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Noncommutative gauge theory for Poisson manifolds

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Abstract

A noncommutative gauge theory is associated to every Abelian gauge theory on a Poisson manifold. The semi-classical and full quantum version of the map from the ordinary gauge theory to the noncommutative gauge theory (Seiberg-Witten map) is given explicitly to all orders for any Poisson manifold in the Abelian case. In the quantum case the construction is based on Kontsevich's formality theorem.

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1 Introduction

Noncommutative geometry naturally enters the description of open strings in a background B-field [1, 2, 3]. The D-brane world volume is then a noncommutative space whose fluctuations are governed by a noncommutative version of Yang-Mills theory [4, 5, 6, 7, 8]. In the case of a constant B-field it has been argued that there is an equivalent description in terms of ordinary gauge theory. From the physics perspective the two pictures are related by a choice of regularization [8, 9]. There must therefore exist a field redefinition – a Seiberg-Witten map [8]. The B-field, if non-degenerate and closed, defines a symplectic structure on the D-brane world volume; its inverse is a Poisson structure whose quantization gives rise to the noncommutativity.

An interesting question arises: Given a gauge theory on a general Poisson manifold – is there always a corresponding noncommutative gauge theory on the noncommutative space – the quantization of the original Poisson manifold? Previously we found this to be true for symplectic manifolds [10].[1] Here we give the construction for an arbitrary Poisson manifold. (The present discussion is much more explicit and complete.) On the way an appropriate generalization of Moser's lemma from symplectic geometry to the Poisson case and its quantization are given. This, we believe, is mathematically interesting in its own right. As in our previous paper [10], we choose to work within the framework of deformation quantization [12, 13, 14]. This allows us to postpone questions related to representation theory, so that we can focus on the algebra. We expect that our results can be used to find derivative corrections to the Born-Infeld action, classifying invariant actions in the spirit of [15].

2 Noncommutative Yang-Mills theory

Here we recall how the gauge theory on a more-less arbitrary noncommutative space was introduced in [16]. The formulation starts with an associative, not necessarily commutative, algebra $\mathcal{A}_x$ over $\mathbb{C}$ freely generated by finitely many generators $\hat{x}^i$ modulo some relations $\mathcal{R}$. $\mathcal{A}_x$ plays role of the noncommutative space-time. The matter fields ψ of the theory are taken to be elements of a left module of $\mathcal{A}_x$ and the infinitesimal gauge transformation induced by $\hat{\lambda} \in \mathcal{A}_x$ is given by the left multiplication (action)

$$\psi \overset{(\hat{\lambda})}{\mapsto} \psi + i\hat{\lambda}\psi. \tag{1}$$

The gauge transformation does not act on the "coordinates" $\hat{x}^i$.

$$\hat{x}^i \overset{(\hat{\lambda})}{\mapsto} \hat{x}^i. \tag{2}$$

[1] For a detailed description of the universal gauge theory of the Weyl-Bundle see, e.g., [11].

The left multiplication of a field by the coordinates $\hat{x}^i$ is not covariant under the gauge transformation

$$\hat{x}^i\psi \overset{(\hat{\lambda})}{\mapsto} \hat{x}^i\psi + i\hat{x}^i\hat{\lambda}\psi, \tag{3}$$

since in general $\hat{x}^i\hat{\lambda}\psi$ is not equal to $\hat{\lambda}\hat{x}^i\psi$. The gauge fields $\hat{A}^i$, elements of $\mathcal{A}_x$, are introduced to cure this. Namely, covariant coordinates

$$\hat{X}^i = \hat{x}^i + \hat{A}^i \tag{4}$$

are introduced.

The gauge transformation is supposed to act on the gauge fields $\hat{A}^i$ in a way that will assure the covariance of $\hat{X}^i\psi$ under the gauge transformation (1),(2). This is achieved by the prescription

$$\hat{A}^i \overset{(\hat{\lambda})}{\mapsto} \hat{A}^i + i[\hat{\lambda}, \hat{x}^i] + i[\hat{\lambda}, \hat{A}^i]. \tag{5}$$

In examples considered in [16] (universal enveloping algebra of a finite dimensional Lie algebra and of the Heisenberg algebra as a special case, quantum plane) also the corresponding field strength $\hat{F}^{ij}$ was introduced. If

$$[\hat{x}^i, \hat{x}^j] = J^{ij}(\hat{x}), \tag{6}$$

then

$$\hat{F}^{ij} = [\hat{X}^i, \hat{X}^j] - J^{ij}(\hat{X}). \tag{7}$$

Of course this is not unique, the choice of ordering in the above formula may effect the definition of $\hat{F}$, but this is not important for covariance. It follows

$$\hat{F}^{ij} \overset{(\hat{\lambda})}{\mapsto} \hat{F}^{ij} + i[\hat{\lambda}, \hat{F}^{ij}], \tag{8}$$

as expected.

This construction covers also the noncommutative generalization of non-Abelian ($GL(N)$ or $U(N)$ if $\mathcal{A}_x$ possesses a $*$–structure) gauge theories, taking the tensor product algebra $\mathcal{A}_x \otimes U(gl(N))$ instead of $\mathcal{A}_x$.

The following question inspired by [8] appears naturally.

Let us assume that our associative algebra $\mathcal{A}_x$ (and this was indeed the case of the examples considered in [16]) can be understood as a deformation quantization of a commutative algebra of functions on some Poisson manifold M. Let $\star$ be the corresponding star product. Let us also assume that we have a (non-Abelian) gauge field A on M.

Does there exist a map SW :

$$A \overset{(SW)}{\mapsto} \hat{A}, \qquad \lambda \overset{(SW)}{\mapsto} \hat{\lambda}(\lambda, A) \tag{9}$$

such that the (non-Abelian) commutative gauge transformation on A

$$A \overset{(\lambda)}{\mapsto} A + d\lambda + i[\lambda, A] \tag{10}$$

is sent by SW into the noncommutative gauge transformation on $\hat{A}$

$$\hat{A}^i \stackrel{(\hat{\lambda})}{\mapsto} \hat{A}^i + i[\hat{\lambda}, \hat{x}^i]_\star + i[\hat{\lambda}, \hat{A}^i]_\star \ ? \tag{11}$$

The commutator in (10) is the matrix one and the commutator in (11) is the star commutator on functions and the matrix one on matrices.

In this paper we give a general and explicit construction of the map SW in the Abelian case. Our main tool is Kontsevich's formality theorem.

The non-Abelian case is more involved and will be treated by similar methods in the sequel to this paper.

3 Classical

Here we formulate the classical analogue of the Seiberg-Witten map between the commutative and noncommutative description of Yang-Mills theory for any Poisson manifold.

Let M be a manifold and F a two-form on M. Let us temporarily assume that F is exact. Later on we will relax this condition, F closed will appear to be good enough for our purposes. In local coordinates we write $A = A_i dx^i$ and $F = F_{ij} dx^i \wedge dx^j$, with $F_{ij} = \partial_i A_j - \partial_j A_i$. Let us further assume that we have on M a one-parameter family of bivector fields $\theta(t) = \frac{1}{2}\theta^{ij}(t)\partial_i \wedge \partial_j$, $t \in [0,1]$, with the explicit t-dependence of the matrix $\theta^{ij}(t)$ given by

$$\partial_t \theta(t) = -\theta(t) F \theta(t), \tag{12}$$

with the initial condition

$$\theta(0) = \theta, \tag{13}$$

where θ is some fixed but otherwise arbitrary Poisson tensor on M. The product on the right in (12) is the matrix one. As above we will often use the same notation for polyvector fields or forms and the corresponding tensors. The formal solution to (12) can be given by the following power series in t (or in θ)

$$\theta(t) = \sum_{n \geq 0} (-t)^n \theta (F\theta)^n. \tag{14}$$

The convergence is not an issue here, because we will work all the time with formal power series in θ. (E.g. if the matrix θ is invertible then in the physical situation of a D-brane in the background of the B-field $B = \theta^{-1}_{ij} dx^i \wedge dx^j$ it simply means that the background is strong.)

It follows from (12), or directly from (14), that $\theta(t)$ continues to be a Poisson tensor for all $t \in [0,1]$. For this only the closedness of F is important. The Poisson bivector field $\theta(t)$ defines an bundle map $T^*M \to TM$ given by $i_{\theta(t)(\omega)}\eta = \theta(t)(\omega, \eta)$ for any one-forms ω and η. Using the Jacobi identity $[\theta(t), \theta(t)] = 0$, with $[,]$

being the Schouten-Nijenhuis bracket of polyvector fields, we can easy verify the t-derivative of $\theta(t)$ is given by a Lie-derivative that

$$\partial_t \theta(t) + [\chi(t), \theta(t)] = 0, \tag{15}$$

where now $\theta(t)$ is understood as a bivector field and

$$\chi(t) = \theta(t)(A) \tag{16}$$

is a vector field that in local coordinates looks like

$$\chi(t) = \theta^{ij}(t) A_i \partial_j.$$

Let us recall that the Schouten-Nijenhuis bracket of two polyvector fields is defined by

$$[\xi_1 \wedge ... \wedge \xi_k, \eta_1 \wedge ... \wedge \eta_l] = \sum_{i=1}^{k} \sum_{j=1}^{l} (-1)^{i+j} [\xi_i, \eta_j] \wedge \xi_1 \wedge ... \wedge \hat{\xi}_i \wedge ... \wedge \hat{\eta}_j \wedge ... \wedge \eta_l$$

$$[\xi_1 \wedge ... \wedge \xi_k, f] = \sum_{i=1}^{k} (-1)^{i-1} \xi_i(f) \xi_1 \wedge ... \wedge \hat{\xi}_i \wedge ... \wedge \xi_k,$$

if all ξ's and η's are vector fields and f is a function.

If f and g are two smooth functions on M with no explicit dependence on t and $\{,\}_t$ denotes the Poisson bracket corresponding to $\theta(t)$ then (15) is rewritten as

$$\partial_t \{f, g\}_t + \chi(t)\{f, g\}_t - \{\chi(t)f, g\}_t - \{f, \chi(t)g\}_t = 0. \tag{17}$$

Both (15) and (17) imply that all the Poisson structures $\theta(t)$ are related by the flow $\rho^*_{tt'}$ of $\chi(t)$: $\rho^*_{tt'}\theta(t') = \theta(t)$. Setting $\rho^* = \rho^*_{01}$ we have in particular

$$\rho^* \theta' = \theta, \tag{18}$$

i.e.,

$$\rho^* \{f, g\}' = \{\rho^* f, \rho^* g\}, \tag{19}$$

where θ' is short for $\theta(1)$. The vector field $\chi(t)$ may not be complete, however ρ^* again has to be understood as a formal diffeomorphism given by formal power series in θ. In this sense we always have a (formal) coordinate change on M which relates the two Poisson structures θ and θ'. Explicitly

$$\rho^* = e^{\partial_t + \chi(t)} e^{-\partial_t} \Big|_{t=0}. \tag{20}$$

Consider now a gauge transformation

$$A \mapsto A + d\lambda. \tag{21}$$

The effect upon $\chi(t)$ will be

$$\chi(t) \mapsto \chi(t) + \chi_\lambda(t), \tag{22}$$

where $\chi_\lambda(t)$ is the Hamiltonian vector field

$$\chi_\lambda(t) = \theta(t)(d\lambda) = [\theta(t), \lambda] \tag{23}$$

and $[\chi_\lambda(t), \theta(t)] = 0$. In local coordinates $\chi_\lambda = \theta^{ij}(t)(\partial_i\lambda)\partial_j$. Correspondingly we use the notation $\rho^*_{\lambda,tt'}$ for the new flow. It follows almost immediately that $\rho^*_\lambda(\rho^*)^{-1} = e^{\partial_t+\chi_\lambda(t)}e^{-\partial_t-\chi(t)}\Big|_{t=0}$ is generated by a Hamiltonian vector field $\theta(d\tilde{\lambda})$ for some $\tilde{\lambda}$. This follows from the Baker-Campbell-Hausdorff identity and the fact that

$$[\partial_t + \theta(t)(A), \theta(t)(df)] = \theta(t)(dg)$$

with $g = \theta(t)(d\lambda, A)$. Having this in mind it is easy to see all the terms coming from B–C–H formula contain only commutators of this type or commutators of two Hamiltonian vector fields which are again Hamiltonian ones. Even more is true: $\rho^*_{\lambda,tt'}(\rho^*_{t't})^{-1}$ for all t and t' is generated by some Hamiltonian vector field for $\theta(t)$.

So the transformation induced by λ takes the form

$$f \overset{(\lambda)}{\mapsto} f + \{\tilde{\lambda}, f\}. \tag{24}$$

It is clear from the above discussion that working only with formal power series in θ we can abandon the exactness condition for F and assume F only closed with the following consequences. The gauge field A and the vector field $\chi(t)$ are given only locally. If A's given in two different local patches are related on their intersection by the gauge transformation (21), then the corresponding local vector fields are related by (22) and the local diffeomorphisms ρ are related by the canonical transformation (24) generated by $\tilde{\lambda}$.

In the case of invertible $\theta(t)$ and of a compact manifold we have the well know lemma of Moser [17].

Let us return to the SW map in classical setting. For this we have to choose some local coordinates x^i on M. Let us write the result of acting by the diffeomorphism ρ^* on the coordinate function x^i in the form [18, 19, 20, 10, 16].

$$\rho^*(x^i) = x^i + A^i_\rho. \tag{25}$$

A_ρ depends as a formal power series in θ on A. Explicitly we have

$$A^i_\rho = (\,e^{\partial_t+\theta^{ij}(t)A_i\partial_j} - 1)\Big|_{t=0} x^i \tag{26}$$

Let us act by the infinitesimal gauge transformation (21) on A. This induces the infinitesimal Poisson map (24) on $\rho^*(x^i)$, which in turn induces a map on A_ρ given by

$$A^i_\rho \overset{(\lambda)}{\mapsto} A^i_\rho + \{\tilde{\lambda}, x^i\} + \{\tilde{\lambda}, A^i_\rho\}. \tag{27}$$

So the map $A \mapsto A_\rho$ can be viewed as the semi-classical version of the SW map which we are looking for.

4 Formality

The existence of a star product on an arbitrary Poisson manifold follows from the more general formality theorem [13]: There exists an L_∞-morphism from the differential graded algebra of polyvector fields into the differential graded algebra of polydifferential operators on M. There is a canonical way to extract a star product $\star$ from such an L_∞-morphism for any formal Poisson bivector field. We will refer to this star product as the Kontsevich star product. Any star product on M is equivalent to some Kontsevich star product.

The differential graded algebra $T_{poly}(M)$ is the graded algebra of polyvector fields on M

$$T^n_{poly}(M) = \Gamma(M, \wedge^{n+1}TM),\ \ n \geq -1,$$

equipped with the standard Schouten-Nijenhuis bracket and differential $d \equiv 0$. An m-differential operator in $D_{poly}(M)$ acts on a tensor product of m functions and has degree $m-1$. The composition $\circ$ on $D_{poly}(M)$ is given by

$$(\Phi_1 \circ \Phi_2)(f_0 \otimes ... \otimes f_{k_1+k_2}) =$$
$$\sum_{i=0}^{k_1}(-1)^{k_2 i}\Phi_1(f_0 \otimes ... \otimes f_{i-1}\otimes \quad (\Phi_2(f_i \otimes ... \otimes f_{i+k_2})) \otimes f_{i+k_2+1} \otimes ... \otimes f_{k_1+k_2})$$

for $\Phi_i \in D^{k_i}_{poly}(M)$ and the Gerstenhaber bracket $[\Phi_1, \Phi_2]$ is then given by

$$[\Phi_1, \Phi_2] = \Phi_1 \circ \Phi_2 - (-1)^{k_1 k_2}\Phi_2 \circ \Phi_1. \tag{28}$$

The differential on $D_{poly}(M)$ is given in terms of the Gerstenhaber bracket as

$$d\Phi = -[\mu, \Phi] \tag{29}$$

where μ is the multiplication of functions: $\mu(f_1 \otimes f_2) = f_1 f_2$.

An L_∞-morphism $U : T_{poly}(M) \to D_{poly}(M)$ is then a collection of skew–symmetric multilinear maps U_n from tensor products of $n \geq 1$ polyvector fields to polydifferential operators of degree $m \geq 0$: $\otimes^n T_{poly}(M) \to D^{m+1}_{poly}(M)$, satisfying the following condition (formality equation) [13, 21]:

$$Q'_1 U_n(\alpha_1, ..., \alpha_n) + \frac{1}{2}\sum_{\substack{I\sqcup J=(1,...,n)\\ I,J\neq\emptyset}} \epsilon_\alpha(I,J) Q'_2(U_{|I|}(\alpha_I), U_{|J|}(\alpha_J)) \tag{30}$$
$$= \frac{1}{2}\sum_{i\neq j}\epsilon_\alpha(i,j,1,...,\widehat{i,j},...,n)(U_{n-1}Q_2(\alpha_i,\alpha_j),\alpha_1,...,\hat{\alpha}_i,...,\hat{\alpha}_j,...,\alpha_n))$$

Here $Q'_1(\Phi) = [\Phi, \mu]$, $Q'_2(\Phi_1, \Phi_2) = (-1)^{|1|(|2|-1)}[\Phi_1, \Phi_2]$, with $|i|$ denoting the degrees of homogeneous polydifferential operators Φ_i and $Q_2(\alpha_1, \alpha_2) = -(-1)^{k_1(k_2+1)}[\alpha_2, \alpha_1]$, where k_i are degrees of homogeneous polyvector fields α_i. Further $|I|$ denotes the numbers of elements of I and $\epsilon_\alpha(I,J)$ is $+1$ or -1 depending on the number of transpositions of odd elements in the permutation of $\{1, ..., n\}$ associated with the partition (I, J). Although it doesn't explicitly enter the formality condition (30) a zero component U_0 can be added to U. By definition it is nonzero only if acting on two functions $f \otimes g$ with the result $U_0(f, g) = fg$.

In the case of $M = \mathbb{R}^d$ Kontsevich gives also a beautiful explicit expression for the formality map U. To reproduce his formula we need to introduce a $(2n+m-2)$–dimensional configuration space $C^{+}_{\{p_1,...,p_n\},\{q_1,...,q_m\}}$. If $\mathcal{H}$ denotes the upper half-plane then $C^{+}_{\{p_1,...,p_n\},\{q_1,...,q_m\}}$ is a quotient of

$$\{(p_1, ..., p_n; q_1, .., q_m)|p_i \in \mathcal{H},\ q_j \in \mathbb{R},\ p_{i_1} \neq p_{i_2} \text{ for } i_1 \neq i_2,\ q_1 < ... < q_m\}$$

by the action of the group $G^{(1)} = \{z \mapsto az + b|a, b \in \mathbb{R}, a > 0\}$ of orientation-preserving affine transformations of the real line.

Then

$$U_n = \sum_{m \geq 0} \sum_{\Gamma \in G_{n,m}} w_\Gamma B_\Gamma. \tag{31}$$

Here the second summation goes over all oriented admissible graphs with n vertices $p_1, ..., p_n$ of the first type and m vertices $q_1, .., q_m$ of the second type.

The rules are: There are no outgoing edges from the second type vertices. There are $k_1, k_2, ..., k_n$ edges starting in the the first type vertices which are ending in either first type vertices again or in the second type vertices. There are no edges starting and ending in the same vertex. The vertices and edges are enumerated in a fashion compatible with the orientation, the edges starting at the first type vertex p_i are labeled by numbers $k_1 + k_2 + ... + k_{j-1} + 1, ..., k_1 + k_2 + k_j$. We denote

$$Star(p_j) = \{\overrightarrow{p_j a_1}, ..., \overrightarrow{p_j a_{k_j}}\} \qquad \overrightarrow{v}_{k_1+...+k_{j-1}+i} = \overrightarrow{p_j a_i}.$$

The weight w_Γ of the oriented graph Γ is defined as an integral over the $(2n+m-2)$-dimensional configuration space $C^{+}_{\{p_1,...,p_n\},\{q_1,...,q_m\}}$

$$w_\Gamma = \int \frac{1}{(2\pi)^{\sum k_i} k_1!...k_n!} d\phi_{\overrightarrow{v}_1} \wedge ... \wedge d\phi_{\overrightarrow{v}_{k_1+...+k_n}}, \tag{32}$$

where

$$\phi_{\overrightarrow{p_j a}} = Arg\left(\frac{a - p_j}{a - \overline{p_j}}\right). \tag{33}$$

If α_1 is of degree $k_1 - 1$, α_2 is of degree $k_2 - 1$,..., α_n is of degree $k_n - 1$, then

$$\begin{aligned} &B_\Gamma(\alpha_1, ..., \alpha_n)(f_1, f_2, ..., f_m) \\ &= \sum D_{p_1} \alpha_1^{i_1 i_2 ... i_{k_1}} ... D_{p_n} \alpha_n^{i_{k_1+...k_{n-1}+1} ... i_{k_1+...+k_n}} D_{q_1} f_1 ... D_{q_m} f_m, \end{aligned} \tag{34}$$

where

$$D_a = \prod_{l, \overrightarrow{v_l} = \overrightarrow{.a}} \partial_{i_l} \tag{35}$$

and the summation runs over repeated indices i_j. Finally

$$U_n = \sum U_{(k_1,..,k_n)}. \tag{36}$$

The weight w_Γ is nonzero only if the degree of the polydifferential operator and the overall degree of the polyvector fields match as

$$m = 2 - n + \sum_{i=1}^{n} (k_i - 1). \tag{37}$$

Only in this case the degree of the form in (32) matches the dimension of the configuration space $C^+_{\{p_1,...,p_n\},\{q_1,...,q_m\}}$. The construction can be globalized to any (formal) Poisson manifold [13].

To make a relation with the deformation quantization a formal parameter, the Planck constant $\hbar$, has to be introduced. If α is a two-tensor, then by the condition (37) $U_n(\alpha, ..., \alpha)$ is a bidifferential operator for every n, i.e., it acts on two functions. If moreover α is a Poisson tensor then the Kontsevich star product $\star$ is defined for f and g, two smooth functions on M, as

$$f \star g = \sum_{n \geq 0} \frac{\hbar^n}{n!} U_n(\alpha, ..., \alpha)(f, g). \tag{38}$$

The associativity of such a star product follows from the formality equation. If we set in (30) $\alpha_i = \alpha$, for $i = 1, .., n$ and take into account the Jacobi identity $[\alpha, \alpha]$ and the condition (37) we see that (30) is equivalent to the $\hbar^n$–order term of the associativity condition for $\star$.

There are some other consequences from the formality theorem, which will be useful later [22]. Here we present them in form convenient for the further use. We adopt the following notation; for any vector field ξ on the Poisson manifold (M, α) we denote as δ_ξ the following formal series in $\hbar$

$$\delta_\xi = \sum_{n \geq 1} \frac{\hbar^{n-1}}{(n-1)!} U_n(\xi, \alpha, ..., \alpha). \tag{39}$$

From (37) we see that this is a differential operator. Its first term equals to ξ. Let us apply the formality equation in the case where $\alpha_1 = \xi$ and $\alpha_2 = ... = \alpha_n = \alpha$. The matching condition (37) says that in this case RHS of the formality equation is a bidifferential operator; so let us act both sides of (30) on two functions f and g. Using (37) also in evaluating the LHS we find:

$$\delta_\xi(f \star g) - \delta_\xi(f) \star g - f \star \delta_\xi(g) = \sum_{n \geq 2} \frac{\hbar^{n-1}}{(n-2)!} U_{n-1}([\xi, \alpha], \alpha, ..\alpha)(f, g). \tag{40}$$

In particular we see that vector fields preserving the Poisson bracket are lifted via δ to derivations of the Kontsevich's star product.

Further, if we use formality equation and the matching condition in the case, $\alpha_1 = f$, $\alpha_2 = ... = \alpha_n = \alpha$, we see that the RHS of (30) is a differential operator. Acting with both sides of the formality equation on a function g we get

$$\frac{1}{\hbar}[\hat{f}, g]_\star = \delta_{[\alpha, f]}(g), \tag{41}$$

where the function $\hat{f}$ is given by a formal power series

$$\hat{f}=\sum_{n\geq 1}\frac{\hbar^{n-1}}{(n-1)!}U_n(f,\alpha,...,\alpha), \tag{42}$$

starting with f. So the Hamiltonian vector fields are lifted to the inner derivations of the star product.

Let us mention that [23] gives a very nice field theoretical interpretation of the formality map.

5 Quantum

With the help of the formality theorem literally everything in the section 3 can be quantized. To be consistent with physics conventions used in the Section 2 we have to replace $\hbar$ by $i\hbar$ everywhere. Let us quantize the Poisson structure $\theta(t)$ (14) on M for any $t\in[0,1]$ via Kontsevich's deformation quantization (38)

$$f\star_t g=\sum_{n\geq 0}\frac{(i\hbar)^n}{n!}U_n(\theta(t),...,\theta(t))(f,g). \tag{43}$$

That way we get a star product $\star_t$ for any t . For any two t-independent functions f and g on M we can take the t-derivative of $f\star_t g$. Then equations (15) and (40) give the quantum version of (17)

$$\partial_t(f\star_t g)+\delta_{\chi(t)}(f\star_t g)-\delta_{\chi(t)}(f)\star_t g-f\star_t\delta_{\chi(t)}(g)=0, \tag{44}$$

with $\delta_{\chi(t)}$ given by (39)

$$\delta_{\chi(t)}=\sum_{n\geq 1}\frac{(i\hbar)^{n-1}}{(n-1)!}U_n(\theta(t)(A),\theta(t),...,\theta(t)). \tag{45}$$

This means that we can relate the star products $\star_t$ at $\star_{t'}$ at two different time instants by $\mathcal{D}_{tt'}$, the "flow" of $\delta_{\chi(t)}$ (or the "quantum flow" of $\chi(t)$). Particularly for $t=0$ and $t'=1$ we have

$$\mathcal{D}=e^{\partial_t+\delta_{\chi(t)}}e^{-\partial_t}\Big|_{t=0}=e^{-\partial_t}e^{\delta_{\chi(t+1)}+\partial_t}\Big|_{t=0}. \tag{46}$$

Let us note that $\mathcal{D}$ is a composition $D\circ\rho^*$ of the classical flow ρ^* and a gauge equivalence $D=\mathcal{D}\circ(\rho^*)^{-1}$ of the star product $\star_\rho$ obtained from $\star'$ by simple action of ρ^* and the star product $\star$.

Finally the gauge transformation (21) is quantized with the help of (24), (41) and (42) as

$$f\overset{(\lambda)}{\mapsto}f+\frac{1}{i\hbar}[\hat{\tilde{\lambda}},f]_\star, \tag{47}$$

where

$$\hat{\tilde{\lambda}}=\sum_{n\geq 1}\frac{(i\hbar)^{n-1}}{(n-1)!}U_n(\tilde{\lambda},\theta,...,\theta) \tag{48}$$

and $\tilde{\lambda}$ is obtained as explained in section 3 from the condition

$$e^{[\theta,\tilde{\lambda}]} = e^{\partial_t+\chi_\lambda(t)}e^{-\partial_t-\chi(t)}\Big|_{t=0} \tag{49}$$

using the B-C-H formula.

The rest is trivial. We write similarly to (25)

$$\mathcal{D}(x^i) = e^{\partial_t+\delta_{\chi(t)}}x^i\Big|_{t=0} = x^i + \hat{A}^i. \tag{50}$$

Again $\hat{A}$ depends as a formal power series in θ on A. Explicitly we have

$$\hat{A}^i = (\exp(\partial_t + \delta_{\theta^{ij}(t)A_i\partial_j}) - 1)x^i\Big|_{t=0} \tag{51}$$

If we act by the infinitesimal gauge transformation (21) on A, this induces now the action of the inner derivation (47) on $\mathcal{D}(x^i)$, which in turn induces a map on $\hat{A}$ given by

$$\hat{A}^i \overset{(\lambda)}{\mapsto} \hat{A}^i + \frac{1}{i\hbar}[\hat{\tilde{\lambda}}, x^i]_\star + \frac{1}{i\hbar}[\hat{\tilde{\lambda}}, \hat{A}^i]_\star. \tag{52}$$

So (51) gives indeed the desired SW map to all orders in θ in the case of a general Poisson manifold. Moreover, using Kontsevich's construction of the formality map U as described in section 4, we can find explicit expressions in local coordinates for $\hat{A}$ and $\hat{\tilde{\lambda}}$ to any order in θ.

To conform completely to the conventions we adopted in section 2, we could have taken $-A$ and $-\lambda$ as the actual classical gauge field and gauge parameter.

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