Noncommutative Local Algebra

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NONCOMMUTATIVE LOCAL ALGEBRA

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The purpose of this work is to introduce the basic facts of noncommutative local algebra and algebraic geometry. Initial objects of the theory are abelian categories (thought as categories of quasi-coherent sheaves). To any abelian category $\mathcal{A}$ a set $\mathbf{Spec}\mathcal{A}$ - the *spectrum* of $\mathcal{A}$ - is assigned, together with canonical topologies on it. Given a topology $\mathfrak{T}$ on $\mathbf{Spec}\mathcal{A}$, canonical or not, there is a naturally defined functor from $\mathcal{A}$ to the category of presheaves on $(\mathbf{Spec}\mathcal{A},\mathfrak{T})$, which sends objects of the category $\mathcal{A}$ into corresponding 'structure' presheaves. The important thing is that, for the canonical topologies, the stalks of the category of quasi-coherent sheaves (we call them the stalks of the category $\mathcal{A}$) are usually much simpler than the category $\mathcal{A}$ itself: they resemble to (are natural generalizations of) categories of modules over local rings.

A short overview of the contents:

In the first section, the notion of the spectrum of an abelian category is introduced and some basic properties of the spectrum are discussed.

The second section is concerned with the localizations at points of the spectrum and the behaviour of the spectrum with respect to arbitrary exact localizations.

In the third section, we introduce and study *local abelian categories* the classical prototypes of which are categories of modules over commutative local rings.

In the fourth section, the obtained results are specified for the category of left modules over an associative ring. This way we recover the introduced in [R1] (cf. also [R2]) *left spectrum* of a ring.

Section 5 deals with basic properties of supports and related to the subsets of the spectrum localizations.

Section 6 is dedicated to the Zariski topology.

In Section 7 some other canonical topologies are discussed.

In Section 8, the associated points of objects of an abelian category are introduced and their properties - straightforward analogs of the classical ones - are established.

Section 9 is concerned with certain functorial properties of the spectrum. We introduce the notion of the spectrum of a functor (- *the relative spectrum*) and study several of related to this notion constructions and facts.

It is worth to mention that the relative spectrum is the main object of investigation in [R4] and [R6] dedicated to applications of the sketched here noncommutative local algebra to the study of representations.

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1. THE SPECTRUM OF AN ABELIAN CATEGORY.

1.1. A preorder in abelian categories. Fix an abelian category $\mathcal{A}$. For any two objects, X and Y, of the category $\mathcal{A}$ we shall write $X \succ Y$ if Y is a subquotient of a coproduct of a finite number of copies of X, i.e. if, for some finite k, there exists a diagram

$$(k)X \longleftarrow U \longrightarrow Y,$$

where the left arrow is a monomorphism, and the right one is an epimorphism; here $(k)X$ denotes a direct sum of k copies of X.

1.1.1. Lemma. *The relation* $\succ$ *is a preorder on* $Ob\mathcal{A}$.

Proof. In fact, let $X \succ Y$, and $Y \succ Z$; i.e. there exist the diagrams

$$(k)X \xleftarrow{\ i\ } U \xrightarrow{\ g\ } Y \qquad (1)$$

and

$$(n)Y \xleftarrow{\ j\ } V \xrightarrow{\ f\ } Z \qquad (2)$$

in which the arrows i, j are monomorphisms, and the arrows g, f are epimorphisms. Evidently, the direct sum of n copies of the diagram (1)

$$(nk)X \xleftarrow{\ (n)i\ } (n)U \xrightarrow{\ (n)g\ } (n)Y$$

is of the same type. Let W be a fibred product of

$$(n)g\colon (n)U \longrightarrow (n)Y \quad \text{and} \quad j : V \longrightarrow (n)Y;$$

and let $i'\colon W \longrightarrow (n)U$ and $h : W \longrightarrow V$ be the canonical projections. Since j is a monomorphism and $(n)g$ is an epimorphism, the arrow i is a monomorphism, and h is an epimorphism. Hence is a monomorpism, and

$$(n)i \circ i' : W \longrightarrow (nk)X$$

$$(n)g \circ h : W \longrightarrow Z$$

is an epimorphism; i.e. $X \succ Z$. ∎

1.1.2. The notation. Denote by $|\mathcal{A}|$ the ordered set of equivalence classes of objects of $\mathcal{A}$ with respect to the relation $\succ$. We save the same symbol, $\succ$, for the induced order on $|\mathcal{A}|$.

1.2. The spectrum of an abelian category. Let M be a nonzero object of the category $\mathcal{A}$. We write $M \in Spec\mathcal{A}$ if, for any nonzero subobject N of M, we have: $N \succ M$. Since $M \succ N$, we can say that $M \in Spec\mathcal{A}$ if and only if it is equivalent with respect to the preorder $\succ$ to any of its nonzero subobjects.

Denote by **Spec**$\mathcal{A}$ the ordered set of equivalence classes (with respect to $\succ$) of elements of $Spec\mathcal{A}$. We call **Spec**$\mathcal{A}$ *the spectrum of the category* $\mathcal{A}$.

1.3. Spectrum and simple objects. Clearly every simple object of the category $\mathcal{A}$ belongs to $Spec\mathcal{A}$. Moreover, we shall see in a moment that two simple objects are equivalent if and only if they are isomorphic.

1.3.1. Proposition. *Let M be a simple object of the category $\mathcal{A}$, and let N be an object of $\mathcal{A}$. Then the following conditions are equivalent:*

(a) N is isomorphic to $(k)M$ for some (finite) k;

(b) $M \succ N$.

Proof. Clearly $(a) \Rightarrow (b)$. Let us show that $(b) \Rightarrow (a)$. By assumption, N is a subquotient of $(l)M$ for some l. Since M is simple, this implies by a standart argument that $N \simeq (n)M$ for some $n \le l$. ∎

1.3.2. Corollary. *Let M and M' be simple objects of the category $\mathcal{A}$. Then $M \succ M'$ if and only if the objects M and M' are isomorphic.*

2. The spectrum and exact localizations.

2.1. Preliminaries about exact localizations. A localization is a functor having a universal property with respect to the class of arrows it inverts (cf. [GZ, I.1.1). Here we are interested in localizations which are exact functors – exact localizations.

Recall that a subcategory $\mathbb{S}$ of the category $\mathcal{A}$ is called *thick* if the following condition holds:

the object M in the exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

belongs to $\mathbb{S}$ if and only if M' and M'' are objects of $\mathbb{S}$. In other words, $\mathbb{S}$ is closed under taking subquotionts and extensions.

2.2. Proposition. *Let $Q: \mathcal{A} \longrightarrow \mathcal{B}$ be an exact localization of an abelian category $\mathcal{A}$. Then, for any $P \in Spec\mathcal{A}$, either $Q(P) = 0$, or $Q(P) \in Spec\mathcal{B}$.*

Proof. 1) Any exact functor is compatible with the preorder $\succ$; in particular, the localization Q is compatible with $\succ$.

2) Let $P \in Spec\mathcal{A} - KerQ$, and let $\varphi : N \rightarrowtail Q(P)$ be a nonzero subobject of $Q(P)$. Since the functor Q is exact, the class $\mathbb{S}$ of all arrows s of $\mathcal{A}$ such that Qs is invertible, admits left and right fractions (cf. [GZ], I.3.4). This implies that there exists a commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{\varphi} & Q(P) \\ u \downarrow & & \downarrow Qs \\ Q(M) & \xrightarrow{Qh} & Q(P') \end{array} \qquad (1)$$

where u and Qs are isomorphisms. Consider the pair of arrows

$$M \xrightarrow{h} P' \xleftarrow{s} P. \qquad (2)$$

Note that s is a monoarrow,

In fact, $Ker(s) \in KerQ$, because Qs is a monomorphism. If $Ker(s) \neq 0$, then, since $P \in Spec\mathcal{A}$, $Ker(s) \succ P$; Therefore P is an object of $KerQ$ which contradicts to the assumption.

Now, since $\varphi: N \longrightarrow Q(P)$ is a nonzero monomorphism, as well as arrows u and Qs (cf. the diagram (1)), Qh is also a monoarrow. Therefore $Ker(h) \in KerQ$. Replacing the arrow h in the diagrams (1) and (2) by the canonical morphism

$$M/Ker(h) \longrightarrow P'$$

and the isomorphism u in the diagram (1) by the composition of u and the isomorphism $Q(M \longrightarrow M/Ker(h)$, we can assume that h is a monomorphism. Consider the pullback of the arrows (2):

$$\begin{array}{ccc} M' & \xrightarrow{\ h'\ } & P \\ {\scriptstyle s'}\downarrow & & \downarrow{\scriptstyle s} \\ M & \xrightarrow{\ h\ } & P' \end{array} \tag{3}$$

By general properties of cartesian squares, the monomorphness of h (resp. s) implies that of h' (resp. s'). Being exact, the functor Q sends the cartesian square (3) into the cartesian square. Hence, the isomorphness of Qs implies that Qs' is an isomorphism.

Now, it follows from the commutativity of (1) that there is unique morphism $\lambda\colon N \longrightarrow Q(M')$ such that

$$Qs'\circ\lambda = u \quad \text{and} \quad Qh'\circ\lambda = \varphi. \tag{4}$$

Moreover, the first of the equalities (4) shows that λ is invertible: $\lambda = Qs'^{-1}\circ u$. In particular, the monomorphism h' is nonzero. Since $P \in Spec\mathcal{A}$, this means that $M' \succ P$ which implies, thanks to the exactness of Q, that $N \simeq Q(M') \succ Q(P)$. ■

2.3. Points of the spectrum and Serre subcategories. Fix an abelian category $\mathcal{A}$.

For any $M \in Ob\mathcal{A}$, consider the full subcategory $\langle M\rangle$ of $\mathcal{A}$ defined as follows: $Ob\langle M\rangle$ consists of all objects N such that the relation $N \succ M$ does not hold.

Note that, for any object M, the subcategory $\langle M\rangle$ contains all subquotients and finite direct sums of copies of any of its objects.

2.3.1. Lemma. *Let $M, M' \in Ob\mathcal{A}$. The following conditions are equivalent:*

(a) $M \succ M'$;

(b) $\langle M'\rangle \subseteq \langle M\rangle$.

Proof. Since we are dealing with full subcategories, the inclusion $\langle M'\rangle \subseteq \langle M\rangle$ is equivalent to that $Ob\langle M'\rangle \subseteq Ob\langle M\rangle$.

Note that $Ob\mathcal{A} - Ob\langle M\rangle$ consists of all objects L of $\mathcal{A}$ such that $L \succ M$. Clearly, since $\succ$ is a preorder (cf. Lemma 1.1.1), $M \succ M'$ if and only if $Ob\mathcal{A} - Ob\langle M\rangle \subseteq Ob\mathcal{A} - Ob\langle M'\rangle$, or, equivalently, if and only if $Ob\langle M'\rangle \subseteq Ob\langle M\rangle$. ■

Thus, the map $M \longmapsto \langle M\rangle$ induces an isomorphism of ordered sets (cf. 1.1.2):

$$|\mathcal{A}| = (|\mathcal{A}|, \succ) \longrightarrow (\{\langle M\rangle \mid M \in Ob\mathcal{A}\}, \supseteq).$$

We shall use this realization of $|\mathcal{A}|$ all the time.

2.3.2. Serre subcategories. For any subcategory $\mathbb{T}$ of an abelian category $\mathcal{A}$, denote by $\mathbb{T}^-$ the full subcategory of $\mathcal{A}$ generated by all objects M such that any nonzero subquotient of M has a nonzero subobject from $\mathbb{T}$.

2.3.2.1. Lemma. *For any subcategory* $\mathbb{T}$ *of an abelian category* $\mathcal{A}$,

(a) the subcategory $\mathbb{T}^-$ *is thick.*

(b) $(\mathbb{T}^-)^- = \mathbb{T}^-$.

(c) $\mathbb{T} \subseteq \mathbb{T}^-$ *if any subquotient of any object of* $\mathbb{T}$ *is isomorphic to an object of* $\mathbb{T}$.

Proof. (a) Suppose that the object M is the exact sequence

$$0 \longrightarrow M' \xrightarrow{\ \mathfrak{i}\ } M \xrightarrow{\ \mathfrak{e}\ } M'' \longrightarrow 0 \tag{1}$$

belongs to $\mathbb{T}^-$; i.e. any nonzero subquotient of M contains a nonzero subobject from $\mathbb{T}$. Since any subquotient of M' or M'' is at the same time a subquotient of M, both M' and M'' have this property.

Conversely, let M' and M'' are objects of $\mathbb{T}^-$. And let L be a nonzero subquotient of M; i.e. there is a diagram

$$M \xleftarrow{\ \iota\ } K \xrightarrow{\ e\ } L,$$

where ι is a monoarrow and e is an epimorphism.

If the composition of the canonical monomorphism

$$\iota': K \cap L \longrightarrow K$$

and e is nonzero, then $L' := im(e \circ \iota')$ is a nonzero subobject of L and a subquotient of M'. Hence L' has a nonzero subobject from $\mathbb{T}$.

If $e \circ \iota' = 0$, then we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \xrightarrow{\ \mathfrak{i}\ } & M & \xrightarrow{\ \mathfrak{e}\ } & M'' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \mathfrak{i}'} & & \\
0 & \longrightarrow & K \cap L & \longrightarrow & K & \longrightarrow & K' & \longrightarrow & 0 \\
 & & & & {\scriptstyle e}\searrow & L & \swarrow{\scriptstyle e'} & &
\end{array}$$

where $\mathfrak{i}'$ is a monoarrow and e' is an epiarrow; i.e. L is a subquotient of M''; hence L has a nonzero subobject from $\mathbb{T}$.

(b), (c). The assertions *(b)* and *(c)* are evident. ∎

Call a subcategory $\mathbb{T}$ of an abelian category $\mathcal{A}$ *a Serre subcategory* if it coincides with $\mathbb{T}^-$.

2.3.3. Proposition. *If an object* P *of the category* $\mathcal{A}$ *belongs to* $Spec\mathcal{A}$, *then* $<P>$ *is a Serre subcategory of* $\mathcal{A}$.

Proof. Suppose that there is an object M of the category $\mathcal{A}$ which belongs to $<P>^-$, but does not belong to $<P>$. The latter means that there is a diagram

$$(n)M \xleftarrow{\ i\ } K \xrightarrow{\ e\ } P,$$

where i is a mono- and e an epimorphism. In other words, P is a subquotient of $(n)M$. According to Lemma 2.3.2.1, $<P>^-$ is a thick category; in particular, $(n)M$ belongs to $<P>^-$. Being a nonzero subquotient of an object from $<P>^-$, the object P has a nonzero subobject from $<P>$. But, since $P \in Spec\mathcal{A}$, any nonzero subobject of P is equivalent to P; hence it cannot belong to the subcategory $<P>$. ■

2.4. Categories with the property (sup). Their Serre subcategories and their spectrum. Consider abelian categories with the following property:

(sup) for any ascending chain Ω *of subobjects of an object* M, *the supremum of* Ω *exists; and for any subobject* L *of* M, *the natural morphism*

$$sup\{X \cap L : X \in \Omega\} \longrightarrow (sup\Omega) \cap L$$

is an isomorphism.

The categories with property (sup) are called otherwise *the categories with exact direct limits.*

2.4.1. Example: Grothendieck categories. Recall that an abelian category $\mathcal{A}$ is called a *Grothendieck category* if it has a set of generators, and, besides, the following condition holds:

AB5. There exists a direct sum of every set of objects of $\mathcal{A}$, and, for any ascending chain Ω of subobjects of an object M and for any subobject N of M, the natural morphism

$$sup\{X \cap N : X \in \Omega\} \longrightarrow (sup\Omega) \cap N$$

is an isomorphism.

Note that the existence of small direct sums garantees the existence of $sup\Omega$ for any family Ω of subobjects of any object M of $\mathcal{A}$, since this

$sup\Omega$ is the image of the canonical arrow

$$\bigoplus_{X \in \Omega} X \longrightarrow M.$$

Thus Grothendieck categories are categories with coproducts, a set of generators and the property (sup).

Recall three important examples of Grothendieck categories:

1) The category $R\text{-}mod$ of left modules over an associative ring R.

2) The category of sheaves of R-modules on an arbitrary topological space.

3) The category of quasi-coherent sheaves on a quasi-compact and quasi-separated scheme.

Note that it is not known if the category of quasi-coherent sheaves on an arbitrary scheme has enough injectives or even all limits ([TT], B.2). But, one can easily see that it has the property (sup).

In fact, the inclusion of the category $Qcoh(\mathbf{X})$ on a scheme $\mathbf{X}$ into the category $\mathcal{O}_{\mathbf{X}}\text{-}mod$ of $\mathcal{O}_{\mathbf{X}}$-modules is a fully faithful exact functor which reflects finite limits and all colimits. This implies that the category $Qcoh(\mathbf{X})$ has all colimits and inherits the property (sup) from $\mathcal{O}_{\mathbf{X}}\text{-}mod$. ■

2.4.2. Example: noetherian categories. An object M of a category $\mathcal{A}$ is called *noetherian* if any set of its subobjects has a maximal element. An abelian category $\mathcal{A}$ is called *noetherian* if it has a set of generators and all its objects are noetherian.

The standart examles of a noetherian category is the category of left modules of finite type over a left noetherian ring and the category of coherent sheaves on a noetherian scheme.

Note that if all objects of an abelian category $\mathcal{A}$ are noetherian, then the category $\mathcal{A}$ enjoys the condition *(sup)*.

This is a consequence of the fact that the condition *(sup)* restricted to finite families of subobjects holds for any abelian category. ■

2.4.3. Lemma. *Let an abelian category* $\mathcal{A}$ *have the property (sup). And let* $\mathbb{T}$ *be a Serre subcategory of the category* $\mathcal{A}$.

Suppose that Ω *is a family of subobjects of an object* M *of the category* $\mathcal{A}$ *such that* $M = sup\Omega$; *and* Ω *is a subset of the subcategory* $\mathbb{T}$. *Then* M *also belongs* $\mathbb{T}$.

Proof. We need to show that the object $M \simeq sup\Omega$ belongs to the subcategory $\mathbb{T}^{-}$.

In fact, let Ω' denote the directed family

$$\{sup\mathfrak{X} \mid \mathfrak{X} \text{ is a finite subset of } \Omega\}$$

of subobjects of M. For any finite subset $\mathfrak{X}$ of Ω, $sup\mathfrak{X}$ is isomorphic to the image of the canonical arrow

$$\bigoplus_{L \in \mathfrak{X}} L \longrightarrow M,$$

Since the subcategory $\mathbb{T}$ is thick (cf. Lemma 2.3.2.1), hence closed under any finite coproducts, $\bigoplus_{L \in \mathfrak{X}} L$ and, therefore, its image in M, $sup\mathfrak{X}$, belong to $\mathbb{T}$.

Let now we be given an arbitrary diagram

$$M \xleftarrow{\ i\ } K \xrightarrow{\ e\ } X,$$

where i is a monoarrow, e is an epiarrow, and $X \neq 0$. For any finite subset $\mathfrak{X}$ of Ω, denote by $K(\mathfrak{X})$ the intersection of K with the subobject $sup\mathfrak{X}$. Thanks to the property *(sup)*, there is a finite subset $\mathfrak{X} \subseteq \Omega$ such that the composition of the embedding $\iota: K(\mathfrak{X}) \longrightarrow K$ and the epimorphism e is nonzero. The image of $e \circ \iota$ is a nonzero object from $\mathbb{T}$.

This shows that $M \in Ob\mathbb{T}^{-}$. ■

Recall that a subcategory $\mathbb{S}$ of a category $\mathcal{A}$ is said to be *coreflective* (resp. *reflective)* if the inclusion functor $\mathbb{S} \longrightarrow \mathcal{A}$ has a right (resp. left) adjoint.

2.4.4. Lemma. *(a) Any coreflective thick subcategory of an abelian category is a Serre subcategory.*

(b) Let $\mathcal{A}$ be an abelian category with the property (sup). Then any Serre subcategory of $\mathcal{A}$ is coreflective.

Proof. (a) Let $\mathcal{A}$ be an arbitrary abelian category; and let $\mathbb{T}$ be a coreflective subcategory of $\mathcal{A}$. Take an arbitrary object M in the subcategory $\mathbb{T}^{-}$. Since $\mathbb{T}$ is coreflective, M has the $\mathbb{T}$-torsion, $\mathbb{T}(M)$, - the biggest among the subobjects of M which belong to $\mathbb{T}$.

If the quotient object $M/\mathbb{T}(M)$ is nonzero, then $M/\mathbb{T}(M)$ has a nonzero subobject, X, from $\mathbb{T}$. Since the subcategory $\mathbb{T}$ is thick, the preimage of X in M is a subobject of M containing properly $\mathbb{T}(M)$ which contradicts to the maximality of $\mathbb{T}(M)$.

Therefore $M/\mathbb{T}(M) = 0$; i.e. $M = \mathbb{T}(M) \in Ob\mathbb{T}$.

(b) Fix a Serre subcategory $\mathbb{T}$ of the category $\mathcal{A}$.

For any object M of $\mathcal{A}$, consider the family $\mathbb{T}\{M\}$ of all subobjects of M which belong to $\mathbb{T}$. According to Lemma 2.4.3, $\mathbb{T}(M) := sup\mathbb{T}\{M\}$ belong to $\mathbb{T}$. Clearly, $\mathbb{T}(M)$ is the $\mathbb{T}$-*torsion of* M - the biggest among the subobjects of M which belong to $\mathbb{T}$.

One can see that the map $M \longmapsto \mathbb{T}(M)$ defines uniquely (once the subobject $\mathbb{T}(M)$ is chosen for each M) the right adjoint to the inclusion functor

$$\mathbb{T} \longrightarrow \mathcal{A}. \quad \blacksquare$$

2.4.5. Corollary. *Let an abelian category* $\mathcal{A}$ *have the property (sup). Then, for any Serre subcategory* $\mathbb{T}$ *of* $\mathcal{A}$, *the embedding* $J_{\mathbb{T}}: \mathbb{T} \longrightarrow \mathcal{A}$ *preserves and reflects colimits.*

In particular, any Serre subcategory of $\mathcal{A}$ *is closed under small coproducts (taken in* $\mathcal{A}$*).*

Proof. In other words, the assertion sounds as follows:

for any small diagram $D: \mathfrak{D} \longrightarrow \mathbb{T}$, $colim(D)$ exists if and only if the colimit of the composition $J_{\mathbb{T}} \circ D$ exists, and the canonical arrow

$$colim(J_{\mathbb{T}} \circ D) \longrightarrow J_{\mathbb{T}}(colim(D))$$

is an isomorphism.

(i) Suppose that $colim(J_{\mathbb{T}} \circ D)$ exists. Denote this colimit by M, and take as Ω the family of images of all canonical arrows

$$J_{\mathbb{T}} \circ D(x) \longrightarrow M, \quad x \in Ob\mathfrak{D}.$$

Clearly the canonical arrow $sup\Omega \longrightarrow M$ is an isomorphism. By Lemma 2.4.3, this means that M belongs to $\mathbb{T}^{-} = \mathbb{T}$.

(ii) If $colim(D)$ exists, then $J_{\mathbb{T}}(colim(D))$ is canonically isomorphic to $colim(J_{\mathbb{T}} \circ D)$.

This follows from the existence of a right adjoint to the functor $J_{\mathbb{T}}$. ■

2.4.6. Note. The assertion *(i)* in the proof of Corollary 2.4.5 is a special case of a more general fact. Namely, it is a consequence of the full faithfulness of the embedding $J_{\mathbb{T}}$, the existence of a right adjoint to $J_{\mathbb{T}}$ functor (cf. Lemma 2.4.4, and Proposition I.1.4 in [GZ]). ■

Corollary 2.4.5 shows that the given here definition of a Serre subcategory coincides with the conventional one in the case of Grothendieck categories.

2.4.7. Proposition. *Let an abelian category* $\mathcal{A}$ *have the property (sup). Then, for any object* V *of* $\mathcal{A}$ *such that* $\langle V\rangle$ *is a Serre subcategory, there is an object* $P \in Spec\mathcal{A}$ *which is equivalent to* V*; i.e.* $\langle V\rangle = \langle P\rangle$.

Proof. Let M be an object of the category $\mathcal{A}$ such that $\langle M\rangle$ is a Serre subcategory. Since $\mathcal{A}$ has the property *(sup)*, the subcategory $\langle M\rangle$ is coreflective (cf. Lemma 2.4.4); i.e. each object L of $\mathcal{A}$ has $\langle M\rangle$-torsion $\langle M\rangle(L)$. Denote by $f\langle M\rangle L$ the quotient object $L/t\langle M\rangle(L)$. Since $\langle M\rangle$ is thick, $f\langle M\rangle L$ is $\langle M\rangle$-torsion free; hence $f\langle M\rangle L$ and L belong or do not belong to $\langle M\rangle$ simultaniously. In particular, M is equivalent to $f\langle M\rangle M$ with respect to $\succ$. Clearly any nonzero subobject of $f\langle M\rangle M$, being $\langle M\rangle$-torsion free, is equivalent to $f\langle M\rangle M$ with respect to $\succ$; i.e. $f\langle M\rangle M \in Spec\mathcal{A}$. ■

2.4.8. Serre subcategories and flat localizations. We call an exact localization $Q: \mathcal{A} \longrightarrow \mathcal{B}$ *flat* if the functor Q has a (necessarily fully faithful) right adjoint functor.

A thick subcategory $\mathbb{T}$ of an abelian category $\mathcal{A}$ is called *localizing* if it is a kernel of a flat localization.

Note that any localizing subcategory is coreflective.

In fact, let Q be a flat localization, $Q^\wedge$ a right adjoint to Q functor, and $\eta: Id_{\mathcal{A}} \longrightarrow Q^\wedge \circ Q$ an adjunction arrow. Then the map $M \longmapsto Ker\eta(M)$ defines a functor which is right adjoint to the inclusion functor $KerQ \longrightarrow \mathcal{A}$.

It is known (cf. [Gab], Ccrollary 3.3.3) that if $\mathcal{A}$ is an abelian category with injective hulls, then the converse is true:

A thick subcategory of an abelian category with injective hulls is localizing if and only if it is coreflective.

This and Lemma 2.4.4 imply the following assertion:

2.4.8.1. Proposition. *Let* $\mathcal{A}$ *be an abelian category with property (sup) and with injective hulls. Then any Serre subcategory of* $\mathcal{A}$ *is localizing.*

2.4.8.2. Corollary. *Suppose that* $\mathcal{A}$ *is an abelian category with propery (sup) and with injective hulls. Then*

a) The map $Q \longmapsto KerQ$ *provides a one-to-one correspondence between the* *equivalence classes of flat localizations of $\mathcal{A}$ and Serre subcategories of the category $\mathcal{A}$.*

b) For any $P \in Spec\mathcal{A}$, the subcategory $<P>$ is localizing.

Note that Corollary 2.4.8.2 is applicable to the case when $\mathcal{A}$ is a Grothendieck category, because any Grothendieck category has both the property *(sup)* and injective hulls.

3. LOCAL ABELIAN CATEGORIES AND LOCALIZATION AT POINTS OF THE SPECTRUM.

Thus, according to Proposition 2.3.3, to any point $<M>$ of **Spec$\mathcal{A}$** an exact localization, $Q_{<M>}: \mathcal{A} \longrightarrow \mathcal{A}/<M>$, corresponds.

Our immediate goal is to show that these localizations at points of the spectrum (or, ruther, quotient categories $\mathcal{A}/<M>$) are as special, as the localizations of categories of modules over a commutative ring at points of the prime spectrum.

3.1. Local abelian categories. A nonzero object M of an abelian category $\underline{A}$ will be called *quasifinal* if $N \succ M$ for any nonzero object N of $\mathcal{A}$.

In other words, a nonzero object M is quasi-final if and only if $<M> =$

$$\{0\} = \bigcap_{N \in Ob\mathcal{A}-\{0\}} <N>.$$

Clearly a quasifinal object of the category $\mathcal{A}$ (if any) belongs to $Spec\mathcal{A}$, and every two quasifinal objects of $\mathcal{A}$ are equivalent.

3.1.1. Definition. An abelian category $\mathcal{A}$ will be called *local* if it possesses a quasifinal object. ■

3.1.2. Lemma. *The following properties of an abelian category $\mathcal{A}$ are equivalent:*

(a) $\mathcal{A}$ is local and has simple objects;

(b) any nonzero object of $\mathcal{A}$ has a simple subquotient, and all simple objects of $\mathcal{A}$ are isomorphic one to another.

Proof. (a) $\Rightarrow$ (b). Let M be a quasifinal object and L a simple object of the category $\mathcal{A}$. Then $L \succ M$ which implies, by Proposition 1.3.1, that M is a coproduct of a finite number of copies of L; hence M is equivalent to L.

Since L in this argument is an arbitrary simple object, we have obtained that all simple objects are equivalent each other which means, according to Corollary 1.3.2, that they are pairwise isomorphic.

The implication *(b)* $\Rightarrow$ *(a)* is evident. ■

3.1.3. Example. It is easy to see that the category *R-mod* of left modules over an associative ring R is local if and only if any two maximal left ideals m and m' are equivalent in the following sense: $m' = (m:x)$ for some $x \in R$, where $(m:x) = \{y \in R \mid yx \in m\}$ by definition. In particular, the category of modules over a commutative ring k is local if and only if the ring k is local. ■

3.2. Local categories and local rings. For any abelian category $\mathcal{A}$, denote by $\mathfrak{z}(\mathcal{A})$ the "center" of $\mathcal{A}$ which is, by definition, the ring of endomorphisms of the identical functor $Id_{\mathcal{A}}$. Clearly the ring $\mathfrak{z}(\mathcal{A})$ is commutative.

3.2.1. Proposition. *Let $\mathcal{A}$ be a local abelian category. Then the ring $\mathfrak{z}(\mathcal{A})$ is local.*

Proof. Let M be a quasi-final object in the category $\mathcal{A}$. *(a) The endomorphism ξ of $Id_{\mathcal{A}}$ is invertible if and only if $\xi(M) \neq 0$.*

Suppose that $Ker\xi(X) \neq 0$ for some object X; and let α be the canonical monomorphism $Ker\xi(X) \longrightarrow X$. The equality

$$0 = \xi(X)\circ\alpha = \alpha\circ\xi(Ker\xi(X))$$

implies that $\xi(Ker\xi(X)) = 0$.

Since M is a quasi-final object, there exists a diagram

$$(I)Ker\xi(X) \xleftarrow{\ i\ } V \xrightarrow{\ e\ } M$$

where i is a monomorphism and e is an epimorphism.

$$0 = \xi((I)Ker\xi(X))\circ i = i\circ\xi(V) \Rightarrow \xi(V) = 0,$$

since i is a monoarrow, and

$$0 = e\circ\xi(V) = \xi(M)\circ e \Rightarrow \xi(M) = 0$$

thanks to the epimorphness of e.

Suppose now that $Cok\xi(X) \neq 0$; and let υ be the canonical epimorphism

$$X \longrightarrow Cok\xi(X).$$

The equalities

$$\xi(Cok(X))\circ \mathfrak{v} = \mathfrak{v}\circ\xi(X) = 0$$

imply that $\xi(Cok\xi(X)) = 0$.

Since M is a quasi-final object, there exists a diagram

$$(n)Cok\xi(X) \xleftarrow{i'} V \xrightarrow{e'} M$$

where i' is a monomorphism and e' is an epimorphism which implies (by the same argument as above) that $\xi(M) = 0$.

Thus, if $\xi(M) \neq 0$, then $\xi(X)$ is an isomorphism for any object X in $\mathcal{A}$; i.e. ξ is an isomorphism.

In particular, $\xi(M)$ is invertible if and only if $\xi(M) \neq 0$.

(b) Thus, the map $\xi \longmapsto \xi(M)$ is an epimorphism of the ring $\mathfrak{z}(\mathcal{A})$ onto the skew field $\{\xi(M) \mid \xi \in \mathfrak{z}(\mathcal{A})\}$. ■

3.3. Localizations at points of the spectrum. Now we are going to get one of the most convincing indications that the chosen here notion of the spectrum is a right one.

3.3.1. Proposition. *Let $\mathcal{A}$ be an abelian category. For any object M of the category $\mathcal{A}$ such that $\langle M\rangle$ is a thick subcategory of $\mathcal{A}$, the quotient category $\mathcal{A}/\langle M\rangle$ is local.*

Proof. Denote by Q the localization $\mathcal{A} \longrightarrow \mathcal{A}/\langle M\rangle$. Fix a nonzero object, X, of the quotient category $\mathcal{A}/\langle M\rangle$. There is an object X' of the category $\mathcal{A}$ such that $X \simeq Q(X')$. Since the object X is nonzero, $X' \notin Ob\langle M\rangle$ which means that $X' \succ M$. The last relation is respected by exact functors. In particular, we have: $X \simeq Q(X') \succ Q(M)$. Thus, $Q(M)$ is a quasi-final object of the category $\mathcal{A}/\langle M\rangle$. ■

3.3.2. Corollary. *For any abelian category $\mathcal{A}$ and any object M from $Spec\mathcal{A}$, the quotient category $\mathcal{A}/\langle M\rangle$ is local.*

Proof. By Proposition 2.3.3, if $M \in Spec\mathcal{A}$, then $\langle M\rangle$ is a thick subcategory. ■

4. THE LEFT SPECTRUM OF A RING.

Let $\mathcal{A}$ be the category R-*mod* of left modules over an associative ring R

with unity. Since each module from *Spec(R-mod)* is equivalent to any of its cyclic submodules, we can restrict ourselves to the modules of the form R/m, where m runs over the set $I_l R$ of left ideals of the ring R. The next step, which we are going to do now, is to translate the defined above notions of preorder $\succ$ and spectrum from the language of modules into the language of left ideals.

4.1. Lemma. *Let* m *and* n *be left ideals of the ring* R. *The relation* $R/m \succ R/n$ *is equivalent to the following condition:*

(#) there exists a finite set y *of elements of the ring* R *such that the ideal* $(m:y) := \{z \in R \mid zy \subset m\}$ *is contained in the ideal* n.

Proof. By definition, the relation $R/m \succ R/n$ means that, for some positive integer k, there exist a submodule N of the module $(k)R/m$ and an epimorphism $f: N \longrightarrow R/n$. Let e be the image of the unity e of the ring R under the canonical epimorphism $R \longrightarrow R/n$; and let z be an element of the module N such that $f(z) = e$. It is clear that the restriction of the epimorphism f onto the cyclic submodule Rz is also an epimorphism. This implies that the annihilator $Ann(z)$ of z is contained in the annihilator $Ann(e')$ of the element e'. But $Ann(e') = n$, and $Ann(z) = (m:y)$, where $y = \{y_1, y_2, ..., y_k\}$ is the set of elements of the ring R such that z is the direct sum of $\pi(y_i)$, $i = 1,...,k$; here π is the canonical map $R \longrightarrow R/m$.

Conversely, if the left ideal $(m:y)$ is contained in the left ideal n for some finite set $y = \{y_1, ..., y_k\}$ of elements of the ring R, then there exists an epimorphism of the generated by direct sum of the elements $\pi(y_i)$, $i = 1,...,k$, submodule of the module $(k)R/m$ onto R/n. ■

We will write $m \to n$ if the left ideals m and n satisfy the condition (#) of Lemma 4.1. Thus, the preorder $\to$ in the category *R-mod* induces the preorder $\to$ in the set $I_l R$ of left ideals of the ring R.

4.2. Proposition. *Let* p *be a left ideal of the ring* R. *The quotient module* R/p *belongs to* *SpecR-mod* *if and only if the following condition holds:*

(*) for any $x \in R - p$, *the left ideal* $(p:x)$ *is equivalent to* p *with respect to the preorder* $\to$; *or, what is the same,* $(p:x) \to p$.

Proof is left to the reader.■

Thus, the set of all left ideals p such that the module R/p belongs to the spectrum of the category *R-mod* coincides with the left spectrum, $Spec_l R$, of the ring R (cf. I.1).

4.3. Localizations at points of the left spectrum of a ring. We have the following picture.

4.3.1. Proposition. *For every ideal p from the left spectrum of the ring R, the localization of the category R-mod at p is naturally realized as a full local subcategory R-mod/<p> of R-mod, or, what is sometimes more convenient, as a full local subcategory G<p>R-mod/<G<p>p> of the category G<p>R-mod.*

Proof. This assertion follows from results of Section I.2 and I.0. We leave the details to the reader. ■

The ring R is called *left local* if the category *R-mod* of left R-modules is local. Since *R-mod* is a category of finite type, it means that all simple left R-modules are isomorphic to each other, or, equivalently, for any two left maximal ideals m and m' of the ring R there exists an element z of R such that $m' = (m:z)$.

4.4. Lemma. *Let* $p \in Spec_l R$. *The following conditions are equivalent:*

(a) The natural module morphism G<p>R/G<p>p ⟶ GF(R/p) is an isomorphism.

(b) The quotient module G<p>R/G<p>p belongs to the (quotient) subcategory G<p>R-mod/<G<p>p>.

(c) The functor G<p> is exact; i.e.

G<p>R-mod/<G<p>p>=G<p>R-mod.

(d) The ring G<p>R is left local, and its left ideal G<p>p is equivalent to a left maximal ideal.

Proof is left to the reader. ■

4.5. Remark. The equivalent conditions of Lemma 4.4 hold for any commutative ring and for any hereditary ring. They hold also for some "good" rings, such as certain rings of differential operators, and some others. But the left spectrum of most rings (and even most among interesting rings) is far from being abundant with points satisfying the conditions of Lemma 4.4. ■

5. SUPPORTS AND LOCALIZING SUBCATEGORIES.

5.1. The topology τ. Fix an abelian category $\mathcal{A}$. Clearly the least requirement on a topology of the spectrum, **Spec**$\mathcal{A}$, of the category $\mathcal{A}$ is that it should be compatible with the preorder $\succ$. This means that the closure of any point $<P>$ of **Spec**$\mathcal{A}$ should contain the set $\mathfrak{s}(<P>)$ of all specializations of that point, i.e. all $<P'> \in$ **Spec**$\mathcal{A}$ such that $<P'> \subseteq <P'>$.

Denote by τ the strongest among the topologies having this property. It is easy to describe τ explicitly: the closure of a subset W with respect to τ is $\bigcup_{<P> \in W} \mathfrak{s}(<P>)$.

One can check that not only the intersection, but also the union of any family of closed in the topology τ subsets is closed. This shows that τ is too strong to be really useful in terms of applications. Still, since any admissible topology lives inside of τ, it is convenient to take τ into account.

5.2. Supports. The *support* of an object M of an abelian category $\mathcal{A}$ is the set $Supp(M)$ of all $<P> \in$ **Spec**$\mathcal{A}$ such that $M \succ P$.

For instance, if $M \in$ **Spec**$\mathcal{A}$, then $Supp(M)$ coincides with the set $\mathfrak{s}(M)$ of specializations of M - the closure of the M in the topology τ (cf. 5.1).

Clearly $Supp(M)$ is closed in the topology τ for any object M.

Note that $Supp(M)$ depends only on the equivalence class, $<M>$, of the object M. So, we could write $Supp(<M>)$ instead of $Supp(M)$.

5.2.1. Lemma. *For any object* M *of the category* $\mathcal{A}$,

$$Supp(M) = \{<P> \in \mathbf{Spec}\mathcal{A} \mid Q_{<P>}M \neq 0\},$$

where $Q_{<P>}$ *is the localization at* $<P>$.

Proof. In fact, by the definition of the support,

$$<P> \in Supp(M) \text{ if and only if } M \notin <P>.$$

On the other hand, $Q_{<P>}M \neq 0$ if and only if $M \notin <P>$. ∎

5.2.2. Proposition. *(a) For any exact short sequence*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0, \qquad (1)$$

$$Supp(M) = Supp(M') \cup Supp(M'').$$

(b) Suppose $\mathcal{A}$ *is a Grothendieck category. If an object* M *is the supremum of a family,* Ξ, *of its subobjects, then*

$$Supp(M) = \bigcup_{X \in \Xi} Supp(X).$$

Proof. *(a)* Since $M \succ M'$, as well as $M \succ M''$, and $\succ$ is a transitive relation, we have the inclusion

$$Supp(M) \supseteq Supp(M') \cup Supp(M'').$$

On the other hand, for any $<P> \in \mathbf{Spec}\mathcal{A}$, the localization $Q_{<P>}$, being an exact functor, sends the exact sequence (1) into the exact sequence

$$0 \longrightarrow Q_{<P>}M' \longrightarrow Q_{<P>}M \longrightarrow Q_{<P>}M'' \longrightarrow 0. \tag{2}$$

If $<P> \in Supp(M)$, then, according to Lemma 5.2.1, $Q_{<P>}M \neq 0$ which implies, thanks to the exactness of (2) that either $Q_{<P>}M' \neq 0$, or $Q_{<P>}M'' \neq 0$.

(b) Now, let $\mathcal{A}$ be a Grothendieck category. Again, we have the inclusion

$$Supp(M) \supseteq \bigcup_{X \in \Xi} Supp(X)$$

for free.

Note now, that, since the functor $Q_{<P>}$ is flat, it sends subobjects into subobjects, and the canonical arrow

$$\sup_{X \in \Xi} Q_{<P>}X \longrightarrow Q_{<P>}(\sup_{X \in \Xi} X) \simeq Q_{<P>}M$$

is an isomorphism. Hence, if $<P> \in Supp(M)$, i.e. $Q_{<P>}M \neq 0$ (cf. Lemma 5.2.1), then $Q_{<P>}X \neq 0$ for some $X \in \Xi$ which means, by Lemma 5.2.1, that $<P>$ belongs to $Supp(X)$ for that particular X. ∎

5.2.3. Corollary. *For any family* Ξ *of objects of a Grothendieck category* $\mathcal{A}$,

$$Supp\Big(\bigoplus_{X \in \Xi} X\Big) = \bigcup_{X \in \Xi} Supp(X).$$

5.2.4. Lemma. *The map* $M \longmapsto \mathbf{Spec}\mathcal{A} - Supp(M)$ *is a functor from the preorder* $|\mathcal{A}| := (Ob\mathcal{A}, \succ)$ *to the preorder* $(Open(\tau), \subseteq)$ *of open subsets of the topological space* $(\mathbf{Spec}\mathcal{A}, \tau)$.

Proof. Since the relation $\succ$ is transitive, the map

$$M \longmapsto Supp(M) := \{<P> \mid M \succ P\}$$

is a contravariant functor from $[\mathcal{A}]$ to the preorder (under $\subseteq$) of closed subsets of the topology τ on $\mathbf{Spec}\mathcal{A}$ which implies the assertion of the lemma. ∎

5.3. Subsets of the spectrum and topologizing subcategories. For any abelian category $\mathcal{B}$, denote by $|\mathcal{B}|$ the order induced by $\succ$; i.e. $|\mathcal{B}|$ can be regarded as the set of full subcategories $\{<M> \mid M \in Ob\mathcal{B}\}$ with the order $\supseteq$ (cf. Lemma 2.3.1).

Call a full subcategory, $\mathbb{T}$, of an abelian category $\mathcal{A}$ *topologizing* if it contains all subquotients (in $\mathcal{A}$) of any of its objects and a coproduct (in $\mathcal{A}$) of any pair of its objects.

Clearly any thick subcategory is topologizing.

5.3.1. Lemma. *Any topologizing subcategory* $\mathbb{T}$ *of a abelian category* $\mathcal{A}$ *defines a subset of* $\mathbf{Spec}\mathcal{A}$, $\mathbb{T} \longmapsto |\mathbb{T}| \cap \mathbf{Spec}\mathcal{A}$, *which is closed in the topology* τ.

Moreover, $|\mathbb{T}| \cap \mathbf{Spec}\mathcal{A} = \mathbf{Spec}\mathbb{T}$.

Proof. In fact, since the category $\mathbb{T}$ is topologizing, it contains with every object X all the objects Y of the category $\mathcal{A}$ such that $X \succ Y$. In particular, the set $|\mathbb{T}| \cap \mathbf{Spec}\mathcal{A}$ is closed in the topology τ.

The equality $|\mathbb{T}| \cap \mathbf{Spec}\mathcal{A} = \mathbf{Spec}\mathbb{T}$ is left to the reader as an exercise. ■

5.3.2. Proposition. *(a) For any subset* $W \subseteq \mathbf{Spec}\mathcal{A}$, *the full subcategory* $\mathcal{A}(W)$ *generated by all objects* M *of the category* $\mathcal{A}$ *such that* $Supp(M) \subseteq W$ *is a Serre subcategory.*

(b) The subcategory $\mathcal{A}(W)$ *coincides with the subcategory* $<W\perp> := \bigcap_{\mathbf{p} \in W\perp} <\mathbf{p}>$, *where* $W\perp := \mathbf{Spec}\mathcal{A} - W$.

(c) If the set W *is closed in the topology* τ, *then (and only then)*

$$\mathbf{Spec}\mathcal{A} \cap |\mathcal{A}(W)| = \mathbf{Spec}\mathcal{A}(W) = W.$$

Proof. (b) It follows from the definitions of *Supp* and $<W\perp>$ that the relation $M \in Ob<W\perp>$ means exactly that

$$Supp(M) \cap W\perp = \varnothing, \text{ i.e. } Supp(M) \subseteq W.$$

(a) One can check that the intersection of any set of Serre subcategories is a Serre subcategory. Thus the assertion *(b)* imply the assertion *(a)*.

(c) For any $<P> \in \mathbf{Spec}\mathcal{A}$, the support of P coincides with the set $\mathfrak{s}<P>$ of specializations of $<P>$ (or, the closure of the 'point' $<P>$ in topology τ). Hence $\mathbf{Spec}\mathcal{A} \cap |\mathcal{A}(W)|$ consists of all $<P> \in W$ such that $\mathfrak{s}<P> \subseteq W$. ■

Thus, we have the diagram:

$$
\begin{array}{ccc}
\{\text{thick subcategories of } \mathcal{A}\} & \searrow & \\
\uparrow & & \{\text{topologizing subcategories of } \mathcal{A}\} \\
\{\text{closed subsets of } (\mathbf{Spec}\mathcal{A}, \tau)\} & \swarrow &
\end{array}
$$

The first arrow determines the localization at any closed subset of $(\mathbf{Spec}\mathcal{A},\tau)$; the second arrow can be used to create new topologies on $\mathbf{Spec}\mathcal{A}$. Namely, choosing a class of topologizing subcategories of $\mathcal{A}$, we obtain a set of subsets in $\mathbf{Spec}\mathcal{A}$ which is considered as a base of closed subsets of a topology.

We are going to use this procedure in Section 6 to define the *Zariski* topology.

5.4. The residue field of a point. Fix any point $\langle P\rangle$ of $\mathbf{Spec}\mathcal{A}$. According to Proposition 5.3.2, $\mathcal{A}(\langle P\rangle^{-}):= \mathcal{A}(Supp(P))$ is a thick subcategory of $\mathcal{A}$. Consider the quotient category

$$\mathcal{K}^{-}\langle P\rangle := \mathcal{A}(\langle P\rangle^{-})/\langle P\rangle.$$

Clearly the category $\mathcal{K}^{-}\langle P\rangle$ is local. Moreover, one can see that it is "zero-dimensional"; i.e. $\mathbf{Spec}\mathcal{K}^{-}\langle P\rangle$ consists of only one point.

Denote by $\mathcal{K}\langle P\rangle$ the full subcategory of the category $\mathcal{K}^{-}\langle P\rangle$ generated by all objects M of $\mathcal{K}^{-}\langle P\rangle$ which are supremum of its subobjects $V \longrightarrow M$ such that $\langle V\rangle = \langle P\rangle$.

We call the category $\mathcal{K}\langle P\rangle$ *the residue category of* $\langle P\rangle$.

One can check that the subcategory $\mathcal{K}\langle P\rangle$ is topologizing which implies that it inherits the nice properties of the category $\mathcal{K}^{-}\langle P\rangle$: it is local and its spectrum consists of only one point.

5.4.1. Lemma. *(a) If one of the categories* $\mathcal{A}/\langle P\rangle$, $\mathcal{K}^{-}\langle P\rangle$ *and* $\mathcal{K}\langle P\rangle$ *has objects of finite type, then the other two also enjoy this property.*

(b) If the quotient category $\mathcal{A}/\langle P\rangle$ *has objects of finite type, then* $\mathcal{K}\langle P\rangle$ *is equivalent to the category of modules over a skew field.*

Proof. (a) A local category has objects of finite type if and only if its quasi-final object is semisimple. Clearly, the latter property holds for all the listed in the assertion *(a)* categories if it holds for one of them.

(b) Since quasi-final objects of $\mathcal{K}\langle P\rangle$ are semisimple, every nonzero object of $\mathcal{K}\langle P\rangle$, being a sum of its simple subobjects, is semisimple. Thus, $\mathcal{K}\langle P\rangle$ is a semisimple category with only one up to isomorphism simple object, say M.

Therefore the functor

$$X \longmapsto \mathcal{K}\langle P\rangle(M,X)$$

from $\mathcal{K}\langle P\rangle$ to the category $K(\langle P\rangle)\text{-}Vec$ of vector spaces over the skew field $K(\langle P\rangle) := \mathcal{K}\langle P\rangle(M,M)$ is an equivalence of categories. ■

We call the field $K(\langle P\rangle)$ from (the proof of) Lemma 5.4.1 *the residue skew field of the point* $\langle P\rangle$.

Clearly the residue skew field of a point is defined uniquely up to isomorphism.

6. Left closed subcategories and Zariski topology.

6.0. Preliminaries about the Gabriel multiplication. For any two subcategories, $\mathbb{X}$, $\mathbb{Y}$ of an abelian category $\mathcal{A}$, define their product $\mathbb{X}\bullet\mathbb{Y}$ as the full subcategory of $\mathcal{A}$ generated by all objects M of $\mathcal{A}$ such that there exists an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

with $M' \in Ob\mathbb{Y}$ and $M'' \in Ob\mathbb{X}$. One can check that if $\mathbb{X}$ and $\mathbb{Y}$ are topologizing subcategories, then such is $\mathbb{X}\bullet\mathbb{Y}$.

Note that, for any three topologizing subcategories, $\mathbb{S}$, $\mathbb{T}$, and $\mathbb{U}$,

$$\mathbb{S}\bullet(\mathbb{T}\bullet\mathbb{U}) = (\mathbb{S}\bullet\mathbb{T})\bullet\mathbb{U}; \quad \text{and} \quad 0\bullet\mathbb{S} = \mathbb{S}\bullet 0 = \mathbb{S}.$$

It follows from definitions that a topologizing subcategory $\mathbb{T}$ is thick if and only if $\mathbb{T}\bullet\mathbb{T} = \mathbb{T}$.

6.1. The Gabriel multiplication and the spectrum. Recall that, for any topologizing subcategory $\mathbb{S}$ of $\mathcal{A}$, the set $\mathbf{V}(\mathbb{S})$ consists of all $\langle P\rangle \in \mathbf{Spec}\mathcal{A}$ such that $P \in Ob\mathbb{S}$.

6.1.1. Lemma. *For any pair* $\mathbb{S}$, $\mathbb{T}$ *of topologizing subcategories of an abelian category* $\mathcal{A}$, *we have:* $\mathbf{V}(\mathbb{S}\bullet\mathbb{T}) = \mathbf{V}(\mathbb{S}) \cup \mathbf{V}(\mathbb{T})$.

Proof. a) Clearly $\mathbb{S} \subseteq \mathbb{S}\bullet\mathbb{T} \supseteq \mathbb{T}$ which implies the inclusion

$$\mathbf{V}(\mathbb{S}\bullet\mathbb{T}) \supseteq \mathbf{V}(\mathbb{S}) \cup \mathbf{V}(\mathbb{T}). \tag{1}$$

b) Let $\langle P\rangle \in \mathbf{V}(\mathbb{S}\bullet\mathbb{T})$; i.e. $P \in Spec\mathcal{A} \cap Ob(\mathbb{S}\bullet\mathbb{T})$. The latter means that there exists an exact sequence

$$0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0$$

in which $P' \in Ob\mathbb{T}$ and $P'' \in Ob\mathbb{S}$.

If $P' \neq 0$, then $P' \succ P$; hence $P \in Ob\mathbb{T}$.

If $P' = 0$, then $P \simeq P''$; i.e. $P \in Ob\mathbb{S}$.

Thus

$$Spec\mathcal{A} \cap Ob(\mathbb{S}\bullet\mathbb{T}) \subseteq (Spec\mathcal{A} \cap Ob\mathbb{S}) \cup (Spec\mathcal{A} \cap Ob\mathbb{T})$$

which implies the inverse to (1) inclusion:

$$\mathbf{V}(\mathbb{S}\bullet\mathbb{T}) \subseteq \mathbf{V}(\mathbb{S}) \cup \mathbf{V}(\mathbb{T}). \quad \blacksquare$$

Identifying $\mathbf{V}(?)$ with $\mathbf{Spec}(?)$, we can rewrite the equality $\mathbf{V}(\mathbb{S}\bullet\mathbb{T}) = \mathbf{V}(\mathbb{S}) \cup \mathbf{V}(\mathbb{T})$ of Lemma 6.1.1 as

$$\mathbf{Spec}(\mathbb{S}\bullet\mathbb{T}) = \mathbf{Spec}(\mathbb{S}) \cup \mathbf{Spec}(\mathbb{T}). \tag{2}$$

6.2. Left closed subcategories. A subcategory $\mathbb{S}$ of an abelian category $\mathcal{A}$ is called *closed* if it is both topologizing and coreflective (in [Gab], IV.4) . We call a subcategory $\mathbb{S}$ of $\mathcal{A}$ *left closed* if it is topologizing and reflective.

6.2.1. Lemma. *Suppose that subcategories* $\mathbb{S}$ *and* $\mathbb{T}$ *of an abelian category* $\underline{A}$ *are closed (resp. left closed). Then the subcategory* $\mathbb{S}\bullet\mathbb{T}$ *is closed (resp. left closed).*

Proof. Since the subcategories $\mathbb{S}$ and $\mathbb{T}$ are topologizing, such is their Gabriel product $\mathbb{S}\bullet\mathbb{T}$ (cf. 6.0).

a) Let $\mathbb{S}$ and $\mathbb{T}$ be closed; i.e. the inclusion functors

$$J_{\mathbb{S}}: \mathbb{S} \longrightarrow \mathcal{A} \quad \text{and} \quad J_{\mathbb{T}}: \mathbb{T} \longrightarrow \mathcal{A}$$

have right adjoints $J_{\mathbb{S}}^{\wedge}$ and $J_{\mathbb{T}}^{\wedge}$ respectively. Following [Gab], denote by $\mathbb{S}$ the functor $J_{\mathbb{S}} \circ J_{\mathbb{S}}^{\wedge}: \mathcal{A} \longrightarrow \mathcal{A}$ which assigns to any object M of $\mathcal{A}$ the biggest among subobjects of M which belong to the subcategory $\mathbb{S}$. For any object M of $\mathcal{A}$, denote by $M_{\mathbb{S},\mathbb{T}}$ the kernel of the composition of epimorphisms

$$M \longrightarrow M/\mathbb{T}M \longrightarrow M/\mathbb{S}(M/\mathbb{T}M)$$

It is clear that $M_{\mathbb{S},\mathbb{T}}$ contains $\mathbb{T}M$ and the quotient object $M/M_{\mathbb{S},\mathbb{T}}$ belongs to the subcategory $\mathbb{S}$; i.e. $M_{\mathbb{S},\mathbb{T}} \in Ob\mathbb{S}\bullet\mathbb{T}$. It is equally evident that $M_{\mathbb{S},\mathbb{T}}$ is the biggest among the subobjects of M which belong to $\mathbb{S}\bullet\mathbb{T}$; i.e. $M_{\mathbb{S},\mathbb{T}} \simeq (\mathbb{S}\bullet\mathbb{T})M$.

b) Note that a subcategory $\mathbb{T}$ of $\mathcal{A}$ is topologizing iff its opposite, $\mathbb{T}^{op}$, is a topologizing subcategory in $\mathcal{A}^{op}$. And also, for any two subcategories $\mathbb{S}$ and $\mathbb{T}$ of $\mathcal{A}$, we have:

$$(\mathbb{S}\bullet\mathbb{T})^{op} = \mathbb{T}^{op}\bullet\mathbb{S}^{op}.$$

Finally, note that a subcategory $\mathcal{S}$ of $\mathcal{A}$ is reflective iff its dual, $\mathcal{S}^{op}$ is a coreflective subcategory of $\mathcal{A}^{op}$.

This shows that the assertion about left closedsubcategories follows from the assertion about closed subcategories. ■

6.2.2. Lemma. *Let an abelian category* $\mathcal{A}$ *have supremums of sets of subobjects (for instance,* $\mathcal{A}$ *is a category with coproducts).*

Then the intersection of any set of left closed subcategories is a left closed subcategory.

Proof. Clearly the intersection of any set of topologizing subcategories of $\mathcal{A}$ is a topologizing subcategory. So, it remains to show that, under the assumption, the reflectiveness stands the intersections.

In fact, let Ω be a set of reflective subcategories of $\mathcal{A}$. Fix an object M of $\mathcal{A}$; and, for any $\mathcal{S} \in \Omega$, denote by $K\mathcal{S}(M)$ the kernel of an adjunction arrow

$$\varepsilon_{\mathcal{S}}(M):\ M \longrightarrow J_{\mathcal{S}}\circ{}^{\wedge}J_{\mathcal{S}}(M).$$

Here, as usual, ${}^{\wedge}J_{\mathcal{S}}$ denotes a left adjoint to the inclusion functor

$$J_{\mathcal{S}}:\ \mathcal{S} \longrightarrow \mathcal{A}.$$

Set $\Omega K(M) := sup\{K\mathcal{S}(M) \mid \mathcal{S} \in \Omega\}$. Note that the quotient object $M/\Omega K(M)$ belongs to the intersection $\bigcap_{\mathcal{S} \in \Omega} \mathcal{S}$.

This follows from the epimorphness of $\varepsilon_{\mathcal{S}}(M)$ for any $\mathcal{S} \in \Omega$, and from the fact that every $\mathcal{S} \in \Omega$, being topologizing, contains all quotients of any of its objects. And $M/\Omega K(M)$ is a quotient of $J_{\mathcal{S}}\circ{}^{\wedge}J_{\mathcal{S}}(M)$ for every $\mathcal{S} \in \Omega$.

On the other hand, if $g: M \longrightarrow V$ is any arrow such that V belongs to $\bigcap_{\mathcal{S} \in \Omega} \mathcal{S}$, then, for any $\mathcal{S} \in \Omega$, the morphism g is (uniquely) represented as a composition $g_{\mathcal{S}}\circ\varepsilon_{\mathcal{S}}(M)$. Therefore the kernel of g contains $K\mathcal{S}(M)$ for all $\mathcal{S} \in \Omega$ which implies that g is represented as a composition of the canonical epimorphism

$$\varepsilon(M):\ M \longrightarrow M/\Omega K(M)$$

and a uniquely defined arrow $g_{\Omega}: M/\Omega K(M) \longrightarrow V$.

This shows that the functor which assigns to an object M of $\mathcal{A}$ the object $M/\Omega K(M)$ and acting correspondingly on morphisms, is left adjoint to the inclusion functor $\bigcap_{\mathcal{S} \in \Omega} \mathcal{S} \longrightarrow \mathcal{A}$. ■

6.2.3. Note. The proof of Lemma 6.2.2 shows that, for any abelian category $\mathcal{A}$, the intersection of a finite number of left closed subcategories is left closed. By duality the same holds for closed subcategories. ■

6.3. Zariski topology. For any abelian category $\mathcal{A}$, denote by $\mathfrak{Z}\mathfrak{T}$ the set of sets $\mathbf{V}(\mathbb{T})$, where $\mathbb{T}$ runs through the set of all left closed subcategories of the category $\mathcal{A}$.

6.3.1. Lemma. *For any abelian category* $\mathcal{A}$, *the set* $\mathfrak{Z}\mathfrak{T}$ *is closed under finite intersections and finite unions.*

If $\mathcal{A}$ *is a category with supremums of sets of subobjects, then* $\mathfrak{Z}\mathfrak{T}$ *is admits arbitrary intersections.*

Proof. By Lemma 6.1.1, $\mathbf{V}(\mathbb{S}) \cup \mathbf{V}(\mathbb{T}) = \mathbf{V}(\mathbb{S}\bullet\mathbb{T})$ for any pair of topologizing subcategories of $\mathcal{A}$. And, according to Lemma 6.2.1, the subcategory $\mathbb{S}\bullet\mathbb{T}$ is left closed if $\mathbb{S}$ and $\mathbb{T}$ are. Hence the set $\mathfrak{Z}\mathfrak{T}$ is closed under finite unions. Clearly

$$\mathbf{V}\Big(\bigcap_{\mathbb{S}\in\Omega}\mathbb{S}\Big) = \bigcap_{\mathbb{S}\in\Omega}\mathbf{V}(\mathbb{S})$$

for any set Ω of topologizing subcategories of $\mathcal{A}$. If Ω is a finite set of left closed subcategories than $\bigcap_{\mathbb{S}\in\Omega}\mathbb{S}$ is also left closed by Lemma 6.2.2 (cf. Note 6.2.3). If Ω is an infinite set of left closed subcategories, but the category $\mathcal{A}$ has supremums of sets of subobjects, then, again by Lemma 6.2.2, the subcategory $\bigcap_{\mathbb{S}\in\Omega}\mathbb{S}$ is left closed. Thus in both cases, $\bigcap_{\mathbb{S}\in\Omega}\mathbf{V}(\mathbb{S})$ belongs to $\mathfrak{Z}\mathfrak{T}$. ■

We define the *Zariski topology* on $\mathbf{Spec}\mathcal{A}$ as the topology $\mathfrak{Z}\mathfrak{t}$ which has $\mathfrak{Z}\mathfrak{T}$ as a base of closed sets. According to Lemma 6.3.1, if the category $\mathcal{A}$ has supremums of sets of subobjects, then $\mathfrak{Z}\mathfrak{T}$ coincides with the set of all closed sets of the Zariski topology.

6.4. Left closed subcategories and the spectrum of a category of modules. Let $\underline{A}$ be the category R-*mod* left modules over a ring R. Let α be a two-sided ideal in R; and let $R\text{-}mod|\alpha$ denote the full subcategory of R-*mod* generated by all R-modules M such that $Ann(M) \supseteq \alpha$.

Clearly the canonical (full) embedding

$$R/\alpha\text{-}mod \longrightarrow R\text{-}mod$$

induces an equivalence of categories $R/\alpha\text{-}mod \longrightarrow R\text{-}mod|\alpha$.

6.4.1. Proposition. *Left closed subcategories of the category R-mod are exactly the subcategories $R\text{-}mod|\alpha$, where α runs through the set IR of all two-sided ideals of the ring R.*

Proof. 1) Clearly $R\text{-}mod|\alpha$ is a left closed subcategory of the category R-*mod*, since

(*a*) it is a topologizing subcategory;

(*b*) the canonical embedding $R\text{-}mod|\alpha \longrightarrow R\text{-}mod$ has a left adjoint functor, $M \longmapsto M/\alpha M \simeq R/\alpha \otimes_R M$.

2) Let $\mathbb{S}$ be a left closed subcategory of the category $\mathcal{A} = R\text{-}mod$,

$$J_{\mathbb{S}}: \mathbb{S} \longrightarrow \mathcal{A}$$

the natural embedding, and $^{\wedge}J_{\mathbb{S}}$ the left adjoint to $J_{\mathbb{S}}$ functor. Since the adjunction arrow

$$\eta = \eta_{\mathbb{S}}: Id \longrightarrow J_{\mathbb{S}} \circ {}^{\wedge}J_{\mathbb{S}}$$

is an epimorphism, the generator $J_{\mathbb{S}}R$ is isomorphic to R/α for some left ideal α. This ideal α is, actually, two-sided.

In fact, the quotient modules $R/(\alpha:x)$, $x \in R\text{-}\alpha$, being cyclic submodules of R/α, are in $\mathbb{S}$.

(Recall that $(\alpha:x) = \{r \in R \mid rx \in \alpha\}$.)

The projection

$$\pi_x: R \longrightarrow R/(\alpha:x)$$

is represented as a composition of

$$\pi_1 = \eta_{\mathbb{S}}(R) : R \longrightarrow R/\alpha$$

and a uniquely defined arrow

$$h_x: R/\alpha \longrightarrow R/(\alpha:x).$$

Since π_x is an epimorphism, h_x is an epimorphism. But the epimorphness of h_x means that $\alpha \subseteq (\alpha:x)$. Therefore, since the element $x \in R\text{-}\alpha$ parametrizing this inclusion is arbitrary, the ideal α is two-sided.

This implies, since the R-module R/α is a generator of the subcategory $\mathbb{S}$, that $\mathbb{S}$ coincides with $R\text{-}mod|\alpha$. ∎

Let Ω be a family of left closed subcategories of R-*mod*. By Proposition

6.2, Ω is a set of the categories $R\text{-}mod|\alpha$, where α runs through a set, X, of two-sided ideals in R.

One can see that $\bigcap_{\alpha \in X} R\text{-}mod|\alpha = R\text{-}mod|(\sum_{\alpha \in X} \alpha)$.

6.4.2. The Zariski topology on the left spectrum. Consider now the left spectrum $Spec_l R$ of the ring R which consists of all left ideals p in R such that the module R/p belongs to $SpecR\text{-}mod$ (cf. Section 4). The canonical surjection

$$Spec_l R \longrightarrow \mathbf{Spec}R\text{-}mod, \quad p \longmapsto <R/p>, \tag{1}$$

allows to transfer the Zariski topology (as any other topology) on $Spec_l R$. Namely, define the Zariski topology, $\tau^\wedge$, on $Spec_l R$ as the weakest topology such that the map (1) is continuous; i.e. the set of closed sets in $\tau^\wedge$ consists of preimages of closed sets in the Zariski topology $\mathfrak{Z}$ on $\mathbf{Spec}R\text{-}mod$. Clearly the preimage of the set $\mathbf{V}(\alpha) := \mathbf{V}(R\text{-}mod|\alpha)$, where α is a two-sided ideal in R, is the set $\mathbf{V}_l(\alpha) = \{p \in Spec_l R \mid \alpha \subseteq p\}$. And the equalities

$$\mathbf{V}(\mathbb{S} \bullet \mathbb{T}) = \mathbf{V}(\mathbb{S}) \cup \mathbf{V}(\mathbb{T}), \quad \mathbf{V}(\bigcap_{\mathbb{S} \in \Omega} \mathbb{S}) = \bigcap_{\mathbb{S} \in \Omega} \mathbf{V}(\mathbb{S})$$

correspond to the equalities

$$\mathbf{V}_l(\alpha\beta) = \mathbf{V}_l(\alpha) \cup \mathbf{V}_l(\beta), \quad \mathbf{V}_l(\sum_{\mathbb{S} \in \Omega} \alpha) = \bigcap_{\mathbb{S} \in \Omega} \mathbf{V}_l(\alpha).$$

6.4.3. The Serre subcategory related to a Zariski closed set. Fix a two-sided ideal α in the ring R. To the closed set $\mathbf{V}(\alpha) := \mathbf{V}(R\text{-}mod|\alpha)$, there corresponds the Serre subcategory, $\mathfrak{L}(\alpha)$, of $R\text{-}mod$ generated by all R-modules M such that

$$Supp(M) \subseteq \mathbf{Spec}(R\text{-}mod|\alpha). \tag{1}$$

(cf. Proposition 5.3.2). The following Lemma can be regarded as an estimate of the difference between $\mathfrak{L}(\alpha)$ and $R\text{-}mod|\alpha$.

6.4.3.1. Lemma. *Suppose that an R-module* M *has the property:*

$$\alpha \subseteq \mathcal{L}(Ann(M))$$

for some two-sided ideal α. *Then* $M \in Ob\mathfrak{L}(\alpha)$.

Here $\mathcal{L}(Ann(M)) :=$ *Levitzki radical of the ideal* $Ann(M) :=$ *the preimage of the biggest locally nilpotent ideal in* $R/Ann(M)$.

Proof. 1) *For any two R-modules,* M *and* M', *the relation* $M \succ M'$ *impli-*

es the inclusion $Ann(M) \subseteq Ann(M')$.

In particular, for any $<P> \in Supp(M)$, $Ann(M) \subseteq Ann(P)$.

In fact, the relation $M \succ M'$ means that there is a diagram

$$(\nu)M \xleftarrow{\ i\ } L \xrightarrow{\ e\ } M'$$

where ν is a positive integer, i is a monoarrow, and e is an epimorphism. Thus, we have:

$$Ann(M) = Ann((\nu)M) \subseteq Ann(L) \subseteq Ann(M').$$

2) By Theorem I.4.10.2, the Levitzki radical $L(Ann(M))$ is the intersection of all the ideals $Ann(P)$, where $<P>$ runs through all elements of the spectrum, **Spec**$(R\text{-}mod)$, such that $Ann(M) \subseteq Ann(P)$.

Thus, we have the inclusion:

$$L(Ann(M)) \subseteq \bigcap_{<P> \in\ Supp(M)} Ann(P)$$

which implies the assertion immediately. ■

6.5. When the Zariski topology has a base of quasi-compact open sets? One of the most important properties of the conventional Zariski topology is the quasi-compactness of affine schemes and the (following from it) existence of a base of quasi-compact open subsets for a general scheme.

We are going to show that, in the noncommutative case, the affine objects - the spectra of categories of modules - are still quasi-compact and have (canonical) base of open compact subsets in the Zariski topology.

Note that the second fact does not follow from the first one, and, certainly, does not imply the existence of a base of quasi-compact open subsets of the topological space (**Spec**$\mathcal{A}$, $\mathfrak{Z}$t) for a general abelian category $\mathcal{A}$, since most of abelian (or even Grothendieck) categories are not locally affine.

The following investigation provides a way to find out if the topological space (**Spec**$\mathcal{A}$, $\mathfrak{Z}$t) has a base of quasi-compact open sets for a wide class of abelian categories.

6.5.1. Lemma. *Let an abelian category* $\mathcal{A}$ *have a generator of finite type. Then any left closed subcategory of* $\mathcal{A}$ *enjoys the same property.*

Proof. (a) Let M be a generator of finite type in $\mathcal{A}$; and let $^\wedge J$ be a left adjoint to the inclusion functor $J_{\mathbb{S}} = J$ of a closed subcategory $\mathbb{S}$ into $\mathcal{A}$. Then $^\wedge J(M)$ is a generator of finite type of the category $\mathbb{S}$.

In fact, $\hat{}J(M)$ is a generator of the category $\mathbb{S}$ because any arrow α from M to $J(V)$ is represented as a composition of the adjunction morphism

$$\eta(M):\ M \longrightarrow J\circ\hat{}J(M)$$

and an arrow $J\alpha'$ for a uniquely defined arrow $\alpha':\hat{}J(M) \longrightarrow V$.

(b) Note that the adjunction arrow $\eta\colon Id_{\mathcal{A}} \longrightarrow J\circ\hat{}J$ is an epimorphism.

In fact, for any object M of the category $\mathcal{A}$, the image of $\eta(M)$, being a subobject of an object from $\mathbb{S}$, is also an object of $\mathbb{S}$. Therefore, thanks to the universal property of $\eta(M)$, the canonical monoarrow

$$Im(\eta(M)) \longrightarrow J\circ\hat{}J(M)$$

is an isomorphism; i.e. $\eta(M)$ is an epimorphism.

(c) Since the adjunction arrow $\eta(M)$ is an epimorphism, and M is of finite type, the object $J\circ\hat{}J(M)$ is of finite type. This implies, thanks to the faithfulness of J, that the object $\hat{}J(M)$ is of finite type. ∎

Let $\mathbb{S}$ be a closed subcategory of a Grothendieck category $\mathcal{A}$, $J_{\mathbb{S}}$ the natural embedding of $\mathbb{S}$ into $\mathcal{A}$, and

$$\eta_{\mathbb{S}}\colon Id_{\mathcal{A}} \longrightarrow J_{\mathbb{S}}\circ\hat{}J_{\mathbb{S}}$$

an adjunction arrow.

For any object V of the category $\mathcal{A}$, set for convenience

$$\mathbb{S}V := J_{\mathbb{S}}\hat{}\circ J_{\mathbb{S}}(V), \quad \text{and} \quad K_{\mathbb{S}}V = Ker\eta_{\mathbb{S}}(V).$$

Thus, we have the short exact sequence

$$0 \longrightarrow K_{\mathbb{S}}M \longrightarrow M \xrightarrow{\eta_{\mathbb{S}}(M)} \mathbb{S}M \longrightarrow 0 \qquad (\mathbb{S})$$

To the inclusion $\mathbb{S} \subseteq \mathbb{S}'$, there corresponds a morphism

$$(\mathbb{S}') \longrightarrow (\mathbb{S})$$

of the exact sequences corresponds such that

the arrow $M \longrightarrow M$ is identical;

the arrow $\mathbb{S}'M \longrightarrow \mathbb{S}M$ is an epimorphism;

the arrow $K_{\mathbb{S}'}M \longrightarrow K_{\mathbb{S}}M$ is (therefore) a monomorphism.

Now, let $\mathcal{A}$ have a generator M of finite type.

Call the closed subcategory $\mathbb{S}$ *finite* if $K_{\mathbb{S}}M$ is of finite type with respect to the subobjects $K_{\mathbb{T}}M$, $\mathbb{S} \subseteq \mathbb{T}$; i.e. for any inductive system of subobjects $K_{\mathbb{T}}M \longrightarrow K_{\mathbb{S}}M$, $\mathbb{T} \in \Omega$, such that $sup\Omega = K_{\mathbb{S}}M$, the arrow $K_{\mathbb{T}}M \longrightarrow K_{\mathbb{S}}M$ is an isomorphism for some $\mathbb{T} \in \Omega$.

It is left to the reader to check that the notion is well defined; i.e. it does not depend on the choice of a generator of finite type.

6.5.2. Proposition. *Let an abelian category $\mathcal{A}$ have the property (sup) and a generator of finite type. And let Ξ be a family of closed subcategories of the category $\mathcal{A}$ such that the intersection, $\mathbb{S}$, of all the categories from Ξ is finite. Then $\mathbb{S}$ is the intersection of a finite number of categories from Ξ.*

Proof. 1) Denote by Ω all possible finite intersections of categories from Ξ.

2) Fix a generator of finite type, M, of the category $\mathcal{A}$. For any $\mathbb{T} \in \Omega$, $J_{\mathbb{T}}M$ is a generator of the subcategory $\mathbb{T}$ (cf. the proof of Lemma 6.5.1).

The exact sequences

$$0 \longrightarrow K_{\mathbb{T}}M \longrightarrow M \xrightarrow{\eta_{\mathbb{T}}(M)} M\mathbb{T} \longrightarrow 0 \qquad (\mathbb{T})$$

$\mathbb{T} \in \Omega$, form an inductive system; and the limit of this inductive system is again an exact sequence (thanks to the property *(sup)*) which we denote by

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0. \qquad (1)$$

Since, for any $\mathbb{T} \in \Omega$, the canonical arrow $\mathbb{T}M \longrightarrow M''$ is an epimorphism, $M'' \in \bigcap_{\mathbb{T} \in \Omega} Ob\mathbb{T} = Ob\mathbb{S}$.

On the other hand, the canonical arrows of short sequences

$$(\mathbb{T}) \longrightarrow (\mathbb{S}), \quad \mathbb{T} \in \Omega, \qquad (2)$$

form a cone. The cone (2) defines a unique arrow from the sequence (1) to the sequence $(\mathbb{S})$. In particular, we have a commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & M'' \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \sigma} \\ M & \longrightarrow & \mathbb{S}M \end{array}$$

Since M'' is an object of $\mathbb{S}$, and $M \longrightarrow \mathbb{S}M$ is the universal arrow, σ is an isomorphism which means that the whole arrow $(1) \longrightarrow (\mathbb{S})$ is an isomorphism. In particular, the object M' in the sequence (1) is isomorphic to $K_{\mathbb{S}}M$.

Since $K_{\mathbb{S}}M$ is the inductive limit of $\{K_{\mathbb{T}}M, \ \mathbb{T} \in \Omega\}$, and $\mathbb{S}$ is finite by assumption, $K_{\mathbb{T}}M \longrightarrow M'$ is an isomorphism for some $\mathbb{T} \in \Omega$. Therefore $\mathbb{T}M \simeq M'' \simeq \mathbb{S}M$; i.e. the subcategories $\mathbb{S}$ and $\mathbb{T}$ coincide. ■

6.5.2.1. Corollary. *Let an abelian category* $\mathcal{A}$ *have the property (sup) and a generator of finite type. Then the topological space* $(\mathbf{Spec}\mathcal{A}, \mathfrak{Zt})$ *is quasi- compact.*

Proof. It is just the application of Proposition 6.5.2 in the case when $\mathbb{S} = \{o\}$. ∎

6.5.3. The Zariski topology in the affine case. Let $\mathcal{A}$ be the category *R-mod* of left modules over an associative ring *R*.

Recall that a closed subset *W* of a topological space *X* is *noetherian* if any family Ω of closed subsets of *X* such that *X* equals to $\bigcap_{Y \in \Omega} Y$ contains a finite subfamily which has the same property. In other words, the closed set *W* is noetherian iff the open set *X* - *W* is quasi-compact.

6.5.3.1. Proposition. *A closed (in the Zariski topology) subset W is noetherian if and only if it coincides with* **Spec**$(R\text{-}mod|\alpha)$ *for some finitely generated two-sided ideal* α.

Proof. *1)* By Proposition 6.4.1, any left closed subcategory of the category *R-mod* equals to $R\text{-}mod|\alpha'$ for some two-sided ideal α'. The left closed subcategory $R\text{-}mod|\alpha$ is finite if and only if the ideal α is finitely generated (as a two-sided ideal).

Therefore, according to Corollary 6.5.2.1, the closed set **Spec**$(R\text{-}mod|\alpha)$ is noetherian for any finitely generated two-sided ideal α.

2) Suppose now that the closed set V = **Spec**$(R\text{-}mod|\alpha')$ is noetherian. The (two-sided) ideal α' is the supremum (union) of an inductive system $\mathfrak{F}(\alpha')$ of its finitely generated two-sided subideals. This implies that **Spec**$(R\text{-}mod|\alpha')$ is the intersection of **Spec**$(R\text{-}mod|\alpha)$, where α runs through the set $\mathfrak{F}(\alpha')$. Since the topological space **Spec**$(R\text{-}mod|\alpha')$ is noetherian, it coincides with **Spec**$(R\text{-}mod|\alpha)$ for some $\alpha \in \mathfrak{F}(\alpha')$. ∎

6.5.3.2. Corollary. *For any associative ring R, the topological space* $(\mathbf{Spec}R\text{-}mod, \mathfrak{Zt})$*, where* $\mathfrak{Zt}$ *is the Zariski topology, is quasi-compact and has a base of quasi-compact open subsets.*

6.5.3.3. Remark. Proposition 6.5.3.1 has been obtained in [R2] (a detailed account is in [R3]) as a corollary of the following, much more subtle, fact (Theorem I.4.10.2):

The intersection of all ideals of the left spectrum of a ring R coinsides with the biggest locally nilpotent ideal in R.

One of the consequences of this theorem is that the topological space $(\mathbf{Spec}R\text{-}mod, \mathfrak{z}t)$ is quasi-homeomorphic to the *Levitzki spectrum of* R which is, by definition, the subspace of the prime spectrum, $\mathbf{Spec}R$, formed by all the prime ideals p in R such that the quotient ring R/p has no nonzero locally nilpotent ideals.

Note that the Levitzki spectrum, $L\mathbf{Spec}R$, is a *sober* space; i.e. any irreducible closed subset of $L\mathbf{Spec}R$ has unique generic point (Theorem I.5.3). ■

7. Some other canonical topologies.

7.1. The central topology. Fix a abelian category $\mathcal{A}$, and denote by $\mathfrak{z}(\mathcal{A})$ the ring of endomorphisms of $Id_{\mathcal{A}}$.

For any $\xi \in \mathfrak{z}(\mathcal{A})$, denote by $\mathcal{A}_\xi$ the full subcategory of $\mathcal{A}$ generated by all those objects M for which $\xi(M) = 0$.

7.1.1. Lemma. *The subcategory $\mathcal{A}_\xi$ is closed and left closed.*

Proof. In fact, the maps

$$M \longmapsto Ker\xi(M) \quad \text{and} \quad M \longmapsto Cok\xi(M), \quad M \in Ob\mathcal{A},$$

are uniquely extended to functors, $Ker\xi$ and $Cok\xi$, from $\mathcal{A}$ to $\mathcal{A}$ which have the canonical morphisms

$$\mathfrak{k}\xi: Ker\xi \longrightarrow Id_{\mathcal{A}} \quad \text{and} \quad \mathfrak{c}\xi: Id_{\mathcal{A}} \longrightarrow Cok\xi.$$

The both functors take values in the subcategory $\mathcal{A}_\xi$.

This follows immediately from the commutative diagram

$$\begin{array}{ccccc} Ker\xi & \xrightarrow{\mathfrak{k}\xi} & Id_{\mathcal{A}} & \xrightarrow{\mathfrak{c}\xi} & Cok\xi \\ {\scriptstyle \xi(Ker\xi)}\downarrow & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi(Cok\xi)} \\ Ker\xi & \xrightarrow{\mathfrak{k}\xi} & Id_{\mathcal{A}} & \xrightarrow{\mathfrak{c}\xi} & Cok\xi \end{array}$$

Denote the corestrictions of the functors $Ker\xi$ and $Cok\xi$ onto $\mathcal{A}_\xi$ by $K\xi$ and $C\xi$ respectively.

Clearly, the functor $K\xi$ is right adjoint to the embedding $J\xi: \mathcal{A}_\xi \longrightarrow \underline{A}$

having the adjunction arrows $\mathfrak{k}\xi$, id, and $C\xi$ is left adjoint to $J\xi$ with the adjunction arrows id, $\mathfrak{c}\xi$.

One can see that $\mathcal{A}_{\xi}$ is a topologizing subcategory of the category $\mathcal{A}$. Therefore, by Lemma 7.1.1, it is both closed and left closed. ∎

7.1.2. Lemma. *For any* $\xi \in \mathfrak{z}(\mathcal{A})$ *and* $<P> \in$ **Spec**$\mathcal{A}$, *either* $\xi(P)$ *is a monomorphism, or* $\xi(P) = 0$.

Proof. Suppose that $Ker\xi(P) \neq 0$, and let $i: Ker\xi(P) \longrightarrow P$ denote the canonical monomorphism. The equalities

$$i \circ \xi(Ker\xi(P)) = \xi(P) \circ i = 0$$

show that $\xi(Ker\xi(P)) = 0$.

On the other hand, since $<P> \in$ **Spec**$\mathcal{A}$, there exists a diagram

$$(l)Ker\xi(P) \xleftarrow{\;\mathfrak{i}\;} V \xrightarrow{\;\mathfrak{e}\;} P$$

for some integer $l \geq 1$ such that $\mathfrak{i}$ is a monomorphism and $\mathfrak{e}$ is an epimorphism. The equalities

$$\mathfrak{i} \circ \xi(V) = \xi((l)Ker\xi(P)) \circ \mathfrak{i} = (l)\xi(Ker\xi(P)) \circ \mathfrak{i} = 0 \circ \mathfrak{i} = 0$$

imply that $\xi(V) = 0$; and it follows from the epimorphness of $\mathfrak{e}$ and the equalities

$$\xi(P) \circ \mathfrak{e} = \mathfrak{e} \circ \xi(V) = \mathfrak{e} \circ 0 = 0$$

that $\xi(P) = 0$. ∎

7.1.3. Corollary. *For any* $<P> \in$ **Spec**$\mathcal{A}$, *the set*

$$(<P>) := \{\xi \in \mathfrak{z}(\mathcal{A}) \mid \xi(P) = 0\}$$

is a prime ideal in the ring $\mathfrak{z}(\mathcal{A})$.

Thus, we have a well defined map

$$\varphi = \varphi_{\mathcal{A}}: \mathbf{Spec}\mathcal{A} \longrightarrow \mathbf{Spec}\mathfrak{z}(\mathcal{A}).$$

Define the *central* topology, $\tau_{\mathfrak{z}}$, on **Spec**$\mathcal{A}$ as the weakest topology for which the map φ is continuous. In other words, the sets

$$V(X) := \{<P> \in \mathbf{Spec}\mathcal{A} \mid \xi(P) = 0 \text{ for every } \xi \in X\},$$

where X runs through the set of the ideals (or subsets) of the ring $\mathfrak{z}(\mathcal{A})$, is the set of closed subsets in the topology $\tau_{\mathfrak{z}}$.

Clearly the sets $V(\xi) = \{<P> \in \mathbf{Spec}\mathcal{A} \mid \xi(P) = 0\}$, where ξ runs through $\mathfrak{z}(\mathcal{A})$, form a basis of closed sets of the topology $\tau_{\mathfrak{z}}$. Since $V(\xi) = \mathbf{V}(\mathcal{A}_{\xi})$ and the subcategory $\mathcal{A}_{\xi}$ is left closed (cf. Lemma 7.1.1), each V_{ξ} is closed in the Zariski topology; i.e. the topology $\tau_{\mathfrak{z}}$ is weaker than the Zariski topology.

7.1.4. Example. Let $\mathcal{A}$ be the category of left modules over an associative ring R, $\mathcal{A} = R\text{-}mod$. It is well known (and easy to check) that the ring $\mathfrak{z}(\mathcal{A})$ is isomorphic to the center $\mathfrak{z}(R)$ of the ring R: the isomorphism $\mathfrak{z}(R) \longrightarrow \mathfrak{z}(\mathcal{A})$ sends an element of $\mathfrak{z}(R)$ into the action of this element on modules.

Now, $\mathbf{Spec}\mathcal{A} \simeq \mathbf{Spec}_l R$, and the corresponding to

$$\varphi_{\mathcal{A}}: \mathbf{Spec}\mathcal{A} \longrightarrow \mathbf{Spec}\mathfrak{z}(\mathcal{A})$$

map

$$\mathbf{Spec}_l R \longrightarrow \mathbf{Spec}\mathfrak{z}(R)$$

assigns to any ideal $\mathbf{p} \in \mathbf{Spec}_l R$ its intersection with the center:

$$\mathbf{p} \longmapsto \mathbf{p} \cap \mathfrak{z}(R).$$

The transfered to $\mathbf{Spec}_l R$ central topology is described by 'zeros' of sets of cental elements: any closed subset is of the form $V_l(X) := \{\mathbf{p} \in \mathbf{Spec}_l R \mid X \subseteq \mathbf{p}\}$ for some subset X of $\mathfrak{z}(R)$.

In particular, the central topology on $\mathbf{Spec}_l R$ has a base of 'principal' open subsets which consists of the sets

$$U(z) := \{\mathbf{p} \in \mathbf{Spec}_l R \mid z \notin \mathbf{p}\},$$

where z runs through $\mathfrak{z}(R)$.

The localization at the open set $U(z)$, $z \in \mathfrak{z}(R)$, coincides with the 'classical' localization

$$M \longmapsto (z)^{-1} R \otimes_R M$$

at the multiplicative set $(z) := \{z^n \mid n \in \mathbb{Z}_+\}$. ■

7.2. The topology τ^*. Another way to define a topology on $\mathbf{Spec}\mathcal{A}$ is to single out a class of objects, $\mathfrak{Cl}$, of the category $\mathcal{A}$ and declare the set $\{Supp(M) \mid M \in \mathfrak{Cl}\}$ a base of closed subsets.

This is the way we define the topology τ^*: by taking as $\mathfrak{C}\mathfrak{l}$ the union of the class $\mathrm{ft}\mathcal{A}$ of all objects of finite type in $\mathcal{A}$ with **Spec**$\mathcal{A}$.

If $\mathcal{A}$ is the category of modules over a commutative ring, then τ^* coincides with the Zariski topology $\mathfrak{Z}\mathfrak{t}$ (and also with the central topology $\mathfrak{z}\tau$).

If $\mathcal{A} = R\text{-}mod$, where the ring R is noncommutative, then τ^* can differ from the Zariski topology $\mathfrak{Z}\mathfrak{t}$ drastically.

For instance, if R is a simple ring, the Zariski topology $\mathfrak{Z}\mathfrak{t}$ is trivial (since there is no non-trivial two-sided ideals), while the topology τ^* is quite ample even in the general case.

In fact, it follows from the definition of τ^* that, for any abelian category $\mathcal{A}$, the closure of any point of the spectrum of $\mathcal{A}$ coincides with the set of specializations of this point; i.e. the closure of a point in the topology τ^* coincides with its closure in the topology τ. It cannot be better.

A draw back is that, even in affine case, $\mathcal{A} = R\text{-}mod$, the topological space $(\mathbf{Spec}\mathcal{A}, \mathfrak{t}^*)$ is not quasi-compact in general. It is, however, in a lot of important special cases.

7.3. The topology $\tau_\mathfrak{s}$. The base of closed subsets of the topology $\tau_\mathfrak{s}$ is the set $\{\mathfrak{s}(<P>) \mid <P> \in \mathbf{Spec}\mathcal{A}\}$ of all the closures of points of the spectrum in topology τ; i.e. the sets

$$\mathfrak{s}(<P>) = \{<P'> \mid <P'> \subseteq <P>\} = \{<P'> \mid P \succ P'\}$$

of specializations of points.

Clearly $\tau_\mathfrak{s}$ is the weakest among the topologies on **Spec**$\mathcal{A}$ having the mentioned above property: the closure of any point of the spectrum coincides with the set of the specializations of that point.

7.4. Structure presheaves. Fix a topology $\mathfrak{T}$ on **Spec**$\mathcal{A}$. To any set $U \in Open\mathfrak{T}$, we assign the Serre subcategory $<U> := \bigcap_{\mathbf{P} \in U} \mathbf{P}$. According to Proposition 5.3.2, the subcategory $<U>$ is generated by all $M \in Ob\mathcal{A}$ such that $Supp(M)$ is contained in the complementary to U closed subset.

To any object M of the category $\mathcal{A}$, we assign a function $M^\sim$ on $Open\mathfrak{T}$ which sends any open set U into the localization of M at $<U>$:

$$M^\sim(U) = Q_{<U>}M.$$

The map $M^\sim$ is functorial in a natural sense and is defined uniquely up to isomorphism. It is called *a structure presheaf associated to the object* M. One can show that presheaves $M^{\sim}$ are actually sheaves: the object $M^{\sim}(U)$ can be reconstructed from local data related to any finite covering of U.

The explicit definitions and reconstruction (globalization) theorems can be found in [R3] for the affine case and in [R7] for the general case.

8. ASSOCIATED POINTS.

Fix an abelian category $\mathcal{A}$. For any object M of $\mathcal{A}$, denote by $Ass(M)$ the set of $<P> \in$ **Spec**$\mathcal{A}$ such that P is a subobject of M, and call the points of $Ass(M)$ *associated to* M *elements of the spectrum.*

Clearly $Ass(M) \subseteq Supp(M)$ *for any* M.

8.1. Example. If $M \in$ **Spec**$\mathcal{A}$, then $Ass(M) = \{<M>\}$. ∎

In general, $Ass(M)$ might by empty.

8.2. Proposition. *(a) For any short exact sequence,*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

$$Ass(M') \subseteq Ass(M) \subseteq Ass(M') \cup Ass(M'').$$

(b) Suppose that $\mathcal{A}$ *has the property (sup). Then*

if Ω *is a directed family of subobjects of an object* M *such that the canonical arrow* $\sup \Omega \longrightarrow M$ *is an isomorphism, then*

$$Ass(M) = \bigcup_{X \in \Omega} Ass(X).$$

Proof. (a) The inclusion $Ass(M') \subseteq Ass(M)$ is obvious.

Let now $<P> \in Ass(M)$, and there is a monoarrow $P \longrightarrow M$.

If $P \cap M'$ is nonzero, then $<P> = <P \cap M'>$; hence

$$<P \cap M'> \in Ass(M').$$

If $P \cap M' = 0$, then the composition of the $P \longrightarrow M$ and the epimorphism $M \longrightarrow M''$ is a monoarrow; i.e. $<P> \in Ass(M'')$.

(b) It follows from *(a)* that $\bigcup_{X \in \Omega} Ass(X) \subseteq Ass(M)$. We need to show that the inverse inclusion is also true.

Let $<P>$ be an arbitrary element of $Ass(M)$ (we assume that P is a subobject of M). Since $\mathcal{A}$ has the property *(sup)*, the canonical arrow

$$\sup_{X \in \Omega}(P \cap X) \longrightarrow P$$

is an isomorphism. In particular, the monoarrow $P \cap X \longrightarrow P$ is nonzero for some $X \in \Omega$. But then $<P> = <P \cap X>$, and, since the other canonical arrow,

$$P \cap X \longrightarrow X,$$

is also a monomorphism, $<P \cap X> \in Ass(X)$. ∎

8.3. Corollary. *Let* $\mathcal{A}$ *be a category with the property (sup). Then, for any family* Ξ *of objects of* $\mathcal{A}$ *such that there is a coproduct* $\bigoplus_{X \in \Xi} X$,

$$Ass\Big(\bigoplus_{X \in \Xi} X\Big) = \bigcup_{X \in \Xi} Ass(X).$$

Proof. It follows from the assertion *(a)* of Proposition 8.2 that, for any objects L and M,

$$Ass(L) \cup Ass(M) \subseteq Ass(L \oplus M) \subseteq Ass(L) \cup Ass(M);$$

i.e.

$$Ass(L \oplus M) = Ass(L) \cup Ass(M).$$

Therefore

$$Ass\Big(\bigoplus_{X \in \Omega} X\Big) = \bigcup_{X \in \Omega} Ass(X)$$

for any finite family of objects Ω. Since the coproduct of an arbitrary family, Ξ, of objects is the inductive limit of coproducts of finite subfamilies of Ξ, the assertion follows from the assertion *(b)* of Proposition 8.2. ∎

8.4. Corollary. *Let* M *be a nonzero object of an abelian category* $\mathcal{A}$. *And let* Ξ *be a finite family of subobjects of an object* M *such that* $\bigcap_{X \in \Xi} X = 0$. *Then* $Ass(M) \subseteq \bigcup_{X \in \Xi} Ass(M/X)$.

Proof. In fact, the canonical map

$$M \longrightarrow \bigoplus_{X \in \Xi} M/X$$

is a monomorphism. ∎

Futher on, we shall assume that $\mathcal{A}$ has the property *(sup)*.

8.5. Proposition. *Let* $M \in Ob\mathcal{A}$, *and let* Φ *be an arbitrary subset of* $Ass(M)$. *There exists a subobject* $L \longrightarrow M$ *such that*

$$Ass(M/L) = \Phi, \quad \text{and} \quad Ass(L) = Ass(M) - \Phi.$$

Proof. Let $\mathfrak{D}$ be the set of monoarrows $M' \longrightarrow M$ such that $Ass(M') \subseteq Ass(M) - \Phi$. Clearly $\mathfrak{D}$ is not empty, since it contains the zero subobject. It follows from Proposition 8.2 that, for any directed subset, Ω, of subobjects of $\mathfrak{D}$, $sup\ \Omega \in \mathfrak{D}$. Therefore, by Zorn's Lemma, there exist a *maximal* subobject, L, in $\mathfrak{D}$.

Now, it is enough (thanks to Proposition 8.2) to show that $Ass(M/L) \subseteq \Phi$.

Let $<P> \in Ass(M/L)$, and $P \longrightarrow M/L$ be a monoarrow. Then the canonical morphism $P' := P \underset{M/L}{\times} M \longrightarrow M$ is a monoarrow too. By Proposition 8.2,

$$Ass(P') \subseteq Ass(L) \cup \{<P>\}.$$

Since L is maximal in $\mathfrak{D}$, $P' \notin \mathfrak{D}$. Therefore $<P> \in \Phi$. ∎

8.5.1. Remark. It follows from the inductiveness of $\mathfrak{D}$ (cf. the proof of Proposition 8.5) that, for any $\Phi \subseteq Ass(M)$ and any subobject $\mathfrak{k}: K \longrightarrow M$ such that $Ass(K) \subseteq Ass(M) - \Phi$, there is a monoarrow $\mathfrak{l}: L \longrightarrow M$ which 'contains' $\mathfrak{k}$ (i.e. $\mathfrak{k}$ is the composition of $\mathfrak{l}$ and a unique monoarrow $K \longrightarrow L$) and has the following properties: $Ass(L) = Ass(M) - \Phi$, $Ass(M/L) = \Phi$, and

$$\mathfrak{l}: L \longrightarrow M$$

is the maximal among the subobjects satisfying to these conditions. ∎

8.5.2. Example. Let $\mathfrak{S}$ be a Serre subcategory of the category $\mathcal{A}$ and M an object from $Ob\mathcal{A} - Ob\mathfrak{S}$. Take $\Phi = Ass(M) - \mathbf{Spec}\mathfrak{S}$.

Clearly the $\mathfrak{S}$-torsion, $\mathfrak{S}M$, of M has the property:

$$Ass(\mathfrak{S}M) = Ass(M) \cap \mathbf{Spec}\mathfrak{S} = Ass(M) - \Phi,$$

Hence there is a subobject $L \longrightarrow M$ which contains $\mathfrak{S}M$ and is maximal with respect to the properties:

$$Ass(L) = Ass(M) \cap \mathbf{Spec}\mathfrak{S},\quad Ass(M/L) = Ass(M) - \mathbf{Spec}\mathfrak{S} \tag{1}$$

(cf. Remark 8.5.1). ∎

8.5.3. Example. Let W be a closed subset of $(\mathbf{Spec}\mathcal{A}, \tau)$, and M an object of $\mathcal{A}$. Set $\Phi = Ass(M) - W$, and take the maximal subobject,

$$M(W) \longrightarrow M,$$

among those subobjects $L \longrightarrow M$ for which $Supp(L) \subseteq W$.

Clearly

$$Ass(M(W)) \subseteq Supp(M(W)) \subseteq Ass(M) \cap W = Ass(M) - \Phi.$$

By Proposition 8.5 (and Remark 8.5.1), there exists a subobject L of M which contains $M(W) \longrightarrow M$ and is maximal with respect to the properties:

$$Ass(L) = Ass(M) \cap W, \quad Ass(M/L) = Ass(M) - W.$$

Note that this example is a special (but important) case of Example 8.5.2: one should take (in 8.5.2) $\mathbb{S} = \mathcal{A}(W)$ - the Serre subcategory generated by all objects X of $\mathcal{A}$ such that $Supp(X) \subseteq W$ (cf. Proposition 5.3.2). ∎

8.5.4. Associated points and exact localizations. Let $Q: \mathcal{A} \longrightarrow \mathcal{B}$ be an exact localization of an abelian category $\mathcal{A}$, M an object of the category $\mathcal{A}$, and $<P> \in Ass(M) - |KerQ|$. We assume that P is a subobject of M. Then, since $Q(P) \in Spec\mathcal{B}$ and Q respects monoarrows, $<Q(P)> \in Ass(Q(M))$.

One can easily check that the map

$$Ass(M) - |KerQ| \longrightarrow Ass(Q(M)), \quad <P> \longmapsto <Q(P)>,$$

is injective.

9. Relative spectra.

Define *the spectrum of a functor* $\mathfrak{F}$ from an abelian category $\mathcal{B}$ to an abelian category $\mathcal{A}$ as the ordered set $\mathbf{Spec}(\mathfrak{F})$ of all pairs $(<M>,<P>) \in \mathbf{Spec}\mathcal{B} \times \mathbf{Spec}\mathcal{A}$ such that there is an $M' \in Ob\mathcal{B}$ satisfying the conditions:

$$<M'> = <M>, \quad \text{and} \quad <P> \in Ass(M').$$

The order in $\mathbf{Spec}(\mathfrak{F})$ is induced from $\mathbf{Spec}\mathcal{B} \times \mathbf{Spec}\mathcal{A}$.

9.1. Example. Clearly $\mathbf{Spec}(Id_{\mathcal{A}})$ coincides with the diagonal in $\mathbf{Spec}\mathcal{A} \times \mathbf{Spec}\mathcal{A}$. In particular, $\mathbf{Spec}\mathcal{A}$ can be naturally identified with $\mathbf{Spec}(Id_{\mathcal{A}})$. ∎

9.2. Example. Let $\phi: A \longrightarrow B$ be a ring morphism and $\mathfrak{F} = \phi_*$ the corresponding functor from $\mathcal{B} = B\text{-}mod$ to $\mathcal{A} = A\text{-}mod$. If the ring A is commutative and noetherian, then the projection

$$\mathbf{Spec}(\phi_*) \longrightarrow \mathbf{Spec}\mathcal{B}$$

is surjective, since, for any nonzero B-module M (in particular, for any $M \in Spec\mathcal{B}$), the set $Ass(\phi_*(M))$ is nonempty. This means that knowing $\mathbf{Spec}(\phi_*)$ we can recover the spectrum of $\mathcal{B}$.

This is not true in general. ∎

Fix a functor $\mathfrak{F}: \mathcal{B} \longrightarrow \mathcal{A}$. It is important to single out, in a natural way, some topologizing (or thick) subcategories $\mathcal{B}'$ of $\mathcal{B}$ such that $Ass(\mathfrak{F}(M)) \neq \varnothing$ for all $<M> \in \mathbf{Spec}\mathcal{B}'$.

The defined below subcategories $[\mathcal{A},\mathfrak{F}]$ and $<\mathcal{A},\mathfrak{F}>$ are most straightforward examples of such natural constructions.

9.3. The subcategory $[\mathcal{A},\mathfrak{F}]$. Fix abelian categories, $\mathcal{A}$ and $\mathcal{B}$, and a functor $\mathfrak{F}: \mathcal{A} \longrightarrow \mathcal{B}$. Denote by $[\mathcal{A},\mathfrak{F}]$ the full subcategory of the category $\mathcal{A}$ which is generated by all objects M such that, for any nonzero subquotient L of M, $Ass(\mathfrak{F}L) \neq \varnothing$.

Note that the restriction of $\mathfrak{F}$ to $[\mathcal{A},\mathfrak{F}]$ is a faithful functor.

9.3.1. Lemma. *If the functor* $\mathfrak{F}: \mathcal{A} \longrightarrow \mathcal{B}$ *is left exact, then* $[\mathcal{A},\mathfrak{F}]$ *is a Serre subcategory of* $\mathcal{A}$.

Proof. In fact, let M is an object of $[\mathcal{A},\mathfrak{F}]^{\Gamma}$. And let L is a nonzero subquotient of M. Since $M \in Ob[\mathcal{A},\mathfrak{F}]^{\Gamma}$, there is a nonzero subobject X of L which belongs to $[\mathcal{A},\mathfrak{F}]$. Therefore $Ass(\mathfrak{F}(X)) \neq \varnothing$. Since $\mathfrak{F}$ is left exact, $\mathfrak{F}(X)$ is a subobject of $\mathfrak{F}(L)$ which implies the inclusion $Ass(\mathfrak{F}(L)) \supseteq Ass(\mathfrak{F}(X))$. ■

9.3.2. Corollary. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be abelian categories and* $\mathfrak{F}$ *a left exact functor from* $\mathcal{A}$ *to* $\mathcal{B}$. *Then*

$$\mathbf{Spec}[\mathcal{A},\mathfrak{F}] = \mathbf{Spec}\mathcal{A} \cap |[\mathcal{A},\mathfrak{F}]|.$$

Proof. According to Lemma 5.3.1, $\mathbf{Spec}\mathbb{T} = \mathbf{Spec}\mathcal{A} \cap |\mathbb{T}|$ for any thick subcategory $\mathbb{T}$. And the category $[\mathcal{A},\mathfrak{F}]$ is thick by the first assertion of Proposition 9.3.1. ■

9.3.3. Example. Clearly the subcategory $[\mathcal{A},Id]$ contains all simple objects of the category $\mathcal{A}$. By Proposition 9.3.1, it contains also all objects of finite length and a lot more. ■

9.3.4. Example. Let $\varphi: B \longrightarrow A$ be a ring morphism, $\mathfrak{F}$ the corresponding functor $A\text{-}mod \longrightarrow B\text{-}mod$. Suppose that the ring B is commutative and noetherian. Then $[A\text{-}mod,\mathfrak{F}] = A\text{-}mod$.

In fact, if M is a module over a commutative noetherian ring, then $Ass(M) = \varnothing$ if and only if $M = 0$ (cf. [BCA], Ch. IV, Corollary 1.1.2). ■

9.3.5. Example. Let $\mathfrak{k}$ be a Lie subalgebra of a Lie algebra $\mathfrak{g}$. Consider the corresponding to the embedding $\mathfrak{k} \longrightarrow \mathfrak{g}$ morphism, $\phi: U(\mathfrak{k}) \longrightarrow U(\mathfrak{g})$, of the universal enveloping algebras, and take as $\mathfrak{F}$ the base-change functor

$$U(\mathfrak{g})\text{-}mod \longrightarrow U(\mathfrak{k})\text{-}mod.$$

The subcategory $[U(\mathfrak{g})\text{-}mod, \mathfrak{F}]$ contains the category $\mathbf{HC}(\mathfrak{g},\mathfrak{k})$ of Harish-Chandra $\mathfrak{g}$-modules with respect to the subalgebra $\mathfrak{k}$.

In fact, by definition (cf. [D], 9.1.4), a $\mathfrak{g}$-module V is a Harish-Chandra module with respect to $\mathfrak{k}$ if the $\mathfrak{k}$-module $\mathfrak{F}V$ is the sum of its irreducible submodules.

Note that if the Lie subalgebra $\mathfrak{k}$ is finite dimensional and commutative (for example, $\mathfrak{k}$ is a Cartan subalgebra of a finite dimensional reductive Lie algebra $\mathfrak{g}$), then, according to Example 9.3.4, $[U(\mathfrak{g})\text{-}mod, \mathfrak{F}] = U(\mathfrak{g})\text{-}mod$. ■

9.4. The subcategory $<\mathcal{A}|\mathfrak{F}>$. Let $\mathfrak{F}: \mathcal{A} \longrightarrow \mathcal{B}$ be a functor. Set

$$<\mathcal{A}|\mathfrak{F}> \ = \ \mathfrak{F}^{-1}([\mathcal{B}, Id]).$$

9.4.1. Lemma. *(a) If the functor* $\mathfrak{F}$ *is exact, then* $<\mathcal{A}|\mathfrak{F}>$ *is a thick subcategory of the category* $\mathcal{A}$.

(b) If the functor $\mathfrak{F}$ *is exact and faithful, then* $<\mathcal{A}|\mathfrak{F}>$ *is a subcategory of the category* $[\mathcal{A},\mathfrak{F}]$ *(cf. 9.3).*

Proof. (a) Since the functor $\mathfrak{F}$ is exact, $\mathfrak{F}^{-1}(\mathbb{T})$ is a thick subcategory of $\mathcal{A}$ for any thick subcategory $\mathbb{T}$ of $\mathcal{B}$. In particular, it follows from Proposition 9.3.1 that the subcategory $<\mathcal{A}|\mathfrak{F}> := \mathfrak{F}^{-1}([\mathcal{B}, Id])$ is thick.

(b) Let $M \in Ob<\mathcal{A}|\mathfrak{F}>$, and let K be a nonzero subquotient of M. Since $\mathfrak{F}$ is exact, $\mathfrak{F}K$ is a subquotient of $\mathfrak{F}M$. And $\mathfrak{F}K \neq 0$ thanks to the faithfulness of $\mathfrak{F}$. Therefore $Ass(\mathfrak{F}K) \neq \emptyset$ which means, by definition (cf. 9.3), that $M \in Ob[\mathcal{A},\mathfrak{F}]$. ■

Note that the functor of Example 9.3.4 (hence that of 9.3.5) is exact and faithful.

The main advantage of the subcategories $<\mathcal{A}|\mathfrak{F}>$ in comparison with the subcategories $[\mathcal{A},\mathfrak{F}]$ is their functoriality. The latter means that, for any quasi-commutative diagram of functors

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{H} & \mathcal{A}' \\ & {\scriptstyle \mathfrak{F}} \searrow \quad \swarrow {\scriptstyle \mathfrak{F}'} & \\ & \mathcal{B} & \end{array}$$

(i.e. $\mathfrak{F} \simeq \mathfrak{F}' \circ H$), the functor H sends objects of the subcategory $\langle \mathcal{A} | \mathfrak{F} \rangle$ into objects of $\langle \mathcal{A}' | \mathfrak{F}' \rangle$.

Thus, we can consider the category $Ex/\mathcal{B}$ the objects of which are exact (additive) functors from abelian categories to $\mathcal{B}$, and morphisms

$$\text{from } \mathfrak{F}: \mathcal{A} \longrightarrow \mathcal{B} \text{ to } \mathfrak{F}': \mathcal{A}' \longrightarrow \mathcal{B}$$

are pairs (H,h), where $H: \mathcal{A} \longrightarrow \mathcal{A}'$ is a functor, h is an isomrophism from $\mathfrak{F}' \circ H$ to $\mathfrak{F}$. The composition is defined in an obvious way:

$$(H',h') \circ (H,h) = (H' \circ H, h \circ h'H).$$

(Clearly what is defined above is a metacategory. To make it category one should consider functors from categories which are equivalent to "small" categories.)

APPENDIX: GABRIEL LOCALIZATIONS.

A.1. Serre subcategories and radical filters. For any subcategory B of the category R-*mod* of left R-modules, denote by $\mathfrak{F}(B)$ the set of all left ideals m of the ring R such that the module R/m belongs to B. If B is a thick subcategory of R-*mod*, then the set $\mathfrak{F}(B)$ turns to be so called *Gabriel* (or *localizing*) *filter*.

By definition, the set F of left ideals of the ring R is a Gabriel filter iff it has the following properties:

(a) if $m \in F$, then for any element $r \in R$, the ideal

$$(m:r) := \{y \in R: yr \in m\}$$

also belongs to F;

(b) if $m \in F$, and n is a left ideal such that $(n:y) \in F$ for each $y \in m$, then n belongs to F.

Conversely, to any set F of left ideals of the ring R one can assign a full subcategory $\mathcal{S}(F)$ of the category R-*mod* formed by all the modules M such that the annihilators of all the elements of M belong to the set F.

One can check that the subcategory $\mathcal{S}(F)$ is thick. Moreover, $\mathcal{S}(F)$ is closed with respect to small coproducts (taken in R-*mod*) which means that $\mathcal{S}(F)$ is a Serre subcategory. It is easy to see that, for any thick subcategory B of

R-mod, the subcategory $\mathfrak{S}(\mathfrak{F}(B))$ coincides with B^{-} (cf. 1.?). In particular, B equals to $\mathfrak{S}(\mathfrak{F}(B))$ if B is a Serre subcategory; i.e. if $B = B^{-}$.

This proves the following assertion:

A.1.1. Lemma. *The map* $\mathfrak{F} : B \longrightarrow \mathfrak{F}(B)$ *establishes a one to one correspondence between the set of Serre subcategories of the category R-mod and the set of Gabriel filters of left ideals of the ring R.*

More explicitly, the restriction of the map $F \longmapsto \mathfrak{S}(F)$ *to the set of Gabriel filters is inverse to the restriction of* $\mathfrak{F}$ *to the set of Serre subcategories.*

A.2. Localizations in terms of radical filters. Let F be a Gabriel filter of left ideals of the ring R; and let *R-mod/F* be a full subcategory of the category *R-mod* formed by all the left modules M such that the canonical map $M \longmapsto R\text{-}mod(m,M)$, which sends an element z of the module M into the arrow $r \longrightarrow rz$, is a bijection for any ideal m from the filter F.

On the other hand, for any R-module M, denote by $'HF(M)$ the direct limit colim$\{R\text{-}mod(n,M)$: $n \in F\}$. The Z-module $'HF(M)$ possesses a natural structure of R-module,

$$g : R \longrightarrow \mathrm{Hom}('HF(M),'HF(M)),$$

such that the canonical map

$$iF(M)\colon M \longrightarrow HF(M):=('HF(M),g)$$

turns to be an R-module morphism. Moreover, the map $M \longrightarrow HF(M)$ is extended to a functor $HF\colon R\text{-}mod \longrightarrow R\text{-}mod$ such that the collection $iF=\{iF(M)\colon M \in \mathrm{Ob}R\text{-}mod\}$ is a functor morphism from Id to HF. Denote the square of the functor HF by GF (- Gabriel functor), and set $jF:=HF(iF)\circ iF$.

A.3. Proposition. *a) The category R-mod/F is equivalent to the quotient category R-mod/S(F).*

b) The functor GF takes values in the subcategory R-mod/F, and the co-restriction of GF onto R-mod/F is a localization. More precisely, for an arbitrary R-module M and a module N from R-mod/F, any R-module morphism $f\colon M \longrightarrow N$ *is uniquely represented as a composition* $f=f'\circ jF(M)$.

A.4. Proposition. *1) There is a unique ring structure on GF(R) such that the canonical R-module morphism* $JF(R)\colon R \longrightarrow GF(R)$ *turns to be a ring morphism.*

2) For any R-module M, there is a unique extension of R-module structure *on GF(M) to GF(R)-module structure.*

3) These extensions (for all M ∈ ObR-mod) define a full and faithful left exact functor 𝒢F: R-mod/F ⟶ GF(R)-mod.

4) The functor 𝒢F is right adjoint to the localization

$$QF: GF(R)\text{-mod} \longrightarrow R\text{-mod}/F$$

at the Serre subcategory SF formed by all GF(R)-modules which are F-torsions as R-modules.

A.5. Remark. It is easy to see that the following conditions are equivalent:

a) The functor

$$\mathcal{G}F: R\text{-mod}/F \longrightarrow GF(R)\text{-mod}$$

is an equivalence of categories.

b) GF is isomorphic to the functor $GF(R)\otimes_R$.

According to [Gab], Corollary V.2.2, the condition *b)* holds if the functor *GF* is exact and the Gabriel filter *F* contains a cofinal subset of finitely generated left ideals.

The exactness of the functor *GF* is garanteed in the following two cases:

1) The filter *F* contains a cofinal subset of projective (as *R*-modules) ideals.

2) There is a multiplicative system S satisfying the left Ore conditions and such that *F* consists of left ideals *m* such that *(m:y)* contains some elements from S for any $y \in R$. ∎

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