Max-Planck-Institut für Mathematik
Bonn

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Deformations

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Preprint Series 2012 (33)

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MPIM 12-33

METABELIAN SL$(n,\mathbb{C})$ REPRESENTATIONS OF KNOT GROUPS, III: DEFORMATIONS

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ABSTRACT. Given a knot K with complement N_K and an irreducible metabelian $\mathrm{SL}(n,\mathbb{C})$ representation $\alpha\colon\ \pi_1(N_K)\to \mathrm{SL}(n,\mathbb{C})$, we establish the inequality $\dim H^1(N_K;sl(n,\mathbb{C})_{ad\,\alpha})\geq n-1$. In the case of equality, we prove that α must have finite image and is conjugate to an $SU(n)$ representation. In this case we show α determines a smooth point ξ_α in the $\mathrm{SL}(n,\mathbb{C})$ character variety, and we use a deformation argument to establish the existence of a smooth $(n-1)$–dimensional family of characters of irreducible $\mathrm{SL}(n,\mathbb{C})$ representations near ξ_α, and a corresponding sub–family of characters of irreducible $SU(n)$ representations of real dimension $n-1$. Both families can be chosen so that ξ_α is the only metabelian character.

Combining this with our previous existence results, we deduce the existence of large families of irreducible $SU(n)$ and $\mathrm{SL}(n,\mathbb{C})$ non-metabelian representation for knots K in homology 3-spheres Σ with nontrivial Alexander polynomial. We then relate the condition on twisted cohomology to a more accessible condition on untwisted cohomology of a certain metabelian branched cover $\widehat{\Sigma}_\varphi$ of Σ branched along K.

1. Introduction

Suppose K is an oriented knot in an integral homology 3-sphere Σ with exterior $N_K=\Sigma^3\smallsetminus\tau(K)$. In [BF08], we show how to construct irreducible metabelian $\mathrm{SL}(n,\mathbb{C})$ representations of the knot group $\pi_1(N_K)$ for any knot K with nontrivial Alexander polynomial. This provides a constructive proof for the existence of irreducible metabelian representations in $\mathrm{SL}(n,\mathbb{C})$, and in this paper we prove a stronger existence result (see Theorem 8) and consider the problem of existence of irreducible non-metabelian $\mathrm{SL}(n,\mathbb{C})$ representations of $\pi_1(N_K)$.

In rank $n=2$, a result of Thurston implies that any irreducible metabelian representation $\alpha\colon\ \pi_1(N_K)\to \mathrm{SL}(2,\mathbb{C})$ can be deformed within the larger space of all (conjugacy classes of) representations, and in fact Theorem 3.2.1 of [CS83] shows the existence of a family of conjugacy classes of irreducible representations near α of dimension ≥ 1. In this paper, we study the character varieties of knot groups in higher rank, with a focus on existence of irreducible metabelian $\mathrm{SL}(n,\mathbb{C})$ representations and their deformations. For instance, given an irreducible metabelian representation $\alpha\colon\ \pi_1(N_K)\to \mathrm{SL}(n,\mathbb{C})$ satisfying a cohomological condition, we establish the existence of an $(n-1)$-dimensional family of conjugacy classes of irreducible non-metabelian $\mathrm{SL}(n,\mathbb{C})$ representations near α.

Date: May 25, 2012.

2010 *Mathematics Subject Classification.* Primary: 57M25, Secondary: 20C15.

Key words and phrases. Metabelian representation, knot group, character variety, deformation.

The first author was supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

In order to more precisely state our results, we introduce notation that will be used throughout the paper.

Given a finitely generated group π, let $R_n(\pi) = \mathrm{Hom}(\pi, \mathrm{SL}(n,\mathbb{C}))$ be the representation variety, which is an affine algebraic set with a natural action of $\mathrm{SL}(n,\mathbb{C})$ by conjugation. The set-theoretic quotient is in general not well-behaved, (e.g. it is typically not Hausdorff), so instead we consider the natural quotient in the category of algebraic sets, which is by definition the character variety $X_n(\pi)$ (see [LM85] for details on the construction of character varieties). Given a representation $\alpha\colon \pi \to \mathrm{SL}(n,\mathbb{C})$, its character is the map $\xi_\alpha\colon \pi \to \mathbb{C}$ defined by $\gamma \mapsto tr\,\alpha(\gamma)$, and setting $t(\alpha) = \xi_\alpha$ defines the quotient map $t\colon R_n(\pi) \to X_n(\pi)$.

For a topological space M, let $R_n(M) = R_n(\pi_1(M))$ and $X_n(M) = X_n(\pi_1(M))$. Given $\alpha\colon \pi_1(M) \to \mathrm{SL}(n,\mathbb{C})$, let $ad\,\alpha$ be its composition with the adjoint representation on the Lie algebra $sl(n,\mathbb{C})$, thus $ad\,\alpha$ determines a $\pi_1(M)$ action on $sl(n,\mathbb{C})$. We let $H^*(M; sl(n,\mathbb{C})_{ad\,\alpha})$ denote the cohomology groups of M with coefficients in $sl(n,\mathbb{C})$ twisted by this action.

Given an irreducible metabelian representation $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(n,\mathbb{C})$, we show that $\dim H^1(N_K; sl(n,\mathbb{C})_{ad\,\alpha}) \geq n-1$ (Proposition 14) and deduce that $\dim X_j \geq n-1$ for any algebraic component $X_j \subset X_n(N_K)$ containing ξ_α (Corollary 16). We then show that if $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(n,\mathbb{C})$ is an irreducible metabelian representation such that $\dim H^1(N_K; sl(n,\mathbb{C})_{ad\,\alpha}) = n-1$, then α has finite image and is conjugate to a unitary representation. The following result gives a local description of the character variety near ξ_α under the assumption $\dim H^1(N_K; sl(n,\mathbb{C})_{ad\,\alpha}) = n-1$.

Theorem 1. *If $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(n,\mathbb{C})$ is an irreducible metabelian representation with $\dim H^1(N_K; sl(n,\mathbb{C})_{ad\,\alpha}) = n-1$, then α has finite image and is therefore conjugate to a unitary representation. Further, we have:*

(i) *The character ξ_α is a smooth point in $X_n(N_K)$, and there exists a smooth complex $(n-1)$–dimensional family of characters of irreducible $\mathrm{SL}(n,\mathbb{C})$ representations near $\xi_\alpha \in X_n(N_K)$.*

(ii) *As a point in $X_{SU(n)}(N_K)$, the character ξ_α is again a smooth point and there exists a smooth real $(n-1)$–dimensional family of characters of irreducible $SU(n)$ representations near $\xi_\alpha \in X_{SU(n)}(N_K)$.*

Both deformation families can be chosen so that ξ_α is the only metabelian character within them.

Deformations of dihedral $\mathrm{SL}(2,\mathbb{C})$ representations were studied by Heusener and Klassen in [HK97], and metabelian representations in $\mathrm{SL}(n,\mathbb{C})$ are their analogues in higher rank. Theorem 1 is established by applying deformation arguments developed for $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{PSL}(2,\mathbb{C})$ by Heusener, Porti, and Suárez Peiró in [HPS01, HP05]. These techniques were extended to $\mathrm{SL}(n,\mathbb{C})$ in [AHJ10], where they were applied to deform reducible metabelian $\mathrm{SL}(3,\mathbb{C})$ representations of knot groups. In this paper, we apply the same technique to the problem of deforming irreducible metabelian characters. In Subsection 4.3, we state the deformation results that are needed to establish Theorem 1, and in Appendix A, we provide detailed arguments for these results, following the treatment given in [HPS01, HP05, AHJ10].

Theorem 1 applies in many cases. For instance, in rank 2, given an irreducible representation $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(2,\mathbb{C})$ such that $\{\alpha(\mu), \alpha(\lambda)\} \not\subset \{\pm I\}$, Thurston proved that any algebraic component of $X_2(N_K)$ containing ξ_α has dimension

$d \geq 1$ (see Theorem 3.2.1 of [CS83]). We will see that every irreducible metabelian SL$(2,\mathbb{C})$ representation α satisfies this condition, and thus $\xi_\alpha \in X_2(N_K)$ can be deformed to an irreducible non-metabelian representation. If one assumes, in addition, that N_K does not contain any closed incompressible surfaces, then it follows from [CS83] that any algebraic component $X_2(N_K)$ has dimension $d = 1$. Knots K whose complements N_K satisfy this condition are called *small*, and we see that Theorem 1 applies to irreducible metabelian representations $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(2,\mathbb{C})$ when K is a small knot in an integral homology 3-sphere. In Subsection 4.6, we show by example that Theorem 1 can also be applied in higher rank.

Note that Theorem 1 does not apply to knots K whose Alexander polynomial $\Delta_K(t)$ has a root which is an n-th root of unity. Indeed, if L_n is the n-fold cyclic branched cover of Σ branched along K, then we have $b_1(L_n) > 0$, and any irreducible metabelian representation $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(n,\mathbb{C})$ will have $\dim H^1(N_K; sl(n,\mathbb{C})_{ad\,\alpha}) > n-1$ (see Proposition 18). The simplest example occurs in rank $n = 6$ for the trefoil knot $K = 3_1$, though other examples can be constructed using Theorem 3.10 of [BF08].

Thus, it is useful to have an alternative criterion for applying Theorem 1, and our next result provides such a criterion in terms of the untwisted cohomology of a certain metabelian branched cover of Σ branched along K.

Theorem 2. *Suppose that n is such that $b_1(L_n) = 0$ (equivalently, suppose the Alexander polynomial $\Delta_K(t)$ has no root which is an n-th root of unity). Suppose further that $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(n,\mathbb{C})$ is an irreducible metabelian representation and $\varphi\colon \pi_1(N_K) \to \mathbb{Z}/n \ltimes H$ is a group homomorphism with H finite and abelian such that α factors through φ. Denote by $\widetilde{N}_\varphi \to N_K$ the covering map corresponding to φ. Then the following hold:*

(i) *$b_1(\widetilde{N}_\varphi) \geq |H|$ and if $b_1(\widetilde{N}_\varphi) = |H|$, then $\dim H^1(N_K; sl(n,\mathbb{C})_{ad\,\alpha}) = n-1$.*
(ii) *The cover $\widetilde{N}_\varphi \to N_K$ extends to a cover $\widehat{\Sigma}_\varphi \to \Sigma$ branched over K.*
(iii) *If $b_1(\widehat{\Sigma}_\varphi) = 0$, then $\dim H^1(N_K; sl(n,\mathbb{C})_{ad\,\alpha}) = n-1$.*

Remark 3. Theorem 2 is a generalization of a result proved for dihedral groups by Boileau and Boyer, see [BB07, Lemma A.2].

Acknowledgments. The authors are grateful to Steve Boyer, Jérôme Dubois, Michael Heusener, and Adam Sikora for many helpful discussions. The first author is happy to acknowledge the Max Planck Institute for Mathematics for its support.

2. Metabelian representations of knot groups

In this section we review the construction of metabelian representations for knot groups from [BF08]. We then use a result of Silver and Williams [SW02] to show existence of irreducible metabelian SL$(n,\mathbb{C})$ representations for all but finitely many ranks n for any knot K whose Alexander polynomial $\Delta_K(t)$ has a root that is not a root of unity.

2.1. **Construction of irreducible metabelian SL$(n,\mathbb{C})$ representations.** Given a group π and a finite dimensional vector space V over $\mathbb{C}$, a representation $\varrho\colon \pi \to \mathrm{Aut}(V)$ is called *reducible* if there exists a proper invariant subspace $U \subset V$, otherwise ϱ is called *irreducible*. We say ϱ is *metabelian* if its restriction $\varrho|_{\pi^{(2)}}$ is trivial, where $\pi^{(2)}$ denotes the second commutator subgroup of π. Equivalently, a metabelian representation is one that factors through the metabelian quotient $\pi/\pi^{(2)}$.

Given a knot $K \subset \Sigma^3$ in an integral homology 3-sphere, let $N_K = \Sigma \smallsetminus \tau(K)$ be the complement and $\widetilde{N}_K$ be the infinite cyclic cover of N_K. Thus $\pi_1(\widetilde{N}_K) = \pi_1(N_K)^{(1)}$ and

$$H_1(N_K; \mathbb{Z}[t^{\pm 1}]) = H_1(\widetilde{N}_K) \cong \pi_1(N_K)^{(1)}/\pi_1(N_K)^{(2)},$$

where we use $\pi^{(n)}$ to denote the n–th term of the derived series of a group π, so $\pi^{(1)} = [\pi, \pi]$ and $\pi^{(2)} = [\pi^{(1)}, \pi^{(1)}]$, and so on. The $\mathbb{Z}[t^{\pm 1}]$–module structure is given on the right hand side by $t^n \cdot g := \mu^{-n} g \mu^n$, where μ is a meridian of K.

Set $\pi := \pi_1(N_K)$ and $H = H_1(N_K; \mathbb{Z}[t^{\pm 1}])$ and consider the short exact sequence

$$1 \to \pi^{(1)}/\pi^{(2)} \to \pi/\pi^{(2)} \to \pi/\pi^{(1)} \to 1.$$

Since $\pi/\pi^{(1)} = H_1(N_K) \cong \mathbb{Z}$, this sequence splits and we get isomorphisms

$$\begin{array}{ccccc} \pi/\pi^{(2)} & \cong & \pi/\pi^{(1)} \ltimes \pi^{(1)}/\pi^{(2)} & \cong & \mathbb{Z} \ltimes \pi^{(1)}/\pi^{(2)} \cong \mathbb{Z} \ltimes H \\ g & \mapsto & (\mu^{\varepsilon(g)}, \mu^{-\varepsilon(g)} g) & \mapsto & (\varepsilon(g), \mu^{-\varepsilon(g)} g), \end{array}$$

where the semidirect products are taken with respect to the $\mathbb{Z}$ actions defined by letting $n \in \mathbb{Z}$ act by conjugation by μ^n on $\pi^{(1)}/\pi^{(2)}$ and by multiplication by t^n on $H_1(N_K; \mathbb{Z}[t^{\pm 1}])$. This demonstrates the following lemma.

Lemma 4. *For any knot K, the set of metabelian representations of $\pi_1(N_K)$ can be canonically identified with the set of representations of $\mathbb{Z} \ltimes H$.*

When it is convenient, we will blur the distinction between metabelian representations of $\pi_1(N_K)$ and representations of $\mathbb{Z} \ltimes H$.

Lemma 4 applies to give a useful classification of the irreducible $\mathrm{SL}(n, \mathbb{C})$ of $\pi_1(N_K)$, and before explaining that, we point out two important and well–known facts that are used frequently:

(i) $H = H_1(N_K; \mathbb{Z}[t^{\pm 1}])$ is finitely generated as a $\mathbb{Z}[t^{\pm 1}]$–module and multiplication by $t - 1$ is an isomorphism.

(ii) There is an isomorphism $H/(t^n - 1) \cong H_1(L_n)$, where L_n denotes the n–fold cyclic branched cover of Σ^3 branched along K.

Suppose $\chi\colon H \to \mathbb{C}^*$ is a character factoring through $H/(t^n - 1)$ and $z \in U(1)$ satisfies $z^n = (-1)^{n+1}$. Given $(j, h) \in \mathbb{Z} \ltimes H$, we set

$$\alpha_{(n,\chi,z)}(j, h) = \begin{pmatrix} 0 & & \dots & z \\ z & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & z & 0 \end{pmatrix}^j \begin{pmatrix} \chi(h) & 0 & \dots & 0 \\ 0 & \chi(th) & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \chi(t^{n-1}h) \end{pmatrix}.$$

It follows that $\alpha_{(n,\chi,z)}$ defines an $\mathrm{SL}(n, \mathbb{C})$ representation that factors over $\mathbb{Z} \ltimes H/(t^n - 1)$ and whose isomorphism type is independent of the choice of z (see [BF08, Section 3]). We write $\alpha_{(n,\chi)}$ for $\alpha_{(n,\chi,z)}$.

Recall that a character $\chi\colon H \to \mathbb{C}^*$ has *order n* if it factors through $H/(t^n - 1)$ but not through $H/(t^\ell - 1)$ for any $\ell < n$. Any character $\chi\colon H \to \mathbb{C}^*$ which factors through $H/(t^n - 1)$ must have order k for some divisor k of n.

Given a character $\chi\colon H \to \mathbb{C}^*$, let $t^i\chi$ be the character defined by $(t^i\chi)(h) = \chi(t^i h)$. The next theorem gives a summary of the results [BF08, Lemma 2.2] and [BF08, Theorem 3.3].

Theorem 5. *Suppose $\chi\colon H \to \mathbb{C}^*$ is a character that factors through $H/(t^n-1)$.*

(i) *$\alpha_{(n,\chi)}\colon \mathbb{Z} \ltimes H \to \mathrm{SL}(n,\mathbb{C})$ is irreducible if and only if the character χ has order n.*

(ii) *Given two characters $\chi, \chi'\colon H \to \mathbb{C}^*$ of order n, the representations $\alpha_{(n,\chi)}$ and $\alpha_{(n,\chi')}$ are conjugate if and only if $\chi = t^k\chi'$ for some k.*

(iii) *For any irreducible representation $\alpha\colon \mathbb{Z} \ltimes H \to \mathrm{SL}(n,\mathbb{C})$ there exists a character $\chi\colon H \to \mathbb{C}^*$ of order n such that α is conjugate to $\alpha_{(n,\chi)}$.*

Remark 6. Note that

$$\alpha_{(n,\chi)}(\mu) = \begin{pmatrix} 0 & \dots & 0 & z \\ z & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & z & 0 \end{pmatrix}$$

is conjugate to the diagonal matrix

$$\begin{pmatrix} z & & & 0 \\ & \omega z & & \\ & & \ddots & \\ 0 & & & \omega^{n-1}z \end{pmatrix},$$

where z satisfies $z^n = (-1)^{n+1}$ and $\omega = e^{2\pi i/n}$. In particular, this shows under $\alpha_{(n,\chi)}$, the meridian is sent to a matrix with n distinct eigenvalues.

2.2. **Existence of irreducible metabelian SL$(n,\mathbb{C})$ representations.** In this section, we apply results of [SW02] to prove a strong existence result for irreducible metabelian SL$(n,\mathbb{C})$ representations of knot groups.

Suppose K is a knot whose Alexander polynomial $\Delta_K(t)$ has a zero which is not a root of unity. Then Kronecker's theorem implies that the Mahler measure m of $\Delta_K(t)$ satisfies $m > 1$. Recall that the Mahler measure of a polynomial $f(t) \in \mathbb{C}[t]$ is defined by the formula

$$m(f) = \exp \int_0^{2\pi} \ln(|f(e^{i\theta}|) d\theta.$$

The next proposition was proved by Silver and Williams in [SW02], and it is an extension of earlier results of Gordon [Gor72, p. 365], González-Acuña–Short [GAS91] and Riley [Ri90].

Proposition 7 (Theorem 2.1, [SW02])**.** *Let K be a knot and let m be the Mahler measure of $\Delta_K(t)$. Then*

$$\lim_{n\to\infty} \frac{\ln \mathrm{Tor}\, H_1(L_n)}{n} = \ln m.$$

We now explain how to apply Proposition 7 to deduce a strengthened existence result for irreducible metabelian SL$(n,\mathbb{C})$ representations for such knots K (cf. Theorems 3.10 and 3.12 of [BF08]).

Theorem 8. *Suppose K is a knot such that $\Delta_K(t)$ has a zero which is not a root of unity. Then the number of distinct conjugacy classes of irreducible metabelian SL$(n,\mathbb{C})$ representations of the knot group increases exponentially as $n \to \infty$. Consequently, for all but finitely many ranks n, there exist irreducible metabelian representations $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(n,\mathbb{C})$.*

Proof. Let K be a knot. Given $n \in \mathbb{N}$, let $r_n \in \mathbb{N} \cup \{\infty\}$ denote the number of distinct conjugacy classes of irreducible metabelian $\mathrm{SL}(n, \mathbb{C})$ representations of the knot group.

Claim. *Let $n \in \mathbb{N}$ and write $n = p_1 p_2 \cdots p_k$ for primes $p_1, \dots, p_k$. Then*

$$r_n \geq \frac{1}{n}\left(|\operatorname{Tor} H_1(L_n)| - \sum_{i=1}^{k} |\operatorname{Tor} H_1(L_{n/p_i})| \right).$$

We write $H = H_1(N_K; \mathbb{Z}[t, t^{-1}])$. Note that given any $k|n$ we have the following commutative diagram

$$\begin{array}{ccc} H/(t^n - 1) & \xrightarrow{\cong} & H_1(L_n) \\ \downarrow & & \downarrow \\ H/(t^k - 1) & \xrightarrow{\cong} & H_1(L_k). \end{array}$$

We pick once and for all a decomposition $H_1(L_n) = F_n \oplus T_n$ where F_n is a free abelian group and T_n is torsion. It follows from Theorem 5 (i) and (ii) that

$$\begin{array}{rcl} r_n & \geq & \frac{1}{n}\#\{\rho\colon\ H_1(L_n) \to T_n \to S^1 \,|\, \rho \text{ does not factor through some } H_1(L_k)\} \\ & = & \frac{1}{n}\#\{\rho\colon\ T_n \to S^1 \,|\, \rho \text{ does not factor through } T_n \to H_1(L_n) \to H_1(L_k)\}. \end{array}$$

Note that the number of characters of a finite group A equals $|A|$. Also note that any map $T_n \to H_1(L_k)$ necessarily factors through $\operatorname{Tor} H_1(L_k)$. The claim is now an immediate consequence of these observations.

Claim. *Suppose $M > 1$ and $n = p_1 p_2 \cdots p_k$ for primes $p_1, \dots, p_k$. Then*

$$\sum_{i=1}^{k} M^{n/p_i} \leq \frac{\ln n}{\ln 2} M^{n/2}.$$

Since each prime factor $p_i \geq 2$, it follows that $p_1 \cdots p_k = n \geq 2^k$. Thus

$$\sum_{i=1}^{k} M^{n/p_i} \leq \sum_{i=1}^{k} M^{n/2} = k M^{n/2} \leq \frac{\ln n}{\ln 2} M^{n/2},$$

and this completes the proof of the claim.

We can now finally turn to the proof of the theorem. Suppose that $\Delta_K(t)$ has a zero which is not a root of unity. Let m be the Mahler measure of $\Delta_K(t)$ and notice that Kronecker's theorem implies $m > 1$.

Suppose $0 < \varepsilon < 1/3$. By Proposition 7, there exists an N such that $n \geq N$ implies

$$\left(m^{1-\varepsilon}\right)^n \leq |\operatorname{Tor} H_1(L_n)| \leq \left(m^{1+\varepsilon}\right)^n.$$

We write

$$D := \sum_{i=1}^{N} |\operatorname{Tor} H_1(L_i)|.$$

Now let $n \geq N$. We factor $n = p_1 p_2 \cdots p_k$ where $p_1, \dots, p_k$ are primes. If we combine the above with the first claim we see that

$$\begin{array}{rcl} r_n & \geq & \frac{1}{n}\left(|\operatorname{Tor} H_1(L_n)| - \sum_{i=1}^{k} |\operatorname{Tor} H_1(L_{n/p_i})| \right) \\ & \geq & \frac{1}{n}\left(m^{(1-\varepsilon)n} - \sum_{i=1}^{k} m^{(1+\varepsilon)n/p_i} - D \right). \end{array}$$

Applying the second claim with $M = m^{1+\varepsilon}$, it follows that

$$r_n \geq \frac{1}{n}\left(m^{(1-\varepsilon)n} - \frac{\ln n}{\ln 2} m^{(1+\varepsilon)n/2} - D\right).$$

This shows that r_n grows exponentially for sufficiently large n. □

3. Twisted homology and cohomology

In this section, we introduce the twisted homology and cohomology groups and give some computations that are used throughout the paper.

3.1. **The adjoint representation.**

In this subsection, we show how, given a metabelian representation, its adjoint representation decomposes as a direct sum of simple representations.

Lemma 9. *Let K be a knot, $n \in \mathbb{N}$, and $\chi\colon H_1(L_n) \to \mathbb{C}^*$ a character. Set $\alpha = \alpha_{(n,\chi)}$ and let $\theta_1\colon \pi_1(N_K) \to \mathrm{GL}(1,\mathbb{C})$ denote the trivial representation. Let $\alpha_n\colon \pi_1(N_K) \to \mathrm{Aut}(\mathbb{C}[\mathbb{Z}/n])$ be the regular representation corresponding to the canonical projection map $\pi_1(N_K) \to \mathbb{Z} \to \mathbb{Z}/n$, and let $ad\,\alpha\colon \pi_1(N_K) \to \mathrm{Aut}(sl(n,\mathbb{C}))$ denote the adjoint representation. Then we have the following isomorphism of representations:*

$$ad\,\alpha \oplus \theta_1 \cong \alpha_n \oplus \bigoplus_{i=1}^{n-1} \beta_{(n,\chi_i)},$$

where χ_i is the character defined by $\chi_i(v) := \chi(v)^{-1}\chi(t^i v)$. Further, if χ is a character of order n, then $\chi_1, \dots, \chi_{n-1}$ are also characters of order n.

Proof. Write $\pi = \pi_1(N_K)$ as before and let $\beta\colon \pi \to \mathrm{Aut}(gl(n,\mathbb{C}))$ denote the adjoint representation of α on $gl(n,\mathbb{C})$, so $\beta(g)(A) = \alpha(g)A\alpha(g)^{-1}$ for $g \in \pi$ and $A \in gl(n,\mathbb{C})$. Note that $gl(n,\mathbb{C}) = sl(n,\mathbb{C}) \oplus \mathbb{C}\cdot I$. It follows immediately that $\beta = ad\,\alpha \oplus \theta_1$ splits off a trivial factor. It therefore suffices to show that

$$\beta \cong \alpha_n \oplus \bigoplus_{i=1}^{n-1} \beta_{(n,\chi_i)}.$$

For $i = 0, \dots, n-1$, let V_i be the set of all matrices (a_{jk}) such that $a_{jk} = 0$ unless $j - k \equiv i \mod n$. It is not difficult to see that the action of π on $gl(n,\mathbb{C})$ restricts to actions on $V_0, V_1, \dots, V_{n-1}$. We equip V_i with the ordered basis $\{e_{i+1,1}, e_{i+2,2}, \dots, e_{i+n,n}\}$, where the indices are taken modulo n. The restriction of β to V_i can then be calculated with respect to this basis and $\alpha_{(n,\chi,z)}(j,h) = z^j\beta_{(n,\chi)}(i,h)$ and the z^i disappears upon conjugation.

Note that χ_0 is the trivial character, and therefore $\beta_{(n,\chi_0)} = \alpha_n$. □

3.2. **Twisted homology and cohomology.**

In this subsection, we recall the twisted homology and cohomology groups and summarize their basic properties.

Let (X,Y) be a pair of topological spaces, V a finite dimensional complex vector space and $\alpha\colon \pi_1(X) \to \mathrm{Aut}(V)$ a representation. Denote by $p\colon \widetilde{X} \to X$ the universal covering and set $\widetilde{Y} := p^{-1}(Y)$. Using the representation, we can regard V as a left $\mathbb{Z}[\pi]$–module, where $\pi = \pi_1(X)$. The chain complex $C_*(\widetilde{X},\widetilde{Y})$ is also a left $\mathbb{Z}[\pi]$–module via deck transformations and we form the twisted cohomology groups

$$H^*(X,Y;V_\alpha) = H_*(\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_*(\widetilde{X},\widetilde{Y}),V)).$$

Using the natural involution $g \mapsto g^{-1}$ on the group ring $\mathbb{Z}[\pi]$, we can view $C_*(\widetilde{X},\widetilde{Y})$ as a right $\mathbb{Z}[\pi]$–module, and we can form the twisted homology groups

$$H_*(X,Y;V_\alpha) = H_*(C_*(\widetilde{X},\widetilde{Y}) \otimes_{\mathbb{Z}[\pi]} V).$$

The groups H^0 and H_0 can be computed immediately from the fundamental group (cf. [HS97, Section VI]):

$$(1) \qquad \begin{array}{rcl} H^0(X;V_\alpha) & = & \{v \in V \mid \alpha(g)v = v \text{ for all } g \in \pi\}, \\ H_0(X;V_\alpha) & = & V/\sim, \text{ where } \alpha(g)v \sim v \text{ for all } v \in V, g \in \pi. \end{array}$$

If M is an n–manifold, then Poincaré duality implies

$$H_i(M;V_\alpha) \cong H^{n-i}(M,\partial M;V_\alpha) \text{ and } H_i(M,\partial M;V_\alpha) \cong H^{n-i}(M;V_\alpha).$$

The next two lemmas are both well–known and therefore stated without proof. For more details, see [FK06, Lemma 2.3].

Lemma 10. *Suppose that V is equipped with a bilinear non–singular form, and that α is orthogonal with respect to this form. Then*

$$H_i(X,Y;V_\alpha) \cong H^i(X,Y;V_\alpha)$$

for any i. The same conclusion holds in the case V has a non–singular hermitian form and α is unitary with respect to this form.

Consider the map defined for $A, B \in sl(n,\mathbb{C})$ by the assignment

$$(A,B) \mapsto -\,tr(AB).$$

This map defines a non–singular, symmetric, bilinear form on $sl(n,\mathbb{C})$ called the Killing form. The next lemma says that the hypotheses of Lemma 10 are satisfied for the adjoint representation.

Lemma 11. *For any $\alpha\colon \pi \to \mathrm{SL}(n,\mathbb{C})$, its adjoint representation $ad\,\alpha\colon \pi \to \mathrm{Aut}(sl(n,\mathbb{C}))$ is orthogonal with respect to the Killing form.*

3.3. **Calculations.** This subsection presents some calculations of twisted homology and cohomology groups that will be used in proving the main results.

Lemma 12. *Let K be a knot, $\chi\colon H_1(L_n) \to \mathbb{C}^*$ a character and $z \in U(1)$. Let $V = \mathbb{C}^n$ and $\alpha = \alpha_{(n,\chi,z)}\colon \pi_1(N_K) \to \mathrm{Aut}(V)$, and set $\widehat{\alpha}$ to be the restriction of α to $\pi_1(\partial N_K)$. If $z^n = 1$, then the following hold:*

$$\begin{array}{rclcl} \dim H^0(\partial N_K;V_{\widehat{\alpha}}) & = & \dim H_0(\partial N_K;V_{\widehat{\alpha}}) & = & 1, \\ \dim H^1(\partial N_K;V_{\widehat{\alpha}}) & = & \dim H_1(\partial N_K;V_{\widehat{\alpha}}) & = & 2, \\ \dim H^2(\partial N_K;V_{\widehat{\alpha}}) & = & \dim H_2(\partial N_K;V_{\widehat{\alpha}}) & = & 1. \end{array}$$

Proof. We let μ and λ be the meridian and longitude of K. Note that $\alpha(\lambda)$ is trivial and that $\alpha(\mu)$ is diagonal with eigenvalues $z, ze^{2\pi i/n}, \ldots, ze^{2\pi i(n-1)/n}$, which are distinct. Note that $\alpha(\mu)$ has precisely one eigenvalue which equals one. A direct calculation using Equation (1) shows that $H^0(\partial N_K, V_{\widehat{\alpha}}) = \mathbb{C}$ and $H_0(\partial N_K, V_{\widehat{\alpha}}) = \mathbb{C}$, and duality gives that $H_2(\partial N_K;V_{\widehat{\alpha}}) = \mathbb{C}$ and $H^2(\partial N_K;V_{\widehat{\alpha}}) = \mathbb{C}$. Since the Euler characteristic of the torus ∂N_K is zero we see that $\dim H^1(\partial N_K;V_{\widehat{\alpha}}) = \dim H_1(\partial N_K;V_{\widehat{\alpha}}) = 2$. □

Lemma 13. *Let K be a knot. For $i = 1, \ldots, \ell$, let $\chi_i\colon H_1(L_n) \to \mathbb{C}^*$ be a non–trivial character and $z_i \in U(1)$ with $z_i^n = 1$. Let $V = \mathbb{C}^{n\ell}$ and consider the representation $\alpha = \bigoplus_{i=1}^{\ell} \alpha_{(n,\chi_i,z_i)}\colon \pi_1(N_K) \to \mathrm{Aut}(V)$. Then the following hold:*

(i) $\dim H^0(N_K;V_\alpha)=0$,
(ii) *if α is orthogonal or unitary with respect to a non–singular form on V, then* $\dim H^1(N_K;V_\alpha)\geq \ell$.

Proof. The first statement is an immediate consequence of Equation (1) and the assumption that χ_i are non-trivial. By Lemma 12 we have $\dim H_1(\partial N_K;V_\alpha)=2\ell$. Now consider the following short exact sequence

$$H^1(N_K;V_\alpha)\longrightarrow H^1(\partial N_K;V_\alpha)\longrightarrow H^2(N_K,\partial N_K;V_\alpha).$$

It follows that either $\dim H^1(N_K;V_\alpha)\geq\ell$ or $\dim H^2(N_K,\partial N_K;V_\alpha)\geq\ell$. But by Poincaré duality and by Lemma 10 the latter also equals $\dim H^1(N_K;V_\alpha)$. □

4. Main Results

In this section we establish the results discussed in the introduction. In §4.1, we present cohomology arguments showing $\dim H^1(N_K;sl(n,\mathbb{C})_{ad\,\alpha})\geq n-1$ for any irreducible metabelian representation $\alpha\colon\ \pi_1(N_K)\to \mathrm{SL}(n,\mathbb{C})$. In §4.2, we prove that any algebraic component X_j of $X_n(N_K)$ has dimension $\dim X_j\geq n-1$, in case X_j contains the character of a regular representation $\alpha\colon\ \pi_1(N_K)\to\mathrm{SL}(n,\mathbb{C})$. This is a generalization to $\mathrm{SL}(n,\mathbb{C})$ of a theorem due to Thurston for $\mathrm{SL}(2,\mathbb{C})$ (see [CS83, Theorem 3.2.1]).

At this point, we make the assumption that $\dim H^1(N_K;sl(n,\mathbb{C})_{ad\,\alpha})=n-1$. Using this condition, we show in §4.3 that every irreducible metabelian character ξ_α is a simple point of the character variety $X_n(N_K)$. In §4.4, we prove that every irreducible metabelian representation $\alpha\colon\ \pi_1(N_K)\to\mathrm{SL}(n,\mathbb{C})$ has finite image and is conjugate to a unitary representation, and we develop $SU(n)$ versions of the earlier results. In §4.5, we give the proofs of Theorems 1 and 2, and in §4.6, we present examples illustrating how to apply these techniques.

4.1. **Cohomology arguments.** Assume now that $\alpha\colon\ \pi_1(N_K)\to\mathrm{SL}(n,\mathbb{C})$ is a representation and let $\widehat{\alpha}\colon\ \pi_1(\partial N_K)\to\mathrm{SL}(n,\mathbb{C})$ denote its restriction to the boundary torus. Throughout much of what follows, we will assume that α is a *regular representation*, meaning that α is irreducible and that the image of $\widehat{\alpha}$ contains a matrix with n distinct eigenvalues. The subset of regular representations is clearly Zariski open in $R_n(N_K)$, and every irreducible metabelian representation of $\pi_1(N_K)$ is regular (see Remark 6).

Choose $g\in\pi_1(\partial N_K)$ so that $\alpha(g)$ has n distinct eigenvalues. Then this matrix is diagonalizable, and any other matrix that commutes with it must lie in the same maximal torus. Since $\pi_1(\partial N_K)\cong\mathbb{Z}\oplus\mathbb{Z}$ is abelian, we see that the stabilizer subgroup of $\widehat{\alpha}$ under conjugation is again this maximal torus. From this, Poincaré duality and Euler characteristic considerations, we conclude that

$$\begin{array}{rcl}\dim H^0(\partial N_K;sl(n,\mathbb{C})_{ad\,\widehat{\alpha}}) &=& n-1,\\ \dim H^1(\partial N_K;sl(n,\mathbb{C})_{ad\,\widehat{\alpha}}) &=& 2(n-1),\ \text{and}\\ \dim H^2(\partial N_K;sl(n,\mathbb{C})_{ad\,\widehat{\alpha}}) &=& n-1.\end{array}$$

We now consider the long exact sequence in twisted cohomology associated with the pair $(N_K,\partial N_K)$. The inclusions

$$(\partial N_K,\varnothing)\stackrel{i}{\hookrightarrow}(N_K,\varnothing)\stackrel{j}{\hookrightarrow}(N_K,\partial N_K)$$

induce the following long exact sequence (coefficients in $sl(n,\mathbb{C})$ twisted by $ad\,\alpha$ or $ad\,\widehat{\alpha}$ understood).

$$\begin{array}{ccccccccc} & 0 & \longrightarrow & H^0(N_K) & \longrightarrow & H^0(\partial N_K) & \longrightarrow & H^1(N_K,\partial N_K) & \\ (2) & & \xrightarrow{j^1} & H^1(N_K) & \xrightarrow{i^1} & H^1(\partial N_K) & \longrightarrow & H^2(N_K,\partial N_K) & \\ & & \xrightarrow{j^2} & H^2(N_K) & \xrightarrow{i^2} & H^2(\partial N_K) & \longrightarrow & H^3(N_K,\partial N_K) & \longrightarrow \ 0. \end{array}$$

Exactness of the middle row implies that

$$\dim H^1(N_K) + \dim H^2(N_K,\partial N_K) \geq \dim H^1(\partial N_K) = 2n-2,$$

and by Poincaré duality and Lemmas 10 and 11, we have that $\dim H^1(N_K) = \dim H^2(N_K,\partial N_K)$. This implies $\dim H^1(N_K) \geq n-1$.

The next proposition shows that the image $\big(i^1\colon\ H^1(N_K)\longrightarrow H^1(\partial N_K)\big)$ has dimension $n-1$, and this should be viewed as an instance of the following general principle. Suppose N is a 3-manifold with boundary $\partial N = \Sigma$ a compact Riemann surface of genus g. Goldman proved that the smooth part of the character variety $X_n(\Sigma)$ carries a natural symplectic structure [Gol84], and a folklore result implies that the image of the restriction $X_n(N) \to X_n(\Sigma)$ is Lagrangian. This idea has been made precise by A. Sikora, who studied this in the general setting of representations into reductive Lie groups in [Si09], under the assumption that ∂X is a connected surface of genus $g \geq 2$. We state and prove analogous results for $\mathrm{SL}(n,\mathbb{C})$ representations of knot complements N_K, which is the main case of interest here.

Proposition 14. *If K is a knot and $\alpha\colon\ \pi_1(N_K) \to \mathrm{SL}(n,\mathbb{C})$ is a regular representation, then the image*

$$\text{image}\,\big(i^1\colon\ H^1(N_K; sl(n,\mathbb{C})_{ad\,\alpha})\longrightarrow H^1(\partial N_K; sl(n,\mathbb{C})_{ad\,\widehat{\alpha}})\big)$$

has dimension $n-1$ and is Lagrangian with respect to the symplectic structure Ω defined below. It follows that

$$\dim H^1(N_K; sl(n,\mathbb{C})_{ad\,\alpha}) \geq n-1.$$

Proof. The fact that image(i^1) has dimension $n-1$ follows easily from a diagram chase of the long exact sequence (2), using the fact that $\widehat{\alpha}(\pi_1(\partial N_K))$ contains an element with n distinct eigenvalues, hence $\dim H^0(\partial N_K; sl(n,\mathbb{C})_{ad\,\widehat{\alpha}}) = n-1 = \dim H^2(\partial N_K; sl(n,\mathbb{C})_{ad\,\widehat{\alpha}})$ and $\dim H^1(\partial N_K; sl(n,\mathbb{C})_{ad\,\widehat{\alpha}}) = 2n-2$.

The symplectic structure Ω on $H^1(\partial N_K; sl(n,\mathbb{C}))$ is defined by composing the cup product with the symmetric bilinear pairing obtained by first multiplying the matrices and then taking the trace:

$$\begin{gathered} sl(n,\mathbb{C}) \times sl(n,\mathbb{C}) \to gl(n,\mathbb{C}) \to \mathbb{C} \\ (A,B) \mapsto A\cdot B \mapsto tr(A\cdot B). \end{gathered}$$

We have already seen that the image(i^1) has dimension $n-1$, so we just need to show that it is isotropic with respect to Ω.

Suppose $x,y \in H^1(N_K; sl(n,\mathbb{C})_{ad\,\alpha})$ and consider the long exact sequence (2) with untwisted coefficients in $\mathbb{C}$. Let $\smile$ denote the combined cup and matrix product, so $x \smile y \in H^2(N_K; gl(n,\mathbb{C})_{ad\,\alpha})$. Using the commutative diagram

$$\begin{array}{ccc} H^1(N_K; sl(n,\mathbb{C})_{ad\,\alpha}) \times H^1(N_K; sl(n,\mathbb{C})_{ad\,\alpha}) & \xrightarrow{\ \smile\ } & H^2(N_K; gl(n,\mathbb{C})_{ad\,\alpha}) \\ \big\downarrow{\scriptstyle i^1\times i^1} & & \big\downarrow{\scriptstyle i^2} \\ H^1(\partial N_K; sl(n,\mathbb{C})_{ad\,\widehat{\alpha}}) \times H^1(\partial N_K; sl(n,\mathbb{C})_{ad\,\widehat{\alpha}}) & \xrightarrow{\ \smile\ } & H^2(\partial N_K; gl(n,\mathbb{C})_{ad\,\widehat{\alpha}}), \end{array}$$

we see that $\Omega(i^1(x), i^1(y)) = tr(i^1(x) \smile i^1(y)) = tr\, i^2(x \smile y)$. This shows $\Omega(i^1(x), i^1(y))$ lies in the image of

$$H^2(N_K;\mathbb{C}) \longrightarrow H^2(\partial N_K;\mathbb{C}), \tag{3}$$

which by exactness of the third row of the long exact sequence (2), now taken with untwisted $\mathbb{C}$ coefficients, equals the kernel of the surjection $H^2(\partial N_K;\mathbb{C}) \to H^3(N_K, \partial N_K;\mathbb{C})$. However, it is not difficult to compute $H^3(N_K, \partial N_K;\mathbb{C}) = \mathbb{C} = H^2(\partial N_K;\mathbb{C})$, and this implies that the map in Equation (3) is the zero map. □

4.2. **Dimension arguments.** In this subsection, we give a lower bound on the dimension of algebraic components of the character variety $X_n(N_K)$ containing a regular representation.

Proposition 15. *If $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(n,\mathbb{C})$ is a regular representation, then any algebraic component $X_j \subset X_n(N_K)$ containing ξ_α has $\dim X_j \geq n-1$.*

Proof. If ξ_α is a smooth point of X_j, then by Proposition 14 we have $\dim X_j = \dim H^1(N_K; sl(n,\mathbb{C})_{ad\,\alpha}) \geq n-1$. Otherwise, we can choose $\beta\colon \pi_1(N_K) \to \mathrm{SL}(n,\mathbb{C})$ a regular representation close to α) so that $\xi_\beta \in X_j$ is smooth. Applying Proposition 14 to β, it follows that $\dim X_j = \dim H^1(N_K; sl(n,\mathbb{C})_{ad\,\beta}) \geq n-1$. □

Since every irreducible metabelian representation is regular, we obtain the following as a direct consequence.

Corollary 16. *If $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(n,\mathbb{C})$ is an irreducible metabelian representation, then any algebraic component X_j of $X_n(N_K)$ containing ξ_α has $\dim X_j \geq n-1$.*

4.3. **Simple points in $X_n(N_K)$.** This subsection presents a smoothness result for irreducible characters which is proved using the powerful deformation argument from [HPS01]. A more detailed explanation of this beautiful argument is presented in Appendix A, following [HPS01, HP05, AHJ10], and the original idea can be traced back to a deep theorem of Artin [Ar68].

Recall that a point $\xi \in X$ in an affine algebraic variety is called a *simple point* if it is contained in a unique algebraic component of X and is a smooth point of that component. The next result, which essentially follows from Theorem 3.2 in [AHJ10], implies that every irreducible metabelian character ξ_α such that $\dim H_1(N_K; sl(n,\mathbb{C})_{ad\,\alpha}) = n-1$ is a simple point of $X_n(N_K)$.

Proposition 17. *If $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(n,\mathbb{C})$ is a regular representation such that $\dim H^1(N_K; sl(n,\mathbb{C})_{ad\,\alpha}) = n-1$, then ξ_α is a simple point in the character variety $X_n(N_K)$.*

Proposition 17 applies to any irreducible metabelian $\mathrm{SL}(n,\mathbb{C})$ representation.

We give a full account of this proposition in the Appendix, and here we briefly explain the basic idea. By irreducibility of α and Luna's étale slice theorem [Lu73], it follows that ξ_α is a simple point of $X_n(N_K)$ if and only if α is a simple point of $R_n(N_K)$. The same is true for $\widehat{\alpha}$, and the hypotheses ensure that $\widehat{\alpha}$ is a simple point of $R_n(\partial N_K)$. The main idea is to construct formal deformations for all (Zariski) tangent vectors and to show their integrability by using the fact that all obstructions project faithfully under projection to ∂N_K, where they are known to vanish by the fact that $\widehat{\alpha}$ is a simple point of $R_n(\partial N_K)$.

4.4. ***SU(n)* results.** This subsection contains the $SU(n)$ analogues of the earlier results on irreducible metabelian representations. We will prove that any irreducible metabelian representation $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(n,\mathbb{C})$ satisfying the condition $\dim H^1(N_K; sl(n,\mathbb{C})_{ad\,\alpha}) = n-1$ has finite image and is therefore conjugate to a unitary representation.

We begin with a few general observations. If π is a finitely generated group and $\alpha\colon \pi \to SU(n)$ is a representation, then we obtain an $\mathrm{SL}(n,\mathbb{C})$ representation by composing α with the inclusion $SU(n) \subset \mathrm{SL}(n,\mathbb{C})$. Irreducibility of α is preserved under this correspondence, and the map $R_{SU(n)}(\pi) \to R_n(\pi)$ descends to a well-defined injective map $X_{SU(n)}(\pi) \longrightarrow X_n(\pi)$ between the two character varieties. Here and in the following, we set $R_{SU(n)}(\pi) = \mathrm{Hom}(\pi, SU(n))$ and use $X_{SU(n)}(\pi)$ to denote the character variety of $SU(n)$ representations of π.

On the level of Lie algebras, the complex Lie algebra $sl(n,\mathbb{C})$ is obtained by tensoring the real Lie algebra $su(n)$ with $\mathbb{C}$, i.e. we have

$$sl(n,\mathbb{C}) \cong su(n) \otimes \mathbb{C}.$$

Thus, for $\alpha\colon \pi \to SU(n)$, we see that for any $i \geq 0$ we have

$$H^i(\pi; sl(n,\mathbb{C})_{ad\,\alpha}) \cong H^i(\pi; su(n)_{ad\,\alpha}) \otimes \mathbb{C}. \tag{4}$$

In the following proposition, we use L_n to denote the n–fold branched cover of Σ^3 branched along K.

Proposition 18. *Suppose $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(n,\mathbb{C})$ is an irreducible metabelian representation. If $\dim H^1(N_K; sl(n,\mathbb{C})_{ad\,\alpha}) = n-1$, then $H_1(L_n)$ is finite. In particular, α has finite image and is conjugate to a unitary representation.*

Proof. It follows from Theorem 5 that we can assume that $\alpha = \alpha_{(n,\chi)}$ for some character $\chi\colon H_1(L_n) \to \mathbb{C}^*$. Let $\theta_1\colon \pi_1(N_K) \to \mathrm{GL}(1,\mathbb{C})$ be the trivial representation and $\alpha_n\colon \pi_1(N_K) \to \mathrm{Aut}(\mathbb{C}[\mathbb{Z}/n])$ be the regular representation corresponding to the canonical projection map $\pi_1(N_K) \to \mathbb{Z} \to \mathbb{Z}/n$. By Lemma 9 we have the following isomorphism of representations:

$$ad\,\alpha \oplus \theta_1 \cong \alpha_n \oplus \bigoplus_{i=1}^{n-1} \beta_{(n,\chi_i)},$$

where $\chi_1,\dots,\chi_{n-1}$ are characters. Clearly θ_1 and α_n are orthogonal representations. Furthermore by Lemma 11 the representation $ad\,\alpha\colon \pi_1(N_K) \to \mathrm{Aut}(sl(n,\mathbb{C}))$ is an isometry with respect to the Killing form. If we equip $sl(n,\mathbb{C})$ with the standard basis and we thus view $ad\,\alpha$ as a representation to $gl(n^2-1,\mathbb{C})$, then it follows from the definition of the Killing form, that $ad\,\alpha$ is an orthogonal representation. It now follows that $\beta := \bigoplus_{i=1}^{n-1} \beta_{(n,\chi_i)}\colon \pi_1(N_K) \to \mathrm{GL}(n(n-1),\mathbb{C})$ is also orthogonal. By Lemma 13 we now have

$$\begin{aligned}
\dim H^1(N_K; sl(n,\mathbb{C})_{ad\,\alpha}) &= \dim H_1(N_K; \mathbb{C}[\mathbb{Z}/n]) - 1 + \dim H^1(N_K; \mathbb{C}^{n(n-1)}_\beta) \\
&= b_1(L_n) + 1 - 1 + \dim H^1(N_K; \mathbb{C}^{n(n-1)}_\beta) \\
&\geq b_1(L_n) + n - 1.
\end{aligned}$$

The condition $\dim H^1(N_K; sl(n,\mathbb{C})_{ad\,\alpha}) = n-1$ now shows that $b_1(L_n) = 0$. Thus $H_1(L_n) = H_1(N_K; \mathbb{Z}[t^{\pm 1}])/(t^n - 1)$ is finite, and this implies α has finite image and is conjugate to a unitary representation. $\square$

Proposition 18 implies that metabelian representations $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(n, \mathbb{C})$ are often conjugate to unitary representations, and for that reason we develop $SU(n)$ versions of the previous results. As the proofs are similar to those already given, we leave the details to the industrious reader.

We begin with the $SU(n)$ version of Proposition 14. Just as in the $\mathrm{SL}(n, \mathbb{C})$ case, we say a representation $\alpha\colon \pi_1(N_K) \to SU(n)$ is *regular* if it is irreducible and if the image of the restriction $\widehat{\alpha}\colon \pi_1(\partial N_K) \to SU(n)$ contains a matrix with n distinct eigenvalues.

Note that the definition of the symplectic form Ω on $H^1(\partial N_K; sl(n, \mathbb{C})_{ad\,\widehat{\alpha}})$ in the proof of Proposition 14 carries over easily to the $SU(n)$ setting, and we use $\Omega_{SU(n)}$ to denote the resulting symplectic form on $H^1(\partial N_K; su(n)_{ad\,\widehat{\alpha}})$.

Proposition 19. *If K is a knot and $\alpha\colon \pi_1(N_K) \to SU(n)$ is a regular representation, then the image*

$$\text{image}\left(i^1\colon\, H^1(N_K; su(n)_{ad\,\alpha}) \longrightarrow H^1(\partial N_K; su(n)_{ad\,\widehat{\alpha}})\right)$$

has real dimension $n-1$ and is Lagrangian with respect to the natural symplectic structure $\Omega_{SU(n)}$. It follows that

$$\dim_{\mathbb{R}} H^1(N_K; su(n)_{ad\,\alpha}) \geq n-1.$$

Next, we present the $SU(n)$ version of Proposition 15. Recall that $X_{SU(n)}(N_K)$ is a real algebraic variety.

Proposition 20. *If $\alpha\colon \pi_1(N_K) \to SU(n)$ is a regular representation, then any algebraic component $X_j \subset X_{SU(n)}(N_K)$ containing ξ_α satisfies* $\dim_{\mathbb{R}} X_j \geq n-1$.

Since all irreducible metabelian $SU(n)$ representations are regular, Proposition 20 applies to give the following as a direct consequence.

Corollary 21. *If $\alpha\colon \pi_1(N_K) \to SU(n)$ is an irreducible metabelian representation, then any algebraic component X_j of $X_{SU(n)}(N_K)$ containing ξ_α has* $\dim_{\mathbb{R}} X_j \geq n-1$.

The final result is an $SU(n)$ version of Proposition 17.

Proposition 22. *If $\alpha\colon \pi_1(N_K) \to SU(n)$ is a regular representation such that* $\dim_{\mathbb{R}} H_1(N_K; su(n)_{ad\,\alpha}) = n-1$, *then ξ_α is a simple point in the character variety* $X_{SU(n)}(N_K)$.

4.5. **Proofs of Theorems 1 and 2.** In this subsection, we prove the two main results from the Introduction.

Proof of Theorem 1. Suppose α is an irreducible metabelian representation with $\dim H^1(N_K; sl(n, \mathbb{C})_{ad\,\alpha}) = n-1$. Applying Proposition 18, we see that α has finite image and hence is conjugate to a unitary representation. Since $\alpha(\mu)$ has n distinct eigenvalues, Proposition 17 applies and gives rise to a smooth complex $(n-1)$–dimensional family of $\mathrm{SL}(n, \mathbb{C})$ characters near $\xi_\alpha \in X_n(N_K)$.

Conjugating, if necessary, we can arrange that α is unitary. In that case, Equation (4) implies that

$$H^1(N_K; sl(n, \mathbb{C})_{ad\,\alpha}) = H^1(N_K; su(n)_{ad\,\alpha}) \otimes \mathbb{C},$$

and it follows that $\dim_{\mathbb{R}} H^1(N_K; su(n)_{ad\,\alpha}) = n-1$. Thus Proposition 22 applies and gives rise to a smooth real $(n-1)$–dimensional family of irreducible characters near $\xi_\alpha \in X_{SU(n)}(N_K)$.

Note that Proposition 18 shows that $b_1(L_n) = 0$, and thus every irreducible metabelian representation $\beta\colon \pi_1(N_K) \to \mathrm{SL}(n,\mathbb{C})$ factors through a finite group. In particular, this shows that up to conjugation there are only finitely many irreducible metabelian $\mathrm{SL}(n,\mathbb{C})$ representations, and their characters give rise to a finite collection of points in the character variety $X_n^*(N_K)$. It follows that we can take either of the two deformation families of conjugacy classes of irreducible representations so that ξ_α is the unique metabelian representation within the family. □

Proof of Theorem 2. Let $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(n,\mathbb{C})$ an irreducible metabelian representation, and $\varphi\colon \pi_1(N_K) \to \mathbb{Z}/n \ltimes H$ a homomorphism such that α factors through φ and with H finite. Set $k = |H|$.

We first consider the cover $p\colon \widetilde{N}_\varphi \to N_K$ corresponding to φ. Note that there exist precisely $k = |H|$ characters $H \to U(1)$. We denote this set by $\{\sigma_1, \dots, \sigma_k\}$, where we assume that σ_1 is the trivial character. It is not difficult to see that the representation $\sigma_1 \oplus \cdots \oplus \sigma_k\colon H \to \mathrm{Aut}(\mathbb{C}^k)$ is isomorphic to the regular representation $H \to \mathrm{Aut}(\mathbb{C}[H])$. We denote the representation $\pi_1(N_K) \to \mathrm{Aut}(\mathbb{C}[\mathbb{Z}/n \ltimes H])$ by φ as well. Then it is straightforward to verify that

$$\varphi \cong \bigoplus_{i=1}^{k} \beta_{(n,\sigma_i)}.$$

In particular, setting $V = \mathbb{C}^{kn}$ and $U = \mathbb{C}^n$, we have

$$b_1(\widetilde{N}_\varphi) = b_1(N; V_\varphi) = \sum_{i=1}^{k} b_1(N; U_{\beta_{(n,\sigma_i)}}).$$

Note that each $\beta_{(n,\sigma_i)}$ is a unitary representation. It now follows immediately from Lemma 13 that $b_1(\widetilde{N}_\varphi) \geq k$. Furthermore, if $b_1(\widetilde{N}_\varphi) = k$ then it follows that $b_1(N; U_{\beta_{(n,\sigma_i)}}) = 1$ for each $i = 1, \dots, k$. Statement (i) now follows immediately from Lemma 9.

We now turn to the proof of (ii). We write $T = \partial N_K$. Note that the image of the restriction $\widehat{\varphi}\colon \pi_1(T) \to \mathbb{Z}/n \ltimes H$ has order n. In particular the preimage of T under the covering $p\colon \widetilde{N}_\varphi \to N_K$ has $k = |H|$ components, and we denote them by $T_1, \dots, T_k$. Note that in each T_i there exist simple closed curves μ_i and λ_i such that $p|_{\mu_i}$ restricts to an n–fold cover of the meridian $\mu \subset T$ and such that $p|_{\lambda_i}$ restricts to a homeomorphism with the longitude λ of T. Note that μ_i, λ_i form a basis for $H_1(T_i)$.

We now denote by $\widehat{\Sigma}_\varphi$ the result of gluing k solid tori $S_1, \dots, S_k$ to the boundary of $\widetilde{N}_\varphi$ such that each μ_i bounds a disk in S_i. The projection map $p\colon \widetilde{N}_\varphi \to N_K$ then extends in a canonical way to a covering map $\widehat{\Sigma}_\varphi \to \Sigma$, branched over K, and that proves (ii).

We finally turn to the proof of (iii). Consider the following Mayer–Vietoris sequence:

$$\bigoplus_{i=1}^{k} H_1(T_i) \to \bigoplus_{i=1}^{k} H_1(S_i) \,\oplus\, H_1(\widetilde{N}_\varphi) \to H_1(\widehat{\Sigma}_\varphi) \to 0.$$

It follows immediately that

$$b_1(\widehat{\Sigma}_\varphi) \geq k + b_1(\widetilde{N}_\varphi) - 2k = b_1(\widetilde{N}_\varphi) - k.$$

In particular if $b_1(\widehat{\Sigma}_\varphi)=0$, then $b_1(\widetilde{N}_\varphi)\leq k$. Applying (i) shows we have equality here and that (iii) holds. □

4.6. **Examples.** In this subsection, we show how to construct deformations of metabelian representations $\alpha\colon\, \pi_1(N_K)\to \mathrm{SL}(n,\mathbb{C})$ in specific situations.

We begin with some general comments about the rank two case. As mentioned in the introduction, by results of Culler and Shalen [CS83], if K is a small knot, then any irreducible metabelian representation $\alpha\colon\, \pi_1(N_K)\to \mathrm{SL}(2,\mathbb{C})$ lies on an algebraic component of $X_2(N_K)$ of dimension one. Since all torus knots and all two-bridge knots are small, this tells us that Theorem 1 applies to many knots in rank two. Interestingly, not all such knots admit irreducible metabelian $\mathrm{SL}(2,\mathbb{C})$ representations. For example, in the notation of Rolfsen's table [Ro76], this occurs for the knots 10_{124} and 10_{153}. Note that 10_{124} is the $(3,5)$–torus knot and is a fibered knot of genus 3, whereas 10_{153} is not a torus knot but it is fibered of genus 4. A simple calculation using [BF08, Theorem 3.7] shows that both knots admit irreducible metabelian representations in $\mathrm{SL}(3,\mathbb{C})$ and $\mathrm{SL}(5,\mathbb{C})$, indeed up to conjugation 10_{124} admits 8 such representations in rank 3 and 16 in rank 5, whereas 10_{153} admits 16 such representations in rank 3 and 24 in rank 5. In both cases, we see that $H_1(L_3)$ and $H_1(L_5)$ are finite, and so the irreducible metabelian characters are isolated points in the character variety $X_n(N_K)$. Proposition 14 applies to show they can be deformed to nearby non-metabelian irreducible representations.

We now investigate situations to which Theorem 1 applies, and for that purpose we will consider a fibered knot K of genus one in a homology 3-sphere Σ. We have in mind the trefoil and the figure eight knot. The trefoil knot has irreducible metabelian representations only in rank $2,3$, and 6. The figure eight knot, on the other hand, has irreducible metabelian $\mathrm{SL}(n,\mathbb{C})$ representations for all but finitely many ranks, which follows directly from Theorem 8. Indeed, the number of conjugacy classes of irreducible metabelian representations for both knots $\alpha\colon\, \pi_1(N_K)\to SU(n)$ can be determined in terms of the orders $|H_1(L_k)|$ taken over all divisors k of n, and direct computation shows that the trefoil has a unique irreducible metabelian representation in ranks 2 and 3, whereas the figure eight has increasingly many as the rank $n\to\infty$. Applying Theorem 3.7 of [BF08], we compute the number of distinct conjugacy classes of irreducible metabelian $SU(n)$ representations for the figure eight knot, and the results for $1\leq n\leq 21$ are listed in Table 1.

The next result shows that any algebraic component of $X_n(N_K)$ containing such a representation has dimension $n-1$. Thus Theorem 1 applies and gives a nice local description of the character variety near these metabelian characters.

Proposition 23. *Suppose K is a fibered knot of genus one in a homology 3-sphere Σ whose n–fold branched cover has $H_1(L_n)$ finite and $\alpha\colon\, \pi_1(N_K)\to \mathrm{SL}(n,\mathbb{C})$ is an irreducible metabelian representation. Then any algebraic component X_j of $X_n(N_K)$ containing ξ_α has* $\dim X_j = n-1$.

Proof. By Proposition 14, we have that $\dim X_j\geq n-1$, so it is enough to show $\dim X_j\leq n-1$. If R_j is the algebraic component of $R_n(N_K)$ lying above X_j, then we will show that $\dim R_j\leq n^2+n-2$. This is sufficient because we know that R_j contains the irreducible representation α, and so the generic fiber of the quotient map $t\colon\, R_j\to X_j$ has dimension n^2-1.

n	#	n	#	n	#
1	1	8	270	15	124,024
2	2	9	640	16	304,290
3	5	10	1500	17	750,120
4	10	11	3600	18	1,854,400
5	24	12	8610	19	4,600,200
6	50	13	20880	20	11,440,548
7	120	14	50700	21	28,527,320

Table 1. The number of conjugacy classes of irreducible metabelian $SU(n)$ representations for the figure eight knot for $1 \leq n \leq 21$

Consider the subset of R_j defined by

$$\widehat{R}_j = \{\varrho \in R_j \mid \varrho \text{ is irreducible and } \varrho(\mu) \text{ has } n \text{ distinct eigenvalues}\}.$$

This is obviously a Zariski open subset, and since $\alpha \in \widehat{R}_j$, it is nonempty. In particular, we see that $\dim \widehat{R}_j = \dim R_j$.

Given $A \in \mathrm{SL}(n, \mathbb{C})$, we use $\Phi_A(t) = \det(tI - A)$ to denote its characteristic polynomial. In general, given $\gamma \in \pi$, the association $\varrho \mapsto \Phi_{\varrho(\gamma)}(t)$ gives an algebraic map $\Phi_{\cdot(\gamma)} \colon R_n(\pi) \longrightarrow \mathbb{C}^{n-1}$, where $\Phi_{\varrho(\gamma)}(t) = t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + (-1)^n$ gives the point $(a_1, \ldots, a_{n-1}) \in \mathbb{C}^{n-1}$.

Taking $\gamma = \lambda$, the longitude of K, we define

$$Z_j = \{\varrho \in \widehat{R}_j \mid \Phi_{\varrho(\lambda)}(t) = (t-1)^n\}.$$

Clearly Z_j is a Zariski closed subset of $\widehat{R}_j$. Since Z_j is obtained by applying $n-1$ algebraic equations, we see that $\dim Z_j \geq \dim R_j - (n-1)$ (see [Sh95, p. 75, Corollary 2]). Furthermore, since $\alpha \in Z_j$, we see that Z_j is nonempty.

For any fibered knot, the commutator subgroup of $\pi_1(N_K)$ is the finitely generated free group given by the fundamental group of the fiber. In the case of a fibered knot of genus one, this group is a free group of rank two, and we obtain the short exact sequence

$$1 \to F_2 \longrightarrow \pi_1(N_K) \longrightarrow \mathbb{Z} \to 1. \tag{5}$$

Taking S to be the fiber surface, then $F_2 = \pi_1(S) = \langle a, b \rangle$ and we can write the longitude as $\lambda = aba^{-1}b^{-1}$. Thus, a given representation $\varrho \colon \pi_1(N_K) \to \mathrm{SL}(n, \mathbb{C})$ is metabelian if and only if its restriction to F_2 is abelian, namely if $\varrho(a)$ and $\varrho(b)$ commute. Since $\lambda = [a, b]$, we see that ϱ is metabelian if and only if $\varrho(\lambda) = I$. This shows that every irreducible metabelian representation in $\widehat{R}_j$ is contained in Z_j, and we will now show the reverse inclusion.

Suppose $\varrho \in Z_j$. Then since $Z_j \subset \widehat{R}_j$, ϱ is irreducible and $\varrho(\mu)$ has n distinct eigenvalues. Thus $\varrho(\mu)$ is contained in a unique maximal torus, which we can arrange by conjugation to be the standard maximal torus of diagonal matrices in $\mathrm{SL}(n, \mathbb{C})$. Since $\varrho(\lambda)$ commutes with $\varrho(\mu)$, it follows that $\varrho(\lambda)$ is also a diagonal matrix. The condition that $\Phi_{\varrho(\lambda)}(t) = (t-1)^n$ implies $\varrho(\lambda) = I$, and the sequence (5) shows that ϱ is necessarily metabelian.

We now make use of the assumption that $H_1(L_n)$ is finite. This implies that, up to conjugation, there are only finitely many irreducible metabelian representations.

Thus the quotient of Z_j by conjugation is a finite collection of points, and since every $\varrho \in Z_j$ is also irreducible, we conclude that $\dim Z_j = n^2 - 1$. Using that

$$\dim R_j = \dim \widehat{R}_j \leq \dim Z_j + (n-1) = n^2 + n - 2,$$

we conclude that $\dim X_j \leq n - 1$. □

Proposition 23 applies to irreducible metabelian representations of the figure eight knot in all ranks (see Table 1), but it only applies to the trefoil in ranks 2 and 3. The only other rank where the trefoil admits irreducible metabelian representations is rank 6, and in that case $H_1(L_6)$ is not finite.

We investigate the general situation of torus knots, and we note that as a consequence of Proposition 3.10 (iii) of [BF08], a (p,q) torus knot K has no irreducible metabelian $\mathrm{SL}(n,\mathbb{C})$ representations if n is relatively prime to p and q. Torus knot groups have the following well-known presentation:

$$\pi_1(N_K) = \langle x, y \mid x^p = y^q \rangle, \tag{6}$$

where the meridian and longitude μ and λ are represented by $\mu = x^s y^r$ and $\lambda = x^p(\mu)^{-pq}$, for $r, s \in \mathbb{Z}$ with $rp + sq = 1$. We choose n to be a divisor of q and work with $SU(n)$ representations for convenience. Then any irreducible metabelian representation $\varrho\colon \pi_1(N_K) \to SU(n)$ will satisfy $\varrho(\mu)^n = I$ and $\varrho(\lambda) = I$, and this implies that $\varrho(x)$ and $\varrho(y)$ are p-th and q-th roots of unity, respectively.

Since $\varrho(x)$ and $\varrho(y)$ are diagonalizable, we can arrange that $\varrho(x)$ is conjugate to $A = \mathrm{diag}(a_1, \ldots, a_n)$ and $\varrho(y)$ is conjugate to $B = \mathrm{diag}(b_1, \ldots, b_n)$, where $a_1, \ldots, a_n$ are p-th roots of unity and $b_1, \ldots, b_n$ are q-th roots of unity. Let C_A and C_B denote the conjugacy classes in $SU(n)$ of A and B, respectively. The eigenspaces of A and B determine partitions $(\alpha_1, \ldots, \alpha_k)$ and $(\beta_1, \ldots, \beta_\ell)$ of n, respectively, and we have

$$\dim C_A = n^2 - (\alpha_1^2 + \cdots + \alpha_k^2) \quad \text{and} \quad \dim C_B = n^2 - (\beta_1^2 + \cdots + \beta_\ell^2).$$

For instance, if A has n distinct eigenvalues, then $\dim C_A = n^2 - n$. In general, $\dim C_A$ and $\dim C_B$ are even numbers between 0 and $n^2 - n$.

The component R_j of $R_{SU(n)}(N_K))$ containing ϱ is just the direct product $C_A \times C_B$, and it follows that $\dim R_j = \dim C_A + \dim C_B$. If A and B can be chosen so that $\dim C_A + \dim C_B = n^2 + n - 2$, then we will be able to apply Theorem 1. This will occur if say A has n distinct eigenvalues and B has one eigenvalue of multiplicity 1 and a second eigenvalue of multiplicity $n-1$. Assuming that R_j contains an irreducible representation, then just as in the proof of Proposition 23, it follows that if $X_j \subset X_{SU(n)}(N_K)$ is the quotient of R_j under conjugation, then $\dim X_j = n - 1$.

For specific examples, consider the torus knots $K = T(2,q)$, where q is a multiple of 3. Then direct calculation shows that any irreducible metabelian representation $\varrho : \pi_1(N_K) \to SU(3)$ has $\dim H^1(N_K; su(3)_{ad\,\varrho}) = 2$ (see Proposition 3.1 of [BHK05], for example). Hence Theorem 1 applies to establish the existence of 2–dimensional deformation families in $X_{SU(3)}(N_K)$ and $X_3(N_K)$.

Appendix A. Deformation arguments

In this appendix, we present the deformation arguments that prove Proposition 17. This material is included for the reader's convenience. The original arguments were given for $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{PSL}(2,\mathbb{C})$ in [HPS01] and [HP05], and they were generalized to $\mathrm{SL}(n,\mathbb{C})$ in [AHJ10]. In what follows, we present detailed arguments for $\mathrm{SL}(n,\mathbb{C})$, focusing on the implications for the character variety $X_n(N_K)$, where $N_K = \Sigma \smallsetminus \tau(K)$ is the complement of a knot in an integral homology 3-sphere.

Proof of Proposition 17. The first step is to show that $\xi_{\widehat{\alpha}}$ is a simple point in $X_n(\partial N_K)$. We do this by comparing the dimension of the cocycles

$$Z^1(\pi_1(\partial N_K); sl(n,\mathbb{C})_{ad\,\alpha})$$

with the local dimension of $R_n(\partial N_K)$ at α, which is defined to be the maximal dimension of the irreducible components of $R_n(\partial N_K)$ containing α.

First, some notation. Given a finitely generated group π and a representation $\alpha\colon \pi \to \mathrm{SL}(n,\mathbb{C})$, let $H^*(\pi; sl(n,\mathbb{C})_{ad\,\alpha})$ denote the cohomology of the group with coefficients in the π–module by $sl(n,\mathbb{C})_{ad\,\alpha}$.

In [We64], Weil observed that there is a natural inclusion of the Zariski tangent space $T^{\mathrm{Zar}}_{\alpha}(R_n(\pi)) \hookrightarrow Z^1(\pi; sl(n,\mathbb{C})_{ad\,\alpha})$ into the space of cocycles, and we will combine this observation with computations of the twisted cohomology of $\pi_1(\partial N_K)$ and $\pi_1(N_K)$.

Because ∂N_K is a $K(\mathbb{Z}\oplus\mathbb{Z},1)$, we have isomorphisms

$$H^*(\partial N_K; sl(n,\mathbb{C})_{ad\,\alpha}) \longrightarrow H^*(\pi_1(\partial N_K); sl(n,\mathbb{C})_{ad\,\alpha}),$$

and the inclusion $N_K \hookrightarrow K(\pi_1(N_K),1)$ induces maps

$$H^i(\pi_1(N_K); sl(n,\mathbb{C})_{ad\,\alpha}) \longrightarrow H^i(N_K; sl(n,\mathbb{C})_{ad\,\alpha})$$

that are isomorphisms when $i=0$ and 1 and injective when $i=2$ (see [HP05, Lemma 3.1]).

Consider the 2-torus ∂N_K with its standard CW–structure consisting of one 0–cell, two 1–cells and one 2–cell. It is straightforward to verify that the spaces of twisted 1-coboundaries and 1-cocycles satisfy

$$\begin{aligned} \dim B^1(\partial N_K; sl(n,\mathbb{C})_{ad\,\widehat{\alpha}}) &= n^2-1-(n-1) = n^2-n, \text{ and} \\ \dim Z^1(\partial N_K; sl(n,\mathbb{C})_{ad\,\widehat{\alpha}}) &= 2(n-1)+n^2-n = n^2+n-2. \end{aligned}$$

Since $\widehat{\alpha}$ sits on an (n^2+n-2)–dimensional component, its local dimension is

$$\dim_{\widehat{\alpha}} R_n(\partial N_K) = n^2+n-2.$$

For arbitrary $\sigma \in R_n(\partial N_K)$, we have

$$\dim_\sigma R_n(\partial N_K) \leq \dim T^{\mathrm{Zar}}_\sigma(R_n(\partial N_K)) \leq \dim Z^1(\partial N_K; sl(n,\mathbb{C})_{ad\,\sigma}).$$

In our case, we have equality throughout, and it follows that $\widehat{\alpha}$ lies on a unique irreducible component of $R_n(\partial N_K)$ and is a smooth point of that component (see [Sh95, §2, Theorem 6]). This shows $\widehat{\alpha}$ is a simple point of $R_n(\partial N_K)$.

The next step is to show that ξ_α is a simple point of $X_n(N_K)$. Consider the long exact sequence (2) in cohomology associated with the pair $(N_K, \partial N_K)$. Irreducibility of α implies that $H^0(N_K; sl(n,\mathbb{C})_{ad\,\alpha}) = 0$, and Lemmas 10, 11 and Poincaré duality give $H^3(N_K,\partial N_K; sl(n,\mathbb{C})_{ad\,\alpha}) = 0$. Since $H^1(N_K; sl(n,\mathbb{C})_{ad\,\alpha}) = \mathbb{C}^{n-1}$ by hypothesis, we see $H^2(N_K,\partial N_K; sl(n,\mathbb{C})_{ad\,\alpha}) = \mathbb{C}^{n-1}$ by Poincaré duality.

Since $H^1(\partial N_K; sl(n,\mathbb{C})_{ad\,\widehat{\alpha}}) = \mathbb{C}^{2(n-1)}$, it follows that the middle row of (2)

$$0 \longrightarrow H^1(N_K) \longrightarrow H^1(\partial N_K) \longrightarrow H^2(N_K,\partial N_K) \longrightarrow 0,$$

is short exact (with coefficients in $sl(n,\mathbb{C})$ twisted by $ad\,\alpha$ or $ad\,\widehat{\alpha}$ understood). Thus $j^1=0$ and $j^2=0$, and further i^1 is injective and i^2 is an isomorphism.

We now explain the powerful technique for deforming representations. It involves the following three steps:

(i) constructing formal deformations,
(ii) proving integrability by showing an infinite sequence of obstructions vanish,
(iii) proving convergence by applying a deep result of Artin [Ar68].

A *formal deformation* of α is a homomorphism $\alpha_\infty\colon \pi \to \mathrm{SL}(n, \mathbb{C}[[t]])$ given by

$$\alpha_\infty(g) = \exp\left(\sum_{i=1}^{\infty} t^i a_i(g)\right)\alpha(g),$$

such that $p_0(\alpha_\infty) = \alpha$, where $p_0\colon \mathrm{SL}(n, \mathbb{C}[[t]]) \to \mathrm{SL}(n, \mathbb{C})$ is the homomorphism given by setting $t = 0$ and where $a_i\colon \pi \to sl(n, \mathbb{C})_{ad\,\alpha}, i = 1, \ldots,$ are 1-cochains with twisted coefficients. By [HPS01, Lemma 3.3], it follows that α_∞ is a homomorphism if and only if $a_1 \in Z^1(\pi; sl(n, \mathbb{C})_{ad\,\alpha})$ is a cocycle, and we call an element $a \in Z^1(\pi; sl(n, \mathbb{C})_{ad\,\alpha})$ *formally integrable* if there is a formal deformation with leading term $a_1 = a$.

Let $a_1, \ldots, a_k \in C^1(\pi; sl(n, \mathbb{C})_{ad\,\alpha})$ be cochains such that

$$\alpha_k(g) = \exp\left(\sum_{i=1}^{k} t^i a_i(g)\right)\alpha(g)$$

is a homomorphism into $\mathrm{SL}(n, \mathbb{C}[[t]])$ modulo t^{k+1}. Here, α_k is called a formal deformation of order k, and in this case by [HPS01, Proposition 3.1] there exists an obstruction class $\omega_{k+1} := \omega_{k+1}^{(a_1,\ldots,a_k)} \in H^2(\pi; sl(n, \mathbb{C})_{ad\,\alpha})$ with the following properties:

(1) There is a cochain $a_{k+1}\colon \pi \to sl(n, \mathbb{C})$ such that:

$$\alpha_{k+1}(g) = \exp\left(\sum_{i=1}^{k+1} t^i a_i(g)\right)\alpha(g)$$

is a homomorphism modulo t^{k+2} if and only if $\omega_{k+1} = 0$.

(2) The obstruction ω_{k+1} is natural, i.e. if $\varphi\colon \pi' \to \pi$ is a homomorphism then $\varphi^*\omega_k := \alpha_k \circ \varphi$ is also a homomorphism modulo t^{k+1} and $\varphi^*(\omega_{k+1}^{(a_1,\ldots,a_k)}) = \omega_{k+1}^{(\varphi^* a_1,\ldots,\varphi^* a_k)}$.

Lemma 24. *Let $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(n, \mathbb{C})$ be an irreducible representation such that $\dim H^1(N_K; sl(n, \mathbb{C})_{ad\,\alpha}) = n - 1$. If the image of the restriction $\widehat{\alpha}\colon \pi_1(\partial N_K) \to \mathrm{SL}(n, \mathbb{C})$ contains an element with n distinct eigenvalues, then every cocycle $a \in Z^1(\pi_1(N_K); sl(n, \mathbb{C})_{ad\,\alpha})$ is integrable.*

Proof. Consider first the commutative diagram:

$$\begin{array}{ccc}
H^2(\pi_1(N_K); sl(n, \mathbb{C})_{ad\,\alpha}) & \xrightarrow{i^*} & H^2(\pi_1(\partial N_K); sl(n, \mathbb{C})_{ad\,\widehat{\alpha}}) \\
\downarrow & & \downarrow \cong \\
H^2(N_K; sl(n, \mathbb{C})_{ad\,\alpha}) & \xrightarrow{\cong} & H^2(\partial N_K; sl(n, \mathbb{C})_{ad\,\widehat{\alpha}}).
\end{array}$$

Here, the horizontal isomorphism on the bottom follows by consideration of the long exact sequence (2), and the vertical isomorphism on the right follows since ∂N_K is a $K(\mathbb{Z} \oplus \mathbb{Z}, 1)$. Further, by [HP05, Lemma 3.3], we know the vertical map on the left is an injection, and this shows i^* is an injection.

We now explain how to prove that every element $a \in Z^1(\pi_1(N_K); sl(n,\mathbb{C})_{ad\,\alpha})$ is integrable. Suppose (by induction) that $a_1, \dots, a_k \in C^1(\pi; sl(n,\mathbb{C})_{ad\,\alpha})$ are given so that

$$\alpha_k(g) = \exp\left(\sum_{i=1}^{k} t^i a_i(g)\right)\alpha(g)$$

is a homomorphism modulo t^{k+1}. Then the restriction $\widehat{\alpha}_k\colon\ \pi_1(\partial N_K) \to \mathrm{SL}(n,\mathbb{C}[[t]])$ is also a formal deformation of order k. On the other hand, $\widehat{\alpha}_k$ is a smooth point of $R_n(\partial N_K)$, hence by [HPS01, Lemma 3.7], $\widehat{\alpha}_k$ extends to a formal deformation of order $k+1$. Therefore

$$0 = \omega_{k+1}^{(i^* a_1, \dots, i^* a_k)} = i^* \omega_{k+1}^{(a_1,\dots,a_k)}.$$

As i^* is injective, the obstruction vanishes, and this completes the proof of the lemma. □

We are now ready to conclude the proof of Proposition 17. Lemma 24 shows that all cocycles in $Z^1(\pi_1(N_K); sl(n,\mathbb{C})_{ad\,\alpha})$ are integrable. Applying Artin's theorem [Ar68], we obtain from a formal deformation of α a convergent deformation (see [HPS01, Lemma 3.6]). Thus α is a smooth point of $R_n(N_K)$ with local dimension $\dim_\alpha R_n(N_K) = n-1$. It follows that α is a simple point of $R_n(N_K)$ and this together with irreducibility of α imply that ξ_α is a simple point of $X_n(N_K)$. □

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