# A $p$-adic property of Fourier coefficients of modular forms of half integral weight


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by

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## Notations and Introduction

Denote by $M_{2k}$ ($S_{2k}$) the space of modular (cusp) forms of weight $2k$ on $SL_2(\mathbf{Z})$. We will write $q$ for $\exp(2\pi i\tau)$, where $\tau$ is the variable on the upper-half complex plane. Denote by $M^+_{k+1/2}$ ($S^+_{k+1/2}$) the "+"-subspaces of the spaces of modular (cusp) forms of half integral weight $k+1/2$. These subspaces were introduced by Kohnen [5].

Throwhow the paper we fix an odd prime $p$.

Let $k$ be an even positive integer.

**Definition 1** *We call a pair* $(p, 2k)$ *supersingular if* $2k \equiv 4, 6, 8, 10$ *or* $14 \bmod p-1$

Note that each pair of the type $(p, \text{even integer})$ is supersingular if $p = 3, 5$ or 7. For each $p$ there exist infinitely many values of $k$ such that the pair $(p, 2k)$ becomes supersingular. For each $k$ there exist a finite nonempty set of appropriate values of $p$.

Denote by $\mathbf{C}_p = \hat{\bar{\mathbf{Q}}}_p$ the Tate's field. We fix once and for all an embedding $i_p : \bar{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$. We will not make difference between elements of $\bar{\mathbf{Q}}$ and their images under $i_p$. The symbol lim will always denote the limit in $\mathbf{C}_p$. We write $\sum^{(p)}$ for an infinite sum considered under $p$-adic topology. Denote by $L_p(s, \chi)$ the $p$-adic $L$-function, where $\chi$ is a Dirichlet character ([4], p.29-30). We put $\zeta^*(s) = L_p(s, \omega^{1-s})$, where $\omega$ is the Teichnüller character.

Following [2] we denote by $H(r, N)$ the generalized class numbers. They coincide with the usual class numbers of binary positive definite quadratic forms when $r = 1$. They are the Fourier coefficients of the unique Eisenstein series

$$\mathcal{H}_{k+1/2} = \zeta(1-2r) + \sum_{N \geq 1} H(k, N) q^N \quad \in M^+_{K+1/2}.$$

One has

$$H(r, N) = L(1-r, \chi) \sum_{d|v} \mu(d)\chi(d) d^{r-1} \sigma_{2r-1}(v/d), \tag{1}$$

where $(-1)^r N = Dv^2$, $D$ is a discriminant of a quadratic field, $\chi$ is the Dirichlet character associated with this quadratic field, $\sigma_{2r-1}(n) = \sum_{d|n} d^{2r-1}$ and $\mu$ is the Möbius function.

Let $-\Delta$ be the discriminant of the imaginary quadratic field $\mathbf{Q}(\sqrt{-\Delta})$. Let $\psi_l$, $l \geq 0$ be the theta series associated with the binary quadratic form

$Q_l$, where

$$Q_l(x,y) = \begin{cases} p^{2l}\frac{\Delta}{4}x^2 + y^2 & \text{if } \Delta \equiv 0 \bmod 4 \\ \frac{p^{2l}\Delta+1}{4}x^2 + xy + y^2 & \text{if } \Delta \equiv -1 \bmod 4. \end{cases}$$

Put $\psi_l = \sum_{x,y \in \mathbf{Z}} q^{Q_l(x,y)} = \sum_{n \geq 0} b_l(n) q^n$.

Let $f = \sum_{n>0} a(n) q^n$ be a cusp Hecke eigenform of weight $k$ on $SL_2(\mathbf{Z})$. Denote by $L_2(s,f)$ its symmetric square:

$$L_2(s,f) = \prod_{r:\ prime} (1 - \alpha_r^2 r^{-s})^{-1} (1 - \alpha_r \beta_r r^{-s})^{-1} (1 - \beta_r^2 r^{-s})^{-1},$$

where $\alpha_r$ and $\beta_r$ are complex numbers such that $\alpha_r + \beta_r = r$ and $\alpha_r \beta_r = p^{k-1}$.

Consider Rankin's convolutions

$$D(s,f,\psi_l) = \sum_{n>0} a(n) b_l(n) n^{-s}$$

The number $D^*(k-1,f,\psi_l) = \pi^{2k-2} L_2(2k-2,f)^{-1} D(k-1,f,\psi_l)$ is algebraic.

In the present paper we prove

**Theorem 1** *Let $(p, 2k-2)$ be a supersingular pair. Let $\chi$ denote the quadratic Dirichlet character associated with $\mathbf{Q}(\sqrt{-\Delta})$. Then*

$$D^*(k-1,f,\psi_0) + (1-\chi(p)p^{1-k}) \sum_{l>0}^{(p)} p^{l(2k-3)} D^*(k-1,f,\psi_l)$$

$$= (1-\chi(p)p^{1-k}) \frac{(2k-2)!}{2^{2k-3}(k-1)(1-p^{2k-3})\zeta(3-2k)} \tag{2}$$

**Remarks**

**1.** It is amusing to notice that the value in the write-hand side of (2) does not depend on the particular choice of the cusp Hecke eigenform $f$ of weight $k$. The dependence on the choice of the discriminant $-\Delta$ is very slight and explicit. Actually, only the value $\chi(p)$ is involved.

**2.** The denominators of the numbers $D^*(k-1,f,\psi_l)$ were studied in [8], Theorem 4. The Rankin's method was used for this purpose. Even the $p$-adic convergence of the series (2) does not follow from this result. This convergence is the peculiarity of our supersingular situation.

We are going to derive theorem 1 from the following

**Theorem 2** *Let $(p, 2k)$ be a supersingular pair. Consider a modular form $\varphi \in M^{+}_{k+1/2}$. Suppose that $\varphi = \sum_{n\geq 0} c(n)q^n$, $c(n) \in \bar{\mathbf{Q}}$ for all $n$. Choose $N$ such that $c(N) \neq 0$.*

*Then*

$$\lim_{r\to\infty} c(p^r N) = c(0)\frac{L_p(1-k,\chi)}{\zeta^*(1-2k)},$$

*where $\chi$ is the quadratic character associated with $\mathbf{Q}(\sqrt{(-1)^k N})$.*

We prove theorem 2 in Chapter 1. In order to illustrate this theorem, we need to consider modular forms of half integral weight which Fourier coefficients are "interesting" numbers. Chapter 2 is devoted to theta series. In Chapter 3 we provide a construction which generates another type of half integral weight modular forms. It allows to prove theorem 1. Since this construction seems to us to be of independent interest we will briefly recall it here.

Consider a modular form $f$ of weight $k$. We suppose that $f$ is a normalized cusp Hecke eigenform. Let $F$ be the Klingen - Eisenstein series associated with $f$. Since $F$ is a Siegel modular form of genus 2, it has a Fourier-Jacobi expansion ([3], Chapter II): $F = \sum_{m\geq 0} \phi_m(\tau, z) exp(2\pi i m\tau')$. Here $\phi_m$ are Jacobi forms of indeces $m$ and the same weight $k$. One has $\phi_0 = f$. Consider the Jacobi form $\phi_1$. It follows from [3], Theorem 5.4 that $\phi_1$ corresponds to a half integral weight modular form $\varphi$ of weight $k - 1/2$. The form $\varphi$ belongs to the Kohnen's "+"-space. The Fourier coefficients of the Siegel modular form $F$ were calculated by Böcherer [1] and Mizumoto [7], [8]. These numbers involve special values of Rankin's convolutions of the modular form $f$ with theta series of weight 1. We will apply theorem 2 to the modular form $\varphi$. It will yield theorem 1.

## Chapter 1

In this chapter we prove theorem 2. First we prepare a few lemmas.

**Lemma 1** *Let $(p, 2k)$ be a supersingular pair. Consider a cusp Hecke eigenform $f = \sum_{n\geq 1} a(n)q^n$ of weight $2k$. Suppose that $a(1) = 1$. Let $K = \mathbf{Q}(a(n)_{n\geq 1})$ be the field extension. Let $\mathfrak{P}$ be a prime ideal in $K$ dividing $p$. Then $\mathfrak{P}$ divides $a(p)$.*

**Remarks**

**1.** K is known to be an algebraic number field.

**2.** This lemma explains the name "supersingular". It means that if $(p, 2k)$ is a supersingular pair, then there are no $p$-ordinary cusp Hecke eigenforms of weight $2k$, i.e. all the cusp Hecke eigenforms are supersingular.

**Proof of lemma 1.**

It is known ([6], Theorem 4.4) that the space $S_k$ possesses a basis over $\mathbf{C}$ which consists of cusp forms with rational integer Fourier coefficients. Let $\varphi_1, \ldots, \varphi_t$, where $t = \dim S_k$ and $\varphi_i = \sum_{n\geq 0} b_i(n)q^n$ be such a basis. It follows that there exist algebraic numbers $\alpha_i, \ldots, \alpha_t$ such that $f = \sum_i \alpha_i\varphi_i$. It follows from [10], Theorem 7 (see also Remark p.216) that $\lim_{n\to\infty} b_i(p^n) = 0$ for each $i$. It yields

$$\lim_{n\to\infty} a(p^n) = \lim_{n\to\infty} \sum_{1\leq i\leq t} \alpha_i b_i(p^n) = 0 \tag{3}$$

If $r \geq 0$ then $a(p^{r+1}) = a(p^r) - p^{2k-1}a(p^{r-1})$, because $f$ is a Hecke eigenform. It follows that $a(p^r) \equiv a(p)^r \bmod p^{2k-1}$. Taking in account (3) we obtain the assertion of lemma 1.

**Lemma 2** *Let $(p, 2k)$ be a supersingular pair. Consider a cusp Hecke eigenform $\Phi \in S^+_{k+1/2}$ Let $\Phi = \sum_{n\geq 1} c(n)q^n$ be its Fourier expansion.*

*Then $lim_{r\to\infty} c(p^r n) = 0$ for each $n > 0$.*

**Proof.**

It is known [5] that one can pick a cusp normalized ($a(1) = 1$) Hecke eigenform $f$ of weight $2k$, $f = \sum_{n\geq 1} a(n)q^n$ such that for $N, n \geq 1$

$$c(n^2 N) = c(N) \sum_{d|n} \mu(d) \left(\frac{N}{d}\right) d^{k-1} a(n/d).$$

In particular we get for $n = p^r$

$$c(p^{2r}N) = c(N)\left(a(p^{2r}) - \left(\frac{N}{p}\right)p^{k-1}a(p^{2r-1})\right)$$

Combining this formulae with (3) we obtain the assertion of lemma 2.

Our next assertion immediately follows from (1). However we formulate it as a separate lemma.

**Lemma 3** *Let $\chi$ be the quadratic character associated with $\mathbf{Q}(\sqrt{(-1)^k N})$. Then*

$$\lim_{r\to\infty} H(k, p^r N) = \frac{L_p(1-k,\chi)}{1-p^{2k-1}}.$$

**Proof of theorem 2.** Consider the basis of the space $M^+_{k+1/2}$ which consists of the finite set of cusp Hecke eigenforms $\varphi_i$ together with $\mathcal{H}_{k+1/2}$. One has

$$\varphi = \frac{c(0)}{\zeta(1-2k)}\mathcal{H}_{k+1/2} + \sum_i \beta_1\varphi_i$$

with some algebraic coefficients $\beta_i$. The assertion of the theorem follows now from lemma 2 and lemma 3.

**Remark**

It is a well-known estimate that the absolute values of Fourier coefficients of a cusp form of even weight increase slower than those of an Eisenstein series of the same weight. One can consider theorem 2 as a $p$-adic analogue of this fact. Roughly speaking, consider a modular form $f = \sum_{n\geq 0} c(n)q^n$. Suppose that $F = \mathcal{G} + \Phi$, where $\mathcal{G} = \sum_{n\geq 0} d(n)q^n$ and $\Phi$ is a cusp form. Then $\lim_{r\to\infty}(c(p^r N) - d(p^r N)) = 0$. We have proven such type of statements both in the integral and in the half integral weight cases. The supersingularity condition is crucial for our argument.

Classically, such type of argument was applied to the Fourier coefficients of theta series. The first illustration of theorem 2 deals with theta series associated with unimodular positive definite quadratic forms.

## Chapter 2

Let $Q$ be an unimodular positive definite quadratic form on a lattice $\Lambda$ of rank $4k$ and $B(x,y)$ the associated bilinear form with $Q(x) = 1/2B(x,x)$.

Let $y \in \Lambda$ be such that $Q(y) = 1$. Then by [3], Theorem 7.1 the function

$$\Theta_{Q,y}(\tau, z) = \sum_{x \in \Lambda} q^{Q(x)} \zeta^{B(x,y)} = \sum_{4n \geq r^2} c(4n - r^2) q^n \zeta^r$$

is a Jacobi form of weight $2k$ and index 1 on $SL_2(\mathbf{Z})$. It is known that the function

$$\Theta_{Q,y}(\tau, 0) = \sum_{x \in \Lambda} q^{Q(x)} = \sum_{n \geq 0} r_Q(n) q^n$$

belongs to $M_{2k}$.

Note that $r_Q(n)$ is the number of representations of an integer $n$ by the form $Q$. $r_Q(0) = 1$. The number

$$c(4n - r^2) = \#\{x \in \Lambda | Q(x) = n, B(x, y) = r\}$$

depends only on $4n - r^2$ by [3], Theorem 2.2. The function

$$\varphi = \sum_{N \geq 0} c(N) q^N$$

belongs to $M^+_{2k-1/2}$. $c(0) = 1$.

In some cases of low weight one can get precise formulae for the numbers $c(N)$ and $r_Q(n)$ (cf. [9], §6 for the integral weight case and [3], p.84-85 for the half integral weight case). One also has the following asymptotics ([9], §6, Cor. 2):

$$r_Q(n) = \frac{4k}{B_{2k}} \sigma_{2k-1}(n) + O(n^k).$$

Application of theorem 2 and lemma 1 to the modular forms $\varphi$ and $f$ implies the following

**Proposition 1** *Let $N$ be a positive integer.*

**a.** *Suppose that $(p, 2k)$ is a supersingular pair. Then*

$$(1 - p) \lim_{r \to \infty} r_Q(p^r N) = \frac{4k}{B_k} \sigma_{2k-1}(N).$$

**b.** *Suppose that $(p, 2k-2)$ is a supersingular pair. Then*

$$(1 - p) \lim_{r \to \infty} c(p^r N) = \frac{L_p(2 - 2k, \chi)}{\zeta(3 - 4k)},$$

*where $\chi$ is the quadratic character associated with $\mathbf{Q}(\sqrt{-N})$.*

One has to compare our argument with the Siegel famous formulae on theta series. Let us reformulate the Siegel's result as an identity of Jacobi forms. For this purpose we briefly recall some notations and definitions from [3]. These notations and definitions will be also useful for us in the next chapter.

Let $k > 2$ be an even number.

Let $(\Lambda_i, Q_i)$ $(1 \le i \le h)$ denote the inequivalent unimodular positive definite quadratic forms of rank $2k$ and $w_i$ the number of automorphisms of $Q_i$. Put $\varepsilon_i = w_i^{-1}/(w_1^{-1} + \ldots + w_h^{-1})$. Following [3], Chapter I, we denote by $E_{k,1}$ the Jacobi - Eisenstein series of weight $k$ and index 1. The $V_l$ operator sends a Jacobi form of index $m$ to a Jacobi form of index $ml$. This operator preserves the weight. Its action on the Fourier expansion coefficients is described by formulae ([3], Theorem 4.2):

$$\phi|V_l = \sum_{n.r} \left( \sum_{a|(n,r,l)} a^{k-1} c(nl/a^2, r/a) \right) q^n \zeta^r.$$

Here $\phi = \sum_{n.r} c(n.r) q^n \zeta^r$ is the Fourier expansion of a Jacobi form $\phi$ of index $m$ and weight $k$.

Now the Siegel's formulae can be written down as ([3], p.87)

$$\sum_{1 \le i \le h} \varepsilon_i \sum_{\substack{y \in \Lambda_i \\ Q_i(y) = m}} \Theta_{Q_i,y}(\tau, z) = (E_{k,1}|V_m)(\tau, z). \tag{4}$$

We are interested in the cases when $m = 0, 1$. If $m = 0$, (4) becomes an identity of modular forms of even weight $k$. If $m = 1$, (4) becomes an identity of Jacobi forms of index 1 and weight $k$. Due to the isomorphism established in [3], Chapter II, one can rewrite it as an identity of modular forms of half integral weight $k - 1/2$. In both cases the modular form which appears in the right hand side is an Eisenstein series. The modular forms which appears in the left hand side of (4), are linear combinations of theta series. We apply our proposition 1 to these theta series. The proposition asserts that for certain $p$ their Fourier coefficients which numbers are divisible by an increasing power of $p$, become $p$-adically close to the appropriate coefficients of the Eisenstein series. (See also the remark after the proof of theorem 2.) It accords with (4) since $\sum_{1 \le i \le h} \varepsilon_i = 1$.

## Chapter 3

This chapter is devoted to the proof of theorem 1.

Let

$$F(Z) = \sum_{T \geq 0} A(T) exp(2\pi i tr(TZ)) \tag{5}$$

be a Siegel modular form of genus 2 and even weight $k$. Put $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$, where $n, r, m \in \mathbf{Z}$ and $n, m, 4nm - r^2 \geq 0$. Rewrite the Fourier expansion 5 as

$$F(\tau, z, \tau') = \sum_{n,m,4nm-r^2 \geq 0} A(n, r, m) exp(2\pi i(n\tau + rz + m\tau')).$$

**Lemma 4** *The numbers $A(n, r, 1)$ depend only on $4n - r^2$.*

*Define the numbers $c(N)$ by*

$$c(N) = \begin{cases} A(n, r, 1) & \textit{if there exists a pair } n, r \textit{ such that } N = 4n - r^2 \\ 0 & \textit{otherwise} \end{cases}.$$

*Then the function $\varphi(\tau) = \sum_{N \geq 0} c(N) q^N$ belongs to $M^+_{k-1/2}$.*

**Proof.**

Consider the Fourier-Jacobi expansion of the Siegel modular form $F$:

$$F(\tau, z, \tau') = \sum_{m \geq 0} \phi_m(\tau, z) \exp(2\pi i m \tau').$$

It follows from [3], Theorem 6.1, that $\phi_m(\tau, z)$ is a Jacobi form of weight $k$ and index $m$. Consider $\phi_1(\tau, z)$:

$$\phi_1(\tau, z) = \sum_{n, 4n - r^2 \geq 0} A(n, r, 1) \exp(2\pi i(n\tau + rz)).$$

$A(n, r, 1)$ depends only on $4n - r^2$ by [3], Theorem 2.2. The last assertion of the lemma follows from [3], Theorem 5.4.

Let us specialize our consideration to the case when the Siegel modular form is a Klingen - Eisenstein series. The following proposition is a specialization of results obtained in [1], [7], [8].

**Proposition 2** *Consider $f \in S_k$. Let*

$$F(Z) = \sum_{T \geq 0} A(T) \exp(2\pi i tr(TZ))$$

*be the Klingen - Eisenstein series associated with $f$.*

*Put $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$, where $n, r, m \in \mathbf{Z}$ and $n, m, 4nm - r^2 \geq 0$, $g.c.d.(n, m, r) = 1$.*

*Suppose that $4nm - r^2 = p^{2\nu}\Delta$, where $-\Delta$ is a fundamental discriminant.*

*For a positive integer $v$ put*

$$\Theta_T = \sum_{x,y} q^{nx^2+rxy+my^2} = \sum_{n \geq 0} b_T(n) q^n,$$

$$\Theta_T^{(v)} = \sum_{n \geq 0} b_T(nv^2) q^n.$$

*Consider the algebraic numbers*

$$D(T, \mu) = \frac{(k-1)(2\pi)^{2k-2}}{2(2k-2)! L_2(2k-2, f)} D(k-1, f, \Theta_T^{(p^\mu)}).$$

*Then*

$$A(T) = L(2-k, \chi) \left( D(T, n) + \sum_{0 \leq l < n} p^{(\nu-\mu)(2k-3)} (1 - \chi(p) p^{1-k}) D(T, l) \right).$$

*Here $\chi$ is the quadratic Dirichlet character associated with $\mathbf{Q}(\sqrt{-\Delta})$, and $L(2-k, \chi)$ is the value at negative integer of the Dirichlet L-function.*

We will use proposition 2 in the special case when $m = 1$. In this case $T = \begin{pmatrix} n & r/2 \\ r/2 & 1 \end{pmatrix}$ is the matrix of a quadratic form from the principal class. In what follows we will not make difference between a binary quadratic form and its matrix.

**Lemma 5** *Suppose that $4n - r^2 = \Delta p^{2\nu}$, $0 \leq \mu \leq \nu$.*

*There exists a binary quadratic form $S$ with discriminant $-\Delta p^{2\nu - 2\mu}$ which belongs to the principal class such that*

$$D(T, \mu) = D(S, 0). \tag{6}$$

**Proof**

Actually we are going to prove that $\Theta_T^{(p^\mu)} = \Theta_S$. It will yield (6). We claim that if $S$ exists then it belongs to the principal class. If $\Theta_T = \sum_{n\geq 0} b_T(n)q^n$ then $\Theta_S = \Theta_T^{(p^\mu)} = \sum_{n\geq 0} b_T(np^{2\mu})q^n$. Since $T$ represents 1, $b_T(1) \neq 0$. It yields $b_T(p^{2\mu}) \neq 0$. It means that $S$ represents 1 and our claim follows. The rest of the proof (the existence of $S$) essentially contains in [1], p.33. We omit it.

Combining lemma 5 with Proposition 2 we get the explicit formulas for the Fourier coefficients $A(n, r, 1)$ of the Klingen - Eisenstein series $F$. Lemma 4 allows to regard these numbers as the Fourier coefficients of a modular form of half integral weight. Application of the theorem 2 to these Fourier coefficients completes the proof of theorem 1.

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