STABLE BUNDLES ON HIRZEBRUCH SURFACES

by

N.P. Buchdahl

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
D - 5300 Bonn 3

Sonderforschungsbereich 40
Theoretische Mathematik
Beringstraße 4
D - 5300 Bonn 1

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N.P. Buchdahl
Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
5300 Bonn 3, Federal Republic of Germany

SUMMARY

An analogue of Beilinson's theorem on the structure of coherent sheaves on $\mathbf{P}_N$ is given for the Hirzebruch surfaces $H_n = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-n)) \longrightarrow \mathbf{P}_1$, from which a monad description of stable 2-bundles with $c_1 = 0$, $c_2 = k$ on H_n is derived. The moduli space of such bundles is explicitly computed in the case $k = 2$, it being shown to be the projectivized $\mathcal{O} \oplus \mathcal{O}(n) \oplus B_n$ bundle over $\mathbf{P}_2$ minus a quadratic hypersurface, where B_n is a certain 2-bundle on $\mathbf{P}_2$ with $c_1(B_n) = n+1$ and $c_2(B_n) = \frac{1}{2}n(n+1)$. For $n = 0$, $B_0 = \mathcal{O} \oplus \mathcal{O}(1)$ and an additional $\mathbf{P}_2$ is removed.

KEY WORDS: stable bundle, Hirzebruch surface, anti-self-dual Yang-Mills field.

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INTRODUCTION

The purpose of this paper is to provide a monad description of stable bundles on the Hirzebruch surfaces $H_n = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-n)) \longrightarrow \mathbf{P}_1$, and to consider in particular the moduli space of stable 2-bundles with $c_1 = 0$, $c_2 = 2$ on H_n.

The description follows the established method for classifying stable bundles on $\mathbf{P}_2$ as presented in [OSS], relying on a generalisation of Beilinson's theorem [B] on the structure of coherent analytic sheaves on P_N .

The motivation for this work is provided by recent results of S.K. Donaldson: in [D2] he proves that an m-bundle E on an algebraic surface X admits an irreducible anti-self-dual $U(m)$ connection iff E is stable (in the sense of Mumford and Takemoto), where the notions of stability and anti-self-duality are linked by a fixed embedding $X \hookrightarrow \mathbf{P}_N$. In [D3] he considers simply-connected smooth 4-manifolds X with even intersection form Q , and he proves that if $b_+(X) = 1$ or 2 , then Q is the intersection form of $S^2 \times S^2$ or $S^2 \times S^2 \# S^2 \times S^2$ respectively. His methods, like those of his earlier paper [D1], involve a deep analysis of the moduli spaces M_k of anti-self-dual $SU(2)$ connections with $c_2 = k$ on X , where X is equipped with a generic metric. In the case $b_+(X) = 1$, the space considered is M_2.

Since H_{2m} (resp. H_{2m+1}) is diffeomorphic to $H_0 = \mathbf{P}_1 \times \mathbf{P}_1$ (resp. $H_1 = \mathbf{P}_2 \# \overline{\mathbf{P}}_2$), determining the moduli space of stable bundles on H_n as n varies corresponds to determining the moduli space of anti-self-dual connections on a fixed bundle over $S^2 \times S^2$ or $\mathbb{C}P_2 \# \overline{\mathbb{C}P}_2$ as the metric varies. For a generic metric on $X = S^2 \times S^2$ or $\mathbb{C}P_2 \# \overline{\mathbb{C}P}_2$, the gauge theoretic prediction for M_2 is that of a smooth 10-manifold with a natural compactification $\overline{M}_2$ such that $\overline{M}_2 \setminus M_2$ is contained in $M_1 \times X \cup S^2X$, where S^2 denotes symmetric product. If the metric is non-generic, cone-like singularities can occur in $\overline{M}_2$ resulting from reductions from $SU(2)$ to $U(1) \times U(1)$. These predictions are indeed well fulfilled, as is indicated in Propositions 1 and 2 below.

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1. BEILINSONS'S THEOREM REVISITED

To fix notation, let $\pi : H_n \longrightarrow \mathbf{P}_1$ be the projection, $\mathcal{O}(-1,0)$ be the tautological bundle of the projectivization $H_n = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(n))$, $(n \geq 0)$, and let $\mathcal{O}(0,-1) := \pi^*\mathcal{O}_{\mathbf{P}_1}(-1)$.

As usual, $\mathcal{O}(p,q) := \mathcal{O}(p,0) \otimes \mathcal{O}(0,q)$, so for example, the canonical bundle of H_n is $\mathcal{O}(-2,n-2)$. Let (z_A, w_B) $A,B = 0,1$ be homogeneous coordinates on H_n with z_A being homogeneous coordinates on P_1 and (w_0, w_1) homogeneous of degrees $(1,0)$ and $(1,-n)$ respectively.

Let $\pi_i : H_n \times H_n \longrightarrow H_n$ be projection onto i-th factor, and set $\mathcal{O}(p,q)(r,s)' := \pi_1^*\mathcal{O}(p,q) \otimes \pi_2^*\mathcal{O}(r,s)$. If $Y := \left\{((z,w),(z',w')) \in H_n \times H_n : z_0 z_1' = z_1 z_0'\right\}$, then $\mathcal{O}(0,1)(0,-1)'|_Y \approx \mathcal{O}_Y$, and let $s \in \Gamma(Y, \mathcal{O}_Y(0,1)(0,-1)')$ be the section corresponding to 1 under this isomorphism. The diagonal Δ in $H_n \times H_n$ is then the zero set of $t := w_0 w_1' - w_1 w_0' s^n \in \Gamma(Y, \mathcal{O}_Y(1,0)(1,-n)')$.

Let R be the extension $0 \longrightarrow \mathcal{O}(1,0)(1,-n)' \longrightarrow R \longrightarrow \mathcal{O}(0,1)(0,1)' \longrightarrow 0$ corresponding to the image $\delta t \in H^1(H_n \times H_n, \mathcal{O}(1,-1)(1,-n-1)')$ of t under the connecting homomorphism from

$$0 \longrightarrow \mathcal{O}(1,-1)(1,-n-1)' \xrightarrow{z \cdot z'} \mathcal{O}(1,0)(1,-n)' \longrightarrow \mathcal{O}_Y(1,0)(1,-n)' \longrightarrow 0 ,$$

where $z \cdot z' := z_0 z_1' - z_1 z_0'$. Since $z \cdot z' \, \delta t = 0$, there is a section $U \in \Gamma(H_n \times H_n, R)$ in the preimage of $z \cdot z' \in \Gamma(H_n \times H_n, \mathcal{O}(0,1)(0,1)')$, and by construction, there is a unique such U whose restriction to Y is the image of $-t$ in $\Gamma(Y, R|_Y)$. It follows $U^{-1}(0)$ is precisely Δ , giving the Koszul resolution

$$(1.1) \qquad 0 \longrightarrow \mathcal{O}(-1,-1)(-1,n-1)' \longrightarrow R^* \xrightarrow{U} \mathcal{O} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0 .$$

If E is a holomorphic bundle on H_n , tensor through (1.1) by $\pi_2^* E$, delete the last term on the right, and take direct images under π_1 . This gives the following H_n-analogue of Beilinson's theorem [B] as presented in [OSS].

<u>LEMMA</u>: For any holomorphic bundle E on H_n , there is a spectral sequence

$$E_1^{p,q} \Longrightarrow E_\infty^{p+q} = \begin{cases} E & \text{if } p+q = 0 \\ 0 & \text{otherwise} \end{cases} ,$$

with $E_1^{p,q} = 0$ if $|p+1| > 1$, $E_1^{0,q} = H^q(E) \otimes \mathcal{O}$, $E_1^{-2,q} = H^q(E(-1,n-1)) \otimes \mathcal{O}(-1,-1)$, and an exact sequence
$\dots \to H^q(E(0,-1)) \otimes \mathcal{O}(0,-1) \to E_1^{-1,q} \to H^q(E(-1,n)) \otimes \mathcal{O}(-1,0) \to \dots$

□

(A different spectral sequence can be obtained by interchanging the roles of π_1 and π_2).

2. STABLE 2-BUNDLES ON H_n

To illustrate applications of the lemma, the case of stable 2-bundles with $c_1 = 0$ and $c_2 = k$ will be considered; let such a bundle E be given.

By the Leray-Hirsch theorem, the cohomology ring of H_n is $H^*(H_n, \mathbf{Z}) = \mathbf{Z}[x,y] / x^2 - nxy, y^2$, where $x = c_1(\mathcal{O}(1,0))$ and $y = c_1(\mathcal{O}(0,1))$; the fundamental class is $xy \in H^4(H_n, \mathbf{Z})$. H_n is embedded in $\mathbf{P}_3$ by
$H_n \ni (z,w) \longmapsto (z_0 w_0, z_1 w_0, z_0^{n+1} w_1, z_1^{n+1} w_1) \in \mathbf{P}_3$, so
$\mathcal{O}_{\mathbf{P}_3}(1)\big|_{H_n} = \mathcal{O}(1,1)$ and it follows that the condition of stability is $H^0(E(p,q)) = 0$ whenever $(n+1)p + q \leq 0$. Using Serre Duality, it follows that

$$(2.1) \qquad H^r(E(p,q)) = 0 \quad \text{for} \quad r = 0,2 \quad \text{if} \quad -n-4 \leq (n+1)p + q \leq 0 \ .$$

The Riemann-Roch formula for E is
$\chi(E(p,q)) = (p+1)(np+2q+2) - k$, so if $K_1 := H^1(E(-2,n-1))$, $K_2 := H^1(E(-1,-1))$, $K_3 := H^1(E(-1,0))$ and $L := H^1(E(-2,n))$, it follows from (2.1) that $\dim K_i = k$ and $\dim L = k+2$. Applying the lemma to $E(-1,0)$ and using (2.1) then gives a monad
$0 \longrightarrow K_1(-1,-1) \longrightarrow E_1^{-1,1} \longrightarrow K_3 \longrightarrow 0$ with cohomology $E(-1,0)$, together with an exact sequence
$0 \longrightarrow K_2(0,-1) \longrightarrow E_1^{-1,1} \longrightarrow L(-1,0) \longrightarrow 0$. Since $H^1(H_n, \mathcal{O}(1,-1)) = 0$, this last sequence splits (but not uniquely unless $n = 0$), and after tensoring through by $\mathcal{O}(1,0)$, the result is a monad

$$(2.2) \qquad M : 0 \longrightarrow K_1(0,-1) \longrightarrow K_2(1,-1) \oplus L \longrightarrow K_3(1,0) \longrightarrow 0$$

with cohomology $E(M) = E$.

Monads of the form (2.2) satisfy the hypotheses of Lemma 4.1.3 of [OSS], implying $E(M) \simeq E(M')$ iff $M \simeq M'$. They also satisfy the hypotheses of Lemma 4.1.7 of [OSS], and since $H^2(\mathrm{End}E) = 0$, a repitition of the analysis there leads to a concrete description of the moduli space of such bundles as a non-singular (4k-3)-dimensional quotient of a subspace of $\mathbb{C}^N$ by a matrix group.

3. THE CASE $c_2 = 2$

The Riemann-Roch formula implies that there are no stable 2-bundles on H_n with $c_1 = 0$, $c_2 = 1$. When $c_1 = 0$ and $c_2 = 2$, the lemma yields a more useful description of E than (2.2). In this case $H^*(E) = 0$, and by using (2.1) together with the lemma applied directly to E , the following exact sequence is immediately obtained:

$$(3.1) \qquad 0 \longrightarrow K_1(-1,-1) \xrightarrow[b]{a} \begin{array}{c} K_2(-1,0) \\ \oplus \\ K_3(0,-1) \end{array} \longrightarrow E \longrightarrow 0 \quad .$$

(Here $K_1 := H^1(E(-1,n-1))$, $K_2 := H^1(E(-1,n))$ and $K_3 := H^1(E(0,-1))$, and all are 2-dimensional vector spaces).

The bundle E is thus determined by a pair $a \in \mathrm{Hom}(K_1 \otimes V, K_2)$, $b = (b_0, b_1) \in \mathrm{Hom}(K_1, K_3) \oplus \mathrm{Hom}(K_1 \otimes S^n V, K_3)$ where, for notational convenience, $\mathbb{C}^2$ has been replaced by a 2-dimensional symplectic vector space V and S^n denotes n-th symmetric tensor product. The pair (a,b) is not completely arbitrary: in order that E in (3.1) be non-singular, it is necessary and sufficient that

$$(3.2) \qquad (a(z), b(z,w)) : K_1 \longrightarrow K_2 \oplus K_3 \quad \text{is injective at each} \quad (z,w) \in H_n \quad ,$$

and moreover the stability criteria must be fulfilled. From the exact sequences $0 \longrightarrow \mathcal{O}(0,-1) \xrightarrow{z} V \longrightarrow \mathcal{O}(0,1) \longrightarrow 0$ and $0 \longrightarrow \mathcal{O}(-1,0) \xrightarrow{w} \mathcal{O} \oplus \mathcal{O}(0,-n)) \longrightarrow \mathcal{O}(1,-n) \longrightarrow 0$ it follows that for any bundle E, $H^0(E(p-1,q)) = 0 = H^0(E(p,q-1))$ if $H^0(E(p,q)) = 0$, so stability in the current context is equivalent to $H^0(E(p,-(n+1)p)) = 0$ for all p. Using $(3.1) \otimes \mathcal{O}(p,-(n+1)p)$ and $(3.1)^* \otimes \mathcal{O}(p,-(n+1)p)$, it is quickly found that almost all of these conditions are automatically satisfied by a <u>bundle</u> E defined by (3.1), and the stability of the bundle can be reduced to

(3.3)

(a) $a : K_1 \to K_2 \otimes V^*$, $a^* : K_2^* \to K_1^* \otimes V^*$ are injective;

(b) for $n = 0$: $b : K_1 \to K_3 \otimes V^*$, $b^* : K_3^* \to K_1^* \otimes V^*$ are injective;

for $n > 0$: $(a,b) : K_1 \otimes S^n V \to K_2 \otimes S^{n-1} V \oplus K_3 \otimes S^n V \oplus K_3$ is injective.

Since $H^p(H_n, \mathcal{O}(-1,0)) = 0 = H^p(H_n, \mathcal{O}(0,-1))$ for all p, an isomorphism $E \simeq E'$ extends to a unique isomorphism of exact sequences $(3\cdot1) \simeq (3\cdot1)'$. It follows that $E \simeq E'$ iff

$$(3.4) \qquad a' = g_2 a g_1^{-1} \ , \ (b_0', b_1') = g_3 (b_0, b_1 + ah) g_1^{-1}$$

for some $g_i \in GL(K_i)$ and $h \in \mathrm{Hom}(K_2 \otimes S^{n-1}V, K_3)$. The moduli space of stable 2-bundles with $c_1 = 0$, $c_2 = 2$ on H_n is thus identified with the set of pairs (a,b) satisfying (3.2), (3.3), modulo the group action (3.4), and it now remains to simplify this description.

4. THE CASE $n = 0$

Fix non-degenerate symplectic forms on each of the vector spaces K_i ; then from $a \in \mathrm{Hom}(K_1 \otimes V, K_2)$, three "determinants" can be formed: $\det_0 a \in S^2V^*$, $\det_1 a \in S^2K_1^*$ and $\det_2 a \in S^2K_2$. These determinants are not independent, for each of the spaces S^2V^*, $S^2K_1^*$, S^2K_2 posseses a non-degenerate symmetric bilinear form $\cdot$ canonically induced by the symplecitc forms on the 2-dimensional vector spaces, and

$$\det_0 a \cdot \det_0 a = \det_1 a \cdot \det_1 a = \det_2 a \cdot \det_2 a \quad .$$

The condition (3.3) (a) is equivalent to $\det_0 a \neq 0$, so a gives rise to a point $[\det_0 a] \in \mathbf{P}(S^2V^*)$ which is independent of $a \longmapsto g_2 a g_1^{-1}$. In fact, the map $[a] \longmapsto [\det_0 a]$ is bijective granted (3.2). For by choosing an appropriate basis for V, it can be supposed that $a = (a_0, a_1)$ for some $a_i \in \mathrm{Hom}(K_1, K_2)$ with $\det_0 a \neq 0$. After fixing an isomorphism $K_1 = K_2$, it can then be supposed that $a_0 = 1$ and a_1 is in Jordan form. If a_1 does not have distinct eigenvalues, then it cannot be of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$. Otherwise $a(z) : K_1 \to K_2$ is the zero map at $z = (-\lambda, 1)$. This can be ruled out because on $\pi^{-1}(z)$, $\det(b(z,w))$ must have a zero, implying that (3.2) fails at some point on $\pi^{-1}(z)$. Thus, in this case, a_1 must have the form $\begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$ and it is now straightforward to verify the bijectivity of $[a] \longmapsto [\det_0 a]$.

The above is valid for all $n \geq 0$, but for $n = 0$, (3.3)(b) implies $\det_0 b \in S^2V^*$ is non-zero, giving the point $[\det_0 b] \in \mathbf{P}(S^2V^*)$ independent of (3.4). Since the moduli space has dimension 5, there remains one parameter to be found, and an obvious guess is the class $D(a,b) := \det_1 a \cdot \det_1 b$. Under the action (3.4), $D(a,b) \longmapsto (\det g_2)(\det g_3)(\det g_1)^{-2} D(a,b)$, so (a,b) gives rise to the point $([\det_0 a], [\det_0 b], D(a,b))$ in the total space of the bundle $\mathcal{O}(1,1)$ over $\mathbf{P}(S^2V^*) \times \mathbf{P}(S^2V^*)$, and this point is independent of (3.4).

The non-singularity condition (3.2) fails iff $\det_1 a$, $\det_1 b$ have a common root; i.e. a vector $k \in K_1$ such that $(\det_1 a)(k \otimes k) = 0 = (\det_1 b)(k \otimes k)$. This can occur iff $(\det_1 a \cdot \det_1 a)(\det_1 b \cdot \det_1 b) = (\det_1 a \cdot \det_1 b)^2$. From this point, it is straight-forward algebra in local coordinates to arrive at the following

PROPOSITION 1: The map $[a,b] \mapsto ([\det_0 a],[\det_0 b],D(a,b))$ defines a bijection from the moduli space of stable 2-bundles with $c_1 = 0$, $c_2 = 2$ over $\mathbf{P}_1 \times \mathbf{P}_1$ with the total space L of the line bundle $\mathcal{O}(1,1)$ over $\mathbf{P}_2 \times \mathbf{P}_2$ minus the hypersurface $H = \{(x,y,z) \in L : x \cdot xy \cdot y = z^2\}$.

□

Concerning the boundary of the moduli space, observe that the map

$S^2(\mathbf{P}_1 \times \mathbf{P}_1) \ni [(z,w),(z',w')] \mapsto (z \otimes z' + z' \otimes z, w \otimes w' + w' \otimes w, 2z \cdot z' w \cdot w) \in L$

is well-defined, and its image is contained in H. It is easily verified that this defines a biholomorphism of $S^2(\mathbf{P}_1 \times \mathbf{P}_1)$ with H.

The space L is not compact, resulting from the non-genericity of the product Fubini-Study metric on $\mathbf{P}_1 \times \mathbf{P}_1$. It can be compactified by adding the semi-stable (non-zero) extensions $0 \to \mathcal{O}(-1,1) \to E \to \mathcal{O}(1,-1) \to 0$ or $0 \to \mathcal{O}(1,-1) \to E \to \mathcal{O}(-1,1) \to 0$ ($\det_0 a = 0$ or $\det_0 b = 0$ respectively) to give the projectivized $3\mathcal{O} \oplus \mathcal{O}(1)$ bundle over $\mathbf{P}_2$. Alternatively, the bundle $\mathcal{O}(1,-1) \oplus \mathcal{O}(-1,1)$ can be added alone, topologically giving a cone over the circle bundle with $c_1 = (1,1)$ on $\mathbf{P}_2 \times \mathbf{P}_2$.

5. THE CASE $n > 0$

The case $n > 0$ can be made to closely resemble the case $n = 0$ in the following way. First, pick a basis for V and a fixed isomorphism $K_1 = K_2$, so $a = (a_0, a_1)$ for some $a_i \in \mathrm{End} K_1$. Since $\det_0 a \neq 0$, one of $\det a_0$, $\det a_1$, $\det(a_0 + a_1)$ must be non-zero, corresponding to three open sets U_0, U_1, U_2 covering $\mathbf{P}(S^2V^*)$. After replacing a by ga for suitable $g \in GL(K_1)$, it can be supposed that $a_0 = 1$, $a_1 = 1$, or $a_0 + a_1 = 1$ as $[\det_0 a] \in U_i$. In particular, a_0 and a_1 then commute, and it follows that the homomorphism $\tilde{b}_1 := (b_1)_{A_1 \ldots A_n} a^{A_1 \ldots A_n} \in \mathrm{Hom}(K_1, K_3)$ is independent of $b_1 \longmapsto b_1 + ha$ for $h \in \mathrm{Hom}(K_1 \otimes S^{n-1}V, K_3)$. Here $a^0 := a_1$, $a^1 := -a_0$ and the summation convention is understood. Since $\det_0 a \neq 0$, the map $\mathrm{Hom}(K_1 \otimes S^n V, K_3) / \mathrm{Hom}(K_1 \otimes S^{n-1} V, K_3) \longrightarrow \mathrm{Hom}(K_1, K_3)$ defined in this way is an isomorphism.

The situation is now essentially the same as that for the case $n = 0$. If, for example, $a_0 = 1$, then it can be assumed that $(b_1)_{A_1 \ldots A_n} = 0$ unless $A_1, \ldots, A_n = 1$ for all A_i, with $(b_1)_{1 \ldots 1} = (-1)^n \tilde{b}_1$. Thus $b(z,w) : K_1 \longrightarrow K_3$ is $w_0 b_0 + (-z_1)^n w_1 \tilde{b}_1 =: b(\tilde{w})$, where $\tilde{b} := (b_0, \tilde{b}_1)$ and $\tilde{w} := (w_0, (-z_1)^n w_1)$. The only problem that can occur is when $z_1 = 0$, but then $a(z) = 1$ and (3.2) is automatically satisfied. The stability condition (3.3)(b) is violated iff there is a vector $k \in K_1$ such that $\tilde{b}_1 k = 0 = b_0 a_1^m k$ for

$m = 0,1,\ldots,n$, but this implies $\det_1 a$, $\det_1 \tilde{b} \in S^2K_1^*$ have a common root and (3.2) fails. Thus (3.3)(b) is a consequence of (3.2) if $n > 0$.

The same analysis as for the case $n = 0$ now carries through with b replaced by $\tilde{b}$ throughout, always bearing in mind the open set U_i to which $\det_0 a$ is regarded as belonging. Over each U_i , the map $[a,\tilde{b}] \longmapsto ([\det_0 a], [\det_0 \tilde{b}, D(a,\tilde{b})])$ is an isomorphism, the only additional consideration being that $\det_0 \tilde{b}$ can be zero, a case which is quickly checked. The non-singularity condition $(\det_0 a \cdot \det_0 a)(\det_0 \tilde{b} \cdot \det_0 \tilde{b}) \neq D(a,\tilde{b})^2$ prevents $D(a,\tilde{b})$ from vanishing in this case.

To complete the overall picture, it remains to determine how the descriptions over each U_i are related. The quantity $\det b_0$ remains unchanged, whereas $\det \tilde{b}_1$ behaves as a point in the fibre of $\mathcal{O}(n)$ over $[\det_0 a] \in \mathbf{P}(S^2V^*)$. $D(a,\tilde{b})$ and the remaining component δ of $\det_0 \tilde{b}$ do not change as nicely, and an explicit calculation in local coordinates reveals that the pair (δ, D) changes point in the fibre of a certain 2-bundle B_n on $P(S^2V^*)$. With some effort, it can be shown that B_n is described by the exact sequence

(5.1) $$0 \longrightarrow (n-1)\mathcal{O} \xrightarrow{A} (n+1)\mathcal{O}(1) \longrightarrow B_n \longrightarrow 0 ,$$

where $A = \left(A_i^j\right)$, $i = 1,\ldots,n-1$ and $j = 1,\ldots,n+1$ is the matrix with $A_i^i = z_0$, $A_i^{i+1} = z_1$, $A_i^{i+2} = z_2$ $1 \leq i \leq n-1$ and

$A_i^j = 0$ otherwise. (Here (z_0, z_1, z_2) are standard homogeneous coordinates on $\mathbf{P}_2$ and if $a_0 = 1$, $z_0 = \det a_0 = 1$, $z_1 = \operatorname{tr} a_1$, $z_2 = \det a_1$). Thus $B_1 = \mathcal{O}(1) \oplus \mathcal{O}(1)$ and B_2 is the holomorphic tangent bundle.

The pair (a,b) thus generates the point $([\det_0 a], [\det b_0, \det \tilde{b}_1, (\delta, D)])$ in $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(n) \oplus B_n)$ over $\mathbf{P}(S^2V^*)$. The quantity $\Delta := (\det_0 a \cdot \det_0 a)(\det_0 \tilde{b} \cdot \det_0 \tilde{b}) - D(a, \tilde{b})^2$ defines a section of $\mathcal{O}(2, n+2)$ over this space, where $\mathcal{O}(-1,0)$ is the tautological bundle of the projectivization. After some checking, the net conclusion is the following

<u>PROPOSITION 2</u>: For $n > 0$, the assignment $[a,b] \longmapsto ([\det_0 a], [\det b_0, \det \tilde{b}_1, (\delta, D)])$ defines a bijection from the moduli space of stable 2-bundles with $c_1 = 0$, $c_2 = 2$ on H_n to the projectivized bundle $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(n) \oplus B_n)$ over $\mathbf{P}_2$ minus the hypersurface $\Delta = 0$, where B_n is given by (5.1) and $\Delta = (\det_0 a \cdot \det_0 a)(\det_0 \tilde{b} \cdot \det_0 \tilde{b}) - D(a, \tilde{b})^2 \in \Gamma(\mathcal{O}(2, n+2))$.

□

The boundary $\Delta = 0$ is biholomorphic to $S^2 H_n$: over the set $\left\{ \left[(z,w), (z',w') \right] \in S^2 H_n \; : \; z_j z_j' \neq 0 \right\}$ for example, the map $S^2 H_n \longrightarrow \{\Delta = 0\}$ is given by

$$\left[(z,w), (z',w') \right] \longmapsto \left(\left[z \otimes z' + z' \otimes z \right], \left[2w_1 w_1', \; 2w_0 w_0' (z_j z_j')^{-n}, \; (w_0 w_1' z_j^{-n} + w_1 w_0' z_j'^{-n}), \; 2(w_0 w_1' z_j^{-n} - w_1 w_0' z_j'^{-n}) \, z \cdot z' (z_j z_j')^{-1} \right] \right).$$

The spaces $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(n) \oplus B_n)$ are all diffeomorphic as n is even or odd, as a quick check on Chern classes shows that $\mathcal{O} \oplus \mathcal{O}(n) \oplus B_n$ is topologically isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(n-1) \oplus B_{n-2}(1)$. It is not clear if the uncompleted moduli spaces are also diffeomorphic.

For $n = 1$, the bundles which are pull-backs from $\mathbf{P}_2$ are those with $\det b_0 \neq 0$.

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