Hidden Stasheff polytopes in algebraic K-theory and in the space of Morse functions

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Introduction.

(0.1) The Stasheff polytope, or associahedron, K_n, is a convex polytope of dimension $n-2$ whose vertices correspond to complete parenthesizings of the product of n factors $x_1, ..., x_n$. It was introduced by J. Stasheff [St] in his study of homotopy associativity for binary multiplications on topological spaces. In this paper we describe a surprising appearance of Stasheff polytopes in two related classical problems which, on the surface, have nothing to do with non-associative or weakly associative binary operations.

(0.2) The first is the problem of higher syzygies among row operations on matrices; a row operation with coefficient a is a left multiplication by the familiar "elementary matrix" $e_{ij}(a)$, see [M]. There are well known Steinberg relations among the $e_{ij}(a)$, the most non-trivial being

$$[e_{ij}(a), e_{jk}(b)] = e_{ik}(ab). \tag{1}$$

Any relation in a group can be drawn as a polygon, and to treat meaningfully higher syzygies, i.e., "relations among relations" we can understand them as 3-dimensional polytopes with boundary composed of such polygons, and so on for higher syzygies (see §1). Now, the relation (1) is represented in this language by a pentagon. It turns out that this pentagon should be seen as K_4, the simplest instance of a Stasheff polytope! More precisely, our first result is that there is a syzygy among the Steinberg relations which has the shape of K_5, the 3-dimensional Stasheff polytope and in fact, this pattern continues to hold in all dimensions. Note that the study of higher syzygies among row operations is, at least, ideally, the aim of algebraic K-theory. However, the existing approaches to higher K-theory follow this ideal only vaguely, with elementary treatment available only for $K_i, i \leq 2$.

(0.3) The second situation in which Stasheff polytopes make an unexpected appearance, is Morse theory. It is well known that a Morse function f on a manifold X which, in addition, satisfies the Smale transversality condition, gives rise to a CW-decomposition of X.

Given a Morse but not Morse-Smale function f, we can form its "bifurcation diagram" whose vertices are topologically different ways to make f Smale by small deformation (formally, they are some regions in the function space, see §4), edges correspond to "smalefications" lying in adjacent regions and so on. Note that the Smale condition prohibits, in particular, a gradient trajectory joining two critical points of the same index. Let us consider a Morse function which has gradient trajectories joining a string of critical points of the same index: $x_1 \to x_2 \to ... \to x_n$ and generic (Smale) apart from this. Our second observation is that the bifurcation diagram for such f is the Stasheff polytope K_{n+1}!

(0.4) The second order syzygies among the Steinberg relations were studied by K. Igusa in terms of "pictures", see [Ig2], [W]. In fact, the Stasheff polytope K_5 can be recognized as the subdivision of the 2-sphere dual to one of Igusa's pictures, but this fact, as well as the pattern of appearance of all higher Stasheff polytopes, has not been noticed before. In §2 we describe a conjectural picture of how all syzygies of all orders should ideally look like: they should be parametrized by some labelled graphs which we call "hierogliphs". We also present in §3 a general construction which to any hierogliph Γ associated a CW-complex $P(\Gamma)$ covered by some subcomplexes called "faces" and in many cases, including the one with Stasheff polytopes, $\mathcal{P}(\Gamma)$ is the desired polytope with faces having the usual meaning. So in these cases we can produce a syzygy directly from the hierogliph. This construction is similar to the prime spectrum of a ring in that vertices of $\mathcal{P}(\Gamma)$ are prime ideals in the category formed by paths in Γ. The faces are also obtained in "algebro-geometric" terms, remindful of the Zariski topology. So in fact, we get a third, purely combinatorial, description of the Stasheff polytope: as the "spectrum" of a certain category.

(0.5) The relation between (0.2) and (0.3), i.e., the relation between algebraic K-theory and Morse theory, is well known in topology: when a 1-parameter family of Morse function hits an elementary catastrophe consisting of two critical points of the same index i being joined by a gradient trajectory, the corresponding CW decomposition changes by a transformation known as "handle sliding"; in the cases when one can identify cellular i-chains with ith homology we get two bases in the homology differing by an elementary matrix. This is at the basis of the approach to algebraic K-theory developed by Cerf [C], Hatcher and Wagoner [HW] and many subsequent works. In particular, Hatcher and Wagoner observed that for a function with two consecutive trajectories joining three critical points of the same index, the bifurcation diagram is the Steinberg pentagon. However, little explicit information seems to be known about bifurcation diagrams for higher-codimensional catastrophes, and the appearance of the Stasheff polytopes in this problem has not been noticed before as well.

(0.6) One explanation of the appearance of the Stasheff polytopes K_n in (0.3) and thus in (0.2), can be given by comparison with Teichmüller theory which studies manifolds of dimension 2. There, the role of the K_n has been known for some time. For instance, in the cell decomposition of the moduli space of pointed curves given by R. Penner [Pe1-2], the links of some cells are Stasheff polytopes. Even more transparent is the relation of K_n with the moduli space of stable pointed curves of genus 0, see, e.g., [K]. The combinatorial formalism of nested and mounting caps of §3 is very similar to that of "secondary structures of the RNA" used by R. Penner and C. Waterman [PW] and motivated by Teichmüller theory. While preparing this paper, we received a preprint [B] of Y. Baryshnikov, who studied the bifurcation diagram for a singular point of a quadratic differential on a Riemann surface and found it to be the product of two Stasheff polytopes. Since the global topological behavior of a quadratic differential involves the interaction of two 1-dimensional foliations, this seems to match very well our statement about Morse functions.

(0.7) We are grateful to Y. Baryshnikov for pointing out the reference [PW] and for sending his paper [B]. The first author would like to acknowledge financial support from NSF grants and A.P. Sloan Research Fellowship as well as from the Max-Planck Institute für Mathematik in Bonn which provided excellent conditions for working on this paper. The second author is supported, in part, by the University of South Florida Research and Creative Scholarship Grant Program under Grant Number 1249932R0.

§1. Syzygies among row operations.

(1.1) Higher syzygies in noncommutative groups. Let G be a group (possibly non-commutative) given by generators $x_i, i \in I$ and relations $r_j = 1, j \in J$ where r_j are some formal expressions in the x_i. We would like to study syzygies (i.e., relations, or dependencies) among the relations r_j. One way of approaching this is as follows.

It is well known that the data $\{x_i, r_j\}$ give rise to a 2-dimensional CW - complex $Z_2 = Z_2(\{x_i, r_j\})$ with the fundamental group $\pi_1(Z_2)$ being G. Explicitly, Z_2 has a unique 0-cell; its 1-cells (loops) X_i are in bijection with the generators x_i and 2-cells R_j are in bijection with the relations r_j. If $r_j = x_{i_1}^{\epsilon_1} ... x_{i_m}^{\epsilon_m}$, $\epsilon_\nu = \pm 1$, then the cell R_j has the shape of an m - gon and its ν-th edge is identified with the 1-cell X_{i_ν} (in the direction specified by ϵ_ν).

(1.1.1) Convention. We depict generators by arrows; the product of generators corresponds to the "composition" of arrows as if they were morphisms in a category. Thus the product of generators x and y is represented by the composite arrow $\bullet \xrightarrow{y} \bullet \xrightarrow{x} \bullet$, and not by $\bullet \xrightarrow{x} \bullet \xrightarrow{y} \bullet$. This means that we choose the group operation in the fundamental group to be such that the product of paths γ and δ is the path going first along δ and then along γ.

In general, the space Z_2 is far from being a model for the classifying space BG, i.e., we have $\pi_i(Z_2) \neq 0$ for $i > 1$. For example, an element of $\pi_2(Z_2)$ can be represented by a polyhedral 2-sphere S which is composed from the 2-cells R_j so that the 1-cells at the boundaries of the R_j match. In this paper we adopt the point of view that such spheres should be regarded as non-commutative syzygies among the R_j. This point of view goes back at least to J.H.C. Whitehead, as it was pointed out by Igusa [Ig2] who called polyhedral 2-spheres in Z_2 "geometric pictures". Let us develop this point of view systematically.

Call a system of such syzygies $S_k, k \in K$, complete if after filling the 2-spheres S_k by 3-balls B_k attached to Z_2, we get a CW - complex Z_3 with $\pi_2(Z_3) = 0$. Similarly, by a (second order) syzygy among the S_i we will mean a polyhedral 3-sphere in Z_3 composed out of the B_k and so on.

Thus *the analog of a full chain of syzygies (free resolution) for a non-commutative group G presented by generators x_i and relations r_j is an explicit CW - model Z of the space BG whose 2-skeleton is the complex $Z_2(\{x_i, r_j\})$ described above.* Such a model gives, in particular, an explicit chain complex for calculating the homology of G.

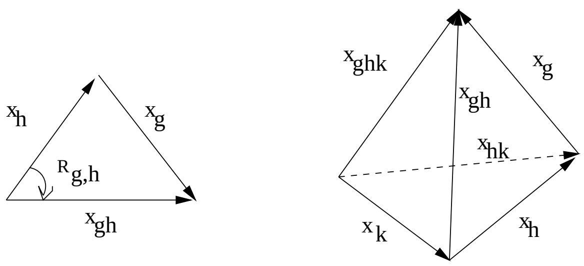

Figure 1: The triangular relation and the tetrahedral syzygy of groups

(1.2) Examples. This very naive point of view on higher non-Abelian syzygies is not so absurd at it might seem to anyone familiar with complicated behavior of higher homotopy groups. Namely, for groups with "good" systems of generators and relations encountered in practice, it is often possible to construct equally good systems of higher syzygies. We start by reviewing three examples.

(1.2.1) Bar-construction: the simplices. Every group G has a "stupid" presentation in which there is one generator x_g for every element $g \in G$ and the relations are given by the multiplication table: $x_g x_h = x_{gh}$. So we have one relation $r_{g,h}$ for each pair of elements (g, h). Each such relation is depicted as a triangle $R_{g,h}$, while the generator x_g is depicted by a segment X_g. For any triple (g, h, k) of elements of G the triangles $R_{g,h}, R_{gh,k}, R_{g,hk}, R_{h,k}$ fit together to form a tetrahedron $S_{g,h,k}$ which is, therefore, a syzygy (Fig.1). Continuing in this way, we get the standard simplicial model for BG (whose n-simplices correspond to n-tuples of elements of G). It is the geometric version of the bar-construction. Of course, this is not the most economical model.

(1.2.2) Koszul complex: the cubes. As another trivial example, we consider the commutative group $\mathbf{Z}^3$ given by generators x, y, z and relations $xy = yx, xz = zx, yz = zy$. The corresponding space Z has three 1-cells corresponding to x, y, z and three 2-cells which have the shape of squares of commutativity. A syzygy among the relations is provided by the boundary of the cube in Fig.2. After attaching to Z this cube we get a CW - decomposition of the torus $T^3 = B(\mathbf{Z}^3)$, so the system of syzygies is complete and there are no further syzigies. This cube is the geometric version of the Koszul complex. One can treat the group $\mathbf{Z}^n$ is the same way.

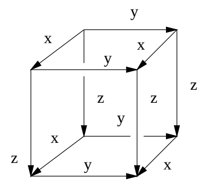

Figure 2: The cubic syzygy for $\mathbf{Z}^3$.

(1.2.3) The braid group: the permutohedra. Consider the braid group Br_n on n strings. It has the presentation by generators $s_1, ..., s_{n-1}$ and the following relations

$$s_i s_j = s_j s_i, |i - j| \geq 2, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, i < n - 1. \tag{1.2.4}$$

Geometrically, they are depicted as squares and hexagons. More generally, one has a series of polytopes known as *permutohedra*: the $(n-1)$-dimensional permutohedron P_n is the convex hull of a generic orbit of the symmetric group S_n in its natural action in $\mathbf{R}^n$, see, e.g., [K]. Thus P_2 is the hexagon. It follows from the results of Deligne [Del] (see also Salvetti [Sa]) that one can continue the above pattern constructing a full chain of syzygies for Br_n consisting permutohedra and their products. The idea of viewing faces of P_n as "higher syzygies" in Br_n is prominent in the paper [MS] by Manin and Schechtman which influenced our outlook considerably.

(1.3) The Steinberg group. Now we consider the main example of interest in the present paper. Let A be a ring. For $a \in A$ and a pair of indices $i, j \leq n$, $i \neq j$ we denote by $e_{ij}(a)$ we the n by n matrix over A with 1's on the diagonal, a at the place (i, j) and zeroes elsewhere. Thus left multiplication by $e_{ij}(a)$ amounts to a row transformation of a matrix. One would like to know all the relations, syzygies etc. among the transformations $e_{ij}(a)$.

The Steinberg group $St(A)$ is obtained by considering those relations among the $e_{ij}(a)$ which can be written off-hand for any A. More precisely (see [M]), $St(A)$ is given by generators $x_{ij}(a)$, $a \in A$, $i, j = 1, 2, 3, ..., i \neq j$ which are subject to the following relations:

(1.3.1) $x_{ij}(a)x_{ij}(b) = x_{ij}(a + b)$;

(1.3.2) $x_{ij}(a)$ commutes with $x_{kl}(b)$, if $i \neq l$ and $j \neq k$;

(1.3.3): $x_{ij}(a)x_{jk}(b) = x_{jk}(b)x_{ik}(ab)x_{ij}(a)$.

In virtue of (1.3.2), the relations (1.3.3) can be written in the more standard form

(1.3.3') $[x_{ij}(a), x_{jk}(b)] = x_{ik}(ab)$, if $i \neq k$.

In this paper we will consider the presentation of $St(A)$ given by the generators $x_{ij}(a)$ and the relations (1.3.1-3).

We would like to find the syzygies among these relations. According to n. 1.1, we represent the relations geometrically, as triangles $T_{ij}(a, b)$, squares $S_{ij,kl}(a, b)$ and pentagons $P_{ijk}(a, b)$ (Fig. 3) whose edges are labelled by the generators, so that the edge corresponding to $x_{ij}(a)$ is denoted by $X_{ij}(a)$. It is convenient to use the following graphical notation for the generators and relations. Namely, we encode the Steinberg generator $x_{ij}(a)$ by an arrow $i \xrightarrow{a} j$ pointing from i to j and carrying the element a. We think of the Steinberg relations as describing the "interaction" of these arrows and encode each relation by a graph composed of arrows, each arrow carrying one or more ring element. We call this graph the hierogliph of the relation, cf. [KV]. On Fig.3 we depict geometrically the Steinberg relations and the corresponding hierogliphs. For instance, the commutativity of $x_{ij}(a)$ and $x_{kl}(b)$ takes place each time when the arrow $i \to j$ (carrying a) and the arrow $k \to l$ (carrying b) do not interact in the sense that they cannot be composed one way or another. This allows for three possible shapes of the hierogliph.

Now, to find a syzygy among the Steinberg relations, we should construct a polytope whose boundary is composed of the above triangles, squares and pentagons in such way that the two labellings of any edge (which is common to two faces) coincide. We will use the hierogliphical notation to encode the syzygies as well.

We start with some obvious syzygies. First, for any $i \neq j$ and any 3 elements $a, b, c \in A$ the triangles $T_{ij}(a, b), T_{ij}(a, b + c), T_{ij}(a + b, c), T_{ij}(b, c)$ fit together to form the boundary of a tetrahedron as in Fig.1. To this tetrahedron we associate the hierogliph $i \xrightarrow{a,b,c} j$.

Second, if we have three pairs $(i, j), (k, l), (m, n)$ such that the generators $x_{ij}(a), x_{kl}(b)$, $x_{mn}(c)$ commute with each other in virtue of the Steinberg relation (1.3.2), then we have a cube as in Fig.2. To such a cube we associate the hierogliph which is the union of the three arrows $i \xrightarrow{a} j$, $k \xrightarrow{b} l$ and $m \xrightarrow{c} n$. The actual shape of the graph may vary but each graph obtained in this way has no pair of composable arrows.

A couple of less trivial syzygies and the corresponding hierogliphs is given in Figs. 4, 5. The polytope in Fig. 4 is the celebrated *Stasheff polytope* (or associahedron) [St] whose

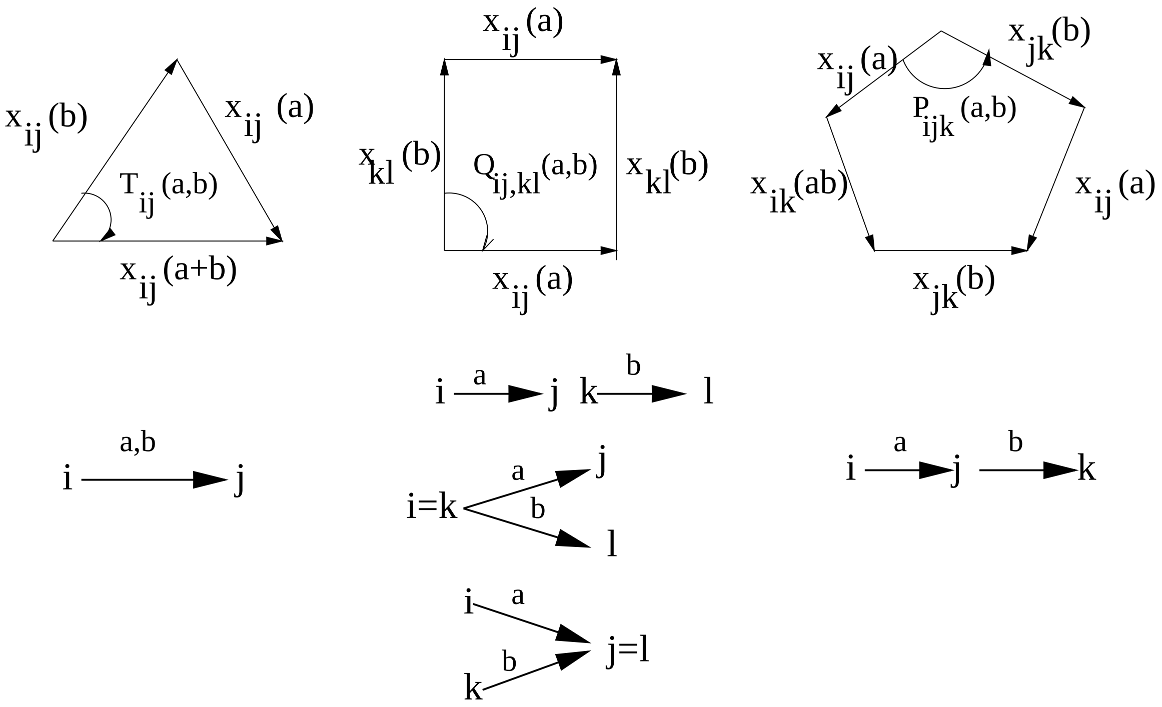

Figure 3: Relations of the Steinberg group

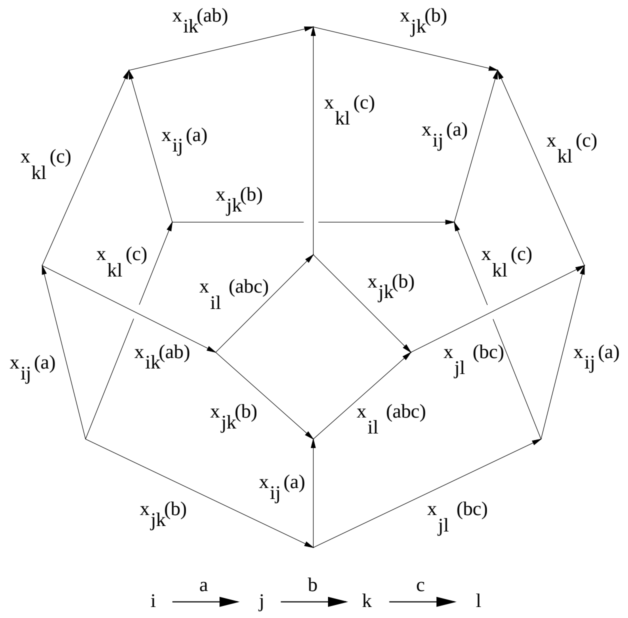

Figure 4: The Stasheff polytope.

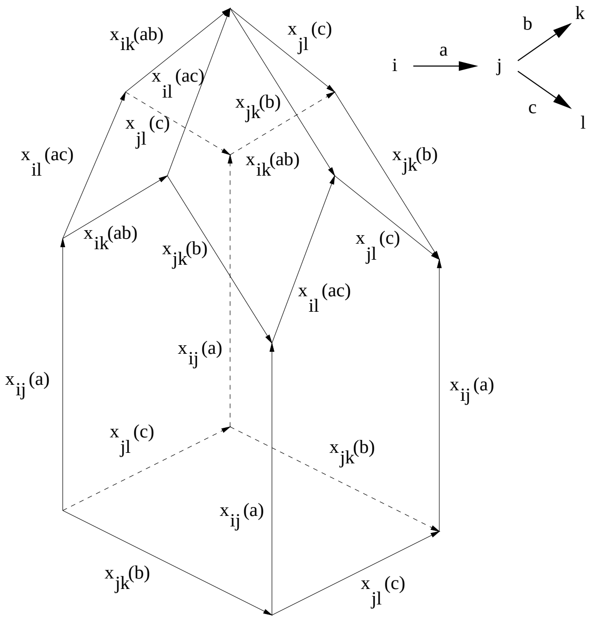

Figure 5: The Chicago building.

vertices correspond to complete parenthesizings of a product of five terms $a_1a_2a_3a_4a_5$. Note that, unlike the appearance of the permutohedron as a syzygy for the braid group, there is no *a priori* reason for the Stasheff polytope to appear in this context: we do not consider any kind of "homotopy associativity" problem, as Stasheff did, but rather a more naive problem of syzygies for row transformations of matrices in linear algebra.

The "Chicago building" on Fig.5 corresponds to the hierogliph in the shape of a fork. There is also another, "dual", building corresponding to the fork with the opposite orientation: we leave its construction to the reader.

There remain a few more patterns for interaction of the arrows $i \xrightarrow{a} j$. One, depicted on Fig.6, corresponds to the graph in the shape of a triangle and is a pentagonal prism (with one square face, the left one in the front row, being actually the union of two triangles expressing the identities $x_{ik}(ab)x_{ik}(c) = x_{ik}(ab+c)$ and $x_{ik}(c)x_{ik}(ab) = x_{ik}(ab+c)$).

The other, which we call the bowtie polytope, corresponds to the hierogliph $i \xrightarrow{a,a'} j \xrightarrow{b} k$ and is depicted on Fig.7. There is also the dual bowtie polytope corresponding to the hierogliph $i \xrightarrow{a} j \xrightarrow{b,b'} k$, which we leave to the reader.

Finally, we have hierogliphs consisting of two non-interacting arrows of which one carries one element of A and the other carries two elements, e.g.,

$$i \xrightarrow{a,a'} j \qquad k \xrightarrow{b} l.$$

To such an hierogliph we associate the triangular prism which is the product of the triangle $T_{ij}(a, a')$ and the interval corresponding to $x_{kl}(b)$.

The geometric syzygies among the Steinberg relations described above can all be recognized (in a different, but equivalent form) in the work of Igusa (see [W] for a published

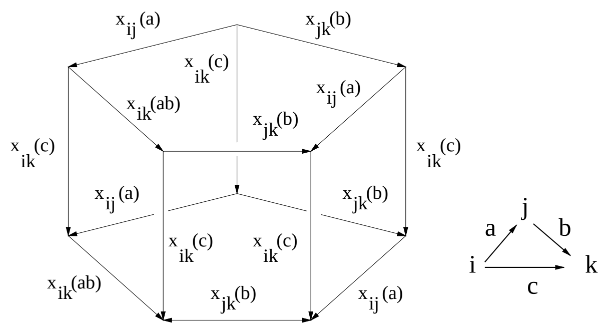

Figure 6: The pentagonal prism.

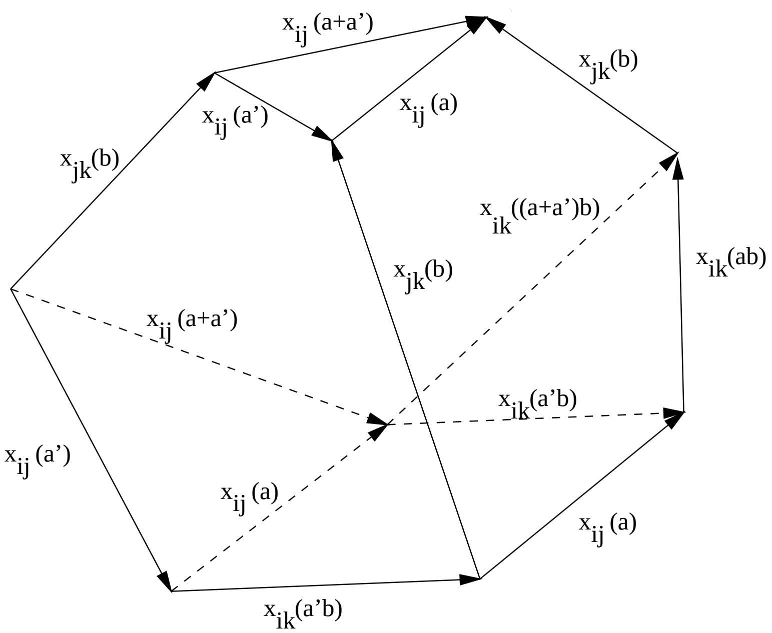

Figure 7: The syzygy corresponding to $i \xrightarrow{a,a'} j \xrightarrow{b} k$: the bowtie polytope.

account). However, the appearance of the Stasheff polytopes in this problem and the ensuing pattern for still higher syzygies was not noticed before. Before analyzing this pattern in general, we spend the rest of this section discussing the 3-dimensional syzygies in more detail.

(1.4) Summary. Properties of the polytopes. Let us summarize what has been done so far and introduce some notation. We have associated generators, relations and syzygies of the Steinberg group (or, in geometric language, polyhedral 1-, 2- or 3-cells) to certain hierogliphs. Formally, a hierogliph Γ is just an oriented graph without oriented loops and at most 3 edges together with labelling of vertices by distinct natural numbers and assignment of an ordered sequence of elements of A to each edge. The weight $\mathrm{wt}(\Gamma)$ of a hierogliph Γ is the total number of ring elements on edges. We consider here only hierogliphs with weight ≤ 3. Thus, to each such hierogliph Γ we have associated a polyhedral ball $P(\Gamma)$ of dimension $\mathrm{wt}(\Gamma)$ in such a way that every face of $P(\Gamma)$ is identified with $P(\Gamma')$ for some other hierogliph Γ'. In particular, the edges of $P(\Gamma)$ are canonically oriented and labelled by the Steinberg generators; if we want to refer to the edge labelled by $x_{ij}(a)$ as to a topological space, we will denote it $X_{ij}(a)$. Let us note the following properties of the $P(\Gamma)$ which all are obvious from the pictures.

(1.4.1) Proposition. *(a) The orientations of edges make the 1-skeleton of $P(\Gamma)$ into an oriented graph without oriented loops which possesses a unique minimal vertex (source) α_Γ) and a unique maximal vertex (sink) ω_Γ.*
(b) There are exactly $\mathrm{wt}(\Gamma)$ edges of $P(\Gamma)$ originating at α_Γ and the Steinberg generators on these edges are precisely $x_{ij}(a_\nu + \ldots + a_r)$, where $i \overset{(a_1,\ldots,a_r)}{\longrightarrow} j$ is an arbitrary arrow of Γ and $1 \leq \nu \leq n$.
(c) Similarly, there are exactly $\mathrm{wt}(\Gamma)$ edges of $P(\Gamma)$ terminating at ω_Γ and their Steinberg generators are $x_{ij}(a_1 + \ldots + a_\nu)$, where $i \overset{(a_1,\ldots,a_r)}{\longrightarrow} j$ and ν are as before.

Call a sequence $l_1, \ldots, l_m$ of distinct edges of Γ *regular*, if, for any ν, the beginnig of l_ν does not coincide with the end of any of the $l_1, \ldots, l_{\nu-1}$. Clearly, any hierogliph Γ admits at least one maximal regular sequence in which each edge enters exactly once.

(1.4.2) Proposition. *The minimal length of an edge path on $P(\Gamma)$ from α_Γ to ω_Γ is equal to the number of edges in Γ. All paths of this minimal length are monotone (i.e., follow the directions on the edges) and are in bijection with maximal regular sequences of arrows of Γ. If $l_1, \ldots, l_m$ is such a sequence and $l_\nu = \{i_\nu \overset{(a_1^\nu,\ldots,a_n^\nu)}{\longrightarrow} j_\nu\}$, then the corresponding edge path consists of $X_{i_\nu,j_\nu}(a_1^\nu + \ldots + a_n^\nu)$, $\nu = 1, \ldots, m$.*

(1.5) Completeness of the syzygies. Since any face of any of the $P(\Gamma)$ is identified with some $P(\Gamma')$, this allows us to glue all the $P(\Gamma)$ ($\mathrm{wt}(\Gamma) \leq 3$) together, getting a 3-dimensional CW-complex $\mathcal{B}_{\leq 3}$.

The next question is how complete this system of syzygies is. In other words, what is $\pi_2(\mathcal{B}_{\leq 3})$?

We should keep in mind that the Steinberg relations themselves do not form a complete system of relations among the elementary matrices $e_{ij}(a)$: the natural homomorphism $St(A) \to GL(A)$ has, in general, nontrivial kernel, which is the Milnor group $K_2(A)$. So we have no reason to expect that our system of syzygies is complete. Instead, we have the following fact.

(1.5.1) Theorem. *The group $\pi_2(\mathcal{B}_{\le 3})$ is isomorphic to $K_3(A)$.*

This is in fact a reformulation of the result of Igusa on description of $K_3(A)$ in terms of his "pictures" (see [W], [Ig 2]). Igusa's result is that $K_3(A)$ is isomorphic to $P(A)/H(A)$ where $P(A)$ is the group of all pictures and $H(A)$ is the subgroup generated by the special pictures. In fact, there is a one-to-one correspondence between Igusa's special pictures and our 3-dimensional syzygies: given one of our polytopes, one can take the CW-decomposition of S^2 dual to that given by the faces of the polytope. This will produce a special picture. More precisely, Igusa's special pictures are in addition labelled by an element of the group $St(A)$. So they correspond to 3-cells in $\tilde{\mathcal{B}}_{\le 3}$, the universal covering of $\mathcal{B}_{\le 3}$. In our language, $P(A)$ is the group of 2-cycles in $\tilde{\mathcal{B}}_{\le 3}$ while $H(A)$ is the group of boundaries of 3-cells, so $K_3(A) = H_2(\tilde{\mathcal{B}}_{\le 3}, \mathbf{Z})$. Now noticing that $H_2(\tilde{\mathcal{B}}_{\le 3}, \mathbf{Z}) = \pi_2(\tilde{\mathcal{B}}_{\le 3})$ by Hurewitz theorem and that, as in the case of any covering, $\pi_2(\tilde{\mathcal{B}}_{\le 3}) = \pi_2(\mathcal{B}_{\le 3})$, we find that $K_3(A) = \pi_2(\mathcal{B}_{\le 3})$ as claimed.

(1.6) Monotone hierogliphs and the triangular group. Let $T_n(A)$ be the group of upper triangular n by n matrices over A with unities on the diagonal. Thus the elementary matrix $e_{ij}(a)$ lies in $T_n(A)$ iff $1 \le i < j \le n$. It is known [M] that the system of Steinberg relations among these particular $e_{ij}(a)$ is complete. In other words, $T_n(A)$ is isomorphic to an abstract group generated by symbols $x_{ij}(a)$, $1 \le i < j \le n$, $a \in A$ which are subject only to the relations (1.3.1-3) involving these symbols.

More generally, call a hierogliph Γ monotone if each time that there is an arrow $i \to j$ in Γ, we have $i < j$. Let $\mathcal{H}_n$ be the set of all monotone hierogliphs (of weight ≤ 3) whose labels on vertices belong to $\{1, ..., n\}$. Then any face of $P(\Gamma), \Gamma \in \mathcal{H}_n$, has the form $P(\Gamma')$ with $\Gamma' \in \mathcal{H}_n$. Thus the subset $\mathcal{B}\mathcal{T}_n^{\le 3} \subset \mathcal{B}_{\le 3}$, defined as the union of $P(\Gamma), \Gamma \in \mathcal{H}_n$, is a CW-subcomplex.

(1.6.1) Theorem. *The space $\mathcal{B}\mathcal{T}_n^{\le 3}$ has $\pi_1 = T_n(A), \pi_2 = 0$, i.e., the above system of syzygies among the uppertriangular Steinberg generators is complete.*

This statement in fact implies Theorem 1.5.1 (or, equivalently, Igusa's result) if one makes use of Volodin's description of K-theory (§2).

Proof: If G is a group acting on a topological space M, we denote by $M//G$ the homotopy quotient of M by G, i.e., $M//G = (EG \times M)/G$, where EG is a contractible space with free G-action. Thus there is a Serre fibration

$$G \to M \to M//G.$$

Suppose that a complete system of syzygies for G is given, starting with generators $(x_i)_{i\in I}$ and relations $(r_j)_{j\in J}$. Let $Z \simeq BG$ be the corresponding CW-complex with 1-cells X_i, 2-cells R_j etc. We think of these cells as being closed topological balls mapping into Z. Then we can construct an explicit model for $M//G$ by using $\tilde{Z}$, the universal cover of Z, as the model for EG, i.e., define $M//G = (\tilde{Z} \times M)/G$. Explicitly, for each $i \in I$ we glue $M \times X_i \simeq M \times [0,1]$ to X by identifying $(m, 0)$ with m and $(m, 1)$ with $x_i m$. Then for each $j \in J$ we attach $M \times R_j$ to the previous space by identifying each $M \times e$, where e is an edge of the polygon

R_j labelled by x_i, with $M \times X_i$, and so on. In particular, if M is a CW-complex and the action of G is by cellular homeomorphisms, then we get an explicit CW model for $M//G$. Its cells are products of cells of M and those of Z with attachment maps given by the G-action.

Consider now the group $T_n(A)$. It acts on the Abelian group A^n in a standard way (matrices act on vectors) and thus we get an action on the classifying space $B(A^n)$.

(1.6.2) Lemma. *The homotopy quotient $B(A^n)//T_n(A)$ is homotopy equivalent to $BT_{n+1}(A)$.*

Proof: We have the exact sequence of groups with Abelian kernel

$$1 \to A^n \xrightarrow{\alpha} T_{n+1}(A) \xrightarrow{\beta} T_n(A) \to 1, \tag{1.6.3}$$

where β is given by forgetting the last row and column of a triangular matrix and α takes $(a_1, ..., a_n) \in A^n$ into the matrix $e_{1,n+1}(a_1)...e_{n,n+1}(a_n)$. The conjugation action of the right term $T_n(A)$ on the left term A^n is just the standard matrix action. This exact sequence gives a Serre fibration

$$T_n(A) \to B(A^n) \to BT_{n+1}(A),$$

which implies our statement.

We will use the standard simplicial model for $B(A^n)$. Thus, according to the convention of (1.2.1), its 1-simplices are denoted by $X_{\mathbf{a}}, \mathbf{a} \in A^n$, its 2-simplices are denoted $R_{(\mathbf{a},\mathbf{b})}, \mathbf{a}, \mathbf{b} \in A^n$ and so on. It will also be convenient for us to view A^n as a subset in $T_{n+1}(A)$, via (1.6.3) and write, for instance, $X_{e_{2,n+1}(a)}$ instead of $X_{(0,a,0,...,0)}$. The action on $T_n(A)$ on $B(A^n)$ is cellular.

Let $(BA)^n$ be the n-fold Cartesian product of the simplicial complex $B(A)$. As a CW-complex, it is glued of polysimplices (i.e., of products of simplices). Of course, $(BA)^n$ is homeomorphic to $B(A^n)$, the latter being obtained by triangulating each polysimplex in a canonical way, but we want to distinguish $B(A^n)$ and $(BA)^n$ as CW-complexes. The action of $T(n, A)$ on $(BA)^n$ is not cellular.

We will now prove Theorem 1.6.1 by induction in n, the case $n = 1$ being trivial. Suppose n is such that the statement of the theorem is true for $\mathcal{BT}_n^{\leq 3}$. In other words, the map $\mathcal{BT}_n^{\leq 3} \to BT_n(A)$ coming from the identification $\pi_1(\mathcal{BT}_n^{\leq 3}) = T_n(A)$, is 2-connected i.e., it induces isomorphisms on $\pi_i, i \leq 2$. Then we can use $\widetilde{\mathcal{BT}_n^{\leq 3}}$, the universal covering of $\mathcal{BT}_n^{\leq 3}$, as an approximation for $ET_n(A)$ in dimensions ≤ 2 and construct the space

$$Q = \left(\widetilde{\mathcal{BT}_n^{\leq 3}} \times B(A^n)\right) \Big/ T_n(A).$$

It is inductively constructed in the way similar to what was described above: we first attach, for any $1 \leq i < j \leq n$ and $a \in A$ a copy of $B(A^n) \times X_{ij}(a) \simeq B(A^n) \times [0, 1]$ to $B(A^n)$ by using the action of $x_{ij}(a)$, then attach products of $B(A^n)$ with any of the 2-cells depicted in Fig.3, and then doing the same with the 3-cells we described in the following figures. In other words, Q is the union

$$Q \quad = \quad \bigcup_{\Gamma \in \mathcal{H}_n} P(\Gamma) \times B(A^n), \tag{1.6.4}$$

with the attaching maps coming from the group action.

By our assumptions and by Lemma 1.6.2, there is a 2-connected map $Q \to B(A^n)//T_n(A) \simeq BT_{n+1}(A)$ and the same of course remains true if we replace Q by its 3-skeleton $Y = \mathrm{sk}_3(Q)$. Let W denote for short the space $\mathcal{BT}_{n+1}^{\leq 3}$. To make the inductive step in the proof of Theorem 1.5.1, it is enough therefore to prove that Y is homotopy equivalent to W.

Let $\mathcal{H}_{n+1}^0 \subset \mathcal{H}_{n+1}$ be the subset of hierogliphs Γ such that any arrow in Γ has the label $(n+1)$ on its endpoint. Note that the 3-skeleton of $(BA)^n$ is realized inside $\mathcal{BT}_{n+1}^{\leq 3}$ as a subcomplex formed by cells $P(\Gamma)$ with $\Gamma \in \mathcal{H}_{n+1}^0$ (these cells are polysimplices). We now want to compare the way $W = \mathcal{BT}_{n+1}^{\leq 3}$ is obtained from the subcomplex $\mathrm{sk}_3((BA)^n)$ by attaching all the other cells $P(\Gamma)$ with the way Y is obtained from $\mathrm{sk}_3(B(A^n))$ by attaching cells, as described in (1.6.4).

As we said, the cells (polysimplices) $P(\Gamma_1), \Gamma_1 \in \mathcal{H}_{n+1}^0$, can be viewed as unions of simplices in $B(A^n)$. Let us call *rough cells* of Y the images of products of these polysimplices with $P(\Gamma_2), \Gamma_2 \in \mathcal{H}_n$ with respect to the map from the decomposition (1.6.4). Of course, the closure of a rough cell may not lie in the union of rough cells of smaller dimension, because the action of $T(n, A)$ on $B(A^{n+1})$ does not preserve the polysimplices. Denote by $Y_a, a \leq 3$, the union of $B(A^n)$ and all the rough cells of dimension $\leq i$.

For any hierogliph $\Gamma \in \mathcal{H}_{n+1}$ we denote by Γ' the sub-hierogliph in Γ formed by all arrows not touching the $(n+1)$th vertex, so $\Gamma' \in \mathcal{H}_n$ and by Γ'' the sub-hierogliph formed by all arrows terminating at the $(n+1)$th vertex, so $\Gamma'' \in \mathcal{H}_{n+1}^0$. Let $W_a \subset W$, $a \leq 3$, be the union of $P(\Gamma)$ with $\mathrm{wt}(\Gamma') \leq a$. Note that there is a bijection between cells $P(\Gamma)$ in W and rough cells in Y of the same dimension, namely to $P(\Gamma)$ we just associate the product $P(\Gamma') \times P(\Gamma'')$.

We now note that there are two distinguished faces of $P(\Gamma)$ isomorphic to $P(\Gamma')$, and same for $P(\Gamma'')$. More precisely, we have the following lemma, in which we use the notations of (1.4).

(1.6.5) Lemma-Definition. *Let $\Gamma \in \mathcal{H}_{n+1}$. Then:*

(a) The edges of $P(\Gamma)$ originating at α_Γ and labelled by $x_{ij}(a_\nu + ... + a_r)$, where $i \xrightarrow{(a_1,...,a_r)} j$ is an arrow of Γ', lie on a unique face of $P(\Gamma)$ isomorphic (together with the labelling of edges) to $P(\Gamma')$ and denoted $P(\Gamma')_\alpha$.

(b) The edges of $P(\Gamma)$ terminating at ω_Γ and labelled by $x_{ij}(a_1 + ... + a_\nu)$, where $i \xrightarrow{(a_1,...,a_r)} j$ is an arrow of Γ', lie on a unique face of $P(\Gamma)$ isomorphic (together with the labelling of edges) to $P(\Gamma')$ and denoted $P(\Gamma')_\omega$.

(c) Similar statements as (a), (b) but with $P(\Gamma'')$ instead of $P(\Gamma')$. The corresponding faces are denoted by $P(\Gamma'')_\alpha$, $P(\Gamma'')_\omega$.

(d) The sink of $P(\Gamma'')_\alpha$ coincides with the source of $P(\Gamma')_\omega$.

The lemma is established by direct inspection of pictures. Note that the sink of $P(\Gamma')_\alpha$ differs, in general, from the source of $P(\Gamma'')_\omega$.

We now use this lemma to define a kind of product structure on $P(\Gamma)$. For any edge e of any hierogliph $\Delta \in \mathcal{H}_n$, let $g_e \in St_{n+1}(A)$ be the corresponding Steinberg generator. If $v \in P(\Delta)$ is a vertex, then let $g_v \in St_{n+1}(A)$ be the product of the g_e for any monotone edge path joining α_Δ and v.

(1.6.6) Lemma. *There exist (non-cellular) homeomorphisms*

$$\varphi_\Gamma : P(\Gamma') \times P(\Gamma'') \to P(\Gamma), \quad \Gamma \in \mathcal{H}_{n+1}$$

with the properties:

(*a*) $$\varphi_\Gamma(\mathrm{sk}_i(P(\Gamma') \times P(\Gamma''))) = \mathrm{sk}_i(P(\Gamma)).$$

(*b*)
$$\varphi_\Gamma(P(\Gamma')\times\{\alpha_{\Gamma''}\}) = P(\Gamma')_\alpha, \quad \varphi_\Gamma(P(\Gamma')\times\{\omega_{\Gamma''}\}) = P(\Gamma'')_\omega, \quad \varphi_\Gamma(\{\alpha_{\Gamma'}\}\times P(\Gamma'')) = P(\Gamma'')_\alpha.$$

(c) If v is a vertex of $P(\Gamma')$ and e is an edge of $P(\Gamma'')$, then $\varphi_\Gamma(\{v\} \times e)$ is a monotone edge path in $P(\Gamma)$ whose composition is $g_v g_e g_v^{-1}$.

Note that g_e for e an edge of $P(\Gamma'')$, always lies in the subgroup $A^n \subset T_{n+1}(A)$, see (1.6.3), so the effect of conjugation is just the action the matrix corresponding to g_v on the vector represented by g_e.

Proof: If Γ' or Γ'' is empty, we set $\varphi_\Gamma = \mathrm{Id}$. If Γ', Γ'' are nonempty, then, since $\mathrm{wt}(\Gamma) \leq 3$, either $P(\Gamma')$ or $P(\Gamma'')$ is a segment.

Consider first the case when $P(\Gamma'')$ is a segment, say $P(\Gamma'') = X_{j,n+1}(a)$. For a vertex $v \in P(\Gamma')$ let v_α (resp. v_ω) be the corresponding vertex of $P(\gamma')_\alpha$ (resp. $P(\Gamma')_\omega$). Note that there is a unique minimal monotone edge path γ_v joining v_α and v_ω. For $v = \alpha_{\Gamma'}$ the path γ_v is just the segment $P(\Gamma')$, in other cases this path can be composite. Note also that the element of $St_{n+1}(A)$ represented by γ_v is precisely $g_v x_{j,n+1}(a) g_v^{-1}$. So we define φ_Γ on each $\{v\} \times P(\Gamma'')$ to identify $P(\Gamma'')$ with γ_v, and then extend φ_Γ on the whole $P(\gamma') \times P(\Gamma'')$ in an arbitrary way so as to satisfy the conditions of the lemma.

If $P(\Gamma'')$ is not a segment and Γ', Γ'' are not empty, then the only possibility is that $\Gamma = \{p \xrightarrow{a} q \xrightarrow{b,b'} (n+1)\}$ and so $P(\Gamma)$ is the dual bowtie polytope similar to Fig.7. In this case $P(\Gamma'')$ is a triangle formed by $X_{q,n+1}(b)$, $X_{q,n+1}(b')$, $X_{q,n+1}(b+b')$, while $P(\Gamma')$ is the segment $X_{pq}(a)$. By making a picture of $P(\Gamma)$, it is immediate to find there the two edge paths

$$X_{q,n+1}(b)X_{p,n+1}(ab)X_{q,n+1}(b')X_{p,n+1}(ab'), \quad X_{q,n+1}(b+b')X_{p,n+1}(ab+ab')$$

both originating at the sink of $P(\Gamma')$ and terminating at ω_Γ. We define φ_Γ on $\omega_{\Gamma'} \times P(\Gamma'')$ to take the sides of this triangle into the paths $X_{q,n+1}(b)X_{p,n+1}(ab)$ etc. (each of length 2), and extend φ_γ from there to satisfy the conditions of the lemma which is thus proved.

Returning to the proof of Theorem 1.6.1, let us now compare W with Y. It is clear that $W_0 = Y_0 = \mathrm{sk}_3((BA)^n))$. Note that if $\mathrm{wt}(\Gamma) = 1$, then either $\Gamma = \Gamma'$, or $\Gamma = \Gamma''$. This means that $W_1 = Y_1$. Further, if $\mathrm{wt}(\Gamma) = 2$, then $P(\Gamma) = P(\Gamma') \times P(\Gamma'')$ except for the case when Γ has the form $i \xrightarrow{b} j \xrightarrow{c} n+1$, in which case $P(\Gamma)$ is a pentagon while $P(\Gamma') \times P(\Gamma'')$ is a square. Note that the attaching map for $\partial P(\Gamma') \times P(\Gamma'') = \{0,1\} \times P(\Gamma'')$ identifies $\{0\} \times P(\Gamma'')$ with $P(\Gamma'')$, i.e., with the edge $X_{e_{j,n+1}(c)}$ of the simplicial complex $B(A^n)$, while the other side, $\{1\} \times P(\Gamma'')$ is identified with the edge $X_{e_{i,n+1}(bc)e_{j,n+1}(c)} \subset B(A^n)$ which is the diagonal of the square formed by the product of edges $X_{e_{i,n+1}(bc)} \times X_{e_{j,n+1}(c)}$ of $(BA)^n$. The difference between W_2 and Y_2 is just that in W_2 the other side is identified not with the diagonal but rather with the composite path formed by two sides of the square, so that $P(\Gamma') \times P(\Gamma'')$ becomes identified with the pentagon $P(\Gamma)$, as in the lemma above. This means that there is a continuous deformation of attaching maps for rough cells of Y_2 into those of W_2 and therefore Y_2 and W_2 are homotopy equivalent.

The deformation of Y_3 into W_3 in achieved in a similar fashion: by using Lemma 1.6.6, it amounts to replacing an edge of the simplicial complex $B(A^n)$ by a possibly composite egde of the polysimplicial complex $(BA)^n$ with the same end points and then doing the same for polyhedral surfaces forming the boundary of the polytopes.

§2. Higher syzygies in the Steinberg group: conjectural general picture.

(2.1) Hierogliphs and syzygies. We now describe the general formalism of hierogliphs. Fix a ring A. By a hierogliph we will mean an oriented graph Γ without oriented loops, equipped with the following additional structure:

(a) An assignment of a positive integer to each vertex of Γ so that all these integers are distinct.

(b) An assignment of a nonempty ordered sequence of elements of A to each edge of Γ.

The number of elements written on the edge of a hierogliph is called the weight of the edge. It is a positive integer. The weight of the whole hierogliph is by definition the sum of weights of all the edges.

A sub-hierogliph in a hierogliph Γ is just a subgraph (given by a subset of edges) with the induced hierogliph structure. We say that two sub-hierogliphs are disjoint if their sets of edges do not intersect. Note that the sets of vertices are allowed to intersect. Two disjoint sub-hierogliphs $\Gamma_1, \Gamma_2 \subset \Gamma$ are said to be non-interacting, if no arrow of Γ_1 can be composed with that of Γ_2, nor any arrow from Γ_2 can be composed with that of Γ_1. An hierogliph will be called irreducible if it can not be represented as a disjoint union of two non-interacting sub-hierogliphs. It is clear that any hierogliph Γ can be decomposed, in a unique way, as a union of non-interacting irreducible hierogliphs called the irreducible components of Γ.

(2.1.1) Conjecture. *For every hierogliph Γ there is a polyhedral ball $P(\Gamma)$ with the following properties:*
(a) The dimension of $P(\Gamma)$ is equal to the weight of Γ. The combinatorial type of $P(\Gamma)$ depends only on the underlying graph of Γ and on weights of the edges.
(b) If $\Gamma = \bigcup \Gamma_i$ is the irreducible decomposition of a hierogliph Γ, then $P(\Gamma) = \prod P(\Gamma_i)$.
(c) The boundary of each $P(\Gamma)$ is composed of the balls $P(\Gamma')$ for some hierogliphs Γ'.
(d) For the hierogliph Γ of the form $\bullet \xrightarrow{(a_1,\dots,a_n)} \bullet$ the polyhedral ball $P(\Gamma)$ is a simplex.
(e) For the hierogliph Γ of the form

$$\bullet \xrightarrow{a_1} \bullet \xrightarrow{a_2} \dots \xrightarrow{a_{n-1}} \bullet$$

(n vertices and $n-1$ arrows) the polyhedral ball $P(\Gamma)$ is the Stasheff polytope K_{n+1} whose vertices correspond to parenthesizings of the product of $n+1$ factors.
(f) For hierogliphs of weight ≤ 3 the balls $P(\Gamma)$ are the same as described in §1.

Let $\mathcal{B}$ be the union of the polyhedral balls $P(\Gamma)$ for all the hierogliphs Γ according to the identifications of their boundaries given by part (c) of the above conjecture. Then the 2-skeleton of $\mathcal{B}$ is the space Z_2 corresponding to the presentation of the Steinberg group. The 3-skeleton of $\mathcal{B}$ is the space $\mathcal{B}_{\leq 3}$ studied in the previous section.

(2.1.2) Conjecture. *We have $\pi_1(\mathcal{B}) = St(A)$ and $\pi_i(\mathcal{B}) = K_{i+1}(A), i \geq 1$. In other words, $\mathcal{B}$ is the homotopy fiber of the natural map $BGL(A) \to BGL^+(A)$.*

(2.2) Stasheff polytope as a higher syzygy. In this paper we will prove that the Stasheff polytopes K_n indeed form higher syzygies among the Steinberg generators: after gluing in the 3-dimensional polytopes as in §1, there will be 4-dimensional Stasheff syzygies among them, then 5-dimensional ones and so on. To formulate the result precisely, recall first of all, that any face of K_{n+1} is a product $K_{m_1+1} \times ... \times K_{m_r+1}$ for some m_ν with $\sum m_\nu = n$, see [St].

Now, for each $n \geq 1$ we will prove:

$(2.2.1)_n$ Theorem. *Let a sequence of elements $a_{12}, ..., a_{n-1,n} \in A$ be given, and denote $a_{ij} = a_{i,i+1}a_{i+1,i+2}...a_{j-1,j}$ for $i < j$. Then one can associate:*
(a) To each vertex $\beta \in K_{n+1}$, a matrix $M_\beta = M_\beta(a_{12}, ..., a_{n-1,n})$ in $T_n(A)$;
(b) To each edge l of K_{n+1}, an orientation and and a pair $1 \leq i_l < j_l \leq n$ so that if, with respect to his orientation, we have $\beta \xrightarrow{l} \beta'$, then

$$M_{\beta'} = e_{i_l,j_l}(a_{i_l,j_l})M_\beta$$

and, moreover,
(c) If a face of K_{n+1} has the form $K_{m_1+1} \times ... \times K_{m_r+1}$, then the elementary matrices associated to edges of different $K_{m_\nu+1}$, commute with each other, while on each $K_{m_\nu+1}$ we have an instance of Theorem $(2.2.1)_{m_\nu}$.

The proof will be given in §3.

(2.3) Monotone hierogliphs. Relation with Volodin K-theory. Let $T_n(A)$ be the group of upper triangular n by n matrices with entries from A with units on the diagonal. For any permutation $\sigma \in S_n$ let $T_n^\sigma(A)$ be the image of $T_n(A)$ under the conjugation by σ, i.e., the group of matrices $\|a_{ij}\|$ such that $a_{ij} = 0$ unless $\sigma(i) \leq \sigma(j)$ and $a_{ii} = 1$. The n - th Volodin space $V_n(A)$ is, by definition, the union, in the simplicial classifying space $BGL_n(A)$, of the subspaces $BT_n^\sigma(A)$. Let $V(A) = \bigcup_n V_n(A)$ be the stable Volodin space. It is known [Su] that $V(A)$ has the homotopy type of the fiber $BGL(A) \to BGL^+(A)$. In other words, Conjecture 2.1.2 can be reformulated by saying that the space $\mathcal{B}$ is homotopy equivalent to $V(A)$. In fact, we can establish a more direct relation between the two spaces.

Note that any hierogliph Γ defines some partial order $<_\Gamma$ on the set of its vertices: we say that $v <_\Gamma w$ if there is a chain of oriented arrows starting at v and ending at w. Thus the polyhedral ball $P(\Gamma)$ represents a higher syzygy among the Steinberg generators which all belong to some subgroup $T_n^\sigma(\mathbf{A})$. We say that Γ is *monotone* if these generators belong to the standard subgroup $T_n(\mathbf{Z})$. In other words, a hierogliph is monotone if for any oriented edge $i \to j$ joining vertives with numbers i and j we have $i < j$. Let $\mathcal{BT}_n$ be the subcomplex is $\mathcal{B}$ consisting of the cells $P(\Gamma)$ for monotone hierogliphs in which, in addition, all the numbers on the vertices are less or equal to n.

Conjecture 2.1.2 can be deduced from the following one.

(2.3.1) Conjecture. *The subcomplex $\mathcal{BT}_n \subset \mathcal{B}$ is the classifying space of the group $T_n(A)$.*

(2.4) The case $A = \mathbf{Z}$: simple hierogliphs and the nilmanifold $T_n(\mathbf{R})/T_n(\mathbf{Z})$. If $A = \mathbf{Z}$, we can consider only the subset of generators $x_{ij} = x_{ij}(1)$ and of Steinberg relations (1.3.2) and (1.3.3) involving these generators. Accordingly, we call a hierogliph *simple* if the sequence of elements of $A = \mathbf{Z}$ written on each edge consists of just one element, the unity. For a simple hierogliph Γ the weight is just the number of edges and the combinatorial type of $P(\Gamma)$ depends, according to conjecture 2.1.1, only on the underlying graph of Γ. The problem of constructing polyhedral balls $P(\Gamma)$ for simple hierogliphs seems easier than the general problem: as we will see in the next section, a candidate for such a $P(\Gamma)$ is in a sense already realized inside the space of Morse functions as a bifurcation diagram of a function with a Γ-pattern of gradient trajectories between critical points of the same index. The only problem is thus to find some combinatorial description of these bifurcation diagrams.

A face of $P(\Gamma)$ for a simple hierogliph Γ may, in fact, not correspond to a simple hierogliph, as we saw, for example, in Fig.6. However, in that case the square formed by the two triangles $T_{ik}(ab, c), T_{ik}(c, ab)$, $a = b = c = 1$, can be collapsed since the two paths on its boundary are identical: $X_{ik}(1)X_{ik}(1)$.

(2.4.1) Conjecture. *(a) For a simple hierogliph Γ all faces of $P(\Gamma)$ corresponding to non-simple hierogliphs, can be collapsed in the sense that the two halves of the boundary of such a cell are identical.*
(b) Let $\overline{\mathcal{B}}$ be the CW-complex obtained by gluing in the cells $P(\Gamma)$ for all simple hierogliphs Γ and collapsing their faces corresponding to non-simple hierogliphs. Then $\overline{\mathcal{B}}$ has homotopy type of the Volodin space $V(\mathbf{Z})$.
(c) Let $\overline{\mathcal{BT}}_n \subset \overline{\mathcal{B}}$ be the union of the $P(\Gamma)$ for all monotone simple hierogliphs Γ with numbers of vertices not exceeding n. Then $\overline{\mathcal{BT}}_n$ is homotopy equivalent to the classifying space of $T_n(\mathbf{Z})$.

Recall that the classifying space $B(T_n(\mathbf{Z}))$ has a particularly nice model: namely, the quotient $T_n(\mathbf{R})/T_n(\mathbf{Z})$. It is a compact manifold of dimension $n(n-1)/2$. Note that the cells of the space $\overline{\mathcal{BT}}_n$ also have dimension $\leq n(n-1)/2$, with the equality holding for just one cell corresponding to the "complete graph" with n vertices $1, ..., n$ and one edge from i to j for any $i < j$. So it is natural to expect that $\overline{\mathcal{BT}}_n$ is in fact homeomorphic to $T_n(\mathbf{R})/T_n(\mathbf{Z})$. In other words, we have the following question.

(2.4.2) Question. *Is there a natural CW decomposition of the manifold $T_n(\mathbf{R})/T_n(\mathbf{Z})$ into $2^{n(n-1)/2}$ cells $P(\Gamma)$ corresponding to all monotone graphs Γ numbered by numbers between 1 and n?*

One possible way to construct such a decomposition would be to exhibit a particularly nice Morse function on $T_n(\mathbf{R})/T_n(\mathbf{Z})$. Another approach might be to try to mimic the inductive proof of Theorem 1.5.1. Namely, Lemma 1.5.2 has the following obvious analog for our chosen model of $BT_n(\mathbf{Z})$. Note that the group $GL_n(\mathbf{Z})$ and, in particular, the subgroup $T_n(\mathbf{Z})$ acts on the n-fold Cartesian product of any Abelian group, in particular, on the n-torus $T^n = (S^1)^n$.

(2.4.3) Lemma. *For each n we have a diffeomorphism*

$$T_{n+1}(\mathbf{R})/T_{n+1}(\mathbf{Z}) \simeq (T_n(\mathbf{R}) \times T^n)/T_n(\mathbf{Z}).$$

Note that T^n has the standard CW-decomposition into cubes and these cubes are precisely $P(\Gamma)$ for simple hierogliphs Γ with the properti that each of their arrows terminates at the vertex $n+1$.

(2.5) The case $A = \mathbf{Z}$: relation to Lie algebra homology. Let us keep the assumption $A = \mathbf{Z}$. Notice that monotone numbered graphs involved in the definition of $\overline{\mathcal{BT}}_n$ are in bijection with arbitrary subsets of the set of pairs $(i, j), i < j$, or, in other words, with subsets

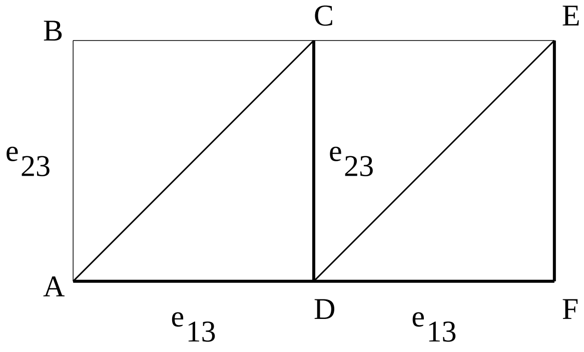

Figure 8: Constructing $T_3(\mathbf{R})/T_3(\mathbf{Z})$.

of the set of positive roots of the Lie algebra $\mathbf{gl}_n(\mathbf{R})$. Let $\mathbf{t}_n$ be the Lie algebra of the group $T_n(\mathbf{R})$, i.e., algebra of strictly upper-triangular real n by n matrices. Its basis is formed by the matrix units $\epsilon_{ij}, i < j$ having 1 at the position (i, j) and 0 elsewhere. Consider the Lie algebra homology $H_\bullet(\mathbf{t}_n, \mathbf{R})$. It is calculated by the Chevalley-Eilenberg complex

$$C_\bullet(\mathbf{t}_n, \mathbf{R}) = \left\{ ... \to \bigwedge^2(\mathbf{t}_n) \to \bigwedge^1(\mathbf{t}_n) \to \bigwedge^0(\mathbf{t}_n) \right\}.$$

Note that a basis in $\bigwedge^m(\mathbf{t}_n)$ is formed by wedge products $\epsilon_{i_1 j_1} \wedge ... \wedge \epsilon_{i_m j_m}$ for all m-element subsets $\{(i_1, j_1), ..., (i_m, j_m)\}$, $1 \leq i_\nu < j_\nu \leq n$ i.e., by monotone numbered graphs labeling the cells from $\overline{\mathcal{BT}}_n$. Thus the chain complex of the CW-complex $\overline{\mathcal{BT}}_n$ (calculating the group homology of $T_n(\mathbf{Z})$) should be "of the same size" as the Chevalley-Eilenberg complex $C_\bullet(\mathbf{t}_n, \mathbf{R})$ calculating the Lie algebra homology. Note that by Malcev's theorem the group-theoretic homology of $T_n(\mathbf{Z})$ with real (or rational) coefficients is the same as the Lie algebra homology of $\mathbf{t}_n$.

(2.6) Example for $T_3(\mathbf{Z})$. In the case $n = 3$ we can answer Question 2.3.3 in the affirmative by direct analysis of the 3-fold $T_3(\mathbf{R})/T_3(\mathbf{Z})$. Denote this threefold simply by B. Let B' be the CW-complex obtained by identifying the faces of the pentagonal prism P in Fig.6 and collapsing the left square in the front row. We want to prove that B is homeomorphic to B'. The argument for that is similar to the proof of Theorem 1.6.1.

For any homeomorphism $\varphi : M \to M$ of a space M its *mapping torus* is defined to be the result of identifications of the two bases of the cylinder $M \times [0, 1]$ according to φ, i.e., $(m, 0) \simeq (\varphi(m), 1)$. In other words, this is the homotopy quotient $M//\mathbf{Z}$ with the $\mathbf{Z}$-action given by powers of φ.

Lemma 2.4.3 says in our case that B is the mapping torus of the automorphism $e_{12} : T^2 \to T^2$. The torus T^2 can be seen as the classifying space of the subgroup in $T_3(\mathbf{Z})$ generated by $e_{13} = e_{13}(1)$ and $e_{23} = e_{23}(1)$. Let us view this torus as obtained by identifying the opposite sides of a square $ABCD$ in the square lattice in $\mathbf{R}^2$, see Fig.8. Then B is obtained by identifying one base $ABCD \times \{0\}$ of the cylinder $ABCD \times [0, 1]$ with $ABCD \times \{0\}$ and the other base with the skew parallelogram $ACDE \times \{1\}$, so that $AB \times \{0\}$ is identified with the diagonal $AC \times \{1\}$ and so on.

Now to get B' we need to do a very similar thing but with the diagonal $[AC]$ replaced by the composite path $[AD] \cup [DC]$ (which is a closed path in T^2 homotopic to the closed path represented by $[AC]$). This in fact describes a map ψ from the pentagonal prism P into B which collapses the square face we mentioned, into the composite segment ADF. It is now clear that ψ does not collapse any other faces and thus provides a CW-decomposition for B identical to the one used to construct B'.

§3. The prime spectrum of a simple hierogliph.

In this section we describe a construction which to any simple hierogliph Γ associates a CW-complex $\mathcal{P}(\Gamma)$ which is directly related to syzygies in the Steinberg group and in many cases is the desired polyhedral ball $P(\Gamma)$ from §2. In particular, this construction agrees with the direct descriptions for $\mathrm{wt}(\Gamma) \leq 3$ given in §1. Also, we will prove that for a linear graph the complex $\mathcal{P}(\Gamma)$ is the Stasheff polytope and will deduce Theorem 2.2.1 from this.

Everywhere in this section the word "hierogliph" will mean "simple hierogliph".

(3.1) Polyhedral complexes and posets. We will use the term "polyhedral complex' to signify what is sometimes called a regular cell complex [LW]. Such complexes differ from general CW-complexes in that for any r-cell σ in a complex X the structure map of the r-ball $f_\sigma : B^r \to X$ (which is always an embedding of the interior of B^r), is required to be an embedding of the whole B^r, so that the image of ∂B^r is an embedded sphere represented as a union of $(r-1)$-cells and so on.

To describe a polyhedral complex X, it is enough to describe the partially ordered set (poset) $\mathcal{F}(X)$ of its (closed) cells, ordered by inclusion. Namely, every poset $(Y, \leq)$ gives a category with the set of objects Y and a unique morphism from x to y existing iff $x \leq y$. The complex X is homeomorphic to $\mathrm{Nerv}(\mathcal{F}(X))$, the simplicial nerve of $\mathcal{F}(X)$ considered as a category. More precisely, $\mathrm{Nerv}(\mathcal{F}(X))$ is the barycentric subdivision of X, and the cells $\sigma \subset X$ are recovered as nerves of subposets $[\sigma] = \{\tau : \tau \leq \sigma\}$.

In order to construct a polyhedral complex, it is thus natural to first describe its poset of cells. Given a poset $\mathcal{F}$, we may ask when there exists a polyhedral ball X with $\mathcal{F} = \mathcal{F}(X)$, and the answer is that the nerve of every $[\sigma]$ should be, topologically, a ball. Posets with this property will be called *ball-like*.

In the rest of this section we will construct some posets asociated to hierogliphs.

(3.2) The path category. Let Γ be a hierogliph, i.e., a finite oriented graph without oriented loops. We will ignore the question of numbering of vertices by integers, working with objects directly indexed by $\mathrm{Vert}(\Gamma)$.

The *path category* $\pi(\Gamma)$ has, by definition, vertices of Γ as objects and oriented edge paths as morphisms. We associate to Γ its *characteristic matrix* $M(\Gamma)$ whose set of indices in *Vert*(Γ) and for two vertices i, j the matrix element $M(\Gamma)_{ij}$ is equal to the cardinality of $\mathrm{Hom}_{\pi(\Gamma)}(i, j)$.

Note that any subcategory in $\pi(\Gamma)$ has the form $\pi(\Gamma')$ for a uniquely defined hierogliph Γ'. We will call an embedding $\pi(\Gamma') \hookrightarrow \pi(\Gamma)$ a *morphism of hierogliphs* $\Gamma' \to \Gamma$.

(3.3) Prime ideals in categories. Let $\mathcal{C}$ be any category. A *left ideal* in $\mathcal{C}$ is a family I of morphisms in $\mathcal{C}$ with the properties:

(3.3.1) No isomorphism lies in I;

(3.3.2) Whenever a composition fg, $f \in \mathrm{Mor}(\mathcal{C}), g \in I$, is defined, the value of this composition lies in I.

If k is a field and $\mathcal{C}$ is a small category, then we can form the algebra $k[\mathcal{C}] = \bigoplus_{i,j\in \mathrm{Ob}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(i, j)$. A left ideal $I \subset \mathcal{C}$ gives, in an obvious way, a left ideal $k[I] \subset k[\mathcal{C}]$ in the usual sense of ring

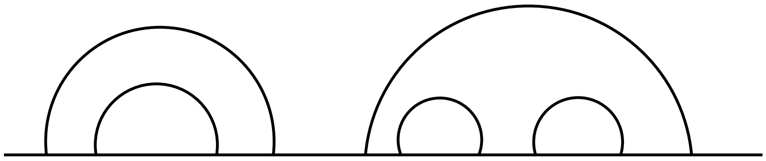

Figure 9: Nested caps

theory.

A left ideal $\wp$ in $\mathcal{C}$ is called *prime*, if the family F of all morphisms in $\mathcal{C}$ which are not in $\wp$, is closed under composition. In this case, by (3.3.1), F contains all the identity morphisms so it is a category.

The conditions on a family of morphisms F to be the complement of a prime ideal, are as follows:

(3.3.3) F is a subcategory;

(3.3.4) If $\gamma\delta \in F$, then $\delta \in F$.

We will call such families of morphisms *admissible*.

(3.4) Examples: corruption orders, nested caps. Let $(Y, \leq)$ be a partially ordered set. We can associate to it a category $\mathcal{C}$ as in (3.1). An admissible family of morphisms in $\mathcal{C}$ is just another partial order $\propto$ on Y with the following properties:

(3.4.1) The order $\propto$ is weaker than $\leq$, i.e., $a \propto b$ implies $a \leq b$.

(3.4.2) If $a \propto c$ and $a \leq b \leq c$ then $a \propto b$.

We will call such orders $\propto$ on a poset $(Y, \leq)$ *corruption orders*. One can imagine a large corporation with set of managers Y and $\leq$ describing the relation of being subordinate. If the system is corrupt, some of the managers can exercise influence on the decisions of their superiors (by bribing or otherwise). This relation is denoted by $\propto$. Clearly, if a can influence decisions of his superior c, then by this, he can influence decisions of anyone subordinate to c (we presume that no manager has any stake in decisions of someone not his superior). This explains our terminology.

Consider the special case when $Y = [n] := \{1, 2, ..., n\}$ with the standard linear order. Let $\mathcal{C}_n$ be the corresponding category. A morphism in the category $\mathcal{C}_n$ is thus a pair of integers $1 \leq i \leq j \leq n$. The algebra $k[\mathcal{C}_n]$ is nothing but the algebra of upper triangular n by n matrices. Corruption orders on $[n]$ have a geometric description in terms of systems of *nested caps*. More precisely, we consider all topologically different ways of arranging n non-intersecting arcs ("caps") in the upper half plane in $\mathbf{R}^2$ whose ends lie on the boundary line $\mathbf{R}$, see Fig. 9. Given such a system, we number the caps according to the x-coordinate of the left end of the arc. So the caps will be denoted $C_1, ..., C_n$ according to this numbering. We get a partial order $\propto$ on $[n]$ defined as follows: $i \propto j$ if C_i contains C_j inside it. Because of our numbering it is clear that $\propto$ satisfies (3.4.1-2). We leave to the reader the proof that this establishes a bijection between corruption orders on $[n]$ and systems of n nested caps.

The next proposition gives the first indication how the "prime spectrum" of $\mathcal{C}_n$ is related

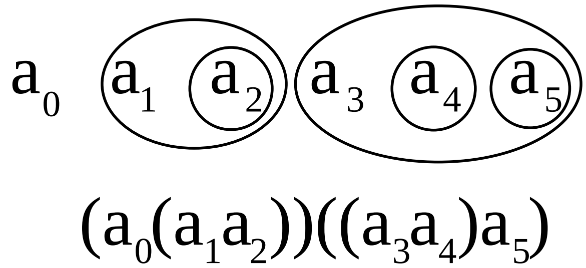

Figure 10: An encircled product and the corresponding parenthesized product

to Stasheff polytopes.

(3.4.3) Proposition. *There is a bijection between prime ideals in $\mathcal{C}_n$ (i.e., corruption orders on $[n]$, or (topologically different) systems of n nested caps), and complete parenthesizings of the product of $n+1$ factors $a_0...a_n$. In particular, the number of such prime ideals in c_{n+1}, the Catalan number.*

Proof: We first insert the letters a_i on the line $\mathbf{R}$ between the points of intersection with the caps. Namely, we write the letter a_0 on the left of the leftmost cap C_1 and write $a_i, i \geq 1$ immediately after the left intersection point of C_i with $\mathbf{R}$. Then we complete each cap to a circle by reflecting with respect to the x-axis. In this way we get an "encircled product", as in Fig.10.

Note that inside each circle we again have an encircled product. We now convert an encircled product into a parenthesized product by induction, assuming that for any encircled product of $\leq n$ letters this has already been done. Look at the outermost circles, i.e., those not contained in any other circles. Suppose that are $C_{i_1}, ..., C_{i_m}$, $i_1 < ... < i_m$. By inductive assumption we already know how to convert the encircled product of the letters inside each C_{i_ν} into a parenthesized product of these letters. Let this parenthesized product be A_ν. Then the parenthesized product which we associate to the whole circle arrangement is just $(...(a_0A_1)A_2)...A_m)$. Thus the arrangement of n caps with disjoint interiors corresponds to the left parenthesizing $(...(a_0a_1)a_2)...a_n)$ and the arrangement of n caps whose interiors form a chain of embedded half-disks, corresponds to the right parenthesizing $a_0(a_1(a_2(...a_n)...)$.

(3.5) The prime spectrum and its edges. Let Γ be an oriented graph as above. Denote by $\mathrm{Spec}(\Gamma)$ the set of all prime ideals in $\pi(\Gamma)$. For a prime ideal $\wp$ it is convenient to denote the correspoding element of $\mathrm{Spec}(\Gamma)$ by $[\wp]$. If F is an admissible system of morphisms which is the complement of $\wp$, then we will also use the notation $[F]$ for $[\wp]$. Introduce the characteristic matrix $M(\wp) = M(F)$ with the set of indices $\mathrm{Vert}(\Gamma)$ and the (i,j)th matrix element equal to the number of morphisms $i \to j$ lying in F.

The categories of the form $\pi(\Gamma)$ have the special property of unique factorization: each morphism (i.e., edge path) can be represented uniquely as composition of "irreducible" morphisms (namely, individual edges), so if we have an equality $ab = cd$ then a, b as well as c, d are compositions of the same edges. This property gives an especially good behavior of prime ideals. Let us note the following.

(3.5.1) Proposition. *If $\wp$ is a prime ideal in $\pi(\Gamma)$ and $S \subset \wp$ is any subset of morphisms then among the prime sub-ideals in $\wp$ which do not meet S, there is a maximal one, which we denote by $\wp \searrow S$.*

Proof: In the dual language this means that if F is an admissible system of paths and S is any set of paths not intersecting F, then there is a minimal admissible system of paths $F[S]$ containing F and S. But such a system is constructed in an obvious way: we define $F[S]$ to consist of all paths obtained from paths in $F \cup S$ by iterated application of two operations: composition of composable paths and taking an initial segment of any path.

Let us also highlight one situation when $F[S]$ or $\wp \searrow S$ can be described more explicitly.

(3.5.2) Definition. *Let $\wp$ be a prime ideal in $\pi(\Gamma)$. Call a morphism $p \in \wp$ irreducible (with respect to $\wp$) if it cannot be represented as a nontrivial composition $p = aq$ with a being any morphism and $q \in \wp$. In other words, a path $p \in \wp$ is irreducible if no initial segment of p lies in $\wp$. Call p co-irreducible (with respect to $\wp$), if no non-trivial composition pu lies in $\wp$. In other words, p is coirreducible, if the intersection of (the left ideal) $\wp$ and the right ideal generated by p consists of p alone. A morphism which is irreducible and co-irreducible, will be called bi-irreducible.*

(3.5.3) Proposition. *(a) Let $p \in \wp$ be irreducible, and set $S = \{p\}$. Let F be the complement of $\wp$. Then $F[\{p\}]$ is obtained by adding to F all morphisms of the form $\alpha p \beta$ with $\alpha, \beta \in F$.*
(b) If p is, in addition, coirreducible, then $F[\{p\}]$ is obtained by adding to F all morphisms of the form αp with $\alpha \in F$. .

Proof: (a) Let F' be the union of F and the morphisms defined above. It is enough to show that F' is admissible, i.e., an initial segment of any morphism γ from F' lies in F'. We need only to consider the case $\gamma = \alpha p \beta$. Let γ' be an initial segment of γ. Then we have three possibilities depending on the length of γ':

(1) γ' is an initial segment of β. In this case $\gamma \in F \subset F'$ because F is admissible.

(2) $\gamma' = p'\beta$ where p' is an initial segment of p. If $p' \neq p$, then $p' \in F$ by irreducibility of p, so $p'\beta \in F$ as well. If $p' = p$, then $p\beta$ is in F' by construction.

(3) $\gamma' = \alpha' p \beta$ where α' is an initial segment of α. In this case $\gamma' \in F$ by construction.

This finishes the proof of (a). To see (b), we need just to notice that if $\beta \neq \mathrm{Id}$, then $p\beta \in F$ by the coirreducibility so $\alpha(p\beta)$ already lies in F.

Now we make $\mathrm{Spec}(\Gamma)$ into the set of vertices of an oriented graph by defining oriented edges $[\wp] \to [\wp']$ issuing from any given vertex $[\wp]$. They correspond to paths $p \in \wp$ which are bi-irreducible with respect to $\wp$. The other end $[\wp']$ of the edge $[\wp] \to [\wp']$ corresponding to p is the ideal $\wp \searrow \{p\}$. We will write $[\wp] \xrightarrow{p} [\wp']$ to indicate p.

The next proposition shows how $\mathrm{Spec}(\Gamma)$ is related to syzygies among elementary matrices.

(3.5.4) Proposition. *If $[\wp] \xrightarrow{p} [\wp']$ is an edge where p is a path from i to j, then the characteristic matrices $M(\wp), M(\wp')$ are related by an elementary transformation: $M(\wp') = e_{ij} M(\wp)$.*

Proof: Immediate corollary of (3.4.3)(b).

(3.6) The complex $\mathcal{P}(\Gamma)$. We now define a class of subsets in $\mathrm{Spec}(\Gamma)$ called *faces*. This construction is remindful of the Zariski topology on the spectrum of a commutative ring.

By definition, faces correspond to pairs $(\wp, \mathcal{C})$ where $\wp$ is a prime ideal in $\pi(\Gamma)$ and $\mathcal{C}$ is any subcategory (i.e., the image of an embedding $\pi(\Gamma') \hookrightarrow \pi(\Gamma)$) whose irreducible morphisms (the images of edges of Γ') are bi-irreducible with respect to $\wp$. We denote the face corresponding to $(\wp, \mathcal{C})$ by $[\wp, \mathcal{C}]$. We will also use notation $[F, \mathcal{C}]$ where F is the admissible system complementary to $\wp$. The vertices of $[\wp, \mathcal{C}]$ are defined to be the points $[\wp \searrow F']$ where F' is an arbitrary admissible system of morpisms in $\mathcal{C}$. Thus each $[P, \mathcal{C}]$ is isomorphic (as a set with a distinguished family of subsets called faces) to $P(\mathcal{C}) = P(\Gamma')$. Also, the intersection with $\wp$ of the prime ideals in $\pi(\Gamma)$ representing vertices of $[P, \mathcal{C}]$ are precisely all the prime ideals in $\mathcal{C}$.

Let $\mathcal{F}(\Gamma)$ be the set of faces in $\mathrm{Spec}(\Gamma)$, ordered by inclusing and set $\mathcal{P}(\Gamma) = \mathrm{Nerv}(\mathcal{F}(\Gamma))$. Proposition 3.5.4 implies:

(3.6.1) Proposition. *If the poset $\mathcal{F}(\Gamma)$ is ball-like, then the polyhedral ball $\mathcal{P}(\Gamma)$ together with its faces represents a higher syzygy among the Steinberg generators.*

The next proposition is proved by inspecting the polytopes in §1.

(3.6.2) Proposition. *For any hierogliph Γ of weight ≤ 3 the poset $\mathcal{F}(\Gamma)$ is ball-like and $\mathcal{P}(\Gamma)$ with its polyhedral ball structure is identified with the polytope $P(\Gamma)$ described in §1.*

(3.7) Stasheff polytopes. We first recall the basic properties of the Stasheff polytope K_{n-1}, see [St]. Consider $n-1$ symbols $a_1, ..., a_{n-1}$. By a complete parenthesizing of the product $a_1...a_{n-1}$ we mean a way of inserting $n-3$ pairs of parentheses so as the product makes sense in any (possibly non-associative) algebra. By a partial parenthesizing we mean a way of inserting some number $k \leq n-3$ of pairs of parentheses which can be extended to a complete parenthesizing. The set of all partial parenthesizings of $a_1...a_{n-3}$ is partially ordered by reverse inclusion: thus, the empty parenthesizing $a_1...a_{n-1}$ is the maximal element while complete parenthesizings are minimal elements.

It is standard that complete parenthesizings of $a_1...a_{n-1}$ are in bijection with triangulations of a convex polygon P_n with n vertices (into triangles whose vertices are among the vertices of the polygon). More generally, call a polygonal decomposition of P_n a decomposition of it into convex polygons with vertices among those of P_n. Polygonal decompositions are partially ordered by refinement (so triangulations are minimal elements with respect to this order while the decomposition consisting of P_n alone, is the maximal element).

The following is the standard property of K_{n-1} and can be in fact considered as a definition.

(3.7.1) Proposition. *The poset of faces of K_{n-1} is naturally identified with the poset of partial parenthesizings of $a_1...a_{n-1}$ and with the poset of polygonal decompositions of P_n.*

(3.7.2) Corollary. *Every face of K_{n-1} is a product of Stasheff polytopes of lower dimensions.*

Indeed, given a polygonal decomposition $\mathcal{Q}$ of P_n into several polygons $Q_1, ..., Q_m$, let n_i be the number of vertices of Q_i minus 1. To complete this decomposition to a triagulation,

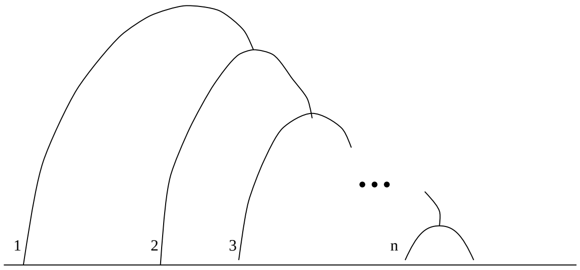

Figure 11: The system Ψ_n.

we should just triangulate each Q_i separately. So the face of K_{n-1} corresponding to $\mathcal{Q}$, is the product $K_{n_1} \times ... \times K_{n_m}$.

Now we can complete the description of the Stasheff polytope in terms of the prime spectrum.

(3.7.3) Theorem. *For a linear hierogliph Γ with n vertices and $(n-1)$ edges, the poset $\mathcal{F}(\Gamma)$ is isomorphic to the poset of faces of K_{n+1}.*

Proof: We already have described vertices of K_{n+1} by systems of n nested caps in Proposition 3.4.3. We now describe the full face lattice. Namely, call a system of *mounting caps* a system of smooth non-self-intersecting arcs ("caps") in $\mathbf{R}^2$ with the following properties:

(a) All the arcs lie in the upper half plane $\{y \geq 0\}$.

(b) The left end of each arc always lies on the horizontal line $\mathbf{R} = \{y = 0\}$.

(c) The right and of each arc lies either on $\mathbf{R}$ or on another arc; the right ends of different arcs are different.

(d) Apart from the situations allowed in (c), the arcs do not intersect.

Thus a system of nested caps is a system of mounting caps. Another example is the system Ψ_n in Fig. 11.

We will identify any two systems of mounting caps differing by an orientation preserving homeomorphism of the upper half-plane. Note that the cups are canonically numbered by abscisses of their left ends.

We now define a binary relation on the set of systems of n mounting cups called *refinement*. Let C be a system of mounting caps such that an arc C_i has its right end point p on another arc C_j $(i < j)$. There are two ways, left or right, of sliding down p. Keeping other arcs fixed topologically, we obtain two *elementary refinements* of Z by sliding down p. Arbitrary refinements are obtained by iterating elementary refinements. For instance, after we slide down p to the left, we may get further refinements. Let C_k be another arc $(k < i)$, such that there is a sequence $C_{k_1} = C_k, C_{k_2}, \cdots, C_{k_m} = C_i$ with the right end point of C_{k_h} lying on $C_{k_{h+1}}$ for any $h = 1, \cdots, m$. Then we can connect the right end point of C_k to C_j, and slide down p to the left, keeping other arcs fixed topologically. The result is a refinement of C.

For example, in Fig. 13, in between the left and left bottom nested caps, there is an arrow labeled 13. To this arrow one can associate a system of mounting caps such that C_1

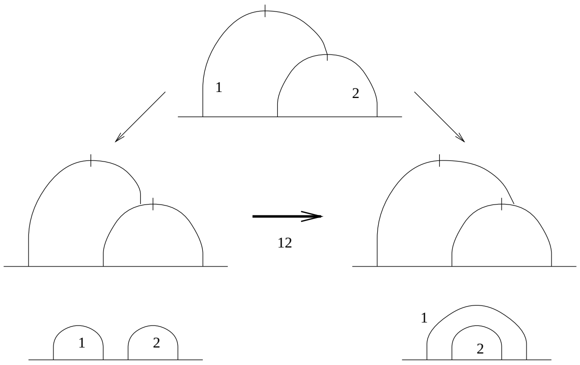

Figure 12: Perturbing mounting caps

has its right end point on C_3, and the right end point of C_2 is slid down to the left of C_3. A refinement is defined by applying this operation consectively.

Clearly, refinement is a partial order on the set of systems of mounting caps. With respect to this order, minimal elements are systems of nested caps while the unique maximal element is the system Ψ_n on Fig. 11.

(3.7.4) Proposition. *The bijection between systems of n nested caps and vertices of K_{n+1} can be extended to an order-preserving bijection between the poset of all systems of n mounting caps and the poset of all faces of K_{n+1}.*

Proof: Since Ψ_n is the unique maximal system of mounting caps, we can reformulate our proposition by saying that there is an order-preserving bijection between refinements of Ψ_n and faces of K_{n+1}. Let us prove this by induction in n.

Given a system C of mounting caps, consider the connected components $Z_1, ..., Z_m$ of the union of all the caps. Each Z_i is isomorphic to some Ψ_{n_i}. Further, to construct a refinement of C, it is enough to construct a refinement of each Z_i separately. So the set of all system of nested caps refining C is the product of similar sets for the $Z_i \simeq \Psi_{n_i}$. By induction this is the set of vertices of $K_{n_1+1} \times ... \times K_{n_m+1}$. This is precisely a face of K_{n+1} (see Corollary 3.7.2). This proves Theorem 3.7.3.

(3.8) Proof of Theorem 2.2.1. Let elements $a_{12}, ..., a_{n-1,n} \in A$ be given. According to Proposition 3.4.3, we can realize vertices of K_{n+1} by systems on n nested circles. Let us now orient each edge of K_{n+1} and label it by a Steinberg generator $x_{ij}(a_{ij})$, $1 \leq i < j \leq n$, as follows. If C and C' are two systems of nested circles forming two ends of an edge in K_{n+1}, then the relative positions of the circles in C and C' are the same except that in one system (say, C) some two circles, say, with numbers i and j, have disjoint interiors while in the other the jth circle is inside the ith one. Note that because of our convention for numbering the circles we have $i < j$. So we orient the edge by putting the system of nested circles where the two circles in questions are disjoint, in the beginning, and take $x_{ij}(a_{ij})$ to be the label of the edge. Fig.13 shows what happens for the pentagon. The verification of the properties required in the theorem, is straightforward.

(3.9) The prime spectrum for the X-graph. As an example of more complicated

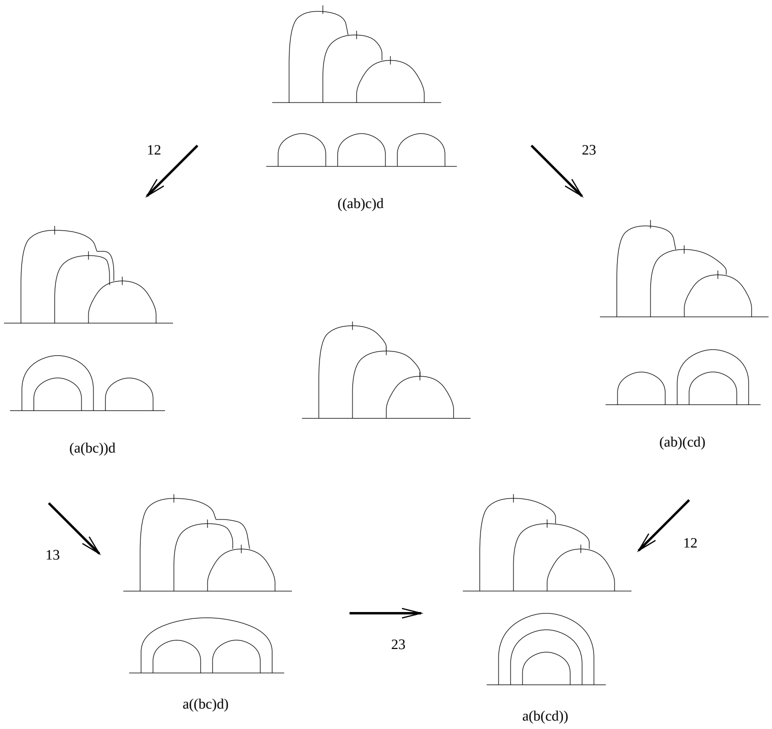

Figure 13: Mounting caps and the graph for the pentagon

behavior of the prime spectrum, consider the case of the graph Γ_X depicted in Fig.14 (we introduce the numbering of vertices for convenient reference).

The poset $\mathcal{F}(\Gamma_X)$ is in this case not ball-like, but for any proper face $\sigma \in \mathcal{F}(\Gamma_X)$, $\sigma \neq \mathrm{Spec}(\Gamma_X)$, the poset $[\sigma]$ is ball-like, so the corresponding balls $\mathrm{Nerv}([\sigma])$ form a CW-complex $\mathcal{P}^0(\Gamma_X)$ such that $\mathcal{P}(\Gamma)$ is just the cone over it. If $\mathcal{P}^0(\Gamma_X)$ was a 3-sphere, then $\mathcal{P}(\Gamma_X)$ would have been a ball. However, a direct analysis of $\mathcal{P}(\Gamma_X)$ shows that it has the following structure. First, there is a 3-dimensional subcomplex U homeomorphic to a 3-ball (i.e., to a 3-sphere with a hole). It is formed by 4 Chicago buildings, 4 dual Chicago buildings, 4 pentagonal prisms and 4 three-dimensional cubes. Next, there is a 4-dimensional cube Q (spanned by the commuting Steinberg generators $x_{14}, x_{15}, x_{24}, x_{25}$) which is attached to the boundary of U along some 2-sphere in the boundary of Q. The resulting complex

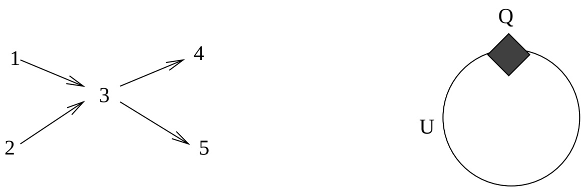

Figure 14: The X-graph

$\mathcal{P}^0(\Gamma_X) = U \cup Q$ looks like a 3-dimensional diamond ring and is symbolically depicted as such in Fig.14.

We shall see in the next section that for any hierogiph Γ a Morse but not Morse-Smale function with a Γ-pattern of trajectories between critical points of same index gives, as its bifurcation diagram, a certain polyhedral ball which is, moreover, embedded into $\mathcal{P}(\Gamma)$. In our example the boundary of the bifurcation diagram will be a 3-sphere in the "diamond ring" $U \cup Q$, and different functions can choose different 3-spheres, i.e., different ways to fill the boundary of U by a polyhedral 3-ball in U.

This example shows that for more complicated hierogliphs in which vertices may possess several incoming and outgoing edges, the complex $\mathcal{P}(\Gamma)$ may be even farther from a ball. However, we would like to conjecture that if a graph Γ does not have a subgraph isomorphic to Γ_X, then $\mathcal{F}(\Gamma)$ is ball-like and in fact the combinatorial ball $\mathcal{P}(\Gamma)$ can be realized as a convex polytope. Note that the constructions of [DP1-2] allow one to associate a convex polytope $M(\Gamma)$ to any (non-oriented) Dynkin graph Γ in such a way that for a linear graph A_n we again get a Stasheff polytope.

§4. The discriminant in the space of Morse functions.

In this section we show that the Stasheff polytope appears as a bifurcation diagram for "Smalefications" of Morse functions. More generally, we relate these bifurcations diagrams with prime spectra of hierogliphs studied in §3.

(4.1) Morse and Morse-Smale functions. Let X be a smooth compact orientable manifold of dimension d with Riemannian metric. A smooth function $f : X \to \mathbf{R}$ is called a Morse function if all its critical points are non-degenerate and the values of f at these points are distinct (the second condition added for later convenience). Thus each critical point x has a well-defined index $i = i_x$, namely the number of negative eigenvalues of the second differential of f at x. By using the Riemannian metric, we form the gradient vector field ∇f on X. To eliminate the direction ambiguity, let us stipulate that the gradient flow decreases the values of f.

For a critical point x we denote by $S^+(x) = S^+(x, f)$ (resp. $S^-(x) = S^-(x, f)$) the stable (resp. unstable) variety of x, i.e., the union of all gradient trajectories which converge for $t \to -\infty$ (resp. for $t \to +\infty$) to x. It is well known that $S^\pm(x)$ are diffeomorphic to Euclidean spaces, $\dim(S^+(x)) = d - i_x, \dim(S^-(x)) = i_x$.

A Morse function f is called a Morse-Smale function, if it satisfies the following transversality condition: for any two critical points x and y the intersection of $S^+(x)$ and $S^-(y)$ is transversal. This implies, in particular, that the dimension of the space of gradient trajectories beginning at x and ending at y, is equal to $i_x - i_y - 1$, if $i_x > i_y$ and is empty when $i_x \leq i_y$.

It is well known that for a Morse-Smale function f the varieties $S^-(x)$ form a CW-decomposition of X. In particular, if f does not have any critical points of index $m\pm1$, and *we have chosen orientations of each* $S^-(x)$, then the m-dimensional cells of this decomposition provide a basis in $H_m(X, \mathbf{Z})$.

(4.2) The discriminant and its strata. Let $\mathcal{F}$ be the space of all smooth functions on X, and $\mathcal{M}$, $\mathcal{MS}$ the subspaces there formed by Morse, resp. Morse-Smale functions. The space $\mathcal{F}$, being an infinite-dimensional vector space, is contractible, and $\mathcal{M}$, $\mathcal{MS}$ are open subsets in this vector space. Denote the complement $\mathcal{F} - \mathcal{MS}$ by Δ and call it the *discriminantal variety*, or simply the discriminant. This is a hypersurface in $\mathcal{F}$ which is highly singular. We are interested in certain "strata" in Δ of finite codimension. A function can lie in Δ, i.e., not be a Morse-Smale function by one of the two reasons: first, it may have complicated critical points and second, the transversality condition may be violated. We will concentrate on the second possibility, and on a particular way of violation of the Smale condition: namely, the presense of gradient trajectories joining critical points of the same index.

More precisely, let Γ be a finite oriented graph without oriented loops (in particular, without edges-loops). Denote by $\Sigma(\Gamma, m)$ the set of those Morse functions for which there exists a Γ-patters on trajectories joining critical points of index m. In other words, $f \in \Sigma(\Gamma, m)$, if there is an embedding $v \mapsto x_v$ of the set of vertices of Γ into the set of critical points of f of index m and of edges of Γ into the set of trajectories (joining the corresponding critical points). Clearly, $\Sigma(\Gamma, m)$ has codimension equal to the number of edges of Γ. On

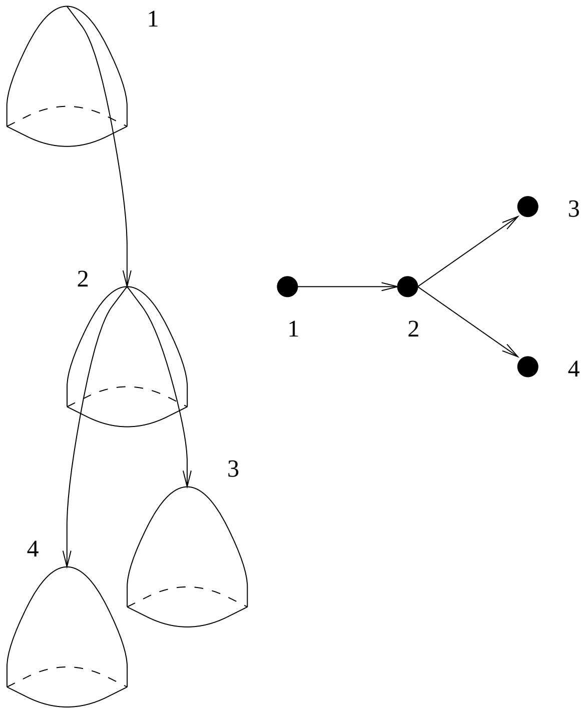

Figure 15: The catastrophe corresponding to a graph

Fig. 15 we depicted a Γ- pattern of critical points of index 2 on a 3-dimensional manifold.

We are now interested in the local structure of Δ in the neighborhood of a generic point $f \in \Sigma(\Gamma, m)$. To understand this structure, we take a small transversal slice T to $\Sigma(\Gamma, m)$ at f. Thus T is a disc of dimension equal to the codimension of our stratum which intersects the stratum only at f. Define the *bifurcation diagram* of $\Sigma(\Gamma, i)$ at f as the CW-complex whose vertices correspond to connected components of $T - \Delta$, the vertices are joined by an edge if the components are adjacent etc. In other words, it is the CW-decomposition of T dual to the decomposition formed by closures of connected components of $T - \Delta$. We will denote this bifurcation diagram $P_f(\Gamma, m)$. In principle, it can depend not only on Γ and m but also on the choice of (generic) f. Note that $P_f(\Gamma, m)$ is always a ball. It can be regarded as a candidate for the conjectural polyhedral ball $P(\Gamma)$ of §2, as we shall presently explain.

(4.3) Handle sliding and elementary transformations. We recall some background material on how the topology of Δ is related to K-theory. Consider the simplest catastrophe, corresponding to the graph $\Gamma = \{\bullet \to \bullet\}$, i.e., presense of just one trajectory joining two critical points with numbers, say, i and j of the same index m. Suppose we have a 1-parameter family f_t, $0 \leq t \leq 1$ of Morse functions crossing the stratum $\Sigma(\Gamma, m)$ only once, for $t = t_0$. Let us number the critical points of index m and retain this numbering during the deformation by continuity. The cell decomposition undergoes a so-called handle sliding transformation, shown in Fig.16 for $n = 2, m = 1$. For any t let $S^-(i)_t$ be the unstable variety for the ith critical point at the moment t. If we assume that there are no critical points of index $m \pm 1$, and choose the orientations of all the unstable varieties, then for each

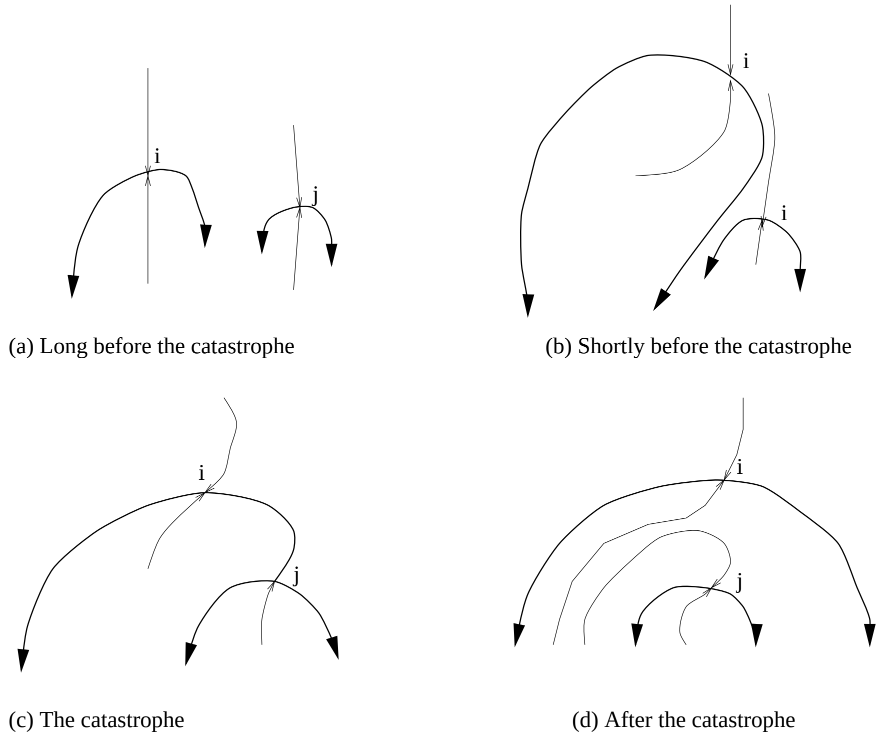

Figure 16: Handle sliding

$t \neq t_0$ we have a basis in $H_m(X)$ given by the unstable varieties of the critical points of index m (thick lines in Fig.16). Let $e_\nu(t)$ be the basis vector corresponding to the νth critical point at the moment t. Then the basis for $t = 1$ differs from the basis for $t = 0$ by an elementary transformation:

$$e_i(1) = e_i(0) + e_j(0), \quad e_\nu(1) = e_\nu(0), \nu \neq i,$$

as it can also be seen from Fig.16: the cell number i "eats" the cell number j.

If now we have a codimension 2 stratum of Δ whose generic points are Morse functions in the above sense (so that only the Smale condition becomes violated), then going around this stratum in a closed path gives a sequence of handle slidings which, being performed one after another, leave the CW decomposition unchanged. In particular, we get a relation among the elementary transformations of bases in the homology. As was observed by Hatcher and Wagoner [HW], we get in this way precisely the Steinberg relations. For instance, going around the stratum corresponding to the catastrophe $i \to j \to k$ gives the pentagonal Steinberg relation $x_{ij}x_{jk} = x_{jk}x_{ik}x_{ij}$.

According to our point of view on non-Abelian syzygies this means that the bifurcation diagrams for higher-codimensional strata $\Sigma(\Gamma, m)$ give higher syzygies among the Steinberg relations. Strangely, the precise structure of these bifurcation diagrams attracted little attention. In particular, the following surprising generalization of the Hatcher-Wagoner observation seems to have been overlooked.

(4.4) Theorem. *Let Γ be the graph of the form $\bullet \to \bullet \to \ldots \to \bullet$ (n vertices and $(n-1)$*

arrows). For any m there is an open set $U \subset \Sigma(\Gamma, m)$ such that for $f \in U$ the bifurcation diagram of stratum $\Sigma(\Gamma, m)$ at f is the Stasheff polytope K_n.

We will prove this theorem later in the section. Here we make some preliminary analysis for the case of arbitrary Γ.

(4.5) Combinatorial invariants of a smalefication: preliminaries. Let $f \in \Sigma(\Gamma, n)$ be a generic Morse function with a Γ-pattern of trajectories among critical points of index m. These points are denoted x_v, where $v \in \mathrm{Vert}(\Gamma)$ is a vertex. We will call a *smalefication* of f a Morse-Smale function g which is a small deformation of f. Thus, in the notation of (4.2), every connected component of $T - \Delta$ gives a well defined topological type of a smalefication. In order to describe the bifurcation diagram in an explicit way, we are now going to define some combinatorial invariants of a smalefication.

Since Γ is supposed not to have oriented loops, there is a natural partial order $\leq_\Gamma$ on $\mathrm{Vert}(\Gamma)$: we say that $v \leq_\Gamma w$, if there is an oriented edge path from v to w. Note that $v \leq_\Gamma w$ implies $f(v) > f(w)$.

Let g be a smalefication of f. By continuity we have a correspondence between critical points of f and g. Let y_v be the critical point of g corresponding to $v \in \mathrm{Vert}(\Gamma)$. Suppose $v \leq_\Gamma w$. Then the unstable and stable varieties $S^-(y_v, g)$ and $S^+(y_w, g)$ for the smalefication do not intersect but are very close to intersecting. So our idea is to look at the relative position of these varieties.

We first assume that Γ is a simple path $\bullet \to \bullet \to \ldots \to \bullet$. Let γ be the union of the gradient trajectories of f joining the critical points x_v. Then the varieties $S^-(y_v, g)$ and $S^+(y_w, g)$ come very close to each other near γ, but are otherwise far away. To analyze the relative position of these varieties, we take a regular value $a \in [g(y_w), g(y_v)]$ and consider the submanifolds

$$S^-(v, w) := S^-(y_v, g) \cap g^{-1}(a), \quad S^+(v, w) := S^+(y_w, g) \cap g^{-1}(a).$$

These subvarieties are in the linking dimension, i.e., their dimension sum to $d - 2 = \dim(g^{-1}(a)) - 1$. So we would like to associate to the pair (v, w) a kind of "local linking number" of $S^\pm(v, w)$, an element of ± 1. This is to be viewed as higher-dimensional analog of saying which critical point "eats" which, like in Fig. 16.

To be consistent, we should take care of choices of orientation. Let us do this in some detail.

(4.6) Reminder on orientations and linking numbers. For a finite-dimensional real vector space V its orientation torsor $o(V)$ is the 2-element set of "directions" of the 1-dimensional real vector space $\bigwedge^{max}(V)$, the top exterior power. An orientation of V is an element of $o(V)$. Clearly, $o(V)$ is a torsor (principal homogeneous space) over the group $\{\pm 1\}$. For a short exact sequence

$$0 \to V' \to V \to V'' \to 0$$

of finite-dimensional real vector spaces we have a natural identification $o(V') \otimes o(V'') \to o(V)$, where $\otimes$ is the tensor product of torsors.

An orientation of a manifold M is a consistent orientation of each tangent space of this manifold. If M is orientable, then the two possible orientations form a torsor over $\{\pm 1\}$ which we denote $o(M)$.

If L, M are two non-intersecting affine subspaces of a real affine space V, such that $\dim(L)+\dim(M) = \dim(V)-1$ and all three spaces L, M, V are equipped with orientations, then we have a well-defined linking index $\langle L, M\rangle_V \in \{\pm 1\}$. It is changed to the opposite by a change of orientation of each of the three ingredients: L, M or V; in particular, it remains unchanged under a simultaneous change of two of the orientations.

More invariantly, without making any choices of orientation, we can say that $\langle L, M\rangle_V$ is not a number but an element of the torsor $o(L)\otimes o(M)\otimes o(V)^{-1}$; then, a choice of the orientations identifies this torsor with $\{\pm 1\}$ and we get a number. Note also that for any torsor T over $\{\pm 1\}$ we can identify T with T^{-1}, so we will not write the inverse sign in the future.

(4.7) Local linking numbers. Return to our situation of a Morse function $f \in \Sigma(\Gamma, m)$. Note that for any critical point $x_v, v \in \mathrm{Vert}(\Gamma)$ we have an identification

$$p_v : o(S^+(x_v, f)) \otimes o(S^-(x_v, f)) \to o(X)$$

coming from the direct sum decomposition of $T_{x_v}X$ into the direct sum of $T_{x_v}S^{\pm}(x_v, f)$. Thus every system $\epsilon = (\epsilon_v)$ of orientations of all the unstable manifolds $S^-(x_v, f)$ together with an overall orientation ζ of X itself give rise to a well defined system $\lambda = (\lambda_v)$ of orientations of the $S^+(x_v)$. More precisely, λ_v is defined so as to have $p_v(\lambda_v \otimes \epsilon_v) = \zeta$. Given a smalefication g of f, with critical points y_v, denote by ϵ'_v, λ'_v the orientations of $S^{\pm}(y_v, g)$ obtained from ϵ_v, λ_v by continuity. The orientations ζ, ϵ'_v and λ_w give rise to orientations of

$$g^{-1}(a), \quad S^-(v,w) = S^-(y_v)\cap g^{-1}(a), \quad S^+(v,w) = S^+(y_w)\cap g^{-1}(a)$$

by trivializing, by means of dg, the 1-dimensional normal bundle to each of these varieties in $X, S^-(y_v), S^+(y_w)$ respectively.

We now have the oriented submanifolds (spheres) $S^{\pm}(v,w) \subset g^{-1}(a)$ which come very close together in the vicinity of one point $\gamma \cap f^{-1}(a)$, see (4.5). In this region we can replace these manifolds by affine spaces, take their linking index, denote it by $l(v,w)$ and call the *local linking index* of the $S^{\pm}(v,w)$. This is an element of the torsor $o(S^+(v,w))\otimes o(S^-(v,w))\otimes o(f^{-1}(a))$. By the choices of orientations ζ, ϵ'_v and λ_w we identify this torsor with $\{\pm 1\}$ and thus view $l(v,w)$ as a number. The collection of these numbers is an obvious combinatorial invariant of the smalefication.

Formally, the numbers $l(v,w)$ depend on the choice of $\epsilon = (\epsilon_v)$ and $\zeta \in o(X)$. However, changing ζ to the opposite orientation of X changes each λ_v to the opposite as well. So in each of the linking numbers $\langle S^+(v,w), S^-(v,w)\rangle_{g^{-1}(a)}$ the orientations of two of the ingredients will be changed and thus the values of the linking numbers will not change. This shows that the $l(v,w)$ do not depend on ζ, but depend only on ϵ. Let us denote them therefore $l_\epsilon(v,w) \in \{\pm 1\}$. If we change just one ϵ_v to the opposite, then the result is that each of the linking numbers involving v will change sign. Thus a simultaneous change of all the orientations ϵ_v to the opposite ones does not affect the $l(v,w)$. So the collection of these linking numbers really depends only on a compatible system of identifications of all the torsors $o(S^-(x_v, f))$. We will now discuss how to construct such a system with good properties and what are the necessary conditions for that.

(4.8) Identifying orientation torsors of unstable manifolds. Let $f \in \Sigma(\Gamma, m)$ be a Morse function. Let v and w be two vertices of Γ such that there is an edge $v \xrightarrow{e} w$, and γ be the gradient trajectory of f going from x_v to x_w. Strictly speaking, the trajectory does not reach x_w at any finite time, but we will include the endpoints x_w and x_w into γ as well and speak about the tangent lines to γ at these points (they are clearly well defined as the limits of the tangent lines at interior points of γ).

Let us look at other trajectories issuing from x_v. All together they form the unstable manifold $S^-(x_v) = S^-(x_v, f)$. But since one of the trajectories hits $x_w \in S^-(x_w, f)$, the closure $\overline{S^-(x_v)}$ of $S^-(x_v)$ in X may have non-trivial intersection with $S^-(x_w)$. In other words, trajectories issuing from x_v and close to γ will asymptotically touch $S^-(x_w)$ and the intersection $\overline{S^-(x_v)} \cap S^-(x_w)$ is formed by such asymptotic points. We are interested only in the germ of this intersection near x_w, or, rather, in its "tangent space". In fact, the following infinitesimal analysis will be sufficient for our purposes (we do not need any details of actual structure of $\overline{S^-(x_v)} \cap S^-(x_w)$, but will define what should be its tangent space by direct construction).

Take an interior point $p \in \gamma$ and look at the tangent space $T_pS^-(x_v) \subset T_pX$. For any $t \in \mathbf{R}_+$ let $E_t : X \to X$ be the time t translation along the gradient flow. Then $(d_pE_t)(T_pS^-(x_v))$ is a subspace in the tangent space to $E_t(p) \in \gamma$. When $t \to \infty$, we have that $E_t(p) \to x_w$. So the limit position

$$L_v^w = \lim_{t\to\infty} (d_pE_t)(T_pS^-(x_v))$$

is an m-dimensional subspace in $T_{x_w}X$. The intersection of this subspace with $T_{x_w}S^-(x_w)$ is the desired "tangent space" we are interested in.

(4.8.1) Proposition. *The intersection of L_v^w with $T_{x_w}S^-(x_w)$ has codimension 1 in $T_{x_w}S^-(x_w)$. The space L_v^w is the direct sum of this intersection and the 1-dimensional subspace $T_{x_w}\gamma$.*

Informally, this means that $\overline{S^-(x_v)} \cap S^-(x_w)$ is a hypersurface in $S^-(x_w)$.

Proof: Take a small neighborhood U of x_w and choose coordinates $(s_1, ..., s_d)$ in U such that the gradient of f is given by $\sum_{i=1}^{d-m} s_i\partial/\partial s_i - \sum_{i=d-m+1}^{d} s_i\partial/\partial s_i$. Thus $S^-(x_w)$ is, in these coordinates, the m-dimensional linear subspace given by $s_1 = ... = s_{d-m} = 0$, while $S^+(x_w)$ is given by $s_{d-m+1} = ... = s_d = 0$. Now if p is any point in U lying on $S^+(x_w)$ and Λ is any linear subspace in T_pX, then it is very easy to analyze the subspace

$$L = \lim_{t\to\infty} (d_pE_t)(\Lambda) \in T_{x_w}X.$$

First of all, this subspace is invariant under the 1-parameter subgroup of linear transformations of $T_{y_w}X$ given by differentials of the E_t. This follows from the definition of L as a limit. The 1-parameter subgroup in question multiplies $s_1, ..., s_{d-m}$ by t^{-1} and $s_{d-m+1}, ..., s_d$ by t. Therefore L, being invariant under such transformations, should have the form $L = L_+ \oplus L_-$ where $L_\pm \subset T_{x_w}S^\pm(y_w)$. Further, $\dim(L_+) = \dim(\Lambda \cap T_pS^+(x_w))$ and thus $\dim(L_-)$ is found as the difference.

In our case $\Lambda = T_pS^-(x_v)$ we have that $\Lambda \cap T_pS^+(x_w) = T_p\gamma$ is 1-dimensional, whence the statement.

Let us denote the hyperplane $L_v^w \cap T_{x_w}S^-(x_w)$ in $T_{x_w}S^-(x_w)$ by M_v^w. Note that the translation along the gradient flow defines an identification of torsors $o(S^-(x_v)) \to o(L_v^w)$, and the direction of the flow identifies $o(L_v^w)$ with $o(M_v^w)$.

Let now $v \to w \to u$ be two consecutive arrows in a graph Γ, and let γ_{vw}, γ_{wu} be the gradient trajectories joining x_v with x_w and x_w with x_u.

(4.8.2) Definition. *A Morse function $f \in \Sigma(\Gamma, m)$ is called generic, if for each two consecutive arrows of Γ as above we have that the hyperplane $T_{x_w}(M_v^w)$ in $T_{x_w}S^-(x_w)$ does not contain the line $T_{x_w}\gamma_{wu}$.*

Let f be a generic Morse function and v, w, u be as above. Then we have a natural identifications

$$o(S^-(x_v)) \to o(M_v^w) \to o(S^-(x_w))$$

of which the first one was constructed above and the second one is obtained by trivializing the 1-dimensional normal bundle to M_v^w in $T_{y_w}S^-(y_w)$ by means of the direction of γ_{wu}. Let α_{vwu} be the composite identification.

Further, suppose that $(-f)$ is a generic Morse function. Then the unstable manifolds for $(-f)$ are just the stable manifolds for f and vice versa. So we get an identification

$$o(S^+(x_u)) = o(X) \otimes o(S^-(x_u)) \to o(S^+(x_w)) = o(X) \otimes o(S^-(x_w)).$$

By tensoring it with the identity map of $o(X)$, we get also an identification $\beta_{vwu} : o(S^-(x_u) \to o(S^-(x_w)$.

Thus for any triple of consecutive vertices of a graph we have an identification of all three orientation torsors of the unstable manifolds of the corresponding critical points.

From now on we assume that the graph Γ is irreducible in the sense of (2.1). This is because for a reducible graph the bifurcation diagrams are just products of the diagrams corresponding to the irreducible components. Consider the binary relation "being connected" on the set of arrows of Γ defined as the symmetric and transitive closure of the relation "being composable". To say that Γ is irreducible (which we assume) is equivalent to saying that any two edges are connected. Therefore for any two vertices z, z' of an irreducible graph Γ there is at least one composite identification $o(S^-(x_z)) \to o(S^-(x_{z'}))$ constructed out of the $\alpha_{vwu}, \beta_{vwu}$. However, we may get more than one such identification. In order to make sure that this does not happen, additional conditions on f are necessary. More precisely, let $w \in \mathrm{Vert}(\Gamma)$ be any vertex. Then for all v such that there is an edge $v \to w$ we have a hyperplane $M_v^w \subset T_{x_w}S^-(x_w)$. These hyperplanes cut out $T_{x_w}S^-(x_w)$ into certain conical chambers (i.e., the connected components of the complement to all the hyperplanes). On the other hand, for any vertex u such that there is an edge $w \to u$ we have a gradient trajectory γ_{wu} issuing from x_w. This trajectory defines the half-line $T^+_{x_w}\gamma_{wu}$ which is the half of the usual tangent line pointing in the direction of the flow.

(4.8.3) Definition. *A Morse function $f \in \Sigma(\Gamma, m)$ is called 1-sided, if all the half-lines $T^+_{x_w}\gamma_{wu}$ for $w \to u$ lie in the same chamber of the complement to the union of the hyperplanes M_v^w for $v \to w$.*

(4.8.4) Proposition. *If both f and $(-f)$ are 1-sided, then all the identifications $\alpha_{vwu}, \beta_{vwu}$ are compatible with each other, i.e., define, for any two vertices v, w, a unique identification $o(S^-(x_v)) \to o(S^-(x_w))$.*

Having constructed the compatible identifications, we get, by the above, well-defined local linking numbers $l(v, w)$.

(4.9) Smalefications and prime ideals. Let $f \in \Sigma(\Gamma, i)$ be a Morse function such that both f and $(-f)$ are 1-sided, and let g be a smalefication of f. Let $p : v \to w$ be any oriented edge path in Γ, and γ the corresponding geometric path in X formed by the gradient trajectories. Then $S^-(v, g)$ and $S^+(w, g)$ come very close to each other in the vicinity of γ. So by the above identification of the orientation torsors we have well-defined local linking numbers $l_p(v, w) \in \{\pm 1\}$. The subscript p is necessary since in general there may be several edge paths from v to w so there is more than one local linking number,

(4.9.1) Proposition. *The collection of all edge paths $p : v \to w$ such that $l_p(v, w) = -1$ forms a prime ideal in the category $\pi(\Gamma)$. Equivalently, the family F formed by paths p such that $l_p(v, w) = +1$, is admissible.*

Proof: Both conditions (3.3.3-4) for F to be admissible involve only linear subgraphs in Γ. So in the proof we can assume that Γ is a linear graph. With this assumption, let us first look at the case $d = \dim(X) = 2$, $m = 1$. Suppose f is a function allowing a chain of n critical points $x_1, ..., x_n$ of index 1 joined by gradient trajectories according to the graph $x_1 \to ... \to x_n$. Thus at every x_i, $i < n$, one of the two outgoing trajectories goes to x_{i+1}. Further, our choice of orientations of the unstable varieties is precisely the one obtained from the orientation of X, if we agree that the trajectory $\gamma_{x_i, x_{i+1}}$ is obtained by making a left turn after coming to x_i along γ_{x_{i-1}, x_i}. In other words, if we consider the union of all the outgoing trajectories of all the x_i, then it is just the system Ψ_n of n mounting caps in Fig.11. Similarly, for every deformation of f the union of outcoing trajectories will be a system of mounting caps refining Ψ_n, in particular, for a Morse-Smale deformation we will have a system of nested caps without intersections. In these terms the condition $l_p(i, j) = 1$ (with p being the unique path joining i and j) means geometrically that the cap number i contains "inside" it the cap number j, so the statement of the Proposition is clear for the case $d = 2$, $m = 1$.

Consider now the general case. As we said, we can assume that Γ is a linear graph, so the system of trajectories has the form $x_1 \to ... \to x_n$. Let us take a 2-dimensional surface ("ribbon") $X' \subset X$ containing all the x_i and all the trajectories $\gamma_{x_i, x_{i+1}}$ (thus the tangent space to X' at x_i, $1 < i < n$, is spanned by the tangent lines of the incoming and the outgoing trajectory). Given a Smaleficiation g of f with critical points y_i (corresponding to the x_i by continuity), we can deform X' along with f into a surface X'' containing the y_i and such that $T_{y_i}X''$ intersects both $T_{y_i}S^{\pm}(y_i, g)$ in 1-dimensional subspaces. So we can identify X' and X'' and view $g|_{X''}$ as a Smalefication of $f|_{X'}$. Then, by quotienting out the overall orientation torsor for the normal bundle of X'' in X, we reduce the calculation of the local linking number $l_p(i, j)$ for g to a similar calculation for $g|_{X''}$, i.e., to the situation we have just considered. This proves the general case.

(4.10) Proof of Theorem 4.4. If Γ is the linear graph, as in the theorem, then any generic Morse function is automatically one-sided, since at the tangent space to every x_v we have just one oriented direction and one hyperplane. We recall the relevant notation from (4.2): so T is a transversal slice to $\Sigma(\Gamma, m)$ at f, and Δ is the discriminantal variety, so the vertices of $P_f(\Gamma, m)$ correspond to chambers, i.e., to connected components of $T - \Delta$. Let $\mathcal{C}$ be the set of chambers.

Note that we can assume that the variety X is a domain in $\mathbf{R}^d$, since we are interested only in what happens in the neighborhood of the critical points and the corresponding trajectories, and this neighborhood can be embedded into $\mathbf{R}^d$.

Assuming that f is generic, Proposition 4.9.1 associates to any chamber C a prime ideal $\wp_C$ in $\pi(\Gamma)$. The resulting map $\mathcal{C} \to \mathrm{Spec}(\Gamma)$ is surjective. To see this, note that any system of nested caps, i.e., a Smalefication of a Morse function $f(t_1, t_2)$ in $\mathbf{R}^2$ can be extended to a Smalefication of a Morse function in $\mathbf{R}^d$ by adding to it the function $\sum_{i=3}^d \epsilon_i t_i^2$ where the signs $\epsilon_i \in \{\pm 1\}$ are chosen to achieve the desired index m. Further, by deforming the 2-dimensional surface X' as in the proof of (4.9.1), together with the functions, we find that each pair of adjacent chambers gives rise to two elements of $\mathrm{Spec}(\Gamma)$ joined by an edge as described in (3.5). This means that we get a map of polyhedral complexes $P_f(\Gamma, m) \to \mathcal{P}(\Gamma) = K_n$ surjective on vertices, and the fact that it is an isomorphism follows by noticing that, similarly to the faces of K_n, each face of $P_f(\Gamma, m)$ is a product of similar bifurcation diagrams corresponding to linear graphs describing trajectories in the corresponding partial Smalefication.

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