INCIDENCE SCHEMES AND A CONJECTURE OF KONTSEVICH

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ABSTRACT. Motivated by the relationship between Feynman amplitudes and multiple zeta values, Kontsevich introduced a polynomial P_G associated to every graph G and conjectured that the number of zeros of P_G over $\mathbf{F}_q$ is a polynomial in q. In studying this conjecture, Stanley introduced a closely related polynomial Q_G and showed that Kontsevich's conjecture is true if and only if the number of zeroes of Q_G over $\mathbf{F}_q$ is a polynomial. Motivated by Stanley's work, Stembridge proposed an *a priori* weaker conjecture: Let A be the adjacency matrix of a simple graph with n vertices. Then the number of $n \times n$ non-degenerate symmetric matrices M over $\mathbf{F}_q$ such that $A_{ij}M_{ij} = 0$ for all i, j is a polynomial in q. Using matroid theory, we prove the existence of a counterexample for all three conjectures, but verify Stembridge's conjecture in several interesting cases. In spite of the counterexample, several interesting questions are raised by the methods of this article.

1. INTRODUCTION

Let G be a graph with vertices $V = V(G)$ and edges $E = E(G)$. We assume that G has neither multiple edges (two edges between the same two vertices) nor loops (an edge connecting a vertex to itself), and we assume that V is finite. Under these conditions one can view E as a subset of the set $\mathrm{Sym}^2 V$ with $\mathrm{Sym}^2 V \subset \mathcal{P}(V)$ equal to the set of subsets of V having exactly two elements. Let $S = S(G) \subset \mathcal{P}(E)$ be the set of spanning trees of G.

For each edge $e \in E$, let x_e denote a formal variable. It is classical in circuit theory (e.g. Maxwell, Kirchoff, etc.) to consider the following

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two polynomials:

$$P_G = \sum_{T\in S} \prod_{x_e \notin T} x_e \tag{1.1}$$

$$Q_G = \sum_{T\in S} \prod_{x_e \in T} x_e. \tag{1.2}$$

Let X_G (resp. Y_G) denote the scheme (a hypersurface) defined by Q_G (resp. P_G) over the integers. That is, $X_G = V(Q_G)$ and $Y_G = V(P_G)$. Motivated by computer calculations of Feynman amplitudes, Kontsevich has speculated that the periods of Y_G are "multiple zeta values" — that is, values of certain generalized zeta functions. And it seems that, given this information on the periods, it is natural to expect that the zeta functions associated to X_G and Y_G are zeta functions of motives of mixed Tate type.

Based on this idea, Kontsevich has made a number theoretic conjecture about the points of Y_G over a finite field. To describe this conjecture, we first make a notational convention: For any scheme X which is separated and of finite type over $\mathbf{Z}$, we let $[X]$ denote the function $q \mapsto \# X(\mathbf{F}_q)$. Thus $[X]$ is a function from the set of primary numbers $\mathcal{Q}$ (i.e. positive integers with only one prime factor) to $\mathbf{Z}$. We will say that X is *polynomially countable* or simply *countable* if $[X]$ is a polynomial in q.

Conjecture 1.1 (Kontsevich)**.** For all graphs G, Y_G is countable.

Using a Maple program Stembridge [Ste98] has verified this conjecture for all graphs with fewer than 12 edges. For certain graphs — e.g. cycles — it is relatively easy to see that the conjecture holds. For example, one sees easily that, for a G a cycle of length n, Y_G is simply $\mathbf{A}^{n-1}$ and, thus, $[Y_G] = q^{n-1}$.

There is an analogous conjecture for X_G, and this has been studied by both Stanley [Sta98] and Stembridge [Ste98].

Conjecture 1.2 (Stanley)**.** For all graphs G, X_G is countable.

We will refer to Conjecture 1.1 as Konsevich's conjecture and, although Conjecture 1.2 was not explicitly stated as a conjecture by Stanley, we will refer to it as Stanley's conjecture.

Stanley has in fact shown that the two conjectures are equivalent. However, knowing Stanley's conjecture for a particular graph G is not known to imply Kontsevich's conjecture for that same G. In fact, knowing Stanley's conjecture for one G alone is not known to imply Kontsevich's conjecture for any other graph. To deduce Kontsevich's conjecture for one G one must know Stanley's conjecture for a rather large number of graph related to G (see [Ste98] Section 4).

There is another conjecture, which is in fact a special case of Stanley's, stated by Stembridge as Conjecture 7.1 of [Ste98]. We now give the conjecture in a slightly different form than in the abstract. Let $V(G) = \{v_1, \dots, v_n\}$ and let $Z_G(\mathbf{F}_q)$ be the set of $n \times n$ nondegenerage symmetric matrices M such that $M_{ij} = 0$ unless $i = j$ or there is an edge from v_i to v_j.

Conjecture 1.3 (Stembridge). $\#Z_G(\mathbf{F}_q)$ is a polynomial in q.

To explain how Stembridge's conjecture is a special case of Stanley's we need to introduce some notation. A vertex v is said to be an *apex* of G if it is connected by an edge to every other vertex in G. If we take a graph G not necessarily having an apex, there is a canonical way of producing a graph G^* with an apex. Namely, one adds a vertex v_0 and connects it to every other vertex v_i by an edge. Conversely, if G is a graph with vertices $\{v_0, \dots, v_n\}$ such that v_0 is an apex, then $G = \Gamma^*$ where Γ is the graph obtained from G by removing v_0 and all edges connected to it.

Using the Matrix-Tree theorem, which we will review in Section 4, Stanley showed that his conjecture for G is equivalent to Stembridge's conjecture for G^*. This was in fact Stembridge's motivation for stating his conjecture.

Given a subgraph F of G let the *complement* $G - F$ of F in G be the graph with the same vertices as G but with all the edges in F removed. Let K_n be the complete graph with n vertices. (I.e, K_n has an edge between any pair of vertices). Let $K_{1,s}$ be the *star* with one vertex connected to s other vertices (see [Sta98] Section 5.4). In [Sta98], Stanley has verified Stembridge's conjecture in the following cases: $G = K_n$ (Theorem 4.1), $G = K_n - K_m$ (Section 5) and $G = K_n - K_{1,s}$ (Theorem 5.4). Here, of course, one must take $m \leq n$ and $s < n$. Using the fact that $(K_n - G)^* = K_{n+1} - G$, one sees that Stanley's conjecture is true for $G = K_n$, $G = K_{n+1} - K_m$ (for $n \geq m$) and $G = K_{n+1} - K_{1,s}$ for $n > s$.

In this paper, we verify Stembridge's conjecture for $G = K_n - F$ where F is any forest (Corollary 9.3). This generalized Stanley's Theorem 5.4. We also show that the truth of Stembridge's conjecture for a rather large set of graphs related to a graph G implies Stanley's conjecture (Theorem 10.2) for G.

However, in the last section, we show that Stembridge's conjecture is false for general G. This we do by showing that, if Stembridge's conjecture were true for all graphs G, then every matroid M representable over a field of characteristic p would have to be representable over all fields.

But suppose we change the definition of what is countable. Consider the following definitions.

1. Call a scheme X (of finite type over $\mathbf{Z}$) countable in each prime if given a prime p, there is a polynomial f_p so that $\#X(\mathbf{F}_q) = f_p(q)$ for every $q = p^l$.
2. Call a scheme X (of finite type over $\mathbf{Z}$) countable for almost all primes, if there is a finite set S of prime numbers, so that $\#X(\mathbf{F}_q)$ is a polynomial in q, for all $q = p^l$ where $p \notin S$.
3. Let C be a class of integer valued functions on the primary positive integers which contains the polynomial functions, closed under multiplication and division. Call a scheme X (of finite type over $\mathbf{Z}$) C-countable if $q \mapsto \#X(\mathbf{F}_q)$ is a function in C.

The methods of this article are equally valid for any definition of countability as above. For example Stembridge's conjecture with countable replaced by countable for almost all primes implies Kontsevich's and Stanley's conjecture with countable replaced by countable for almost all primes. At this point we do not know if Stembridge's conjecture holds for almost all primes.

The paper is organized as follows: In Section 2, we discuss certain schemes which will be basic for the paper. In Section 3, we prove a basic result about polynomial countability. In Section 4, we review Stanley's use of the Matrix-Tree theorem and we reformulate Stembridge's conjecture slightly. In Section 5, we introduce the incidence schemes $A_G(s,r,k)$ for $s,r,j \in \mathbf{N}$ from the title of the paper. Using the Matrix-Tree theorem and the basic result of Section 3, we show in Theorem 5.5 that X_G is countable if and only if $A_G(m,m,m)$ is countable for $m = \#V(G) - 1$. In Section 6, we establish certain relations among the functions $[A_G(s,r,k)]$ for varying s, r, and k. These relations enable us to show in Section 7 that all of the $[A_G(s,r,k)]$ are determined by certain smaller subsets (for example, the $[A_G(s,s,s)]$). Using these results, we show in Section 8 that Stembridge's conjecture is true if and only if the varieties $A_{G^*}(s,r,k)$ are countable for every graph G.

In Section 5, we introduce a scheme $J_G(s)$ which is a union of certain $A_G(s,r,k)$. And in Section 7 it will be seen that the countability of the $J_G(s)$ is equivalent to the countability of all the $A_G(s,r,k)$. It turns out that in the case that G is a forest complement (i.e. $G = K_n - F$ with F a forest) the countability of $J_{G^*}(s)$ is easy to verify. We do this in Section 9 thereby proving in Corollary 9.3 that Z_G is countable for G a forest complement. In Section 10, we compare incidence schemes for G with incidence schemes corresponding to a various schemes associated to G. By doing this, we show that, if Stembridge's conjecture is true for a certain set of graphs related to G, then Stanley's conjecture is true for G.

Finally, in Section 11, we combine the work of all the previous sections to prove the existence of counterexamples to all three conjectures.

1.1. **Notational.** If X is a finite set, we write $\#X$ or $|X|$ interchangeably for the number of points in X. When counting points of a scheme X over a field $\mathbf{F}_q$ we often view X simply as a set without reflecting this in the notation. It should be clear from the context when we mean X and when we mean $X(\mathbf{F}_q)$.

We are constantly working with graphs G with vertex set $V = \{v_1, \dots, v_{n_G}\}$ and we constantly need to refer to the number $n_G - 1$. To compactify the notation we therefore set $m_G = n_G - 1$. When there is only one graph G under discussion we drop the G in the subscript and write simple n for $|V(G)|$ and m for $n - 1$.

In this draft, we take a somewhat cavalier attitude to what constitutes a scheme over $\mathbf{Z}$. We find that is is clearer to define a scheme X by simply giving its K points for K a field. Of course, this is not enough (or rather heretical to algebraic geometers). We promise to rectify this in the final version. We intend that this be accessible to combinatorialists, working on the problem. Aside from the use of Weil conjectures in section 3, the fact that Grassmannians are countable, the methods and the results of this article should be accessible to a wide audience.

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2. Basic Schemes

Let R be a ring and E, F free R-module of ranks e and f respectively. $\mathrm{Hom}(E,F)_R$ (or simply $\mathrm{Hom}(E,F)$ when R is obvious — and for us it will be almost always be $\mathbf{Z}$) is then a scheme which is isomorphic to $\mathbf{A}^{ef}$. We write $\mathrm{Hom}_r(E,F)$ for the locally closed subscheme of $\mathrm{Hom}(E,F)$ whose K points — for K a field over R — consist of morphisms $\psi : E\otimes K \to F\otimes K$ of rank r. We mention that, when L is an arbitrary ring over R, the L points of $\mathrm{Hom}_r(E,F)$ are the maps $\psi : E \otimes L \to F \otimes L$ with locally free rank $f - r$ cokernel. Of course, $\mathrm{Hom}_e(E,E) = \mathrm{GL}(E)$.

We define $\mathrm{Bil}(E)$ to be the scheme over R of symmetric bilinear forms on E. This is isomorphic to $\mathbf{A}^{\binom{e+1}{2}}$ and it is a closed subscheme of $\mathrm{Hom}(E, E^\vee)$. We define $\mathrm{Bil}_r(E)$ to be the intersection of $\mathrm{Bil}(E)$ with $\mathrm{Hom}_r(E, E^\vee)$. For K a field, $\mathrm{Bil}_r(E)(K)$ is the set of rank r bilinear forms on $E \otimes K$. We write Bil_r^n for $\mathrm{Bil}_r(\mathbf{Z}^n)$. For a set S, we write E^S or $\mathrm{Fun}(S, E)$ for the scheme $\mathrm{Hom}(R^S, E)$. The L points of this scheme for any ring L over R is the set of maps f from S to $E \otimes L$. We write $(E^S)_r$ or $\mathrm{Fun}_r(S, E)$ for the scheme $\mathrm{Hom}_r(R^S, E)$.

Write $\mathrm{Gr}(r, F)$ for the Grassmanian of r-planes in F. This is defined in EGA [GD67]. We remark that there is a natural map $\mathrm{Hom}_r(E, F) \to \mathrm{Gr}(r, F)$ which for K valued points sends a map to its image. For $\mathrm{Fun}_r(S, E)$ this map is written as $f \mapsto \langle f \rangle$.

As mentioned above, we always take $R = \mathbf{Z}$ unless otherwise specified. For future reference, we remark that all of the above schemes are countable and we give their associated polynomials.

$$[\mathrm{GL}_n] = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1}) \tag{2.1}$$

$$[\mathrm{Gr}(a, b)] = \frac{[\mathrm{GL}_b]}{[\mathrm{GL}_a][\mathrm{GL}_{b-a}]q^{a(b-a)}} \tag{2.2}$$

$$[\mathrm{Hom}_r(E, F)] = [\mathrm{Gr}(r, e)][\mathrm{Gr}(r, f)][\mathrm{GL}_r] \tag{2.3}$$

The first two of the above are well known and the last is easy. That the rational functions give are actually polynomials can either be checked by hand or one can apply Theorem 3.1 of the next section.

More difficult is the following formula which is taken from [Sta98] who attributes it to [Mac69].

$$[\mathrm{Bil}_r^n] = \begin{cases} \prod_{i=1}^{s} \frac{q^{2i}}{q^{2i}-1} \cdot \prod_{i=0}^{2s-1}(q^{n-i} - 1), & 0 \le r = 2s \le n, \\ \prod_{i=1}^{s} \frac{q^{2i}}{q^{2i}-1} \cdot \prod_{i=0}^{2s}(q^{n-i} - 1), & 0 \le r = 2s + 1 \le n \end{cases} \tag{2.4}$$

In sections 10 and 11, we will need to countability of one other type of scheme. But it is more easily expressed as the dimension of a vector space.

Let

$$A = F^0 A \supset F^1 A \supset \cdots \supset F^{r-1} A \supset F^r A = 0 \tag{2.5}$$

be a filtered vector space over a field $\mathbf{F}$. We write $F^\bullet A$ for the filtered vector space. And we say that r is the *length* of the filtration. Let

$$\begin{aligned} a_i &= \dim(F^i A - F^{i+1} A), \\ a^i &= \dim F^i A. \end{aligned}$$

If $F^\bullet B$ is a another filtered vector space, we write $\mathrm{Hom}(F^\bullet A, F^\bullet B)$ for the $\mathbf{F}$-vector space of maps $f \in \mathrm{Hom}(A, B)$ such that $f(F^i A) \subset F^i B$.

Proposition 2.1. $\dim \operatorname{Hom}(F^\bullet A, F^\bullet B) = \sum a_i b^i$.

Proof. The proof is by induction on the length r of $F^\bullet A$. The result is obvious when the length is 0 or 1. Now let $W^\bullet A$ be the filtered space with $W^i A = F^{i+1} A$, and $W^\bullet B$ be the filtered space with $W^i B = F^{i+1} B$. $W^\bullet A$ has length $r-1$, and we have an exact sequence

$$\operatorname{Hom}(A/F^1 A, B) \rightarrowtail \operatorname{Hom}(F^\bullet A, F^\bullet B) \twoheadrightarrow \operatorname{Hom}(W^\bullet A, W^\bullet B),$$

where the first Hom is of unfiltered vector spaces. The result then follows by induction. ☐

3. Polynomial Countability

Let S be an infinite set and let R be a ring. We define $\operatorname{Ae}(S, R)$ to be the quotient of the ring of functions $\operatorname{Fun}(S, R)$ obtained by identifying any two functions f and g which agree outside of a finite set. We let $\psi : \operatorname{Fun}(S, R) \to \operatorname{Ae}(S, R)$ be the quotient map. We write $f = g$ a.e. if $\psi(f) = \psi(g)$ and note that $\psi(f)$ is well-defined for any f defined outside of a finite set. If $S \subset \mathbf{Q}$ we note that there is an injection $\mathbf{Q}[x] \to \operatorname{Ae}(S, \mathbf{Q})$.

If f is a function on S and R is a subset of $\mathbf{Q}(q)$, we say that $f \in R$ a.e. if there is a (necessarily unique) rational function $g \in R$ such that $f = g$ a.e. When X is a scheme of finite type over $\mathbf{Z}$, we consider the function $[X]$ as defined on the set $S = \mathcal{Q}$. The goal of this section is to prove the following:

Theorem 3.1. *If X is a scheme of finite type over $\mathbf{Z}$ and $[X] \in \mathbf{Q}(q)$ a.e. then X is countable.*

The theorem is a slight generalization of Theorem 2.2 of [Sta98]. Its proof, which will occupy the rest of this section, is also a generalization of Stanley's proof based on Dwork's theorem [Dwo60].

Suppose now that X is a scheme of finite type over $\mathbf{F}_p$. We write $p^{\mathbf{Z}^+} = \{p^n \mid n \in \mathbf{Z}^+\}$ and we write $[X]_p$ for the function $p^n \mapsto \#X(\mathbf{F}_{p^n})$. We say that X is countable on $p^{\mathbf{Z}^+}$ if $[X]_p \in \mathbf{Z}[q]$. It is easy to see that Theorem 3.1 follows directly from the following:

Theorem 3.2. *If X is a scheme of finite type over $\mathbf{F}_p$ and $[X]_p = f(q)$ a.e. on $p^{\mathbf{Z}^+}$ for some function $f \in \mathbf{Q}(q)$ then X is countable on $p^{\mathbf{Z}^+}$.*

We begin the proof of Theorem 3.2 with an easy lemma.

Lemma 3.3. *If $[X]_p \in \mathbf{Q}(q)$ a.e. then $[X]_p \in \mathbf{Q}[q]$ a.e.*

Proof. Actually, since $[X]_p(q) \in \mathbf{Z}$ for all q, this follows from a more general statement: If S is an infinite subset of $\mathbf{Z}$ and $f : S \to \mathbf{Z}$ is any function such that $f \in \mathbf{Q}(q)$ a.e. then $f \in \mathbf{Q}[q]$ a.e.

To see this, set $f = a/b$ with $a, b \in \mathbf{Z}[q]$. Then $a = db + r$ with $\deg r < \deg b$ and with $d, r \in \mathbf{Q}[q]$. Pick $N \in \mathbf{Z}^+$ such that $Nd, Nr \in \mathbf{Z}[q]$. Then $Na/b(q) = Nd(q) + Nr(q)/b(q) \in \mathbf{Z}$ for all q. Thus $Nr(q)/b(q) \in \mathbf{Z}$ for all q. This, however, that $r = 0$. For otherwise, since $\deg r < \deg b$, $\lim_{q\to\infty} Nr(q)/b(q) = 0$ and thus $Nr(q)/b(q) = 0$ for all q sufficiently large. And this itself implies that $r = 0$. $\square$

Thus in proving Theorem 3.2 we may assume that $f(q) \in \mathbf{Q}[q]$. We will prove that $f(q) \in \mathbf{Z}[q]$ and that $[X]_p(q) = f(q)$ for all $q \in p^{\mathbf{Z}^+}$ simultaneously using the rationality of the zeta function proved by Dwork [Dwo60].

We first recall that, for X a scheme of finite type over $\mathbf{F}_p$, the zeta function of X is defined as

$$Z(X,t) = \exp(\sum_{k\geq 1} \#X(\mathbf{F}_p^n)t^n/n). \tag{3.1}$$

Dwork showed that $Z(X,t)$ is a rational function. It follows, since $Z(X,t)$ is also an exponential, that there are nonzero complex numbers $\alpha_1, \ldots, \alpha_r$ and integers $a_1 \ldots, a_r$ such that

$$Z(X,t) = \prod_{i=1}^{r} (1-\alpha_i t)^{-a_i}. \tag{3.2}$$

Now, let $\operatorname{Div} \mathbf{G}_m$ be the set of all divisors on $\mathbf{G}_m$, that is, the set of all finite formal sums $\sum a_i[\alpha_i]$ with $a_i \in \mathbf{Z}$ and $\alpha_i \in \mathbf{C} - \{0\}$. $(\operatorname{Div} \mathbf{G}_m) \otimes \mathbf{Q}$ is then the set of all such formal sums with $a_i \in \mathbf{Q}$. For $D \in (\operatorname{Div} \mathbf{G}_m) \otimes \mathbf{Q}$ we let $\deg D = \sum a_i$. For $f = a\prod(x-\beta_i)^{b_i} \in \mathbf{C}(x)^\times$ we write $\operatorname{div} f$ for the divisor $\sum_{\beta_i\neq 0} b_i[\beta_i]$.

For any $D \in (\operatorname{Div} \mathbf{G}_m) \otimes \mathbf{Q}$, we write $N_D(k)$ for the (finite) sum $\sum a_i \alpha_i^k$. We write

$$f_D(t) = \sum_{k\geq 1} N_D(k)t^{k-1}, \tag{3.3}$$

$$g_D(t) = \sum_{k\geq 1} N_D(k)t^k/k, \tag{3.4}$$

$$Z_D(t) = \exp(\sum_{k\geq 1} N_D(k)t^k/k). \tag{3.5}$$

We remark that if Δ_D is the maximum of the absolute values of the α_i then all of the above expressions converge for $|t| < 1/\Delta_D$. If we pick $0 < R_D < 1/\Delta_D$ then all of the above expressions converge absolutely for $|t| < R_D$. In the region $|t| < R$, it is then easy to see that

$$f_D(t) = \sum \frac{a_i \alpha_i}{1 - \alpha_i t} \tag{3.6}$$

$$g_D(t) = -\sum a_i \log(1 - \alpha_i t). \tag{3.7}$$

The first equation follows by summing the geometric series $f_D(t)$ and the second follows by integrating this series termwise. Here log is a single valued function for $|t| < R$ with $\log(1) = 0$. It follows that, for $D \in \operatorname{Div} \mathbf{G}_m$,

$$Z_D(t) = \prod (1 - \alpha_i t)^{-a_i} \tag{3.8}$$

Thus Dwork's theorem can be phrased as saying that for any scheme of finite type over $\mathbf{F}_p$ there exists a $D \in \operatorname{Div} \mathbf{G}_m$ such that

$$Z(X, t) = Z_D(t). \tag{3.9}$$

Moreover, taking the logarithm of both sides of Equation 3.9, one sees that

$$[X]_p(p^k) = N_D(k) \tag{3.10}$$

for all k.

For $D \in \operatorname{Div} \mathbf{G}_m$ we extend the definition of f_D and Z_D to rational functions on the entire complex plane using Equations 3.6 and 3.8.

Note that D can be obtained from the rational function Z_D by taking divisors. In fact,

$$D = -\operatorname{div} Z(D, 1/t). \tag{3.11}$$

Now for any $D_1, D_2 \in (\operatorname{Div} \mathbf{G}_m) \otimes \mathbf{Q}$ and integers m_1, m_2 it easy to see that

$$N_{m_1 D_1 + m_2 D_2}(k) = m_1 N_{D_1}(k) + m_2 N_{D_2}(k) \tag{3.12}$$

Thus

$$f_{m_1 D_1 + m_2 D_2} = m_1 f_{D_1} + m_2 f_{D_2}, \tag{3.13}$$

$$g_{m_1 D_1 + m_2 D_2} = m_1 g_{D_1} + m_2 g_{D_2}, \tag{3.14}$$

$$Z_{m_1 D_1 + m_2 D_2} = Z_{D_1}^{m_1} Z_{D_2}^{m_2}. \tag{3.15}$$

Proposition 3.4. *Let $D_1, D_2 \in (\operatorname{Div} \mathbf{G}_m) \otimes \mathbf{Q}$. If $N_{D_1}(k) = N_{D_2}(k)$ for almost all k, then $D_1 = D_2$.*

Proof. Multiplying D_1 and D_2 by a suitable integer and using Equation 3.12, it is easy to see that it suffices to consider $D_1, D_2 \in \operatorname{Div} \mathbf{G}_m$. Suppose then that $N_{D_1} = N_{D_2}$ a.e. with $D_1, D_D \in \operatorname{Div} \mathbf{G}_m$. Set $D = D_1 - D_2$. Then, by Equation 3.14, $g_D(t) = tp(t)$ for some polynomial p. Thus $Z_D(t) = \exp(tp(t))$. On the other hand, $Z_D(t) \in \mathbf{C}(t)$

by Equation 3.8. Since exponentials grow faster than polynomials, it follows that $p(t) = 0$. Thus $g_D = 0$ which implies that $Z_D = 1$, and, by Equation 3.11, this implies that $D = 0$. $\square$

Now we are ready to prove Theorem 3.2. Let $Z(X,t) = Z_D(t)$ with $D \in \operatorname{Div} \mathbf{G}_m$ as provided by Dwork's theorem. As remarked earlier, by Lemma 3.3 we can assume that $f \in \mathbf{Q}[q]$. Thus we write $f(q) = \sum b_i q^i$ with $b_i \in \mathbf{Q}$. Set $E = \sum b_i [p^i] \in (\operatorname{Div} \mathbf{G}_m) \otimes \mathbf{Q}$. We then have

$$f(p^k) = N_E(k) \tag{3.16}$$

for almost all k. On the other hand, from Equation 3.10, we have that

$$f(p^k) = N_D(k) \tag{3.17}$$

for almost all k. By Proposition 3.4, we then have that $E = D$ and thus that $f \in \mathbf{Z}[q]$. Then, by equation 3.10, we find that, for all k, $\#X(\mathbf{F}_{p^k}) = f(p^k)$. Thus $[X]_p$ is a polynomial.

We finish the section with one Corollary of Proposition 3.4 which will be conceptually useful in section 7.

Corollary 3.5. *If X and Y are two schemes of finite type over $\mathbf{F}_p$ with $[X]_p = [Y]_p$ a.e. then $[X]_p = [Y]_p$.*

Proof. By Dwork's theorem, we have divisors $D_1, D_2 \in \operatorname{Div} \mathbf{G}_m$ such that $[X]_p(p^k) = N_{D_1}(k)$ and $[Y]_p(p^k) = N_{D_2}(k)$. Thus, by Proposition 3.4, $D_1 = D_2$ and, hence, $[X]_p = [Y]_p$. $\square$

A direct consequence of Corollary 3.5 is the following

Corollary 3.6. *If X and Y are two schemes of finite type over $\mathbf{Z}$ and $[X] = [Y]$ a.e. then $[X] = [Y]$.*

4. The Matrix Tree Theorem

Stanley's positive results mentioned in the introduction were mainly consequences of the *Matrix-Tree Theorem* of Kirchoff, Borchardt and Sylvester which gives an expression of the polynomial P_G as the determinant of a symmetric matrix. As this theorem is also basic to our results we briefly describe it.

4.1. For an edge e and a vertex v, we write $v \in e$ to mean that the edge e meets the vertex v. Pick an ordering of the vertices of G so that $V(G) = \{v_1, \ldots, v_n\}$. For $v_i, v_j \in V(G)$, we write e_{ij} for the set $\{v_i, v_j\}$. We then write $e_{ij} \in E$ to mean that there is an edge in G from v_i to v_j. We write x_{ij} for the variable $x_{e_{ij}}$ when $e_{ij} \in E$. To simplify the notation we define $x_{ij} = 0$ for $e_{ij} \notin E$.

Let $L = L_{ij}$ be the $n \times n$ matrix defined by

$$L_{ij} = \begin{cases} \sum_{k=1}^{n} x_{ik} & \text{if } i = j \\ -x_{ij} & \text{if } i \neq j \end{cases}$$

Let L_0 be L with the last row and the last column removed. Stanley calls L the *generic Laplacian matrix of* G and L_0 the *reduced generic Laplacian.*

Theorem 4.2 (The Matrix-Tree Theorem). $P_G = \det L_0$.

Let X'_G denote the complement of X_G in $\mathbf{A}^E$. Clearly X'_G is countable if and only if X_G is. Stanley's use of the Matrix-Tree Theorem is based on the following important consequence.

Theorem 4.3. *Assuming that v_n is an apex, $[X'_G](q)$ is equal to the number of $(n-1) \times (n-1)$ nonsingular symmetric matrices M over $\mathbf{F}_q$ such that $M_{ij} = 0$ whenever $i \neq j$ and $e_{ij} \notin E$.*

Now let G be a graph with or without an apex and with vertices $\{v_1, \ldots, v_n\}$. Let Z_G be the sub-scheme of GL_n whose K points for K a field are the non-singular symmetric matrices M_{ij} such that $M_{ij} = 0$ whenever $e_{ij} \notin E$ and $i \neq j$. Taken together Theorem 4.3 and Conjecture 1.2 for G^* imply that Z_G is countable as, in fact, $[Z_G] = [X'_{G^*}]$. This leads to the following Conjecture 7.1 of [Ste98] which we present here as a slight reformulation (in terms of the scheme Z_G) of Conjecture 1.3.

Conjecture 4.4. For every graph G, the scheme of Z_G is countable.

Thus clearly Conjecture 4.4 (and thus Conjecture 1.3) is a special case of 1.2. As advertised in the introduction, we will show in Section 10 that they are actually equivalent.

5. Incidence Schemes

We now introduce the incidence schemes from the title of this paper. Here the remarks on the nature of this preliminary version given in the introduction are in full force. We, therefore, follow a slightly dubious method of defining schemes and maps between them.

Definition 5.1. Let $r, k \in \mathbf{N}$, let W be a lattice and let G be a graph as in paragraph 4.1. We write $A_G(W)$ (resp. $A^*_G(W)$) for the closed subscheme of $\mathrm{Bil}(W) \times \mathrm{Fun}(V, W)$ consisting of pairs (Q, f) satisfying the following conditions (resp. satisfying (b)):

(a) $\sum_{i=1}^{n} f(v_i) = 0$,
(b) $Q(f(v_i), f(v_j)) = 0,\ e_{ij} \notin E$.

We then write $A_G(W,r,k)$ (resp. $A_G^*(W,r,k)$) for the locally closed subscheme of $A_G(W)$ (resp. $A_G^*(W)$) consisting of pairs $(Q,f) \in Bil_r(W) \times \mathrm{Fun}_k(V,W)$. We write $A_G(s,r,k)$ (resp. $A_G^*(s,r,k)$) for $A_G(\mathbf{Z}^s,r,k)$ (resp. $A_G^*(\mathbf{Z}^s,r,k)$). We drop the G in the subscript when no confusion can result.

The $A_G(W,r,k)$ (resp. $A_G^*(W,r,k)$) form a stratification of $A_G(W)$ (resp. $A_G^*(W)$) by locally closed subschemes. We write $A_G(W,r)$ (resp. $A_G^*(W,r)$) for $\bigcup_k A_G(W,r,k)$ (resp. $\bigcup_k A_G^*(W,r,k)$).

The action of $\mathrm{GL}(W)$ on W gives rise an action on $A_G(W,r,k)$ (resp. $A_G^*(W,r,k)$. Let T_G be the torus $\mathrm{Fun}(V,\mathbf{G}_m)$. Then T_G acts on $A^*(W,r,k)$ by $\lambda(Q,f) \mapsto (Q,\lambda f)$. We leave the verification of the following easy but important statement to the reader.

Claim 5.2. The map $A_G^*(s,r,k) \to A_{G^*}(s,r,k)$ sending (Q,f) to (Q,g) with $g(v_i) = f(v_i)$ for $i \in [1,n]$ and $g(v_0) = -\sum_{i=1}^n f(v_i)$ is an isomorphism.

5.3. Note that $A_G(s,r,k) = \emptyset$ unless $0 \le k < n$ and $0 \le r,k \le s$. Also note that $A_G(s,r,0) = \mathrm{Bil}_r^s$. There are several other important choices of s,r and k which are important enough to single out with special notation. To compactify the notation let $m = n-1$ — a convention that we will also use later in the paper. We write

$$\begin{array}{llrllr} K_G(s) & \text{for} & A_G(s,s,s), & K_G^*(s) & \text{for} & A_G^*(s,s,s), \\ J_G(s) & \text{for} & A_G(s,s), & J_G^*(s) & \text{for} & A_G^*(s,s), \\ H_G(s) & \text{for} & A_G(m,s,m), & H_G^*(s) & \text{for} & A_G^*(n,s,n). \end{array}$$

Remark 5.4. The idea of the notation is as follows: The $K_G(s)$ (resp. the $J_G(s)$ or the $H_G(s)$) will be seen in Section 7 to determine all other $A_G(t,r,k)$. We believe that the $J_G(s)$ are, however, the easiest to compute. And we do compute them in Section 9 for certain G. On the other hand, the $H_G(s)$ enable induction from one graph to another. We single out the $K_G(s)$ because they naturally come up in proofs as "building blocks" for all other $A_G(s,r,k)$.

Note that the notation has been chosen so that, for example, $H_{G^*}(s) = H_G^*(s)$. We keep this in all of our notation so that we can always move a $*$ superscript down by adding an apex to the graph in question.

Notice also that we have picked $H_G^*(s)$ to conform to the notation $h(G,s)$ that Stanley gives in Section 5 of [Sta98]. In particular, for G a graph with an apex, $[H_G^*(s)] = [\mathrm{GL}_m]h(G,s)$. We do not make use of this fact and leave its verification to the reader. Its proof is very similar to the proof of the next Theorem.

Theorem 5.5. (a) $K_G(m) \cong X_G' \times \mathrm{GL}_m$.

(b) *$K_G(m)$ is countable if and only if X'_G is.*

Proof. We first remark that (b) follows directly from (a), Theorem 3.1, and the countability of GL_m.

To prove (a) we let $W = \mathbf{Z}^m$ and view GL_m as the scheme of frames F on W. We have a map $\psi : K(m) \to X'_G \times \mathrm{GL}_m$ given by setting $x_{ij} = -Q(f(v_i), f(v_j))$ for $e_{ij} \in E$ and letting the frame $F = (f(v_1), \ldots, f(v_m))$.

Let $(x, F) \in X_G \times \mathrm{GL}_m$ with $F = (w_1, \ldots, w_m)$. The inverse of ψ is the map ϕ which takes a pair (x, F) to the unique bilinear from determined by setting

$$Q(w_i, w_j) = \begin{cases} \sum_{k=1}^n x_{ik} & i = j, \\ -x_{ij} & e_{ij} \in E \\ 0 & \text{else,} \end{cases} \qquad f(v_i) = \begin{cases} w_i & i < n \\ -\sum_{k=1}^m w_i & i = n \end{cases}$$

One must of course check that the ranges of ψ and ϕ are as advertised. This is done using The Matrix-Tree Theorem, and we leave this to the reader. It is then easy to check that ψ and ϕ are inverses. □

Corollary 5.6. $K_G^* = Z_G \times \mathrm{GL}_n$.

Proof. Since $K_G^* = K_{G^*}$ and $Z_G = X'_{G^*}$, the corollary follows from the theorem. □

6. Reduction Formulae

In this section we give various formulae which allow us to reduce questions about $A_G(s, r, k)$ (resp. $A_G^*(s, r, k)$) for given s, r or k to questions where s, r or k is smaller. This allows us to set up inductions in favorable cases. We give the proofs and even the statements only for the $A_G(s, r, k)$ leaving to the reader the trivial task of making the analogous statements for the $A_G^*(s, r, k)$ using the identity $A_G^*(s, r, k) = A_{G^*}(s, r, k)$.

Let Q be a fixed bilinear form on $\mathbf{F}_q^{d_1}$ with rank r_1. Then let $C_Q(d_2, r_2, d_1, r_1)$ be the number of ways to extend Q to a form on $\mathbf{F}_q^{d_2}$ of rank r_2. Stanley attributes the following theorem to MacWilliams [Mac69].

Theorem 6.1.

$$C_Q(d_1 + 1, r_2, d_1, r_1) = \begin{cases} q^{r_1} & r_2 = r_1 \\ q^{r_1+1} - q^{r_1} & r_2 = r_1 + 1 \\ q^{d_1+1} - q^{r_1+1} & r_2 = r_1 + 2 \\ 0 & \textit{otherwise.} \end{cases}$$

We use the Theorem inductively to determine the $C_Q(d_2, r_2, d_1, r_1)$ inductively using the fact that (for $d_2 > d_1$)

$$C_Q(d_2,r_2,d_1,r_1) = \sum_{j=0}^{2} C_Q(d_2,r_2,d_1+1,r_1+j)C_Q(d_1+1,r_1+j,d_1,r_1). \tag{6.1}$$

Corollary 6.2.

(a) *$C_Q(d_2, r_2, d_1, r_1)$ is a polynomial in q which is independent of Q. (We may therefore write $C(d_2, r_2, d_1, r_1)$ leaving the Q out of our notation).*

(b) *$C_Q(d_2, r_2, d_1, r_1) \neq 0$ iff $d_2 \geq r_2$, $d_1 \geq r_1$, and $0 \leq r_1 \leq r_2 \leq r_1 + 2(d_2 - d_1)$.*

Proof. (a) follows directly from the recursion formula 6.1.

The necessity of the first two inequalities of (b) are obvious for dimension reasons (the rank of a bilinear form can not be greater than the dimension of the ambient space). The third inequality follows from formula 6.1 by induction.

We prove the sufficiency of the the inequalities in (b) by induction on $i = d_2 - d_1$ using formula 6.1. We do not actually need this for the rest of the paper so the reader may safely skip the proof.

For $i = 0$ sufficiency is obvious. For $i = 1$ the sufficiency results from the fact that $C(d_1 + 1, r_2, d_1, r_1) = 0$ iff $r_2 \in [r_1, r_1 + 2]$ when $d_1 \neq r_1$ and iff $r_2 \in [r_1, r_1 + 1]$ when $d_1 = r_1$.

Now suppose sufficiency is known for $d_2 - d_1 < i$ and assume that (d_2, r_2, d_1, r_1) satisfies the conditions in (b) with $d_2 = d_1 + i$ and $r_2 = r_1 + k$. By formula 6.1, $C(d_2, r_2, d_1, r_1) \neq 0$ if there is a j such that both

1. $C(d_1 + i, r_1 + k, d_1 + 1, r_1 + j) \neq 0$ and
2. $C(d_1 + 1, r_1 + j, d_1, r_1) \neq 0$.

One computes that (2) is satisfied whenever $j \leq d_1 - r_1 + 1$. Using the the induction hypothesis, we see that (1) is satisfied for

$$k - 2i + 2 \leq j \leq \min(k, d_1 - r_1 + 1).$$

So we need only show that $k - 2i + 2 \leq \min(k, d_1 - r_1 + 1)$. That $k - 2i + 2 \leq k$ only says that $i \geq 1$ which we are of course assuming. And $k - 2i + 2 \leq d_1 - r_1 + 1$ iff $(d_2 - d_1) + (d_2 - r_2) \geq 1$ which then follows from the fact that $d_2 \geq r_2$. $\square$

We are now ready to give our reduction theorems. We keep the notation of paragraph 4.1 and the convention that $m = n - 1$ and write $A(s, r, k)$ for $A_G(s, r, k)$.

Theorem 6.3.
$$[A(s,r,k)] = [\mathrm{Gr}(k,s)] \sum_j C(s,r,k,j)[A(k,j,k)].$$

Proof. In this proof and the next one we will pick a base field $\mathbf{F}_q$ at the beginning and then, for any scheme X we encounter, write X instead of $X(\mathbf{F}_q)$.

Write $W = \mathbf{F}_q^s$. For every map $f \in W^V$ let $\langle f \rangle$ denote the span of the $f(v_i)$. The map $(Q,f) \mapsto \langle f \rangle$ fibers the set $A(s,r,k)$ over the $\mathrm{Gr}(k,s)$. The fiber over a subspace $U \subset W$ is then the set $A(s,r,U)$ of $(Q,f) \in A(s,r,k)$ such that $\langle f \rangle = U$. The transitivity of the GL_s action on $\mathrm{Gr}(k,s)$ shows that the fibers all of the same number of points. Thus for any U
$$\#A(s,r,k) = \#\,\mathrm{Gr}(k,s) \cdot \#A(s,r,U). \tag{6.2}$$

Now let $A(s,r,U,j)$ be the set of $(Q,f) \in A(s,r,U)$ such that $Q|_U$ has rank j. This decomposes $A(s,r,U)$ into disjoint subsets. Consider the map
$$\begin{array}{rcl} p_U : A(s,r,U,j) & \to & A(U,j,k) \text{ given by} \\ (Q,f) & \mapsto & (Q|_U, f). \end{array}$$
The fiber of p_U above a given $(\overline{Q},f) \in A(U,j,k)$ is $C_{\overline{Q}}(s,r,k,j)$. Thus
$$\#A(s,r,U,j) = C(s,r,k,j) \cdot \#A(k,j,k) \tag{6.3}$$
Summing over all the j in Equation 6.3 and substituting the result into Equation 6.2 we obtain the desired result. □

Theorem 6.4. $[A(s,r)] = q^{(s-r)m}[\mathrm{Gr}(r,s)][J(r)]$.

Proof. Write $W = \mathbf{F}_p^s$ and let $\Psi : A(W,r) \to \mathrm{Gr}(s-r,W)$ be the map associating to every (Q,f) the kernel $\ker Q$ of Q. The fiber of Ψ over a subspace $U \subset W$ is the set $A(W,r)_U$ consisting of all $(Q,f) \in A(W,r)$ with $Q|_U = 0$. The transitivity of the action of $\mathrm{GL}(W)$ on $\mathrm{Gr}(s-r,W)$ show then that
$$\#A(W,r) = \#Gr(s-r,W) \cdot \#A(W,r)_U \tag{6.4}$$

Let $T = W/U$ and let $\pi : W \to T$ be the quotient map. Q reduces in an obvious way to a form $\overline{Q}$ on T. In fact $Q \mapsto \overline{Q}$ is a one-one correspondence between bilinear forms on W with kernel U and non-degenerate bilinear forms on T. We map $p : A(W,r)_U \to A(T,r)$ by sending (Q,f) to $(\overline{Q}, \pi \circ f)$. The fiber of p above a pair $(\overline{Q},g) \in A(T,r)$ is then identified with the set of $f : V \to W$ such that $\pi \circ f = g$ and $\sum_{i=1}^n f(v_i) = 0$. This is simply the affine translate by (any representative of) g of the vector space S of maps $h : V \to U$ satisfying

$\sum_{i=1}^{n} f(v_i) = 0$. And S is isomorphic to $U^{n-1} \cong \mathbf{A}^{(s-r)(n-1)}$. Thus we find that

$$\#A(W,r)_U = q^{(s-r)m} \cdot \#A(T,r) \tag{6.5}$$

As $A(T,r) = A(r,r) = J(r)$, putting together the two fibrations gives the desired result. □

7. Generators for the $[A(s,r,k)]$

Let $\mathrm{Fun}(\mathcal{Q},\mathbf{Z})$ denote the space of functions from the set of primary positive integers to $\mathbf{Z}$. Let $\mathcal{M}$ be the subgroup of $\mathrm{Fun}(\mathcal{Q},\mathbf{Z})$ generated by the functions $[X]$ for X a separated scheme of finite type over $\mathbf{Z}$. Since $[X \times Y] = [X] \times [Y]$, $\mathcal{M}$ is a subring of $\mathrm{Fun}(\mathcal{Q},\mathbf{Z})$. Note that $\mathbf{Z}[q]$ is a subring of $\mathcal{M}$.

There is a natural map

$$\mathfrak{R} : \mathbf{Q}(q) \otimes_{\mathbf{Z}[q]} \mathrm{Fun}(\mathcal{Q},\mathbf{Z}) \to \mathrm{Ae}(\mathcal{Q},\mathbf{Q})$$

defined by sending $r \otimes f$ to the almost everywhere defined function $q \mapsto r(q)f(q)$.

Proposition 7.1. *$\mathfrak{R}$ restricted to $\mathbf{Q}(q) \otimes_{\mathbf{Z}[q]} \mathcal{M}$ is an inclusion. Thus $\mathbf{Q}(q) \otimes_{\mathbf{Z}[q]} \mathcal{M}$ is a subring of $\mathrm{Ae}(\mathcal{Q},\mathbf{Q})$.*

Proof. Suppose that $\sum (r_i/s) \otimes [X_i] \in \mathcal{M} \otimes_{\mathbf{Z}[q]} \mathbf{Q}(q)$ with $r_i, s \in \mathbf{Z}[q]$, and

$$\sum \frac{r_i(q)}{s(q)} [X_i](q) = 0$$

for almost all q. It follows that $\sum r_i(q)[X_i](q) = 0$ for almost all q.

Now for any $r = \sum a_j q^j \in \mathbf{Z}[q]$ let

$$\begin{aligned} r^{\geq} &= \sum_{\{i \mid a_i \geq 0\}} a_i q^i, \\ r^{<} &= -\sum_{\{i \mid a_i < 0\}} a_i q^i. \end{aligned}$$

Thus $r = r^{\geq} - r^{<}$. Now, for any $t = \sum a_i q^i$ with $a_i \geq 0$, there is a scheme $X(t)$ of finite type over $\mathbf{Z}$ such that $[X(t)] = t$. Thus, for any r, we have schemes $X(r^{\geq})$ and $X(r^{<})$ such that $r = [X(r^{\geq})] - [X(r^{<})]$.

We have now that

$$\sum r_i^{\geq}(q)[X_i](q) = \sum r_i^{<}(q)[X_i](q) \tag{7.1}$$

for almost all q. Setting

$$\begin{aligned} X^{\geq} &= \coprod X(r_i^{\geq}) \times X_i, \\ X^{<} &= \coprod X(r_i^{<}) \times X_i, \end{aligned}$$

it follows from Corollary 3.6 that $[X^{\geq}] = [X^{<}]$ and thus both sides of Equation 7.1 are equal. It then follows that $\sum r_i \otimes [X_i] = 0$ and, thus, that $\sum (r_i/s) \otimes [X_i] = 0$. □

For a graph G let $M(G)$ be the $\mathbf{Z}[q]$ submodule of $\mathcal{M}$ generated by the $[A_G(s,r,k)]$. Let $M(G)_t$ be the submodule of $M(G)$ generated by the $[A_G(s,r,k)]$ for $s \leq t$. Theorem 6.3 shows that $[A_G(s,r,k)] \in M(G)_k$. Thus we have a finite filtration

$$M(G) = M(G)_m \supset M(G)_{m-1} \supset \ldots \supset M(G)_0 = \mathbf{Z}[q]$$

In this section we show that the $K_G(t)$ (resp. the $J_G(t)$) for $t \leq s$ span $M(G)_s$. We also show that the $H_G(s) \in M(G)_s$ and then that the $H_G(t)$ for $t \leq s$ span $M(G)_s \otimes_{\mathbf{Z}[q]} \mathbf{Q}(q)$. Using this last fact we will show in section 8 that, if G has an apex, Conjecture 4.4 implies that $M(G) = \mathbf{Z}[q]$. Thus for any graph Γ, $A_\Gamma^*(s,r,k)$ must be countable if Stanley's conjecture holds.

Theorem 7.2. *$[A_G(s,r,k)]$ is in the $\mathbf{Z}[q]$-module $\tilde{M}$ generated by the $[K_G(t)]$ for $t \leq \min\{s,r,k\}$.*

Proof. By induction, we can assume the result holds for $s < s_0$. (It obviously holds for $s = 0$). Theorem 6.3 shows that $[A(s_0,r,k)]$ is in the $\mathbf{Z}[q]$-module generated by the $[A(k,j,k)]$. By induction we can then assume that $k = s_0$ and it suffices to prove the theorem for $[A(k,j,k)]$. And, of course, we assume that $j < k$ as otherwise the theorem is a tautology.

But now $[A(k,j,k)] = [A(k,j)] - \sum_{i=1}^{k-1}[A(k,j,i)]$. By induction and and application of Theorem 6.3, the result holds for the $[A(k,j,i)]$ since $i < k = s_0$. And by Theorem 6.4 $[A(k,j)]$ is $\mathbf{Z}[q]$-multiple of $[A(j,j)]$. That $[A(j,j)] \in \tilde{M}$ is then known by the inductive hypothesis. And this proves the theorem. □

Corollary 7.3. (a) *$M(G)_t$ is generated by the $[K_G(s)]$ for $s \leq t$.*
(b) *$[A(s,r,k)] \in M(G)_t$ for $t = \min\{s,r,k\}$.*
(c) *$[H_G(s)] \in M(G)_s$.*

Proof. (a) and (b) follow directly from Theorem 6.3 and (c) follows from (b). □

Theorem 7.4.

(a) *$M(G)_t$ is generated by the $[J_G(s)]$ for $s \le t$.*
(b) $[J_G(s)] \equiv [H_G(s)] \mod M(G)_{s-1}$.

Proof. (b) follows from the equation

$$[K(s)] = [J(s)] - \sum_{i=1}^{s-1} [A(s,s,i)] \tag{7.2}$$

and Corollary 7.3 part (b). (a) then follows by induction on s and Corollary 7.3 part (c). ☐

Now we show that the $[H_G(s)]$ span $M(G) \otimes_{\mathbf{Z}[q]} \mathbf{Q}(q)$. We first wish to note a few computational facts.

Lemma 7.5. (a) $C(s,r,k,r) = q^{r(s-k)}$.
(b) $[A(s,r,k)] \equiv [\mathrm{Gr}(k,s)]q^{r(s-k)}[A(k,r,k)] \mod M_{r-1}$.

Proof. (a) is proved by induction using formula 6.1. For (b), write out $[A(s,r,k)]$ using Theorem 6.3 and notice that all terms of the form $C(s,r,k,j)[A(k,j,k)]$ for $j \neq r$ are in M_{r-1} by Theorem 7.2. Then use part (a) to compute the coefficient of $A(k,r,k)$. ☐

For $p \in \mathbf{Z}[q]$ let $\deg(p)$ be the degree of p as a polynomial in q. We make the convention that $\deg 0 = -\infty$. Theorem 6.4 says that $[A(s,r)] = q^{m(s-r)}[\mathrm{Gr}(r,s)][J(r)]$. We write $p(s,r)$ for $q^{m(s-r)}[\mathrm{Gr}(r,s)]$ and note that $\deg p(s,r) = m(s-r) + r(s-r) = (m+r)(s-r)$.

Lemma 7.6. *For all s, r, k with $s \le m$, there is a polynomial $p(s,r,k) \in \mathbf{Z}[q]$ such that*
(a) $[A(s,r,k)] \equiv p(s,r,k)[J(r)] \mod M_{r-1}$*;*
(b) $\deg p(s,r,k) < \deg p(s,r)$ *for $s \neq k$;*
(c) $\deg p(s,r,k) = \deg p(s,r)$ *for $s = k$.*

Proof. Note first that the result is trivial unless $k, r \le s$ as otherwise we can take $p(s,r,k) = 0$.

We prove all three statements simultaneously, reasoning by induction on s. The result is trivial when $s = 0$ as $p(0,0,0) = 1$. For a given s, we first assume that $k < s$. Then, by (b) of Lemma 7.5, $A(s,r,k) \equiv \mathrm{Gr}(k,s)q^{r(s-k)}A(k,r,k) \mod M_{r-1}$. Note that $\deg(\mathrm{Gr}(k,s)q^{r(s-k)}) = (k+r)(s-k)$. By induction, $A(k,r,k) \equiv p(k,r,k)J(r) \mod M_{r-1}$ with $\deg p(k,r,k) = (m+r)(k-r)$. Thus $A(s,r,k) \equiv p(s,r,k)J(r)$ with

$$\begin{aligned}
\deg(p(s,r,k)) &= (m+r)(k-r) + (k+r)(s-k) \\
&= (m+r)(s-r) - (m+r)(s-k) + (k+r)(s-k) \\
&= (m+r)(s-r) - (m-k)(s-k)
\end{aligned}$$

This verifies the statement for $k < s \le m$.

For $k = s$, we then have (modulo M_{r-1})

$$\begin{aligned} [A(s,r,s)] &\equiv [A(s,r)] - \sum_{k=1}^{s-1}[A(s,r,k)] \\ &\equiv p(s,r)[J(r)] - \sum_{k=0}^{s-1} p(s,r,k)[J(r)] \end{aligned}$$

As $\deg(p(s,r)) = (m+r)(s-r)$ and $\deg p(s,r,k) < (m+r)(s-r)$ for $k < s$ the result follows. ☐

Theorem 7.7.
(a) *The $[H_G(t)]$ for $t \leq s$ generate $M_s \otimes_{\mathbf{Z}[q]} \mathbf{Q}(q)$.*
(b) *If the $H_G(t)$ are all countable then $M(G) = \mathbf{Z}[q]$.*

Proof. Since the $[J(t)]$ for $t < s \leq m$ generate $M(G)_s$ (a) follows from Lemma 7.6. (b) then follows from Theorem 3.1. ☐

8. Necessity

We now show that if Stembridge's conjecture (a consequence of Kontsevich's) is true then all $A^*_G(s,r,k)$ are countable for every G.

In this section and for the rest of the paper we will often consider several different graphs G simultaneously. To handle this notationally we make the following convention: For any graph G we now write n_G for $|V(G)|$ and m_G for $n_G - 1$.

Lemma 8.1.

$$[A^*_{G^*}(s,r,k)] = q^k[A^*_G(s,r,k)] + (q^s - q^{k-1})[A^*_G(s,r,k-1)].$$

Proof. Let $W = \mathbf{F}_q^s$. Let $f(V(G^*)) \to W$ with $\langle f \rangle$ a k-dimensional subspace. The span of $f|_{V(G)}$ is either a k or a $k-1$ dimensional subspace. Counting the possibilities proves the lemma. ☐

The following Lemma is a translation of Theorem 5.1 of [Sta98]

Lemma 8.2. *For $r \leq n_G$,*

$$[H^*_{G^*}(r)] = a_G(r)[H^*_G(r)] + b_G(r)[H^*_G(r-1)] + c_G(r)[H^*_G(r-2)]$$

with

$$\begin{aligned} a_G(r) &= q^{n_G+r}(q^{n_G+1}-1) \\ b_G(r) &= q^{n_G+r-1}(q^{n_G+1}-1)(q-1) \\ c_G(r) &= q^{n_G}(q^{n_G+1}-1)(q^{n_G}-q^{r-1}) \end{aligned}$$

all nonzero polynomials in q.

Proof. By the previous Lemma,

$$[H^*_{G^*}(r)] = (q^{n_G+1} - q^{n_G})[A^*_G(n_G+1, r, n_G)].$$

Now applying Theorem 6.3 to $[A^*_G(n_G+1, r, n_G)]$ and expanding out $[\mathrm{Gr}(n_G, n_G+1)]$ in terms of q gives the result. One checks by inspection that a, b and c are nonzero. □

For a graph G, define $G(m)$ inductively by setting $G(0) = G$ and $G(m) = G(m-1)^*$. Before stating the next theorem, we remind the reader that $H_G(m_G) = K_G(m_G) \cong \mathrm{GL}_{m_G} \times X'_G$. Thus $[H_{G(i)}(m_G+i)] = [\mathrm{GL}_{m_G+i}][X'_{G(i)}]$

Lemma 8.3. *If $G = \Gamma^*$ is a graph with an apex, then $[J_{G(i)}(s)] = q^{is}[J_G(s)]$.*

Proof. The statement will follow by induction on i once we prove that $[J_{G^*}(s)] = q^s[J_G(s)]$ or equivalently that $[J^*_{\Gamma^*}(s)] = q^s[J^*_\Gamma(s)]$. Let $\Gamma = \{w_1, \dots, w_m\}$ and $G = \{w_0, \dots, w_n\}$. Then we define a map $\phi : J^*_G(s) \to J^*_\Gamma(s)$ by $(Q, f) \mapsto (Q, f|_{V(\Gamma)})$. It is easy to see that the fiber of ϕ above any point in $J^*_\Gamma(s)(\mathbf{F}_q)$ is precisely $\mathbf{F}_q^s$. □

Theorem 8.4. *Let $G = \Gamma^*$ be a graph with an apex and let $N(G)$ be the module over $\mathbf{Q}(q)$ generated by the $[X_{G(i)}]$ for $i \geq 0$. Then $M(G) \otimes \mathbf{Q}(q) = N(G)$.*

Proof. We first prove that $M(G) \subset N(G)$. For this it suffices to prove the following statement: For any G with an apex and $0 \leq i \leq m_G$, $H_G(m_G - i) \in N(G)$.

The statement clear for $i = 0$. We prove it by induction on i. Assume then that it holds for $i < j$. Then set $r = m_{G(1)} - (j-1)$. By the induction hypothesis, $H_{G(1)}(r) \in N(G(1)) \subset N(G)$.

But then by Lemma 8.2

$$\begin{aligned} H_{G(1)}(r) &= a_\Gamma(r) H_G(m_G - (j-2)) \\ &+ b_\Gamma(r) H_G(m_G - (j-1)) + c_\Gamma(r) H_G(m_G - j). \end{aligned} \tag{8.1}$$

Thus both sides of equation 8.1 are in $N(G)$. But the first two terms on the right hand side are also in $N(G)$ by induction. Since $c(r)$ is a nonzero polynomial, it follows that $H_G(m_G - j) \in N(G)$ and this proves that $M(G) \otimes_{\mathbf{Z}[q]} \mathbf{Q}(q) \subset N(G)$.

To show that $N(G) \subset M(G)$ we remark that each $M(G)$ contains all of the $J_G(s)$ and, thus, by Lemma 8.3 it contains all of the $J_{G(i)}(s)$. Thus by Theorem 7.4 it contains $M(G(i))$ for all i. And thus it contains $X_{G(i)}$ for all i. □

Corollary 8.5. *Let G be as in Theorem 8.4. Then, for any $i \geq 0$, $N(G(i)) = M(G) \otimes_{\mathbf{Z}[q]} \mathbf{Q}(q)$.*

Proof. Lemma 8.3 in fact shows that, for G with an apex, $M(G(i)) \otimes \mathbf{Q}(q) = M(G) \otimes \mathbf{Q}(q)$ for any i. Thus $M(G) \otimes_{\mathbf{Z}[q]} \mathbf{Q}(q) = M(G(i)) = N(G(i))$. □

Corollary 8.6. *Stembridge's conjecture implies that all $A^*_G(s, r, k)$ are countable.*

Proof. By Theorem 8.4, Stembridge's conjecture implies that $N(G^*) = \mathbf{Z}[q]$ for any graph G. Thus the $A^*_G(s, r, k) = A_{G^*}(s, r, k)$ are all countable. □

9. Complements of Complete Graphs

Every graph G is contained in a natural way in the complete graph $K_{V(G)} = K_{n_G}$. We let G^o denote the graph $K_{n_G} - G$. Note that $G^o(i) = K_{n_G+i} - G$.

Now $J^*_{G^o}(s)(\mathbf{F}_q)$ is the set of all $(Q, f) \in \mathrm{Bil}_s(\mathbf{F}^s) \times \mathrm{Fun}(V, \mathbf{F}^s)$ such that $Q(f(v_i), f(v_j)) = 0$ for $e_{ij} \in E(G)$. To simplify the notation, we write $P(G, s)$ for $J^*_{G^o}(s)$.

Theorem 9.1. *If the $P(G, s)$ are countable for $0 \leq s \leq n_G$ then*
(a) *$M(G^{o*}) = \mathbf{Z}[q]$.*
(b) *For every $n > n_G$, $X_{K_n - G}$ is countable.*

Proof. (a): The hypothesis is that the $J^*_{G^o}(s) = J_{G^{o*}}(s)$ are countable for $0 \leq s \leq n_G$. It then follows from Theorem 7.4 that $M(G^{o*})$ is countable. (b) now follows from (a), Theorem 8.4 and Theorem 3.1. □

We are now going to prove that the $P(G, s)$ are countable whenever G is a forest. To do so we need to introduce a few operations on graphs.

Let $v \in V(G)$. We obtain a graph $I_v(G)$ by adding one edge e connected to v and one new vertex w connected to e. Ie, we insert an edge at v. Clearly, a graph is tree if and only if it can be obtained from the graph with one vertex by successive applications of I_v for various v.

We define $D(G)$ to be the graph obtained from G by simply adding a vertex w disjoint from $V(G)$ and letting $E(D(G)) = E(G)$. A graph is a forest if and only if it can be obtained from the empty graph $\emptyset$ by successive applications of D and I_v. We write D_n for $D^n\emptyset$ — the graph with n disjoint vertices.

Lastly, we define R_v to be the graph obtained from G by deleting v and all edges meeting it. We mention that if G is a forest and v is any vertex in G, R_vG will always be a forest.

Theorem 9.2. *Let G be a graph with $v \in V(G)$.*
(a) $[P(D_n, s)] = q^{ns}[\mathrm{Bil}_s^s]$.
(b) $[P(DG, s)] = q^s P(G, s)$.
(c) $[P(I_v, s)] = q^{s-1}\left(P(G, s) + (q-1)P(R_v G, s)\right)$

Proof. For (b) let w be the vertex in $V(DG) - V(G)$. Then the map $P(DG, s) \to P(G, s) \times \mathbf{F}_q^s$ given by $(Q, f) \mapsto (Q, f|_{V(G)}, f(w))$ is an isomorphism. For (a) assume first that $n = 0$. Then tracing through the definitions one sees that $P(\emptyset, s) = \mathrm{Bil}_s^s$. The rest of (a) follows by induction from (b).

For (c) we work over $\mathbf{F}_q$ and consider the map $\pi : P(I_v(G), s) \to P(G, s)$ given by $(Q, f) \mapsto (Q, f|_{V(G)})$. The fiber of π above a point $(Q, g) \in P(G, s)$ depends on whether $g(v)$ is 0 or not. Let $P^0(G, s)$ (resp. $P^\times(G, s)$) be the set where $g(v) = 0$ (resp. $g(v) \neq 0$). Above a point $(Q, g) \in P^0(G, s)$ the fiber of π will have q^s points. Above a point $(Q, g) \in P^\times(G, s)$ the fiber will have q^{s-1} points since Q is non-degenerate. Thus

$$|P(I_v(G), s)| = q^{s-1}|P^\times(G, s)| + q^s|P^0(G, s)| \tag{9.1}$$

The result now follows from the observation that

$$|P^0(G, s)| = |P(R_v(G, s))|.$$

□

The next corollary follows directly from Theorem 9.1 and Theorem 9.2.

Corollary 9.3. *Let F be a forest with r vertices contained in a complete graph K_s with $s > r$. Let $G = K_s - F$. Then X_G is countable.*

10. Stembridge's Conjecture

Remark 10.1 (Guide for the next section). This paper developed out of an attempt to prove Kontsevich's conjecture. In the process, we found a proof that Stembridge's conjecture implied Stanley's. To us, this was an important step because Stembridge's conjecture seemed more natural then Stanley's in terms of our incidence schemes.

However, the same methods that we used to show that Stembridge's conjecture implied Stanley's eventually led us to a counterexample to all three conjectures.

In this section, we present the proof that Stembridge's conjecture if it were true would imply Stanley's. And in the next section we present the proof that, in fact, Stembridge's conjecture itself is false. This may seem unnecessary, and, of course, logically speaking it is. But the methods used in this section are still interesting for the study of weaker versions of Kontsevich's conjecture. And they play a large role in the counterexample of the next section. That is why we have decided to reproduce this proof intact for this MPI preprint.

As mentioned in the section 4, Conjecture 4.4 is a special case of Conjecture 1.2. The goal is to show that Conjecture 4.4 actually implies Conjecture 1.2. In other words, we will prove the following

Theorem 10.2. *Stanley's conjecture is true if and only if it is true for all graphs with apexes.*

Since Conjecture 1.2 is equivalent to Conjecture 1.1, this will show that actually that all three conjectures are equivalent. We warn the reader though that we do not show that the truth of Conjecture 4.4 for a particular graph G implies the truth of Conjecture 1.2 for that same graph. Rather we must assume the truth of Conjecture 4.4 for a set of graphs related to G to assure that Conjecture 1.2 is true for G.

In this section we will more or less abuse notation and define our various schemes by giving their $\mathbf{F}$ points for $\mathbf{F}$ a field.

Let $V = V(G) = \{v_1, \ldots, v_n\}$ and let $m = n - 1$. Our basic tool is the map

$$\begin{array}{rcl} K^*(m) & \stackrel{\phi}{\to} & \mathbf{P}^{m-1} \\ (Q, f) & \mapsto & \{\alpha \in \mathbf{F}^m | \sum \alpha_i f(v_i) = 0\} \end{array}$$

Let $T_G = \mathbf{G}_m^V$. Then ϕ is T_G equivariant provided the action of T_G on $\mathbf{P}^{m-1}$ is chosen appropriately. The orbits of T_G on $\mathbf{P}^{m-1}$ are in $1-1$ correspondence with subsets $\emptyset \neq H \subset V(G)$ with the correspondence sending H to the orbit of the characteristic function χ_H seen as an element of $\mathbf{F}^n$. The orbit stabilizer is $\mathbf{G}_m T_{G-H}$ with the first $\mathbf{G}_m$ factor acting diagonally. Thus the orbit is isomorphic to $\mathbf{G}_m^{|H|-1}$. Hence

$$[K^*(m)] = \sum_{\emptyset \neq H \subset V(G)} [\mathbf{G}_m]^{|H|-1} [\phi^{-1}(\chi_H)].$$

Now $\phi^{-1}\chi_H = \{(Q, f) \in K^*(m) | \sum_{v_i \in H} f(v_i) = 0\}$. In particular, $\phi^{-1}\chi_V = K(m)$. Thus to prove Theorem 10.2 it suffices to show that the $\phi^{-1}(\chi_H)$ are all countable for $H \subsetneqq V(G)$. We will actually show that $\phi^{-1}(\chi_H)$ is countable for all $H \subset V(G)$ (including $V(G)$ itself). But we will do this by induction on the H.

Let

$$\begin{array}{rcl} K_G^H(m) & = & \left\{ (Q, f) \in K_G^*(m) \,\middle|\, \begin{array}{l} \exists \alpha : H \to \mathbf{F}^\times \text{ s.t.} \\ \sum \alpha(v) f(v) = 0, \end{array} \right\} \\ \overline{K}_G^H(m) & = & \left\{ (Q, f) \in K_G^*(m) \,\middle|\, \begin{array}{l} \exists \alpha : H \to \mathbf{F} \text{ s.t.} \\ \sum \alpha(v) f(v) = 0, \ \alpha \neq 0 \end{array} \right\} \end{array}$$

We remark that $K_G^H(m)$ is the T_G orbit of $\phi^{-1}(\chi_H)$.

Proposition 10.3. *The following are equivalent:*

(a) *For all H, $\phi^{-1}(\chi_H)$ is countable.*
(b) *For all H, $K^H(m)$ is countable.*
(c) *For all H, $\bar{K}^H(m)$ is countable.*

Proof. The first equivalence follows from the fact that $[K^H(m)] = [\mathbf{G}_m^{|H|-1}][\phi^{-1}(\chi_H)]$. (It is thus unnecessary to quantify over all H in this statement). (b)⇒(c) follows from the equality $\overline{K}^H(m) = \coprod_{J\subset H} K^J$. This equality also implies that $[K^H] = [\overline{K}^H] - \sum_{J\subsetneqq H}[K^J]$. Thus to prove (c)⇒(b) it suffices to know (c)⇒(b) for $J \subsetneqq H$. As $K^H(m) = \overline{K}^H(m)$ when H has one element, this follows by induction. □

Our strategy now is to show that Stembridge's conjecture implies that the $\overline{K}_G^H(m)$ are countable for arbitrary graphs G. Recall that Stembridge's conjecture implies that for any G the $A_G^*(s,r,k)$ are all countable (Corollary 8.6).

Keeping H fixed, we let $J(k,a,b)$ be the variety of all $(Q,f) \in J_G^*(k)$ such that

1. $\dim\langle f(V)\rangle = a$,
2. $\dim\langle f(H)\rangle = b$.

Note then that $J(m,m,|H|-1) = \overline{K}^H(m)$. Note also that $J(k,a,b) = \emptyset$ unless $0 \le b \le a \le k$.

Under the assumption that Conjecture 1.2 is true, we will now prove that all $J(k,a,b)$ are countable.

Let $G(r,s)$ be the graph obtained by adjoining $r+s$ vertices $S = \{x_1,\dots,x_r,y_1,\dots,y_s\}$ all joined to each other (by an edge) such that the x_i are joined to no vertices in G and such that the second y_i are joined to all vertices in G that are not in H. Let $X = \{x_1,\dots,x_r\}$.

Suppose $C \supset D$ are two $\mathbf{F}$-vector spaces of ranks c and d respectively. Write $M(r,s,C,D)$ for the number of maps $\phi : S \to C$ such that $\phi(X) \subset D$. From equation 2.1, we know that $M(r,s,C,D) = q^{(rd+sc)}$.

Proposition 10.4.

$$[J^*_{G(r,s)}(k)] = \sum_{0\le i\le j\le k} q^{ri+sj}[J(k,k-i,k-j)] \tag{10.1}$$

Proof. To simplify the notation for the proof we write $Z(i,j)$ for the set of $(Q,f) \in J^*_{G(r,s)}(k)$ such that $\dim\langle f(V(G))\rangle = k-i$ and $\dim\langle f(H)\rangle = k-j$. Also, for $Q \in \mathrm{Bil}(\mathbf{F}^k)$ and $C \subset \mathbf{F}^k$, we write $C^{\perp Q}$ for the space perpendicular to C under the Q.

We then have a map

$$\begin{array}{rcl} Z(i,j) & \xrightarrow{\pi} & J(k,k-i,k-j) \\ (Q,f) & \mapsto & (Q, f|_{V(G)}) \end{array}$$

Suppose $(Q,g) \in J(k,k-i,k-j)$. Then $\pi^{-1}(Q,g)$ consists of the set of all (Q,f) where f is an arbitrary element of $\mathrm{Fun}(V(G) \coprod S, \mathbf{F}^k)$ such that $f|_{V(G)} = g$, $f(X) \in g(V(G))^{\perp Q}$ and $f(S) \in g(H)^{\perp Q}$. It follows that, for $\mathbf{F} = \mathbf{F}_q$

$$\begin{array}{rcl} |\pi^{-1}(Q,g)| & = & M(r,s,g(H)^{\perp Q}, g(V(G))^{\perp Q}) \\ & = & q^{ri+sj}. \end{array}$$

Summing over the $Z(i,j)$ finishes the proof. □

Assuming Stembridge's conjecture is true, all $J^*_{G(r,s)}(k)$ are countable. This follows from Corollary 8.6. Thus to show that the $J(k,a,b)$ are countable it will suffice to know that as r and s vary the equations we get from 10.1 determine the $J(k,k-i,k-j)$. Since $J(k,k-i,k-j)$ is a variety it will then be forced to be countable by Theorem 3.1. That the equations from 10.1 are determined follows from the following.

Lemma 10.5. *Let $S_N = \{0,\ldots,N-1\}^2$. For every $x \in S_N$, let $f_x : S_N \to \mathbf{Q}(q)$ be the function given by $f_x(y) = q^{x\cdot y}$ (where $x \cdot y$ is the standard inner product). Then the functions f_x as x ranges through S_N form a basis for the space of functions F_N from S_N to $\mathbf{Q}(q)$.*

Proof. We define a map $h : \mathbf{Q}[q] \to \mathbf{C}$ by sending q to $e^{2\pi i/N}$. Set $\ker h = I$. Set $E_N = \mathrm{Fun}(S_N, \mathbf{Q}[q])$. Then $E_N \otimes_{\mathbf{Q}[q]} \mathbf{C} = \mathrm{Fun}(S_N, \mathbf{C})$. Write this tensor product as $E_{N\mathbf{C}}$. We put a non-degenerate Hermitian inner product on $E_{N\mathbf{C}}$ by setting

$$< f, g >= \frac{1}{N^2} \sum_{z \in S_N} f(z)\overline{g(z)} \tag{10.2}$$

where we think of f and g as functions from S_N to $\mathbf{C}$. It is easy to see that under this inner product the $h(f_x)$ form an orthonormal basis as x ranges through S_N.

Let M_N be the $\mathbf{Q}[q]$ module generated by the f_x as x ranges through S_N. We then have that $E_N = M_N + IE_N$. It then follows from Nakayama's lemma that $M_N \otimes_{\mathbf{Z}[q]} \mathbf{Q}(q) = E_N \otimes_{\mathbf{Z}[q]} \mathbf{Q}(q)$. Since the dimension of F_N is N^2 which is precisely equal to the number of f_x as x runs through S_N, this proves the lemma. □

Theorem 10.6. *Under the assumption that Stembridge's conjecture holds, all $J(k,k-i,k-j)$ are countable for any graph G and any subgraph H.*

Proof. Since the set of i and j such that $0 \leq i \leq j \leq k$ is contained in S_{k+1}. The equations in 10.1 for $(r, s) \in S_{k+1}$ suffice by lemma 10.5 to determine the $J(k, k-i, k-j)$. □

Since $J(m, m, |H|-1) = \overline{K}^H(m)$ Theorem 10.2 now follows from Proposition 10.3 and the fact that $K_G(m) = K_G^{V(G)}(m)$.

11. Bipartite Graphs and Counterexample to Kontsevich's Conjecture and all related conjectures

We want to nail down the counterexample. The approach below is the shortest route to present the counterexample. This may not be the best thing to do, because there are actual theorems hidden here (Motives of complements of Bipartite graphs "=" motives of matroids). Also the hypotheses may not be minimal.

We assume that Stanley's conjecture is true is true for all Graphs.

First we generalize theorem 10.5.

Let G be a graph, V the set of its vertices, U the set of subsets of V. For π a function $U \mapsto \mathbf{Z}$, define

Definition 11.1. $J(k, \pi)$ to be the variety of all $(Q, f) \in J_G^*(k)$ such that

$\dim\langle f(H)\rangle = \pi(H)$, for all $H \in U$.

More generally if W is a subset of U, For π a function $W \mapsto \mathbf{Z}$, define

Definition 11.2. $J(k, W, \pi)$ to be the variety of all $(Q, f) \in J_G^*(k)$ such that

$\dim\langle f(H)\rangle = \pi(H)$, for all $H \in W$.

This generalizes the varieties of section 10, Using the same ideas we will show that

Theorem 11.3. *Under the assumption that Stembridge's conjecture holds, all $J(k, W, \pi)$ are countable for any graph G, any subset W of U and any function $\pi : W \mapsto \mathbf{Z}$.*

Proof. Induct on the number of elements in W.

If W is empty, the result is clear (taking into account theorems of section 8) Now suppose $W' = W \bigcup \{H\}$. Note that W has one fewer element than W' . We can write for $\pi : W \mapsto \mathbf{Z}$.

$$[J_{G(0,s)}(k, W, \pi)] = \sum_{0 \leq i \leq k} q^{si}[J(k, W', \pi_i)] \tag{11.1}$$

where $\pi_i : W' \mapsto \mathbf{Z}$. is defined to be π on W and $k - i$ on the subset H.

The proof is similar to a similar equation in 10.5. We can break up $[J_{G(0,s)}(k, W, \pi)]$ into pieces where the rank of the vectors in H is $k - i$, the s added vectors are forced to lie in the space orthogonal to the subspace spanned by H, and hence a space of dimension i and hence q^{is} possibilities of choosing the s new vectors.

Now consideration of the Vandermonde determinant, completes the proof by induction. To elaborate this point imagine that we are trying to solve for $[J(k, W', \pi_i)]$ in terms of $[J_{G(0,s)}(k, W, \pi)]$ using the above equations for $s = 0$ till $s = k$. Then by Cramer's rule this can be solved if the determinant of the matrix whose $(i, j)th$ entry is q^{ij}. That is, the ith row is just powers of q^i. This is exactly the Vandermonde determinant and is nonzero since different powers of q do not coincide. □

Let G be a complete graph, in this case it is easy to see that if π a function $U \mapsto \mathbf{Z}$, then

$[J(k, \pi)] = [Q(k)][L(k, \pi)]$ where

1. $Q(k)$ =variety of nondegenerate symmetric bilinear forms on a k dimensional space, by results quoted earlier this is countable.
2. $L(k, \pi)$ =variety of $f \in Fun(V, \mathbf{F}_q^k)$ so that $\dim\langle f(H)\rangle = \pi(H)$, for all $H \in U$.

(Looking at the definition of $J(k, \pi)$ makes it clear that Q does not enter in the definition of the J's for the complete graph.) Also note that only the vertex set V is needed for the definition of the L's.

Therefore it is easy to see that Stembridge's conjecture implies that the L's are countable. But examples of L's which are not countable are available from the theory of matroids. These can be found in Chapter 9 of [Wel76]. Instead of defining a matroid, we describe the L which is not countable. This is called the Fano Matroid. Let $V = \{1, 2, 3, 4, 5, 6, 7\}$,$k = 3$. $W = U$, and to define π,

1. If $card(H)$=1, then $\pi(H) = 1$.
2. If $Card(H)$=2 then $\pi(H) = 2$.
3. If $Card(H)$=3 then $\pi(H) = 3$ UNLESS $\pi = \{1, 2, 6\}$ or $\pi = \{1, 4, 7\}$or $\pi = \{1, 3, 5\}$ or $\pi = \{2, 3, 4\}$ or $\pi = \{2, 5, 7\}$ or $\pi = \{3, 6, 7\}$ or $\pi = \{4, 5, 6\}$. In the 7 cases above define π to equal 2.
4. If $Card(H) = i$ where $i > 3$ then define $\pi(H) = 3$.

It is shown (and not difficult, uses simple projective geometry), that L has no points over fields of characteristic $\neq 2$, and has points over every field of Characteristic 2. If it were countable then looking at the prime 2 we see that this polynomial is non zero, and hence non zero for every sufficiently large prime, but it has no points over fields of characteristic p for $p \neq 2$. Contradiction.

We also add that the "pathology" can be arranged to occur over any prime.

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