RATIONAL POINTS ON FANO VARIETIES

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En souvenir d'un séminaire de Yuri I. Manin à Moscou, où il était question de points rationnels.

Let $X \subset \mathbb{P}^n$ be a smooth projective variety defined over a field k of characteristic 0, with complement $U := \mathbb{P}^n \setminus X$. Let $\kappa \in \mathbb{N}$. Then the condition

$$\text{Hodge Type } H_c^m(U) \geq \kappa, \text{ for all } m \geq 0 \tag{0.1}$$

on the Hodge type of relative de Rham cohomology, that is the largest integer a for which the Hodge filtration fulfills

$$F^a H_c^m(U) = H_c^m(U), \tag{0.2}$$

is implied by the condition that $H_c^m(U)$ is supported in codimension κ. The converse is one example of the Hodge conjecture: if (0.2) is fulfilled, then there should be a codimension $\geq \kappa$ subvariety $Z \subset \mathbb{P}^n$ with $\mathrm{codim}_X X \cap Z \geq \kappa$ fulfilling

$$H_c^m(U) = H^m(\mathbb{P}^n, X) \xrightarrow{\text{restriction}=0} H^m(\mathbb{P}^n \setminus Z, X \setminus Z \cap X). \tag{0.3}$$

Using a decomposition of the diagonal as initiated by S. Bloch in [1], (0.3) is implied by the condition

$$CH_0(X \times_k K) = \ldots = CH_{\kappa-1}(X \times_k K) = \mathbb{Q} \tag{0.4}$$

on the Chow groups for any algebraically closed field extension $K \supset k$. Thus

$$(0.4) \Rightarrow (0.3) \Rightarrow (0.1). \tag{0.5}$$

The converse

$$(0.1) \Rightarrow (0.4) \tag{0.6}$$

is one example of the Bloch-Beilinson conjectures asserting that Hodge theory controls the Chow groups even after base extension. (In particular, it implies the Hodge conjecture in this specific instance).

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Let now $X \subset \mathbb{P}^n$ be a projective variety over the finite field $\mathbb{F}_q$, with complement $U := \mathbb{P}^n \setminus X$. Let $\kappa \in \mathbb{N}$. Then the congruence condition

$$|X(\mathbb{F}_q)| \equiv |\mathbb{P}^n(\mathbb{F}_q)| \bmod q^\kappa \tag{0.7}$$

on the number of rational points is equivalent to the condition that the reciprocal roots and poles of the zeta function $\zeta(U,t)$ defined by

$$\frac{\zeta'(U,t)}{\zeta(U,t)} = \sum_{s\geq 1} |U(\mathbb{F}_{q^s})| t^{s-1} \tag{0.8}$$

be divisible by q^κ as algebraic integers. If X is smooth, then the Weil conjectures proven by P. Deligne together with Grothendieck-Lefschetz trace formula imply that this condition is equivalent to the eigenvalues of the geometric Frobenius acting on ℓ-adic cohomology $H^m_c(\overline{U}, \mathbb{Q}_\ell)$ being divisible by q^κ as algebraic integers for all $m \geq 0$

$$\text{Eigenvalues Frob. on } H^m_c(\overline{U}, \mathbb{Q}_\ell) \cong q^\kappa \cdot \text{ alg. int.} \tag{0.9}$$

This is implied by the condition (0.3) in étale cohomology

(0.10)

$$H^m_c(\overline{U}, \mathbb{Q}_\ell) = H^m(\overline{\mathbb{P}^n}, \overline{X}, \mathbb{Q}_\ell) \xrightarrow{\text{restriction}=0} H^m(\overline{\mathbb{P}^n \setminus Z}, \overline{X \setminus Z \cap X}, \mathbb{Q}_\ell).$$

The converse is one example of the Tate conjectures, which asserts then that this should be equivalent to the condition that $H^m_c(\overline{U}, \mathbb{Q}_\ell)$ be supported in codimension κ. Thus

$$(0.10) \Rightarrow (0.9) \Rightarrow (0.7). \tag{0.11}$$

Again, using (a generalization of) S. Bloch's decomposition of the diagonal, one has

$$(0.10) \Leftarrow (0.4). \tag{0.12}$$

The converse

$$(0.7) \Rightarrow (0.4) \tag{0.13}$$

would then be an extension of the Bloch-Beilinson conjectures saying that over finite fields, the zeta function controls the Chow groups even after base extension.

One applies this hierarchy of ideas to obtain the following theorem:

Theorem 0.1 ([4]). *Let X be a smooth projective variety defined over the finite field $\mathbb{F}_q$. Let us assume that for all algebraically closed fields $K \supset \mathbb{F}_q$ one has $CH_0(X \times_k K) = \mathbb{Q}$. Then the number of $\mathbb{F}_q$-rational points of X is equal to 1 modulo q.*

One applies this to smooth Fano varieties, that is smooth geometrically irreducible varieties X over a field with ω_X^{-1} ample. According to Campana and Kollár-Miyaoka-Mori, they are rationally connected over any such K, thus a fortiori $CH_0(X \times_k K) = \mathbb{Q}$.

Corollary 0.2 ([4]). *If X is a smooth Fano variety over the finite field $\mathbb{F}_q$, then the number of its $\mathbb{F}_q$-rational points is equal to 1 modulo q.*

That a smooth Fano variety over a finite field has a rational point is a positive answer to a conjecture by S. Lang and Yu. Manin. The original proof in [4] was done using rigid cohomology instead of étale cohomology, and is nicely reported on in [3], together with other applications of slope arguments. However, our proof has two weaknesses: it does not address higher κ's nor does it address singular varieties. For higher κ's, given (0.11) and (0.13), the problem is that one is not able to compute Chow groups. More precisely, Campana/Kollár-Miyaoka-Mori's theorem linking negativity of ω_X and triviality of CH_0 does not have a version for higher CH_m. For singular varieties, one no longer has the Weil conjectures at one's disposal, thus (0.9) implies (0.7) but in general the converse is false. The theorem of Ax-Katz provides a large class of singular varieties for which (0.7) is fulfilled: let $X \subset \mathbb{P}^n$ be defined by r equations of degrees $d_1 \geq \ldots \geq d_r$ and let κ be the integral part of the rational number $\frac{n-d_1-\ldots-d_r}{d_1}$. Then (0.7) holds true. A first step is

Theorem 0.3 ([5]). (0.9) *holds true as well.*

Yet the proof uses both ℓ-adic and p-adic methods, which is not too encouraging. Except again for the q congruence, that is for $\kappa = 1$, where the correct motivic cohomology is defined in [2] and a motivic statement implying (0.9) is proven.

References

[1] Bloch, S. : Lectures on algebraic cycles, Duke University Mathematics Series, IV, Durham, N. C. (1980).

[2] Bloch, S.; Esnault, H.; Levine, M.: Decomposition of the diagonal and eigenvalues of Frobenius for Fano hypersurfaces, preprint 2003, 16 pages.

[3] Chambert-Loir, A.: Points rationnels et groupes fondamentaux: applications de la cohomologie p-adique [d'après P. Berthelot, T. Ekedahl, H. Esnault..], Séminaire Bourbaki Mars 2003.

[4] Esnault, H.: Varieties over finite fields with trivial Chow group of 0-cycles have a rational point, Invent. math. **151** (2003), 187-191.

[5] Esnault, H.; Katz, N.: Cohomological divisibility and point count divisibility, preprint 2003, 10 pages.

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