. Recently, new examples of n-end catenoids have been found by [Kar], [L], [Xu], [Ross1], [Ross2], [Kat] and [UY]. For any n-end catenoid x, each flux vector φ_j is proportional to the limit normal vector $\nu(q_j)$ with respect to the end q_j, and the scalar $w(q_j) := \varphi_j / 4\pi\nu(q_j)$ is called the *weight* of the end q_j. In this case, the flux formula can be rewritten as follows.

$$\sum_{j=1}^{n} 4\pi\, w(q_j)\, \nu(q_j) = 0.$$

It should be remarked that $w(q_j)$ may take a negative value.

We consider the inverse problem of the flux formula proposed in [Kat] and [KUY] as follows:

Problem. *For given unit vectors $v := \{v_1, \dots, v_n\}$ in $\mathbf{R}^3$, and nonzero real numbers $a := \{a^1, \dots, a^n\}$ satisfying $\sum_{j=1}^{n} a^j v_j = 0$ (we call such a pair (v, a) flux data), is there a (non-branched) n-end catenoid $x : \mathbf{C} \cup \{\infty\} \setminus \{q_1, \dots, q_n\} \to \mathbf{R}^3$ such that $\nu(q_j) = v_j$ and a_j is the weight at the end q_j?*

We remark that Kusner also proposed a similar question (see [Ross1]). Rosenberg and Toubiana [RT] found solutions with branch points in the category that the Gauss map is of degree 1. But if one wishes non-branched solutions, the degree of their Gauss map must be $n-1$, which is the case just treated in this paper.

The problem is not exactly affirmative. By the classification of Lopez [L], we can see that the answer for $n \leq 3$ is "Yes" except for the case when two of $\{v_j\}_{j=1}^n$ coincide. Moreover, for $n \geq 4$, some obstructions exist as closed conditions in the space of flux data as shown in our previous paper [KUY]. In spite of these obstructions, the authors also showed in [KUY] that the inverse problem is true for almost all flux data (v, a) when $n = 4$. In this paper, we treat the case $n \geq 5$ and show the following theorem:

Theorem. *For each integer $n \geq 3$, the problem is solved for almost all flux data.*

In Section 1, we reduce the inverse problem to seeking a sampling point satisfying certain non-degeneracy conditions. Two lemmas in Appendix A are applied to complete the reduction. In Section 2, we shall give a proof of Theorem. However, required technical calculations are done in Section 3 and Appendix B.

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1. Reduction

As shown in the previous paper, the inverse problem of the flux formula reduces to finding solutions of a system of algebraic equations:

Theorem 1.1. ([KUY]) *Let (v, a) be a pair of unit vectors $v = \{v_1, \dots, v_n\}$ $(n \geq 4)$ in $\mathbf{R}^3$ and nonzero real numbers $a = \{a^1, \dots, a^n\}$ satisfying the balancing condition:*

$$\sum_{j=1}^{n} a^j v_j = 0. \tag{1.1}$$

Then there is an evenly branched n-end catenoid $x : \mathbf{C} \cup \{\infty\} \setminus \{q_1, \dots, q_n\} \to \mathbf{R}^3$ $(q_j \neq \infty)$ such that the induced metric is complete at the end q_j, $\nu(q_j) = v_j$ and a^j is the weight at the end q_j $(j = 1, \dots, n)$, if and only if there exist complex numbers $b^1, \dots, b^n$ satisfying the following conditions:

$$\begin{cases} b^j \displaystyle\sum_{\substack{k=1 \\ k \neq j}}^{n} b^k \frac{p_j - p_k}{q_j - q_k} = a^j & (1.2) \\ b^j \displaystyle\sum_{\substack{k=1 \\ k \neq j}}^{n} b^k \frac{\overline{p_j} p_k + 1}{q_j - q_k} = 0 & (1.3) \end{cases} \qquad (j = 1, \dots, n),$$

where $p_j := \sigma(v_j)$, $\sigma : S^2 \to \mathbf{C} \cup \{\infty\}$ *is the stereographic projection, and we assume* $p_j \neq \infty$.

Moreover, the surface x *has no branch points if and only if the two the polynomials*

$$Q(z) := \sum_{j=1}^{n} b^j \prod_{\substack{k=1 \\ k \neq j}}^{n} (z - q_k), \tag{1.4}$$

$$P(z) := \sum_{j=1}^{n} p_j b^j \prod_{\substack{k=1 \\ k \neq j}}^{n} (z - q_k) \tag{1.5}$$

are mutually prime and one of them has degree $n - 1$.

Remark 1.2. When $p_j = rq_j$, the theorem reduces to the results in the first author [Kat]. In this case the system (1.2) and (1.3) reduces to

$$\begin{cases} rb^j \displaystyle\sum_{\substack{k=1 \\ k \neq j}}^{n} b^k = a^j \\ b^j \displaystyle\sum_{\substack{k=1 \\ k \neq j}}^{n} b^k \frac{|r|^2 \overline{q_j} q_k + 1}{q_j - q_k} = 0 \end{cases} \qquad (j = 1, \ldots, n).$$

As seen in [Kat], the surface has no branch point if and only if $\beta := \sum_{j=1}^{n} b^j \neq 0$. By using the relation $P(z)/Q(z) = rz - r\beta / \left(\sum_{j=1}^{n} b^j/(z - q_j)\right)$, it is also checked directly from the last condition of the theorem.

Remark 1.3. The position of the ends $\{q_1, \ldots, q_n\}$ in the source domain $\mathbf{C} \cup \{\infty\}$ has the freedom of Möbius transformations. For example, the following normalization is possible:

$$q_1 = 1, \qquad q_{n-1} + q_{n-2} = 0, \qquad q_n = 0.$$

Remark 1.4. The system of the equations (1.2) and (1.3) has another expression

$$\begin{cases} b^j \displaystyle\sum_{\substack{k=1 \\ k \neq j}}^{n} b^k \frac{1}{q_j - q_k} = a^j \frac{\overline{p_j}}{|p_j|^2 + 1}, & (1.6) \\ b^j \displaystyle\sum_{\substack{k=1 \\ k \neq j}}^{n} b^k \frac{p_j + p_k}{q_j - q_k} = a^j \frac{|p_j|^2 - 1}{|p_j|^2 + 1}. & (1.7) \end{cases}$$

Moreover we may replace (1.6) by

$$p_j b^j \sum_{\substack{k=1 \\ k\neq j}}^{n} b^k \frac{p_k}{q_j - q_k} = -a^j \frac{p_j}{|p_j|^2+1}. \tag{1.8}$$

In fact, if we set

$$\gamma_j := b^j \sum_{\substack{k=1 \\ k\neq j}}^{n} b^k \frac{1}{q_j - q_k}, \quad \delta_j := b^j \sum_{\substack{k=1 \\ k\neq j}}^{n} b^k \frac{p_k}{q_j - q_k} \qquad (j = 1, \dots, n),$$

then (1.2) and (1.3) are written as

$$p_j\gamma_j - \delta_j = a^j, \qquad \gamma_j + \overline{p_j}\delta_j = 0.$$

It is equivalent to the relations

$$\gamma_j = a^j \frac{\overline{p_j}}{|p_j|^2+1}, \qquad p_j\gamma_j + \delta_j = a^j \frac{|p_j|^2-1}{|p_j|^2+1},$$

that is (1.6) and (1.7). On the other hand,

$$p_j\gamma_j = a^j \frac{|p_j|^2}{|p_j|^2+1} = a^j \frac{|p_j|^2-1}{|p_j|^2+1} + \frac{a^j}{|p_j|^2+1} = p_j\gamma_j + \delta_j + \frac{a^j}{|p_j|^2+1},$$

which yields (1.8).

Theorem 1.1 produces many n-end catenoids as seen in [Kat] and [KUY]. First, we fix our attention to the equation (1.3). We consider a matrix

$$A_p := \left(\frac{\overline{p_j} p_k + 1}{q_j - q_k} \right)_{j,k=1,\dots,n}, \tag{1.9}$$

where the diagonal components are interpreted as 0. Then the vector ${}^t(b^1, \dots, b^n)$ belongs to the kernel of the matrix A_p. As shown in the later sections, it is reasonable to expect that the rank of the matrix A_p is generically $n-1$. In this case, ${}^t(b^1, \dots, b^n)$ should be proportional to any column vector of the cofactor matrix $\widetilde{A}_p$ of A_p. (By the definition, $A_p\widetilde{A}_p = \widetilde{A}_pA_p = (\det A_p)I$ holds.) So we set

$$b_p(q) = {}^t(b_p^1(q), \dots, b_p^n(q)) := \text{the } n\text{-th column of the cofactor matrix } \widetilde{A}_p(q).$$

Now we projectify the problem: For fixed $p := (p_1, \dots, p_n) \in \mathbf{C}$, define a rational map between two complex projective spaces

$$\mathcal{F}\!\ell_p = [f_p^1, \dots, f_p^n] : \mathbf{P}^{n-1} \to \mathbf{P}^{n-1}$$

by

$$f_p^j(q_1,\dots,q_n) := b_p^j(q)\sum_{k\neq j} b_p^k(q)\frac{p_j-p_k}{q_j-q_k} \qquad (j=1,\dots,n). \tag{1.10}$$

We set

$$f\ell_p^j(q) := \Delta(q)^5\cdot f_p^j(q),$$

where $\Delta(q)$ is the difference product defined by

$$\Delta(q_1,\dots,q_n) := \prod_{j>k}^{n}(q_j-q_k). \tag{1.11}$$

It is easily seen that each $f\ell_p^j$ is a homogeneous polynomial in $q_1,\dots,q_n$ and $\mathcal{F}\ell_p$ has another expression

$$\mathcal{F}\ell_p = [f\ell_p^1,\dots,f\ell_p^n].$$

This projective formulation is reasonable in the following two senses:

- Any homothety of n-end catenoids changes their weights $(a^1,\dots,a^n)$ only by a constant multiplication. It allows us to projectify the image of $\mathcal{F}\ell_p$.
- Changing coordinates of n-end catenoids by homothetic transformations corresponds to complex multiplications of $(q_1,\dots,q_n)$. (See Remark 1.3.) It allows us to projectify the domain of $\mathcal{F}\ell_p$.

Since p_j is the stereographic image of v_j, the balancing condition (1.1) is rewritten as

$$\sum_{j=1}^{n}\frac{|p_j|^2-1}{|p_j|^2+1}a^j = 0, \qquad \sum_{j=1}^{n}\frac{\overline{p_j}}{|p_j|^2+1}a^j = 0.$$

We define a subspace $\mathcal{W}_p^{n-4}$ in $\mathbf{P}^{n-1}$ by

$$\mathcal{W}_p^{n-4} := \left\{[a^1,\dots,a^n]\in\mathbf{P}^{n-1}\,;\sum_{j=1}^{n}\frac{|p_j|^2-1}{|p_j|^2+1}a^j=0,\sum_{j=1}^{n}\frac{\overline{p_j}}{|p_j|^2+1}a^j=0,\sum_{j=1}^{n}\frac{p_j}{|p_j|^2+1}a^j=0\right\}.$$

We will show that for open dense $p\in\mathbf{C}^n$, the image of the map $\mathcal{F}\ell_p$ is open dense in $\mathcal{W}_p^{n-4}$, and next show that it covers open dense subset of the totally real set $\mathcal{W}_{\mathbf{R}} = \{[a]\in\mathcal{W}_p^{n-4};\, a_j\in\mathbf{R}\}$. Then the image of the map $\mathcal{F}\ell_p$ contains $[a]\in\mathcal{W}_{\mathbf{R}}$ for almost all flux data (p,a), and Theorem in Introduction is obtained. If $\mathcal{F}\ell_p$ is a holomorphic map and there is a point at which the rank of $d\mathcal{F}\ell_p$ is $n-4$, the surjectivity of the map follows by the proper mapping theorem. (See [GR].) But unfortunately, the map $\mathcal{F}\ell_p$ is singular on the set $\bigcap_{j=1}^{n} Z(f\ell_p^j)$, where $Z(f\ell_p^j)$ is the set of zeros of $f\ell_p^j$. As shown below, we will overcome this difficulty by a usual blowing up process.

From here, assume $\dim\langle v_1,\dots,v_n\rangle = 3$, where $v_j := \sigma^{-1}(p_j)$ and σ is the stereographic projection. Then clearly $\dim\mathcal{W}_p^{n-4} = n-4$. We remark here that $\dim\mathcal{W}_p^{n-4} = n-4$ holds for open dense $p\in\mathbf{C}^n$. Now we have the following lemma:

Lemma 1.5. *For each $p \in \mathbf{C}^n$, the following relation holds:*

$$\mathcal{F}\ell_p\Big(Z(\lambda_p) \setminus \bigcap_{j=1}^{n} Z(f\ell_p^j)\Big) \subset \mathcal{W}_p^{n-4},$$

where λ_p is the determinant of the matrix $\Delta \cdot A_p$ and $Z(\lambda_p)$ is the set of zeros of the homogeneous polynomial λ_p.

(Proof.) Let $q \in \mathrm{Z}(\lambda_p) \setminus \bigcap_{j=1}^{n} \mathrm{Z}(f\ell_p^j)$. If $\Delta(q) = 0$, then it is easy to see that $q \in \bigcap_{j=1}^{n} \mathrm{Z}(f\ell_p^j)$. Hence $\Delta(q) \neq 0$, and we get (1.2) with $b^j = b^j(q)$ $(j = 1, \ldots, n)$. Recall Remark 1.4. Then the assertion of the lemma immediately follows by summing up (1.7), (1.6) and (1.8) for $j = 1, \ldots, n$. (q.e.d.)

We define an $(n-1)$-matrix J_p by

$$(1.12) \quad J_p := \left((f_p^n)^2 \left\{ \frac{\partial \det A_p}{\partial q_n} \cdot \frac{\partial \mathring{f}_p^k}{\partial q_j} - \frac{\partial \det A_p}{\partial q_j} \cdot \frac{\partial \mathring{f}_p^k}{\partial q_n} \right\} \right)_{j,k=1,\ldots,n-1},$$

where

$$\mathring{f}_p^j := \frac{f_p^j}{f_p^n} \qquad (j = 1, \ldots, n-1).$$

The matrix J_p has a direct expression

$$J_p = \left(\frac{\partial \det A_p}{\partial q_n} \cdot \left\{ \frac{\partial f_p^k}{\partial q_j} \cdot f_p^n - f_p^k \cdot \frac{\partial f_p^n}{\partial q_j} \right\} - \frac{\partial \det A_p}{\partial q_j} \cdot \left\{ \frac{\partial f_p^k}{\partial q_n} \cdot f_p^n - f_p^k \cdot \frac{\partial f_p^n}{\partial q_n} \right\} \right)_{j,k=1,\ldots,n-1}.$$

The following proposition plays an important role to establish Theorem in Introduction.

Proposition 1.6. *Suppose that there exist $u_0 \in \mathbf{C}^n$ and a point $c = [c_1, \ldots, c_n] \in \mathbf{P}^n$ satisfying the following conditions:*

(1) *$c_1, \ldots, c_n$ are all distinct;*

(2) *The rank of the matrix $A_{u_0}(c)$ is $n-1$;*

(3) *$\frac{\partial \det A_{u_0}}{\partial q_n}$ does not vanish at $q = c$;*

(4) *The rank of the matrix $J_{u_0}(c)$ is $n-4$;*

(5) *Two polynomials $P(z)$ and $Q(z)$ defined in* (1.5) *and* (1.4) *associated with the data $(q,p) = (c, u_0)$ and $b = b_{u_0}(c)$ are mutually prime and one of them has degree $n-1$;*

(6) $f^j_{u_0}(c) \neq 0$ $(j = 1, \ldots, n)$;

(7) $c_j \neq 0$ $(j = 1, \ldots, n-1)$.

Then there exists an open dense subset $U \subset \mathbf{C}^n$ and an open dense subset Ω_p of the totally real set $\mathcal{W}_{\mathbf{R}} = \{[a] \in \mathcal{W}_p^{n-4}\,;\, a_j \in \mathbf{R}\}$ such that, for any $p \in U$ and $[a] \in \Omega_p$, there exists an (non-branched) n-end catenoid with the flux data (p, a).

By the proposition, the inverse problem of the flux formula can be solved for almost all flux data if one succeeds to take such a point c. This will be done in the next section. The outline of the proof of the proposition is as follows.

By the condition (4), at least one $(n-4)$-submatrix S_{u_0} of J_{u_0} is of rank $n-4$. Let $1 \leq j_1 < j_2 < \cdots < j_{n-4} < n$ be the indices of the columns of the submatrix S_{u_0}, and $\{m_1, m_2, m_3\}$ their complement, namely $\{m_1, m_2, m_3\} = \{1, \ldots, n-1\} \setminus \{j_1, \ldots, j_{n-4}\}$. By Remark 1.3, we may restrict the flux map into the following subspace of $\mathbf{P}^{n-1}$ containing the sampling point c:

$$\mathcal{V}^{n-3} := \{[q_1, \ldots, q_n] \in \mathbf{P}^{n-1}\,;\, c_{m_2} q_{m_1} - c_{m_1} q_{m_2} = 0,\ c_{m_3} q_{m_1} - c_{m_1} q_{m_3} = 0\}.$$

Now we define a homogeneous polynomial in $q_1, \ldots, q_n$ by

$$H_p(q) := \Delta(q)^2 \frac{\partial \det A_p}{\partial q_n}(q) \cdot \det\left(\Delta(q)^\ell S_p(q)\right) \cdot R_p(q) \cdot \prod_{j=1}^{n} f\ell^j_p(q) \cdot \prod_{k=1}^{n-1} q_k,$$

where ℓ is chosen sufficiently large so that $\det(\Delta(q)^\ell S_p(q))$ is a homogeneous polynomial in $q_1, \ldots, q_n$, and R_p is the resultant of the two polynomials $P(z)$ and $Q(z)$ of degree $n-1$ defined by (1.5) and (1.4). (It can be easily shown that R_p is also a homogeneous polynomial with respect to q. Or one may replace R_q by the resultant of $P(q_1 z)$ and $Q(q_1 z)$.) Then by the conditions (1)-(7), $c \in \mathcal{V}^{n-3}$ satisfies $H_{u_0}(c) \neq 0$. We prove the following

Lemma 1.7. *The subset*

$$U := \{p \in \mathbf{C}^n\,;\, Z(\lambda_p) \cap \mathcal{V}^{n-3} \not\subset Z(H_p)\}$$

is open dense in $\mathbf{C}^n$, where $\lambda_p = \det(\Delta \cdot A_p)$ is the homogeneous polynomial defined in Lemma 1.5.

(Proof.) Obviously U is an open subset of $\mathbf{C}^n$. Suppose that U is not dense in $\mathbf{C}^n$. Then there exists an open subset V such that

$$\mathrm{Z}(\lambda_p|_{\mathcal{V}^{n-3}}) \subset \mathrm{Z}(H_p|_{\mathcal{V}^{n-3}}) \qquad (p \in V). \tag{1.13}$$

Since $\mathcal{V}^{n-3} \cong \mathbf{P}^{n-3}$, by Lemma A.1 in Appendix, (1.13) holds for any $p \in \mathbf{C}^n$ such that $\lambda_p \not\equiv 0$. But this contradicts the fact that $\lambda_{u_0}(c) = 0$, $\lambda_{u_0} \not\equiv 0$ and $H_{u_0}(c) \neq 0$. (q.e.d.)

Roughly speaking, if $\mathcal{F}\ell_p$ has no singularities and is of maximal rank, then it is surjective and we can find a pair $(q, b_p(q))$ satisfying (1.2) and (1.3). But unfortunately, $\mathcal{F}\ell_p$ has singularities on $\bigcap_{j=1}^n \mathrm{Z}(f\ell_p^j)$. For this reason, we define a new variety $\hat{\mathcal{V}}^{n-3}$ and a map $\widehat{\mathcal{F}\ell_p}$: $\hat{\mathcal{V}}^{n-3} \to \mathcal{W}_p^{n-4}$ instead of $\mathcal{V}^{n-3}$ and $\mathcal{F}\ell_p$ as follows. First we consider an algebraic set

$$\begin{aligned}\mathcal{Y}^{n-3} = \Big\{ &([q_1, \dots, q_n], [a^1, \dots, a^n]) \in \mathbf{P}^{n-1} \times \mathbf{P}^{n-1} \, ; \\ &c_{m_2} q_{m_1} - c_{m_1} q_{m_2} = 0, \; c_{m_3} q_{m_1} - c_{m_1} q_{m_3} = 0, \\ &a^j f\ell_p^k = a^k f\ell_p^j \qquad (j, k = 1, \dots, n), \\ &\sum_{j=1}^n \frac{|p_j|^2 - 1}{|p_j|^2 + 1} a^j = 0, \sum_{j=1}^n \frac{p_j}{|p_j|^2 + 1} a^j = 0, \sum_{j=1}^n \frac{\overline{p_j}}{|p_j|^2 + 1} a^j = 0 \\ &\qquad\qquad (j = 1, \dots, n) \Big\},\end{aligned}$$

and define two canonical projections:

$$\begin{aligned}\pi &: \mathcal{Y}^{n-3} \ni ([q], [a]) \mapsto [q] \in \mathcal{V}^{n-3}, \\ \pi' &: \mathcal{Y}^{n-3} \ni ([q], [a]) \mapsto [a] \in \mathcal{W}_p^{n-4}.\end{aligned}$$

These two projections are both well-defined on $\mathcal{Y}^{n-3}$. Let $\hat{\mathcal{V}}^{n-3}$ be the algebraic closure of the set

$$\hat{\mathcal{V}}_{\mathrm{reg}}^{n-3} := \pi^{-1} \left(\mathcal{V}^{n-3} \setminus \bigcap_{j=1}^n \mathrm{Z}(f\ell_p^j) \right). \tag{1.14}$$

We denote the restriction of the first projection π to $\hat{\mathcal{V}}^{n-3}$ also by π. We remark that $\pi|_{\hat{\mathcal{V}}_{\mathrm{reg}}^{n-3}} : \hat{\mathcal{V}}_{\mathrm{reg}}^{n-3} \to \mathcal{V}^{n-3} \setminus \bigcap_{j=1}^n \mathrm{Z}(f\ell_p^j)$ is bijective. On the other hand, we denote the restriction of the second projection π' to $\hat{\mathcal{V}}^{n-3}$ by

$$\widehat{\mathcal{F}\ell_p} : \hat{\mathcal{V}}^{n-3} \to \mathcal{W}_p^{n-4}.$$

The map $\mathcal{F}\ell_p \circ \pi$ is well-defined on $\hat{\mathcal{V}}_{\mathrm{reg}}^{n-3}$, and coincides with the map $\widehat{\mathcal{F}\ell_p}$.

Lemma 1.8. *For each $p \in U$ satisfying $\dim \mathcal{W}_p^{n-4} = n - 4$, there exists an irreducible component $\hat{X}^{n-4}$ of the algebraic set $Z(\lambda_p \circ \pi) \cap \hat{\mathcal{V}}^{n-3}$ such that $H_p \circ \pi$ is not identically zero on $\hat{X}^{n-4}$. In addition, the restriction of the lifted flux map $\widehat{\mathcal{F}\ell_p}|_{\hat{X}^{n-4}} : \hat{X}^{n-4} \to \mathcal{W}_p^{n-4}$ is surjective.*

(Proof.) Suppose that $Z(\lambda_p \circ \pi) \cap \hat{\mathcal{V}}^{n-3} \subset Z(H_p \circ \pi)$. Since H_p is identically zero on the singular set $\bigcap_{j=1}^n Z(f\ell_p^j)$, it follows that

$$Z(\lambda_p) \cap \mathcal{V}^{n-3} \subset Z(H_p).$$

But this contradicts Lemma 1.7. Hence there exists an irreducible component $\hat{X}^{n-4}$ of the algebraic set $Z(\lambda_p \circ \pi) \cap \hat{\mathcal{V}}^{n-3}$ such that $H_p \circ \pi$ is not identically zero on $\hat{X}^{n-4}$. We set

$$X^{n-4} := \pi(\hat{X}^{n-4}).$$

Now we take a point $x_0 \in X^{n-4}$ such that $H_p(x_0) \neq 0$. Consequently, we have $x_0 \notin \bigcap_{j=1}^n Z(f\ell_p^j)$ and so $\mathcal{F}\ell_p(x_0) \in \mathcal{W}_p^{n-4}$ exists. We remark here that m_1-th component of x_0 in the homogeneous coordinate is not equal to zero. Now we take a coordinate of $\mathbf{P}^{n-1}$ around x_0 defined by

$$\begin{aligned} \varphi : \mathbf{C}^{n-1} \ni x &= (x_1, \ldots, x_{m_1-1}, x_{m_1+1}, \ldots, x_n) \\ &\mapsto q = [x_1, \ldots, x_{m_1-1}, 1, x_{m_1+1}, \ldots, x_n] \in \mathbf{P}^{n-1}. \end{aligned}$$

Since we chose x_0 so that $H_p(x_0) \neq 0$, it holds that the derivative $\frac{\partial \det A_p}{\partial q_n}$ does not vanish at x_0. So by the implicit function theorem, there exists a function Q_n defined on a sufficiently small neighborhood of x_0 such that

$$\begin{aligned} &\lambda_p(x_1, \ldots, x_{m_1-1}, 1, x_{m_1+1}, \ldots, x_{n-1}, Q_n(x)) \\ &\quad = \det A_p(x_1, \ldots, x_{m_1-1}, 1, x_{m_1+1}, \ldots, x_{n-1}, Q_n(x)) = 0. \end{aligned}$$

Since

$$x_{m_1} = 1, \quad x_{m_2} = \frac{c_{m_2}}{c_{m_1}}, \quad x_{m_3} = \frac{c_{m_3}}{c_{m_1}} \qquad \text{on} \quad \mathcal{V}^{n-3},$$

$(x_{j_1}, \ldots, x_{j_{n-4}})$ is considered as a local coordinate system of the variety X^{n-4} around the regular point x_0. Since

$$\frac{\partial Q_n}{\partial x_{j_\ell}} = -\frac{\partial \det A_p}{\partial q_{j_\ell}} / \frac{\partial \det A_p}{\partial q_n} \qquad (\ell = 1, \ldots, n-4)$$

holds, one can easily check that the condition $\det S_p(x_0) \neq 0$ implies that the matrix

$$\left(\frac{\partial(\mathring{f}_p^k \circ \varphi)}{\partial x_{j_\ell}} + \frac{\partial Q_n}{\partial x_{j_\ell}} \frac{\partial(\mathring{f}_p^k \circ \varphi)}{\partial x_n} \right)_{k=1,\ldots,n-1;\, \ell=1,\ldots,n-4}$$

is of rank $n-4$ at x_0. Hence the Jacobi matrix of $\mathcal{F}\ell_p$ is of rank $n-4$ at x_0, and so is that of $\widehat{\mathcal{F}\ell_p}$ at $\pi^{-1}(x_0)$. Thus by the proper mapping theorem, $\widehat{\mathcal{F}\ell_p}(\hat{X}^{n-4})$ is an analytic subset of dimension $n-4$ in the same dimensional complex projective space $\mathcal{W}_p^{n-4}$. Hence $\widehat{\mathcal{F}\ell_p}(\hat{X}^{n-4}) = \mathcal{W}_p^{n-4}$. (q.e.d.)

Lemma 1.9. *Let* $\mathcal{W}_{\mathbf{R}} = \{[a] \in \mathcal{W}_p^{n-4}\,;\, a_j \in \mathbf{R}\}$. *Then*

$$\left\{\mathcal{W}_p^{n-4} \setminus \widehat{\mathcal{F}\ell_p}(Z(H_p \circ \pi) \cap \hat{X}^{n-4})\right\} \cap \mathcal{W}_{\mathbf{R}}$$

is an open dense subset in $\mathcal{W}_{\mathbf{R}}$.

(Proof.) By the proper mapping theorem and the theorem of Chow, $\widehat{\mathcal{F}\ell_p}(\mathrm{Z}(H_p \circ \pi))$ is an algebraic subset of $\mathcal{W}_p^{n-4}$. Thus it is a closed subset in $\mathcal{W}_p^{n-4}$ in the usual topology. Hence $\left\{\mathcal{W}_p^{n-4} \setminus \widehat{\mathcal{F}\ell_p}(\mathrm{Z}(H_p \circ \pi) \cap \hat{X}^{n-4})\right\} \cap \mathcal{W}_{\mathbf{R}}$ is an open subset in $\mathcal{W}_{\mathbf{R}}$. Suppose that it is not dense in $\mathcal{W}_{\mathbf{R}}$. We may assume that $\widehat{\mathcal{F}\ell_p}(\mathrm{Z}(H_p \circ \pi) \cap \hat{X}^{n-4})$ is common zeros of some homogeneous polynomials $\bigcap_{j=1}^{r} \mathrm{Z}(h_j)$. Then there exists an open subset in $\mathcal{W}_p^{n-4}$ on which each h_j is identically zero. Since $\mathcal{W}_{\mathbf{R}}$ is a totally real subset of the complex projective space $\mathcal{W}_p^{n-4}$, by Lemma A.2 in Appendix we have

$$h_1 = \cdots = h_r = 0 \qquad \text{on} \quad \mathcal{W}_p^{n-4}.$$

This implies that $\widehat{\mathcal{F}\ell_p}(\mathrm{Z}(H_p \circ \pi) \cap \hat{X}^{n-4}) = \mathcal{W}_p^{n-4}$. So it holds that

$$\begin{aligned} n-4 &= \dim \mathcal{W}_p^{n-4} = \dim \widehat{\mathcal{F}\ell_p}(\mathrm{Z}(H_p \circ \pi) \cap \hat{X}^{n-4}) \\ &\leq \dim \mathrm{Z}(H_p \circ \pi) \cap \hat{X}^{n-4} \leq \dim \hat{X}^{n-4} = n-4. \end{aligned}$$

By the irreducibility of $\hat{X}^{n-4}$, we have $\mathrm{Z}(H_p \circ \pi) \cap \hat{X}^{n-4} = \hat{X}^{n-4}$. But this contradicts the fact that $H_p(x_0) \neq 0$. (q.e.d.)

(Proof of Proposition 1.6) Let p be a point in U satisfying $\dim \mathcal{W}_p^{n-4} = n-4$. As we mentioned before, $\dim \mathcal{W}_p^{n-4} = n-4$ holds on an open dense subset of $\{p \in \mathbf{C}^n\}$. Then for any

$$[a] \in \left(\mathcal{W}_p^{n-4} \setminus \widehat{\mathcal{F}\ell_p}(\mathrm{Z}(H_p \circ \pi) \cap \hat{X}^{n-4})\right) \cap \mathcal{W}_{\mathbf{R}},$$

there exists $x \in X^{n-4} \setminus \mathrm{Z}(H_p)$ such that $\mathcal{F}\ell_p(x) = [a]$ by Lemma 1.8 and Lemma 1.9. Since $f\ell_p^j(x) \neq 0$ and also the resultant $R_p(x)$ does not vanish, $(x, b_p(x))$ induces an n-end catenoid with the flux data (p, a) by Theorem 1.1. (q.e.d.)

For the later application, the following modification of Proposition 1.6 is needed: Recall here that any elements of the matrices A_p and J_p are rational functions in $p_1, \ldots, p_n, \bar{p}_1, \ldots, \bar{p}_n$ and $q_1, \ldots, q_n$. Let $\check{A}_p$ and $\check{J}_p$ be the matrices obtained by replacing $\bar{p}_n$ by p_n, namely

$$\check{A}_p := A_p(p_1, \ldots, p_n, \bar{p}_1, \ldots, \bar{p}_{n-1}, p_n, q_1, \ldots, q_n), \tag{1.15}$$

$$\check{J}_p := J_p(p_1, \ldots, p_n, \bar{p}_1, \ldots, \bar{p}_{n-1}, p_n, q_1, \ldots, q_n), \tag{1.16}$$

and let $\check{b}_p^j$ (resp. $\check{f}_p^j$, $\check{\mathcal{W}}_p^{n-4}$) be the vector (resp. function, set) obtained by replacing $\bar{p}_n$ in b_p^j (resp. f_p^j, $\mathcal{W}_p^{n-4}$) by p_n.

Proposition 1.10. *Suppose that there exist $u_0 \in \mathbf{C}^n$ and a point $c = [c_1, \ldots, c_n] \in \mathbf{P}^n$ satisfying the following conditions:*

(1) *$c_1, \ldots, c_n$ are all distinct;*

(2) *The rank of the matrix $\check{A}_{u_0}(c)$ is $n-1$;*

(3) *$\frac{\partial \det \check{A}_{u_0}}{\partial q_n}$ does not vanish at $q = c$;*

(4) *The rank of the matrix $\check{J}_{u_0}(c)$ is $n-4$;*

(5) *Two polynomials $P(z)$ and $Q(z)$ defined in (1.5) and (1.4) associated with the data $(q, p) = (c, u_0)$ and $\check{b} = \check{b}_{u_0}(c)$ are mutually prime and one of them has degree $n-1$;*

(6) *$\check{f}^j_{u_0}(c) \neq 0$ $(j = 1, \ldots, n)$;*

(7) *$c_j \neq 0$ $(j = 1, \ldots, n-1)$.*

Then there exists an open dense subset $U \subset \mathbf{C}^n$ and an open dense subset Ω_p of the totally real set $\mathcal{W}_{\mathbf{R}} = \{[a] \in \check{\mathcal{W}}_p^{n-4}\,;\, a_j \in \mathbf{R}\}$ such that, for $p = (p_1, \ldots, p_n) \in U$ satisfying $p_n \in \mathbf{R}$ and $[a] \in \Omega_p$, there exists an (non-branched) n-end catenoid with the flux data (p, a).

(Proof.) The proof of Proposition 1.6 works even if we replace $\bar{p}_n$ by p_n. When p_n is real, $\check{A}_p$, $\check{J}_p$, $\check{\mathcal{F}\ell}_p$ and $\check{\mathcal{W}}_p^{n-4}$ coincide with A_p, J_p, $\mathcal{F}\ell_p$ and $\mathcal{W}_p^{n-4}$ respectively. (q.e.d.)

Remark 1.11. To solve the inverse problem of the flux formula, we may assume that $p_n \in \mathbf{R}$ since by a suitable rotation in $\{(x, y, z) \in \mathbf{R}^3\}$, we can choose that v_n is in the xz-plane. By the above modification of Proposition 1.6, the parameter p_n (=the stereographic image of v_n) can be treated as a complex analytic parameter.

2. Finding a regular point of the flux map

In the previous section, we reduced our inverse problem to finding a regular point of the flux map. However, the following difficulties arise in this process.

- As seen in [Kat] and [KUY], n-end catenoids with many symmetries are easy to construct. But unfortunately, they are not expected to be a regular point of the flux map because of their symmetries.

- If we take a less symmetric n-end catenoid, the computation of the rank of the flux map is very complicated and hard to calculate even by computer.

To avoid these difficulties, we first take an n-end catenoid with many symmetries, and next consider a perturbation of it which attains the desired properties.

Set $m := n - 1$. First we consider a 1-parameter family of $(m+1)$-end catenoids given in [Kat];

$$(2.1)\quad \begin{cases} p_j := r\zeta^{j-1} \quad (j = 1, \dots, m), \quad p_{m+1} := 0, \\ a^1 = \dots = a^m := \dfrac{m-1}{2} r(r^2+1), \quad a^{m+1} := \dfrac{m(m-1)}{2} r(r^2-1), \\ q_j := \zeta^{j-1} \quad (j = 1, \dots, m), \quad q_{m+1} := 0, \\ b^1 = \dots = b^m := 1, \quad b^{m+1} := \dfrac{m-1}{2}(r^2-1), \end{cases}$$

where $r > 0$, $r \neq 1$ and $\zeta := \exp(2\pi\sqrt{-1}/m)$. In fact, they are $(m+1)$-end catenoids without branch points by Remark 1.2, and are invariant under the action of the cyclic group Z_m. Unfortunately, as we shall see below, $J_p(q) = \check{J}_p(q) = 0$ holds for any of these examples, namely they all are singular points of the flux maps. However, we will show that there exists a regular point near them.

Note here that the matrix $A_p(q)$ (defined in (1.9)) for the example above is given by

$$(2.2)\qquad A_p(q) = \begin{pmatrix} 0 & \frac{1+r^2\zeta^1}{q_1-q_2} & \cdots & \frac{1+r^2\zeta^{m-1}}{q_1-q_m} & \frac{1}{q_1-q_{m+1}} \\ \frac{1+r^2\zeta^{-1}}{q_2-q_1} & 0 & \cdots & \frac{1+r^2\zeta^{m-2}}{q_2-q_m} & \frac{1}{q_2-q_{m+1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1+r^2\zeta^{-(m-1)}}{q_m-q_1} & \frac{1+r^2\zeta^{-(m-2)}}{q_m-q_2} & \cdots & 0 & \frac{1}{q_m-q_{m+1}} \\ \frac{1}{q_{m+1}-q_1} & \frac{1}{q_{m+1}-q_2} & \cdots & \frac{1}{q_{m+1}-q_m} & 0 \end{pmatrix}.$$

Now, We consider a 1-parameter family of matrices

$$(2.3)\qquad A(q,\mu) := \begin{pmatrix} 0 & \frac{1+\mu\zeta^1}{q_1-q_2} & \cdots & \frac{1+\mu\zeta^{m-1}}{q_1-q_m} & \frac{1}{q_1-q_{m+1}} \\ \frac{1+\mu\zeta^{-1}}{q_2-q_1} & 0 & \cdots & \frac{1+\mu\zeta^{m-2}}{q_2-q_m} & \frac{1}{q_2-q_{m+1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1+\mu\zeta^{-(m-1)}}{q_m-q_1} & \frac{1+\mu\zeta^{-(m-2)}}{q_m-q_2} & \cdots & 0 & \frac{1}{q_m-q_{m+1}} \\ \frac{1}{q_{m+1}-q_1} & \frac{1}{q_{m+1}-q_2} & \cdots & \frac{1}{q_{m+1}-q_m} & 0 \end{pmatrix}.$$

By comparing (2.2) with (2.3), we have $A(q, r^2) = A_p(q)$ for p as in (2.1). When we evaluate it at $q = q^0 := (1, \zeta^1, \dots, \zeta^{m-1}, 0)$, we have

$$
A(q^0,\mu) = \begin{pmatrix} 0 & \frac{1+\mu\zeta^1}{1-\zeta^1} & \cdots & \frac{1+\mu\zeta^{m-1}}{1-\zeta^{m-1}} & 1 \\ \frac{1+\mu\zeta^{-1}}{\zeta^1-1} & 0 & \cdots & \frac{1+\mu\zeta^{m-2}}{\zeta^1-\zeta^{m-1}} & \zeta^{-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1+\mu\zeta^{-(m-1)}}{\zeta^{m-1}-1} & \frac{1+\mu\zeta^{-(m-2)}}{\zeta^{m-1}-\zeta^1} & \cdots & 0 & \zeta^{-(m-1)} \\ -1 & -\zeta^{-1} & \cdots & -\zeta^{-(m-1)} & 0 \end{pmatrix}. \tag{2.4}
$$

We remark that the matrix $A(q^0, \mu)$ has the simplest form when $\mu = -1$. The following lemma holds.

Lemma 2.1. *The $(m+1)$-matrix $A(q^0, \mu)$ is of rank m except for finite values of $\mu \in \mathbf{R}$. Moreover $A(q^0, \mu)$ has a 0-eigenvector given by*

$$
{}^t\left(1, \dots, 1, \frac{m-1}{2}(\mu - 1)\right).
$$

(Proof.) The second assertion is easily checked. Hence the rank of the matrix $A(q^0, -1)$ is at most m. Moreover, it is easy to see that the rank of the matrix $A(q^0, -1)$ is m. Since each component of $A(q^0, \mu)$ is a polynomial in μ, the first assertion is obtained. (q.e.d.)

Remark 2.2. Similarly, a 0-eigenvector of ${}^tA(q^0, \mu)$ is given by

$$
{}^t\left(1, \dots, 1, \frac{1}{2}\{2\mu - (m-1)(\mu+1)\}\right).
$$

Proposition 2.3. *The following identity holds.*

$$
\frac{\partial \det A}{\partial q_j}(q^0, \mu) = 0 \qquad (j = 1, \dots, m+1).
$$

(Proof.) We denote the cofactor matrix of $A(q, \mu)$ by $B(q, \mu)$. By Lemma 2.1 and Remark 2.2, it can be easily checked that $B(q^0, \mu)$ is written in the form $B(q^0, \mu) = f(\mu)S(\mu)$, where $f(\mu)$ is a polynomial in μ satisfying $f(-1) = 1$,

$$
S(\mu) := \begin{pmatrix} 1 & \cdots & 1 & \psi(\mu) \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 1 & \psi(\mu) \\ \varphi(\mu) & \cdots & \varphi(\mu) & \varphi(\mu)\cdot\psi(\mu) \end{pmatrix}, \tag{2.5}
$$

and $\varphi(\mu)$ and $\psi(\mu)$ are explicitly given by

$$\varphi(\mu) := \frac{m-1}{2}(\mu-1), \qquad \psi(\mu) := \frac{1}{2}\Big\{2\mu - (m-1)(\mu+1)\Big\}.$$

Note here that

$$\frac{\partial \det A}{\partial q_j}(q,\mu) = \mathrm{Tr}\left(\frac{\partial A}{\partial q_j}(q,\mu)\cdot B(q,\mu)\right)$$

always holds for any j. Denote the (k,ℓ)-component of the matrix $A(q,\mu)$ by $\alpha_{k\ell}(q,\mu)$. Then we have

$$\frac{\partial \alpha_{k\ell}}{\partial q_j}(q^0,\mu) = \begin{cases} -\dfrac{1+\mu\zeta^{\ell-j}}{(\zeta^{j-1}-\zeta^{\ell-1})^2} & (k=j;\ell=1,\dots,m;\ell\neq j) \\ -\zeta^{-2(j-1)} & (k=j;\ell=m+1) \\ \dfrac{1+\mu\zeta^{j-k}}{(\zeta^{k-1}-\zeta^{j-1})^2} & (k=1,\dots,m;k\neq j;\ell=j) \\ \zeta^{-2(j-1)} & (k=m+1;\ell=j) \\ 0 & \text{elsewhere} \end{cases}$$

for $j=1,\dots,m$, and

$$\frac{\partial \alpha_{kl}}{\partial q_{m+1}}(q^0,\mu) = \begin{cases} \zeta^{-2(k-1)} & (k=1,\dots,m;\ell=m+1) \\ -\zeta^{-2(\ell-1)} & (k=m+1;\ell=1,\dots,m) \\ 0 & \text{elsewhere} \end{cases}$$

for $j=m+1$.

For $j=1,\dots,m$, by using the formula above, we have

$$\begin{aligned}
\frac{\partial \det A}{\partial q_j}(q^0,\mu) &= \mathrm{Tr}\left(\frac{\partial A}{\partial q_j}(q^0,\mu)\cdot B(q^0,\mu)\right) \\
&= \sum_{\substack{k=1\\k\neq j}}^{m} f(\mu)\frac{\partial \alpha_{kj}}{\partial q_j}(q^0,\mu) + \sum_{\substack{\ell=1\\ \ell\neq j}}^{m} f(\mu)\frac{\partial \alpha_{j\ell}}{\partial q_j}(q^0,\mu) \\
&\quad + \frac{\partial \alpha_{jm+1}}{\partial q_j}(q^0,\mu)f(\mu)\varphi(\mu) + \frac{\partial \alpha_{m+1j}}{\partial q_j}(q^0,\mu)f(\mu)\psi(\mu) \\
&= f(\mu)\left\{\sum_{\substack{k=1\\k\neq j}}^{m}\frac{1+\mu\zeta^{j-k}}{(\zeta^{k-1}-\zeta^{j-1})^2} - \sum_{\substack{\ell=1\\ \ell\neq j}}^{m}\frac{1+\mu\zeta^{\ell-j}}{(\zeta^{j-1}-\zeta^{\ell-1})^2} - \zeta^{-2(j-1)}\varphi(\mu) + \zeta^{-2(j-1)}\psi(\mu)\right\} \\
&= f(\mu)\zeta^{-2(j-1)}\left\{\sum_{k=1}^{m-1}\frac{1+\mu\zeta^{-k}}{(1-\zeta^k)^2} - \sum_{k=1}^{m-1}\frac{1+\mu\zeta^{k}}{(1-\zeta^k)^2} - (m-2)\mu\right\} \\
&= \mu f(\mu)\zeta^{-2(j-1)}\left\{\sum_{k=1}^{m-1}\frac{1+\zeta^k}{\zeta^k(1-\zeta^k)} - (m-2)\right\} \\
&= \mu f(\mu)\zeta^{-2(j-1)}\left\{\sum_{k=1}^{m-1}\frac{1}{\zeta^k} + \sum_{k=1}^{m-1}\frac{2}{1-\zeta^k} - (m-2)\right\} \\
&= \mu f(\mu)\zeta^{-2(j-1)}\left\{-1+(m-1)-(m-2)\right\} = 0.
\end{aligned}$$

On the other hand, for $j = m+1$, we have

$$\begin{aligned}\frac{\partial \det A}{\partial q_{m+1}}(q^0,\mu) &= \operatorname{Tr}\left(\frac{\partial A}{\partial q_{m+1}}(q^0,\mu)\cdot B(q^0,\mu)\right)\\ &= \sum_{k=1}^{m}\zeta^{-2(k-1)}f(\mu)\varphi(\mu) - \sum_{\ell=1}^{m}\zeta^{-2(\ell-1)}f(\mu)\psi(\mu)\\ &= f(\mu)(\varphi(\mu)-\psi(\mu))\sum_{k=1}^{m}\zeta^{-2k} = 0.\end{aligned}$$

This completes the proof. (q.e.d.)

By Lemma 2.1 and Proposition 2.3, it follows that $J_{rq^0}(q^0) = 0 \quad (r \in \mathbf{R})$. Therefore, we try to perturb a sampling point. To do this, we consider an m-matrix $\Gamma_{m+1}(\mu)$ by

$$\Gamma_{m+1}(\mu) := \left(\frac{\partial^2 \det A}{\partial q_1 \partial q_{m+1}}(q^0,\mu)\cdot\frac{\partial(f^k/f^{m+1})}{\partial q_j}(q^0,\mu) - \frac{\partial^2 \det A}{\partial q_1\partial q_j}(q^0,\mu)\cdot\frac{\partial(f^k/f^{m+1})}{\partial q_{m+1}}(q^0,\mu)\right)_{j,k=1,\dots,m},$$

where we denote the (j,k)-component of the cofactor matrix $B(q,\mu)$ by $\beta_{jk}(q,\mu)$, and set

(2.6)

$$\begin{aligned} f^k(q,\mu) &:= \beta_{k\,m+1}(q,\mu)\left(\sum_{\substack{j=1\\ j\neq k}}^{m}\beta_{j\,m+1}(q,\mu)\frac{\zeta^{k-1}-\zeta^{j-1}}{q_k-q_j} + \beta_{m+1\,m+1}(q,\mu)\frac{\zeta^{k-1}}{q_k-q_{m+1}}\right)\\ &\qquad (k=1,\dots,m),\\ f^{m+1}(q,\mu) &:= \beta_{m+1\,m+1}(q,\mu)\sum_{j=1}^{m}\beta_{j\,m+1}(q,\mu)\frac{-\zeta^{j-1}}{q_{m+1}-q_j}.\end{aligned}$$

(Compare with the definition of the matrix $J_p(q)$ and $f_p^k(q)$.) We prove the following

Theorem 2.4. *Suppose that there exists a positive number μ such that the matrix $\Gamma_{m+1}(\mu)$ $(n = m+1 \geq 5)$ is of rank $m-3(=n-4)$ and*

$$\frac{\partial^2 \det A}{\partial q_1 \partial q_{m+1}}(q^0,\mu) \neq 0. \tag{2.7}$$

Then, for each of almost all flux data, there exists an n-end catenoid with the flux data.

Till now, we fix the parameter p_{m+1} at

$$p_{m+1} = 0.$$

Let us now move the parameter p_{m+1}.

Lemma 2.5. *Let $\mu \neq 1$ be a positive real number such that $f(\mu) \neq 0$, where $f(\mu)$ is a polynomial given by (2.5). Then*

$$\frac{\partial \det \check{A}_p(q)}{\partial p_{m+1}} \neq 0$$

at the point $q = q^0 = (1, \zeta^1, \ldots, \zeta^{m-1}, 0)$ for $p = \sqrt{\mu} q^0$, where $\check{A}_p(q)$ is defined in (1.15).

(Proof.) We denote the cofactor matrix of $\check{A}_p(q)$ by $\check{B}_p(q)$. Since $\check{A}_{\sqrt{\mu}q^0}(q) = A_{\sqrt{\mu}q^0}(q)$ for any $\mu > 0$, it holds that $\check{B}_{\sqrt{\mu}q^0}(q) = B_{\sqrt{\mu}q^0}(q)$ and in particular, we have $\check{B}_{\sqrt{\mu}q^0}(q^0) = B_{\sqrt{\mu}q^0}(q^0) = B(q^0, \mu)$. Then we have

$$\left.\frac{\partial \det \check{A}_p(q^0)}{\partial p_{m+1}}\right|_{p=\sqrt{\mu}q^0} = \mathrm{Tr}\left(\left.\frac{\partial \check{A}_p(q^0)}{\partial p_{m+1}}\right|_{p=\sqrt{\mu}q^0} \cdot B_{\sqrt{\mu}q^0}(q^0) \right).$$

Since

$$\text{the } (j,k)\text{-component of } \left.\frac{\partial \check{A}_p(q^0)}{\partial p_{m+1}}\right|_{p=\sqrt{\mu}q^0} = \begin{cases} \zeta^{-2(j-1)} & (j = 1, \ldots, m; k = m+1) \\ -1 & (j = m+1; k = 1, \ldots, m) \\ 0 & \text{elsewhere,} \end{cases}$$

by (2.5), we have

$$\begin{aligned} &\mathrm{Tr}\left(\left.\frac{\partial \check{A}_p(q^0)}{\partial p_{m+1}}\right|_{p=\sqrt{\mu}q^0} \cdot B(q^0, \mu) \right) \\ &= f(\mu)\left\{ \varphi(\mu) \sum_{k=1}^{m} \zeta^{-2(k-1)} - (m-1)\psi(\mu) \right\} \\ &= -(m-1) f(\mu)\psi(\mu) = \frac{(m-1)^2}{2}(\mu - 1) f(\mu) \neq 0. \end{aligned}$$

Now the assertion is clear. (q.e.d.)

(Proof of Theorem 2.4.) Since $f(\mu)$ is a polynomial in μ and $f(\mu) \not\equiv 0$, by our assumptions and Lemmas 2.1 and 2.5, we can choose a positive number μ such that $f(\mu) \neq 0$, rank $\check{A}_{\sqrt{\mu}q^0}(q^0) = m$, rank $\Gamma_{m+1}(\mu) = m - 3$,

$$\frac{\partial^2 \det \check{A}_{\sqrt{\mu}q^0}}{\partial q_1 \partial q_{m+1}}(q^0) \neq 0 \qquad \text{and} \qquad \left.\frac{\partial \det \check{A}_p(q^0)}{\partial p_{m+1}}\right|_{p=\sqrt{\mu}q^0} \neq 0.$$

Throughout this proof, we fix the parameters except for q_1 and p_{m+1} to the same values as $q = q^0$ and $p = \sqrt{\mu} q^0$:

$$\begin{aligned} &p_j = \sqrt{\mu}\zeta^{j-1} \qquad (j = 1, \ldots, m), \\ &q_j = \zeta^{j-1} \qquad (j = 2, \ldots, m), \qquad q_{m+1} = 0. \end{aligned}$$

Regard $\det \check{A}_p(q)$ as a function with respect to only q_1 and p_{m+1}, and apply the implicit function theorem to the point $(q_1, p_{m+1}) = (1, 0)$. Then there exist an open neighborhood $U \subset \mathbf{C}$ of $1 \in \mathbf{C}$ and a complex analytic function $p_{m+1} = p_{m+1}(q_1) : U \to \mathbf{C}$ such that $p_{m+1}(1) = 0$ and

$$\det \check{A}_p\Big|_{p_{m+1}=p_{m+1}(q_1)} = 0 \qquad (q_1 \in U).$$

Since $\operatorname{rank} \check{A}_{\sqrt{\mu}q^0}(q^0) = m$, $\operatorname{rank} \check{A}_p|_{p_{m+1}=p_{m+1}(q_1)} = m$ holds also for q_1 near 1.

Since $\hat{A} = A$ at $p = \sqrt{\mu}q^0$, by Lemma 2.3, we have

$$\frac{\partial \det \check{A}_{\sqrt{\mu}q^0}}{\partial q_j}(q^0) = 0 \qquad (j = 1, \ldots, m+1).$$

On the other hand, the assumption (2.7) yields

$$\frac{\partial \det \check{A}_p}{\partial q_{m+1}}\Big|_{p_{m+1}=p_{m+1}(q_1)} \neq 0$$

for any $q_1 \neq 1$ enough close to 1. Therefore we have

$$\begin{aligned}
&\lim_{q_1 \to 1} \left(\left\{ \frac{\partial(\check{f}_p^k/\check{f}_p^{m+1})}{\partial q_j} - \frac{\frac{\partial \det \check{A}_p}{\partial q_j}}{\frac{\partial \det \check{A}_p}{\partial q_{m+1}}} \cdot \frac{\partial(\check{f}_p^k/\check{f}_p^{m+1})}{\partial q_{m+1}} \right\}\Bigg|_{p_{m+1}=p_{m+1}(q_1)} \right)_{j,k=1,\ldots,m} \\
&= \left(\left\{ \frac{\partial(\check{f}_p^k/\check{f}_p^{m+1})}{\partial q_j} - \frac{\frac{\partial^2 \det \check{A}_p}{\partial q_1 \partial q_j}}{\frac{\partial^2 \det \check{A}_p}{\partial q_1 \partial q_{m+1}}} \cdot \frac{\partial(\check{f}_p^k/\check{f}_p^{m+1})}{\partial q_{m+1}} \right\}\Bigg|_{p=\sqrt{\mu}q^0; q=q^0} \right)_{j,k=1,\ldots,m} \\
&= \left(\frac{\partial^2 \det \check{A}_{\sqrt{\mu}q^0}}{\partial q_1 \partial q_{m+1}}(q^0) \right)^{-1} \Gamma_{m+1}(\mu),
\end{aligned}$$

and hence

$$\begin{aligned}
&\operatorname{rank} \check{J}_p\Big|_{p_{m+1}=p_{m+1}(q_1)} \\
&= \operatorname{rank} \left(\left\{ \frac{\partial(\check{f}_p^k/\check{f}_p^{m+1})}{\partial q_j} - \frac{\frac{\partial \det \check{A}_p}{\partial q_j}}{\frac{\partial \det \check{A}_p}{\partial q_{m+1}}} \cdot \frac{\partial(\check{f}_p^k/\check{f}_p^{m+1})}{\partial q_{m+1}} \right\}\Bigg|_{p_{m+1}=p_{m+1}(q_1)} \right)_{j,k=1,\ldots,m} \\
&= m - 3 = n - 4
\end{aligned}$$

for any q_1 as above.

Since the initial sampling point $q = q^0$, $p = \sqrt{\mu}q^0$ is chosen from the data which realizes a non-branched n-end catenoid ($n = m+1$), $\Delta(q^0) \neq 0$ and $q_j^0 \neq 0$ ($j = 1, \ldots, m$), the other conditions in Proposition 1.10 are also satisfied for q_1 near 1. Now, by Remark 1.11, we have proved the theorem. (q.e.d.)

Thus we will get our main theorem in Introduction, if the matrix $\Gamma_{m+1}(\mu)$ is of rank $m-3 (= n-4)$ and (2.7) holds for some $\mu > 0$, which will be shown in the next section.

3. Computation of $\Gamma_{m+1}(\mu)$

In this section, we compute the matrix $\Gamma_{m+1}(\mu)$ defined in the previous section, and show that it is of rank $m-3$ for almost all $\mu > 0, \neq 1$.

(Computation of $\frac{\partial f^k}{\partial q_j}(q^0,\mu)$) As before, we write $A(q,\mu) =: (\alpha_{k\ell})_{k,\ell=1,\dots,m+1}$ and $B(q,\mu) =: (\beta_{k\ell})_{k,\ell=1,\dots,m+1}$. By (2.6), (2.5) and straightforward calculations, we have, for any $k = 1,\dots,m$,

$$(3.1) \qquad \frac{\partial f^k}{\partial q_j} = f\psi\left[(m-1+\varphi)\frac{\partial \beta_{k\,m+1}}{\partial q_j} + \sum_{\substack{\ell=1\\ \ell\neq k}}^{m+1}\frac{\partial \beta_{\ell\,m+1}}{\partial q_j} + f\psi\zeta^{1-j}\eta_1\right]$$

at (q^0,μ), where

$$\eta_1(\mu) := \begin{cases} -\frac{m-1}{2} - \varphi(\mu) & (j=k) \\ \frac{1}{\zeta^{k-j}-1} & (j=1,\dots,m;\, j\neq k) \\ \zeta^{j-k}\varphi(\mu) & (j=m+1), \end{cases}$$

and for $k = m+1$,

$$(3.2) \quad \frac{\partial f^{m+1}}{\partial q_j} = f\psi\left[m\frac{\partial \beta_{m+1\,m+1}}{\partial q_j} + \varphi\sum_{\ell=1}^{m}\frac{\partial \beta_{\ell\,m+1}}{\partial q_j} - \begin{cases} f\varphi\psi\zeta^{1-j} & (j=1,\dots,m) \\ 0 & (j=m+1) \end{cases}\right].$$

Hence we have only to compute $f(\mu)$ and $\frac{\partial \beta_{k\,m+1}}{\partial q_j}(q^0,\mu)$. Denote the first $m\times m$-submatrix of $A(q^0,\mu)$ by $A^0(\mu)$. Clearly $f\varphi\psi = \beta_{m+1\,m+1} = \det A^0$. Set $C_1 :=\mathrm{diag}[1,\zeta^1,\dots,\zeta^{m-1}]$. Since C_1A^0 is a cyclic matrix whose (j,k)-component is equal to $(1+\mu\zeta^{k-j})/(1-\zeta^{k-j})$, and whose diagonal components vanish, it can be diagonalized as ${C_2}^{-1}C_1A^0C_2 =\mathrm{diag}[\psi_1,\dots,\psi_m]$, where

$$C_2 := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \zeta^1 & \zeta^2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ \zeta^{m-1} & \zeta^{2(m-1)} & \cdots & 1 \end{pmatrix}$$

and the eigenvalues $\psi_1,\dots,\psi_m$ of C_1A^0 are given by

$$\begin{aligned} \psi_\ell(\mu) &= \sum_{k=2}^{m}\frac{1+\mu\zeta^{k-1}}{1-\zeta^{k-1}}(\zeta^\ell)^{k-1} \\ &= \begin{cases} \left(\ell-\frac{m-1}{2}\right)\mu + \left(\ell-\frac{m+1}{2}\right) & (\ell=1,\dots,m-1) \\ -\frac{m-1}{2}\mu + \frac{m-1}{2} & (\ell=m). \end{cases} \end{aligned}$$

Now we have

$$f\varphi\psi = (-1)^{m-1}\prod_{\ell=1}^{m}\psi_\ell.$$

Note here that $\psi_1 = \psi$ and $\psi_m = -\varphi$ and that $\psi_\ell(\mu) \neq 0$ holds for any $\mu > 0, \neq 1$ ($\ell = 1, \dots, m$).

To compute the derivatives $\frac{\partial B}{\partial q_j}(q^0, \mu)$ of the cofactor matrix $B(q, \mu)$, we apply the formula (B.2) in Appendix B by putting $X := E_{m+1}$, where E_{m+1} is the $(m+1)$-matrix given by

$$E_{m+1} := \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

For $A_t(q, \mu) = A(q, \mu) + tE_{m+1}$, we have already shown that

$$\det A(q^0, \mu) = \frac{\partial \det A}{\partial q_j}(q^0, \mu) = 0$$

in Lemma 2.1 and Proposition 2.3. Moreover we have

$$\mathrm{Tr}(E_{m+1} \cdot B(q^0, \mu)) = f(\mu)\varphi(\mu)\psi(\mu) \neq 0.$$

Thus we may apply (B.2), and get the following identity

$$\begin{aligned} \frac{\partial B}{\partial q_j} &= \frac{1}{f\varphi\psi}\left\{\mathrm{Tr}\left(\frac{\partial A}{\partial q_j} \cdot \left.\frac{\partial Y_t}{\partial t}\right|_{t=0}\right) \cdot B \right. \\ &\qquad \left. - \left.\frac{\partial Y_t}{\partial t}\right|_{t=0} \cdot \frac{\partial A}{\partial q_j} \cdot B - B \cdot \frac{\partial A}{\partial q_j} \cdot \left.\frac{\partial Y_t}{\partial t}\right|_{t=0}\right\} \end{aligned} \tag{3.3}$$

at (q^0, μ), where $Y_t(\mu)$ is the cofactor matrix of $A(q^0, \mu) + tE_{m+1}$. The first $m \times m$-components of $\frac{\partial}{\partial t}|_{t=0} Y_t(\mu)$ is given as the cofactor matrix of the first $m \times m$-components of $A(q^0, \mu)$, that is

$$\begin{aligned} \det A^0 \cdot (A^0)^{-1} &= f\varphi\psi \cdot C_2 \mathrm{diag}[\psi_1{}^{-1}, \dots, \psi_m{}^{-1}] C_2{}^{-1} C_1 \\ &= \frac{f\varphi\psi}{m}\left(\zeta^{k-1}\sum_{\ell=1}^{m} \zeta^{(j-k)\ell}\psi_\ell^{-1}\right)_{j,k=1,\dots,m} \\ &=: \frac{f\varphi\psi}{m} Y^0, \end{aligned}$$

and the other components of $\frac{\partial}{\partial t}|_{t=0} Y_t(\mu)$ vanish. Namely

$$\left.\frac{\partial}{\partial t}\right|_{t=0} Y_t(\mu) = \begin{pmatrix} & \frac{f\varphi\psi}{m} Y^0 & & 0 \\ & & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Therefore we have

$$(3.4)\quad \left(\frac{\partial \beta_{k\,m+1}}{\partial q_j}\right)_{k=1,\dots,m+1} = \frac{f\psi}{m}\left\{\mathrm{Tr}\left(\frac{\partial A}{\partial q_j}\cdot Y^0\right)\cdot I - Y^0\cdot\frac{\partial A}{\partial q_j}\right\}\begin{pmatrix}1\\ \vdots\\ 1\\ \varphi\end{pmatrix}$$

at (q^0,μ). Recall here the values of $\frac{\partial \alpha_{k\ell}}{\partial q_j}(q^0,\mu)$ computed in the proof of Proposition 2.3. Now, by direct computation, we have

$$(3.5)\quad \frac{\partial \beta_{k\,m+1}}{\partial q_j}(q^0,\mu) = -f(\mu)\psi(\mu)\zeta^{1-j} \times \begin{cases} (1-\frac{1}{2m}\eta_2(\mu)) & (k,j=1,\dots,m)\\ \varphi(\mu) & (k=m+1; j=1,\dots,m)\\ \zeta^{1-k}\varphi(\mu)\psi_{m-1}(\mu)^{-1} & (k=1,\dots,m; j=m+1)\\ 0 & (k=j=m+1),\end{cases}$$

where

$$\eta_2(\mu) := \begin{cases} \frac{m(m-1)}{2} + \frac{\psi_1(\mu)}{\mu+1}\left\{m-1+(m+\varphi(\mu))\sum_{\ell=1}^{m-1}\psi_\ell(\mu)^{-1}\right\} & (k=j)\\ \frac{m}{\zeta^{k-j}-1} + \frac{\psi_1(\mu)}{\mu+1}\left\{-1+(m+\varphi(\mu))\sum_{\ell=1}^{m-1}\zeta^{(k-j)\ell}\psi_\ell(\mu)^{-1}\right\} & (k\neq j).\end{cases}$$

Putting it into (3.1) and (3.2), we get

$$(3.6)\quad \frac{\partial f^k}{\partial q_j}(q^0,\mu) = -f(\mu)^2\psi(\mu)^2\zeta^{1-j} \times \begin{cases} 2(m-1+\varphi(\mu)) - \frac{m-2+\varphi(\mu)}{2m}\eta_2(\mu) - \eta_1(\mu) & (k,j=1,\dots,m)\\ (2m+1)\varphi(\mu) & (k=m+1; j=1,\dots,m)\\ 0 & (k=1,\dots,m+1; j=m+1).\end{cases}$$

In particular, we have

$$\Gamma_{m+1}(\mu) = (f^{m+1})^{-2}\frac{\partial^2 \det A}{\partial q_1\partial q_{m+1}}\cdot\left(f^{m+1}\frac{\partial f^k}{\partial q_j} - f^k\frac{\partial f^{m+1}}{\partial q_j}\right)_{k,j=1,\dots,m}$$

at (q^0,μ).

(Computation of $\frac{\partial^2 \det A}{\partial q_1\partial q_{m+1}}(q^0,\mu)$) First we compute

$$\frac{\partial^2 \det A}{\partial q_1\partial q_{m+1}}(q^0,-1) = \mathrm{Tr}\left(\frac{\partial^2 A}{\partial q_1\partial q_{m+1}}(q^0,-1)\cdot B(q^0,-1) + \frac{\partial A}{\partial q_1}(q^0,-1)\cdot\frac{\partial B}{\partial q_{m+1}}(q^0,-1)\right).$$

It is easy to see that,

$$\frac{\partial^2 \alpha_{k\ell}}{\partial q_1\partial q_{m+1}}(q^0,-1) = \begin{cases} -2 & (k=1;\ell=m+1)\\ 2 & (k=m+1;\ell=1)\\ 0 & \text{elsewhere.}\end{cases}$$

On the other hand, we have

$$\frac{\partial}{\partial t}\bigg|_{t=0} Y_t(-1) = \begin{pmatrix} 2-m & \zeta^1 & \cdots & \zeta^{m-1} & 0 \\ 1 & (2-m)\zeta^1 & \cdots & \zeta^{m-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \zeta^1 & \cdots & (2-m)\zeta^{m-1} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

By putting these values into (3.3), we have

$$(3.7)\ \frac{\partial \beta_{k\ell}}{\partial q_{m+1}}(q^0,-1) = \begin{cases} -(m-1)\zeta^{1-k} + \zeta^{1-\ell} & (k,\ell = 1,\ldots,m) \\ (m-1)\zeta^{1-k} & (k = 1,\ldots,m; \ell = m+1) \\ -(m-1)\zeta^{1-\ell} & (k = m+1; \ell = 1,\ldots,m) \\ 0 & (k = \ell = m+1). \end{cases}$$

Now, by a straightforward calculation, we have

$$(3.8) \qquad \frac{\partial^2 \det A}{\partial q_1 \partial q_{m+1}}(q^0,-1) = m(m-1) \neq 0.$$

Since $\frac{\partial^2 \det A}{\partial q_1 \partial q_{m+1}}(q^0,\mu)$ is a polynomial in μ, it does not vanish for any μ except for finite values.

(Computation of the rank of $\Gamma_{m+1}(\mu)$) For any $\mu > 0, \neq 1$ such that $\frac{\partial^2 \det A}{\partial q_1 \partial q_{m+1}}(q^0,\mu) \neq 0$, define a cyclic matrix

$$\Gamma^0_{m+1} := -\frac{1}{(f\psi)^2}\left(\frac{\partial f^k}{\partial q_j} - \frac{f^k}{f^{m+1}}\frac{\partial f^{m+1}}{\partial q_j}\right)_{k,j=1,\ldots,m} \cdot C_1.$$

Then it is clear that the rank of Γ_{m+1} is equal to the rank of Γ^0_{m+1}. The (k,j)-component γ_{kj} of Γ^0_{m+1} is given by

$$\gamma_{kj} = -\frac{m-1+\varphi}{m} - \frac{m-2+\varphi}{2m}\eta_2 - \eta_1,$$

and the eigenvalues $\chi_1,\ldots,\chi_m$ of Γ^0_{m+1} are given by

$$\begin{aligned} \chi_\ell(\mu) &= \sum_{j=1}^m \gamma_{1j}(\mu)(\zeta^\ell)^{j-1} \\ &= \begin{cases} -\frac{(\mu+1)\{(m-1)\mu+m+1\}(\ell-1)(\ell-m+1)}{4\psi_\ell(\mu)} & (\ell = 1,\ldots,m-1) \\ 0 & (\ell = m). \end{cases} \end{aligned}$$

Now it is clear that $\chi_\ell(\mu) \neq 0$ for $\ell = 2,\ldots,m-2$, and Γ^0_{m+1} is of rank $m-3$. Consequently, Γ_{m+1} is of rank $m-3$ for any $\mu > 0, \neq 1$ except for finite values.

Now, by Theorem 2.4, we get the following theorem:

Theorem 3.1. *For almost all given unit vectors $v = \{v_1, \ldots, v_n\}$ $(n \geq 5)$ in $\mathbf{R}^3$, and nonzero real numbers $a = \{a^1, \ldots, a^n\}$ satisfying $\sum_{j=1}^n a^j v_j = 0$, there is a (non-branched) n-end catenoid $x : \mathbf{C} \setminus \{q_1, \ldots, q_n\} \to \mathbf{R}^3$ such that $\nu(q_j) = v_j$ and a_j is the weight at the end q_j.*

This theorem and the results for $n \leq 4$ ([L], [KUY]) imply our main theorem in Introduction.

Appendix A

In this appendix, we give two lemmas on real analytic families of algebraic equations which are applied in the proof of Proposition 1.6.

Lemma A.1. *Let $\{f_p(q_1, \ldots, q_n)\}_{p \in \mathbf{R}^\ell}$ and $\{g_p(q_1, \ldots, q_n)\}_{p \in \mathbf{R}^\ell}$ be two real analytic families of polynomials on $\mathbf{C}$ of degree bounded by m. Suppose that there exists a non-empty open subset U such that*

$$Z(f_p) \subset Z(g_p) \qquad (p \in U). \tag{A.1}$$

Then (A.1) *holds for all $p \in \mathbf{R}^\ell$ such that $f_p \not\equiv 0$.*

(Proof.) For each $p \in \mathbf{R}^\ell$, since the degree of f_p is bounded by m, $Z(f_p) \subset Z(g_p)$ if and only if $(g_p)^m$ is divided by f_p. We operate a differential operator

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial q_1^{\alpha_1} \cdots \partial q_n^{\alpha_n}}$$

into the rational function $\varphi_p := (g_p)^m / f_p$. Let $\mathcal{N}^\alpha(\varphi_p)$ be a polynomial formally defined as

$$\mathcal{N}^\alpha(\varphi_p) := (f_p)^{|\alpha|+1} \cdot D^\alpha \varphi$$

which is the numerator part of $D^\alpha \varphi$.

Now we fix an element $p_0 \in \mathbf{R}^\ell$ such that $f_p \not\equiv 0$, and choose an element $q_0 \in \mathbf{C}^n$ such that $f_{p_0}(q_0) \neq 0$. Since f_p is real analytic with respect to the parameter p, we can take a subdomain V of U such that $f_p(q_0) \neq 0$ for all $p \in V$, and φ_p is a polynomial on $\mathbf{C}$ of degree bounded by m^2 for any $p \in V$. Hence for any multi-index $|\alpha| > m^2$, we have $\mathcal{N}^\alpha(\varphi_p)(q_0) = 0$ for $p \in V$. By the real analyticity with respect to the parameter p, we have $\mathcal{N}^\alpha(\varphi_{p_0})(q_0) = 0$ for $|\alpha| > m^2$. Since $f_{p_0}(q_0) \neq 0$, we get $D^\alpha \varphi(q_0) = 0$ for $|\alpha| > m^2$. Thus φ_{p_0} is also a polynomial on $\mathbf{C}$. (q.e.d.)

The following lemma is easily proved by using the Cauchy-Riemann equation.

Lemma A.2. *Let $\mathcal{W}_0$ be a totally real subset of $\mathbf{P}^{n-1}$ defined by*

$$\mathcal{W}_0 := \{[a^1, \dots, a^n] \in \mathbf{P}^{n-1};\ a^j \in \mathbf{R}\ (j = 1, \dots, n)\}.$$

Let h be a homogeneous polynomial on $\mathbf{C}$. If h is identically zero on a non-empty open subset in $\mathcal{W}_0$, then $h \equiv 0$ on $\mathbf{P}^{n-1}$.

Appendix B

Let A be an $n \times n$ matrix. The cofactor matrix B of A is the matrix satisfying the identity $BA = AB = \det A \cdot I$. In this appendix, we give an identity which is useful to compute a differential of the cofactor matrix of a singular matrix.

Let Ω be a domain in $\mathbf{C}$ containing the origin, and $A(q) : \Omega \to M(n, \mathbf{C})$ a smooth map into the set of all $n \times n$ matrices. Let $B(q)$ be the cofactor matrix of $A(q)$. We set $A := A(0)$ and $B := B(0)$. Suppose that

$$\det A = \left.\frac{\partial}{\partial q}\right|_{q=0} \det A(q) = 0. \tag{B.1}$$

Then the following lemma holds.

Lemma B.1. *Let X be an $n \times n$ matrix such that $Tr(XB) \neq 0$. Then the following identity holds:*

$$\begin{aligned} \frac{\partial B}{\partial q}(0) &= \frac{1}{Tr(XB)} \left\{ Tr\left(\frac{\partial A}{\partial q}(0) \cdot \left.\frac{\partial Y_t}{\partial t}\right|_{t=0} \right) \cdot B \right. \\ &\qquad \left. - \left.\frac{\partial Y_t}{\partial t}\right|_{t=0} \cdot \frac{\partial A}{\partial q}(0) \cdot B - B \cdot \frac{\partial A}{\partial q}(0) \cdot \left.\frac{\partial Y_t}{\partial t}\right|_{t=0} \right\}, \end{aligned} \tag{B.2}$$

where Y_t is the cofactor matrix of $A + tX$.

(Proof.) We set $A_t(q) := A(q) + tX$, and denote by $B_t(q)$ its cofactor matrix. We have the following Taylor expansions:

$$\begin{aligned} A_t(q) &= (A + tX) + q\frac{\partial A}{\partial q}(0) + o(q), \\ B_t(q) &= Y_t + q\frac{\partial B_t}{\partial q}(0) + o(q). \end{aligned}$$

Since $A_t(q)B_t(q) = \det A_t(q) \cdot I$, we have by taking the first degree terms that

$$\left.\frac{\partial}{\partial q}\right|_{q=0} \det A_t(q) \cdot I = \frac{\partial A}{\partial q}(0) \cdot Y_t + (A + tX) \cdot \frac{\partial B_t}{\partial q}(0).$$

Since

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \det(A+tX) = \mathrm{Tr}(XB) \neq 0,$$

$A+tX$ is non-singular around $t=0$. Hence we have

$$\begin{aligned}\frac{\partial B_t}{\partial q}(0) &= (A+tX)^{-1}\left(\left.\frac{\partial}{\partial q}\right|_{q=0} \det A_t(q)\cdot I - \frac{\partial A}{\partial q}(0)\cdot Y_t\right) \\ &= \frac{\left.\frac{\partial}{\partial q}\right|_{q=0} \det A_t(q)\cdot Y_t - Y_t\cdot \frac{\partial A}{\partial q}(0)\cdot Y_t}{\det(A+tX)}.\end{aligned}$$

Apply de L'Hospital rule to the right-hand side of $\frac{\partial B}{\partial q}(0) = \lim_{t\to 0} \frac{\partial B_t}{\partial q}(0)$. Then we get the equality (B.2). (q.e.d.)

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