ON THE DISCRIMINANT OF THE ARTIN-COMPONENT

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Introduction.

By a normal surface singularity (V,p) we understand the germ of a normal complex surface V at the singular point p. Let $\pi: M \to V$ be the minimal resolution. Laufer,[11], has shown that there exists a 1-convex flat map $\omega: \mathfrak{M} \to R$ over a complex manifold R of dimension $m = \dim_{\mathbb{C}} H^1(M; \Theta_M)$ which represents the semi-universal deformation of the germ of M at the exceptional set E, see also [7]. If (V,p) is rational, i.e. $R^1\pi_* \mathcal{O}_M = 0$, then ω simultaneously blows down to a deformation of (V,p). This procedure yields a holomorphic map germ $\Phi: (R,0) \to (S,0)$ where $(S,0)$ denotes the base space of the semi-universal deformation $\vartheta: (\mathcal{V},p) \to (S,0)$ of given rational singularity. Results of Artin,[1], say that the blowing down map Φ is finite and that the germ of the image $S_a := \Phi(R)$ is an irreducible component of the deformation space $(S,0)$ which is also called the Artin-component. The aim of this paper is to study the base change given by Φ, the discriminant $\Delta_a := \Delta \cap S_a$ of the Artin-component, and the singularities of the fibers corresponding to generic points of Δ_a.

Basic examples are provided by the rational double points (RDP's) which arise as singularities of quotients of $\mathbb{C}^2$ by actions of finite subgroups of $SL_2(\mathbb{C})$. Let Γ be the weighted dual graph associated to the minimal resolution of such singularity. Then it is well known that Γ corresponds uniquely to the Dynkin diagrams which classify those simple Liealgebras having root systems with only roots of equal length. It is the work of Brieskorn,[2], which makes this connection more precise. In particular it turns out that the map-germ $\Phi: (R,0) \to (S,0)$ may be represented by a Galois covering whose group of automorphisms is the Weyl group of the corresponding Liealgebra. Further the discriminant $\Delta \subset S$ of the semi-universal deformation δ is an irreducible hypersurface such that the fiber over a generic point of Δ has an ordinary double point as its only singularity.

Our main result, Theorem 2, generalizes these results to arbitrary rational singularities. It has been conjectured by Wahl, [17], at least for the deformation theory taking place on the category of artin (respectively complete) local $\mathbb{C}$-algebras. One point was to prove smoothability of certain (divisorial) cycles, Theorem 1, which heavily depends on our main result in [8]. The other parts of Theorem 2 have been already stated in [17] but Wahl's approach only works well for the formal deformation theory. Thus it seems to be worthwhile to present a complete proof in the analytic context.

Notations and conventions: We write $h^i(X;\mathcal{F}) := \dim_{\mathbb{C}} H^i(X;\mathcal{F})$ and use the standard symbols $\mathcal{O}, \Theta, \Omega^k$ in order to denote the sheaf of germs of holomorphic functions, the tangent sheaf and the sheaf of differential k-forms. There will be no systematic distinction between germs and spaces representing them whenever there is no serious likelihood of confusion.

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§1. Smoothing of cycles.

1.1 Let $\pi: M \to V$ be a resolution of a normal surface singularity (V,p) with exceptional set E. A cycle D on M is a divisor on M which is given by an integral linear combination of the exceptional components $E_1, \ldots, E_r$. By simplifying notations, the corresponding compact (non-reduced) curve $(\mathrm{supp}(D), \mathcal{O}_D)$ will be also denoted by D.

1.2 Let $\varphi: \mathfrak{M} \to Q$ be a flat map which represents a deformation of the germ (M,E) over the germ of a complex space Q at a distinguished point 0. We may always assume that φ is a 1-convex map and that each fiber $\mathfrak{M}_q$ is a strictly pseudoconvex manifold with a well defined exceptional set E_q, [14]. By $\mathcal{E}$ we denote the union of the exceptional sets E_q, $q \in Q$, provided with the reduced complex structure.

<u>1.3</u> One says that a positive cycle D on M <u>lifts</u> to the germ (Q,0) if there exists a complex subspace $\mathcal{D}$ of $\mathcal{M}$ such that, after possibly shrinking of Q, the restriction $\lambda := \varphi|\mathcal{D} : \mathcal{D} \to Q$ is a deformation of $D = \mathcal{D}_0$ which is also called a <u>lifting</u> of D over (Q,0). Equivalently, a lifting of D is given by a relative Cartier divisor $\mathcal{D}$ on $\mathcal{M}$ whose intersection with $M = \mathcal{M}_0$ gives D . This concept yields a contravariant functor $\mathcal{L}_D(-)$ from the category of germs of complex spaces to the category of sets which is defined by

$$\mathcal{L}_D((Q,0)) := \text{set of equivalence classes of deformations of } (M,E) \text{ over } (Q,0) \text{ together with a lifting of D}$$

Let Def((M,E),-) denote the deformation functor of (M,E), and let $\mathbb{C}_1$ denote the 0-dimensional germ $(0,\mathbb{C}\langle t\rangle/(t^2))$. Then it turns out that, via the well-known identification of $\mathrm{Def}((M,E),\mathbb{C}_1)$ with $H^1(M;\Theta_M)$, we have a natural isomorphism

$$\mathcal{L}_D(\mathbb{C}_1) \xrightarrow{\simeq} \mathrm{Ker}(\,\gamma^*: H^1(\Theta_M) \to H^1(\,\mathcal{O}_D(D)))$$

where γ^* is induced by the homomorphism $\gamma : \Theta_M \to \mathcal{O}_D(D)$ of sheaves which can be locally described as follows. If θ is a vector field near a point $x \in M$ and $f(z)$ is a local defining equation for D near x, then $\gamma(\Theta) = \theta(f)$, compare [17],[10],[7].

Furthermore it can be shown without any difficulties that there exists a semi-universal formal lifting, i.e. $\mathcal{L}_D(-)$ has a hull in the sense of Schlessinger on the category of 0-dimensional complex spaces. Unfortunately it is not known whether there exists a lifting which is semi-universal with respect to germs of complex spaces of arbitrary dimension. To avoid this unpleasant difficulty at least in the case of reduced parameter spaces, Laufer ,[10], has introduced a weaker notion of a lifting.

<u>1.4</u> Suppose that Q is reduced. Then a positive cycle D <u>weakly lifts</u> to the germ (Q,0) if for each $q \in Q$, q near 0, there is a (necessarily unique) cycle $D_q > 0$ on $\mathcal{M}_q$ such that D and D_q are homologous in $\mathcal{M}$. Note that the family of cycles $\{D_q\}, q \in Q$, also called a <u>weak</u> <u>lifting</u> of D, does not define in general a Cartier divisor on $\mathcal{M}$. But this is true if Q is smooth. So a positive cycle lifts to a smooth space if and only if it weakly lifts. By semi-continuity $\chi(\mathcal{O}_q) = \chi(D)$ for a lifting of D. Hence, using

a resolution of given parameter space Q, it can be readily seen that $\chi(D_q) = \chi(D)$ and $D_q \cdot D_q = D \cdot D$ for a weak lifting $\{D_q\}$ of D over (Q,0) ,too . Now the point is that to each deformation $\varphi: \mathfrak{M} \to Q$ of (M,E) over a reduced space Q there exists a maximal reduced subspace $(Q_D, 0) \subset (Q,0)$ to which a given positive cycle D weakly lifts ,[10; Proposition 2.7] .

<u>1.5</u> Let $\omega: \mathfrak{M} \to R$ represent the semi-universal deformation of the minimal resolution germ (M,E). Recall that R is smooth of dimension $m = h^1(\Theta_M)$. Then, via the Kodaira-Spencer isomorphism $T_0R \xrightarrow{\simeq} H^1(M; \Theta_M)$, we may identify

$$T_0 R_D \cong \mathcal{L}_D(\mathbb{C}_1) ,$$

compare the arguments in [7 ;Satz 11.6]. Standard arguments in deformation theory show that the obstruction space $ob(\mathcal{L}_D)$ for the functor $\mathcal{L}_D(-)$ is given by

$$ob(\mathcal{L}_D) \cong H^1(D; \mathcal{T}^1_D) ,$$

where $\mathcal{T}^1_D$ is the sheaf of germs of infinitesimal deformations of D. Furthermore ,if supp(D) is connected and

$h^1(\mathcal{T}^1_D) = 0$, then
(i) R_D is smooth
(ii) fibers of $\lambda: \mathfrak{D} \to R_D$ are generically smooth
(iii) codim $R_D = h^1(\mathcal{O}_D(D))$,

see [17],[10],[7 ;Satz 11.8]. Clearly, $h^1(\mathcal{T}^1_D) = 0$ if D is reduced.

<u>1.6</u> From now on assume $h^0(\mathcal{O}_D) = 1$,e.g. take D to be the fundamenta cycle. Note that the vanishing of $H^1(\mathcal{T}^1_D)$ implies that $h^0(\mathcal{O}_D) = 1$ if supp(D) is connected,[7]. Then straightforward computations show that

$$h^1(\mathcal{T}^1_D) = h^1(\mathcal{O}_{D-D_{red}}(D)) ,$$

where $D_{red} = \sum E_i$, $E_i \subset$ supp(D) . Thus it is easy to find examples of D (even in case of rational singularities of multiplicity ≥ 4) which $H^1(\mathcal{T}^1_D)$ does not vanish for. Now a major problem is to find useful weaker conditions that guarantee that D is smoothable over R_D. The first step should be to find non-obstructed first-order liftings of D. Our basic tool is provided by the following easy result.

1.7 Proposition. Suppose D admits a decomposition $D = \sum k_i \cdot D_i$, $1 \leq i \leq s$, such that $h^1(\tau^1_{D_i}) = 0$ for $1 \leq i \leq s$. Then R_D contains an irreducible component of codimension

$$\leq \sum_{1 \leq i \leq s} h^1(\mathcal{O}_{D_i}(D_i)) .$$

Using 1.5, the proof is clear since $R_D \supset \bigcap_{1 \leq i \leq s} R_{D_i}$ by hypothesis.

1.8 Without loss of generality assume $D_{red} = E$. Then

$$R_D \supseteq \bigcap_{1 \leq i \leq r} R_{E_i} =: \Sigma$$

where the R_{E_i}'s are smooth subspaces of R of codimension $= h^1(\mathcal{O}_{E_i}(E_i))$ which transversally intersect in a smooth subspace Σ of dimension equal to $h^1(\Theta_M(\log E))$, see 1.5 and [10]. But note that, if M is a good resolution, Σ is the moduli space for the functor of equitopological deformations of M introduced by Laufer, [9], see also [7; Proposition 11.14.3]. Hence ω induces a locally trivial deformation of each positive cycle Y over Σ. Thus D cannot be smoothable over Σ.

1.9 Definition. Assume that $D_{red} = E$. Then a decomposition $D = \sum k_i \cdot D_i$, $1 \leq i \leq s$, is called a good decomposition of D if

$$\sum_{1 \leq i \leq s} \operatorname{codim} R_{D_i} < \operatorname{codim} \Sigma = h^1(\Theta_M) - h^1(\Theta_M(\log E))$$

1.10 Remarks. a) If D admits a good decomposition, then the codimension of each irreducible component of R_D is $<$ codim Σ because Σ is smooth.

b) With respect to the smoothing problem it is very important to find conditions that guarantee the existence of a good decomposition. In [8; §3] we gave an affirmative answer to this problem in case of the fundamental cycle of a rational or minimally elliptic singularity. The proof is very technical and it seems to be extremely difficult to find good decompositions in more general cases.

Theorem 1 Suppose (V,p) is a rational singularity. Let $\omega : \mathfrak{M} \to R$ be a flat 1-convex map which represents the semi-universal deformation of the minimal resolution germ (M,E) of (V,p). If D is a positive cycle on M with $\chi(D) = 1$, then we have :

(i) If R_D' is an irreducible component of R_D, then, for generic $t \in R_D''$, the cycle D_t is smooth and is the (full) exceptional set of $\mathfrak{M}_t$, $t \neq 0$.

(ii) $\dim R_D' = h^1(\Theta_M) + 1 + D \cdot D$

(iii) R_D is smooth if D is almost reduced , i.e. D is reduced at the non -2 curves .

Remark. It is most likely to expect that the R_D's are irreducible but we cannot yet prove it .

1.11 Corollary. Assumptions as in Theorem 1. Then D is the only positive cycle which weakly lifts to R_D' and satisfies $\chi(D) = 1$.

Proof. Take Y to be an arbitrary positive cycle with $\chi(Y) = 1$ whic weakly lifts to R_D' . Then Y weakly lifts to a cycle Y_t on $\mathfrak{M}_t$ wher t is a generic point on R_D' as in (i). Since $\chi(Y_t) = 1$ and D_t is the full exceptional set of $\mathfrak{M}_t$, we observe that $Y_t = D_t$. Hence Y and D are homologous in M. But this is only true if $D = Y$ because the intersection form on M is negative definite.

1.12 Proof of Theorem 1. The last statement follows immediately from 1.5 and 1.6 , see also [10]. So it remains to prove (i) and (ii First let us assume that D admits a good decomposition. Then we can continue as follows.

Clearly we may assume that D is supported on the full exceptiona set E . Recall that $\chi(D)$ equals 1 if and only if D appears as part of a computation sequence for the fundamental cycle Z on M. Hence $h^0(\mathcal{O}_D) = 1$ and there are only finitely many positive cycles Y on M which satisfy $Y \leq D$ and $\chi(Y) = 1$. We do induction on this number N to verify the first statement.

If $N \leq 6$, then D is automatically reduced and we are done,see 1. Otherwise it follows from our additional hypothesis,the openess property of the semi-universal deformation of (M,E) and Theorem 3.6

in [12] that, possibly after a finite base change, there exists a non-equitopological, 1-convex deformation $\Psi : \mathcal{N} \to B$ of $M = \mathcal{N}_0$ with a smooth 1-dimensional parameter space over each irreducible component R_D^{t} of R_D such that the support of D_t is the (full) exceptional set of $\mathcal{N}_t$, $t \in B$. Thus

$$N_t < N \quad \text{for } t \neq 0 ,$$

where N_t denotes the number of positive cycles F on $\mathcal{N}_t$ which satisfy $\chi(F) = 1$ and $F \leq D_t$. For otherwise it can be readily seen that each cycle $Y \leq D$ with $\chi(Y) = 1$ lifts to B. Hence Ψ would be an equitopological deformation ; a contradiction . Therefore, by induction, statement (i) is true for D_t, $t \neq 0$, and hence also for D because of the openess property of the semi-universal deformation of (M,E).

Now let D_t be a weak lifting of D over an irreducible component $R_D^{!}$ of R_D. Let ω_t denote the deformation of $\mathcal{M}_t$ over (R,t), t near 0, induced by ω. The openess property says that the corresponding Kodaira-Spencer map $\varrho_t : T_tR \to H^1(\Theta_{\mathcal{M}_t})$ is surjective. Thus, if t is a generic point, D_t lifts to a smooth subgerm (R_{D_t}, t) of (R,t) of dimension

$$d = \dim \operatorname{Ker}(\gamma_t^* \circ \varrho_t : T_tR \to H^1(\Theta_{\mathcal{M}_t}) \to H^1(\mathcal{O}_{D_t}(D_t))$$
$$= h^1(\Theta_M) - h^1(\Theta_{\mathcal{M}_t}) + \dim \operatorname{Ker} \gamma_t^* \quad ,$$

see 1.3. But $\dim \operatorname{Ker} \gamma_t^* = h^1(\Theta_{\mathcal{M}_t}) + 1 + D_t \cdot D_t$,compare 1.5 . Since $D_t \cdot D_t = D \cdot D$ and $(R_{D_t}, t) = (R_D^!, t)$, we obtain the equality we were looking for .

So the proof is complete as soon as we can check the existence of a good decomposition of D. Again, we do induction on N . As before, we are done if $N \leq 6$. Now given D with $N > 6$. Then there is an irreducible component of D, say E_k, such that $\chi(D-E_k) = 1$ and $E_k \cdot (D-E_k) = 1$. We claim that $D = D-E_k + E_k$ is a good decomposition. By induction and previous arguments,

$$\operatorname{codim} R_{D-E_k} + \operatorname{codim} R_{E_k} = -1 - (D-E_k) \cdot (D-E_k) - 1 - E_k \cdot E_k = - D \cdot D .$$

Now suppose that $-D \cdot D = \operatorname{codim} \Sigma = \sum (e_i - 1)$,$1 \leq i \leq r$, where $e_i = -E_i \cdot E_i$. Then it would follow that $-Z \cdot Z = \operatorname{codim} \Sigma$ since $D \cdot D \geq Z \cdot Z$ as it can be readily seen. But this is impossible because Z admits a good decomposition, [8].

Remark. Since Z admits a good decompositon, it follows from our discussion in 1.8 that

$$\mathrm{mult}(V,p) = -Z\cdot Z < \sum_{1\le i\le r}(e_i-1)$$

It would be interesting to have a direct proof of this inequality.

§2. The Main Result.

2.1 Let $\varphi:\mathfrak{M}\to Q$ be a 1-convex map which represents a deformation of the minimal resolution germ (M,E) of a rational singularity (V,p). Consider the unique relative Stein-factorization

$$\begin{array}{ccc}\mathfrak{M} & \xrightarrow{\tau} & \mathfrak{X}\\ & \varphi\searrow \ \ Q \ \ \swarrow\tilde{\varphi} & \end{array}$$

i.e. $\mathfrak{X}$ is a normal Stein space and τ is a proper, surjective holomorphic map such that $\tau_*\mathcal{O}_{\mathfrak{M}} = \mathcal{O}_{\mathfrak{X}}$ and τ is biholomorphic on $\mathfrak{M}-\mathcal{E}$. Since (V,p) is rational, it is known, [14], that $\tilde{\varphi}$ is flat and $\tau|\mathfrak{M}_t:\mathfrak{M}_t\to\mathfrak{X}_t$ is the Stein factorization of $\mathfrak{M}_t$, $t\in Q$. Hence $(\mathfrak{X}_0,x)$, $x := \tau(E)$, is isomorphic to (V,p) and $\tilde{\varphi}:\mathfrak{X}\to Q$ defines a deformation of (V,p). We say $\tilde{\varphi}$ arises by simultaneously blowing down of φ. Conversely, given a deformation $\delta:\mathcal{Y}\to Q$ of the rational singularity (V,p) such that δ is isomorphic over Q to the relative Stein factorization of a deformation $\varphi:\mathfrak{M}\to Q$ of M. Then we call the diagram $\begin{array}{ccc}\mathfrak{M} & \xrightarrow{\tau} & \mathcal{Y}\\ & \varphi\searrow Q \swarrow\delta & \end{array}$ a simultaneous resolution of δ.

2.2 One can generalize the construction above as follows. Let $A\subset E$ be an exceptional subset (not necessarily connected). Then φ induces a 1-convex deformation $f:\mathfrak{N}\to Q$ of a strictly pseudoconvex neighborhood N of A. Blowing down of A yields a 1-convex normal space M^* which has singularities corresponding to the connected components of A. Let E^* be the exceptional set of M^*. Then it is a rather easy exercise to show that the relative Stein factorization with respect to f extends (after possibly shrinking of $\mathfrak{M}$) to a commutative diagram $\begin{array}{ccc}\mathfrak{M} & \xrightarrow{\tau^*} & \mathfrak{M}^*\\ & \varphi\searrow Q \swarrow\varphi^* & \end{array}$

where τ^* is a proper holomorphic map and φ^* is a 1-convex deformation of M^* inducing the deformation $\tilde{f}$ of the singularities of M^*

We say φ^* arises by <u>partially blowing down of</u> φ <u>relative</u> A .

<u>2.3</u> Let $\vartheta : \mathcal{V} \to S$ denote the semi-universal deformation of (V,p). Replace φ by the semi-universal deformation $\omega : \mathcal{M} \to R$ of (M,E). Then ω blows down simultaneously to a deformation $\tilde{\omega} : \mathcal{W} \to R$ of (V,p). Hence there exists a map-germ $\Phi : (R,0) \to (S,0)$ uniquely determined up to first order which yields the cartesian diagram

$$\begin{array}{ccccc} \mathcal{M} & \xrightarrow{\pi} & \mathcal{W} & \to & \mathcal{V} \\ & \searrow^{\omega} & \downarrow \tilde{\omega} & & \downarrow \vartheta \\ & & R & \xrightarrow[\Phi]{} & S \end{array}$$

By a result of Artin,[1], one knows that Φ is a finite map and that the image $\quad S_a := \Phi(R)$

is an irreducible component of S, the so called Artin-component. Via the identification. $T_0 R \xrightarrow{\cong} H^1(M; \Theta_M)$, the kernel of the tangent map $T_0\Phi : T_0 R \to T_0 S$ can be identified with the local cohomology group $H^1_E(M; \Theta_M)$. It turns out that

$$h^1_E(\Theta_M) = \# \{ -2 \text{ curves on } M \} \quad .$$

Hence Φ is a local embedding if M does not contain any -2 curve. For more details compare [15].

<u>2.4</u> <u>Proposition.</u> Let $\Delta \subset S$ be the discriminant of the semi-universal deformation ϑ of a rational singularity (V,p), and let $\Delta_a := \Delta \cap S_a$ be the discriminant of the Artin-component. Then

$$\Delta_a = \bigcup \Delta_D \quad , D \in \Lambda_+ \ ,$$

where $\Delta_D := \Phi(R_D)$ and Λ_+ is the set of positive cycles D on the minimal resolution M with $\chi(D) = 1$.

<u>Proof.</u> Let $R^* := \bigcup R_D \ , D \in \Lambda_+$. Suppose there is a $t \in R - R^*$, t near 0, such that the exceptional set E_t of $\mathcal{M}_t$ is non empty. Let C be an irreducible component of E_t. Then C must appear in some irreducible component $\mathcal{E}'$ of $\mathcal{E} \subset \mathcal{M}$. Let $Q := \omega(\mathcal{E}')$. It follows from [10; Theorem 2.1] that $\mathcal{E}'$ defines a weak lifting of a cycle $Y \in \Lambda_+$ over Q. But this gives a contradiction. Hence the fibers of ω are Stein manifolds over $R - R^*$,and we are done.

<u>2.5</u> The matrix $-(E_i \cdot E_j)$, $1 \le i,j \le r$, defines an inner product $\langle \ , \ \rangle$

on $H := H_2(M;\mathbb{R})$. Consider the finite sets

$$\Lambda_+^! = \{ D \in \Lambda_+ \mid D\cdot D = -2\} \quad \text{and} \quad \Lambda = \Lambda_+ \cup \Lambda_- \text{ with } \Lambda_- = \{ -D \mid D \in \Lambda$$

It is easy to see that each element of $\Lambda_+^!$ is supported on an exceptional subset of E which blows down to a RDP .Associating to each element of Λ its fundamental class, we may consider Λ as a subset of H . Each element D of $\Lambda' := \Lambda_+^! \cup \Lambda_-^!$ defines a reflection , say s_D, at the hyperplane H_D orthogonal to D which is given by

$$(2.5.1) \qquad s_D(x) = x - \langle x,D\rangle \cdot D \quad , \; x \in H \; .$$

<u>Notations.</u> (i) By W we denote the subgroup of GL(H) which is generated by the reflections s_D ,$D \in \Lambda'$. It is easy to check that s_D sends $\Lambda_+ - \{D\}$ into itself and D to -D .

(ii) Let A denote the union of all -2 curves on M, and let A_1, ..,A_n be the connected components of A. We call A_i a RDP-configuration and A the maximal RDP-configuration on M.

<u>2.6</u> <u>Lemma.</u> (i) If (V,p) is a RDP, then $\Lambda = \Lambda'$ and Λ is a root system of H with Λ_+ as the set of positive roots. The group W is the Weylgroup of this root system, and the associated Dynkin-diagram is given by the weighted graph Γ associated to the minimal resolution M .

(ii) Suppose (V,p) is rational. By W_i, $1 \le i \le n$, denote the Weyl-groups corresponding to the RDP-configurations A_i on M. Let H_i, $1 \le i \le n$, be the subspaces of H generated by the irreducible components of A_i . By restriction one obtains a faithful representation

$$W \to GL\Big(\bigoplus_{1 \le i \le n} H_i \Big) \; .$$

Furthermore the induced action is equivariantly isomorphic to the action of $\prod_{1 \le i \le n} W_i$ on $\bigoplus_{1 \le i \le n} H_i$.

(iii) $s(Y_1)\cdot s(Y_2) = Y_1 \cdot Y_2$ for $s \in W$ and Y_1, Y_2 .

<u>Proof.</u> The first part is an easy exercise, see also [17; Lemma 6.6]. The statements in (ii) follow immediately from (i) and (2.5.1). Therefore, to prove (iii), it suffices to check the identity for reflections s_D where D is carried on a -2 curve. But this can be easily done.

<u>Theorem 2</u> Let $\pi: M \to V$ be the minimal resolution of a rational surface singularity (V,p).

(i) There is an analytic W-action on $(R,0)$ such that

$$s(R_D) = R_{s(D)} \quad \text{for } s \in W \text{ and } D \in \Lambda_+ .$$

(ii) $\Phi:(R,0) \to (S_a,0)$ is a Galois covering and its group of automorphism is W. Further, S_a is smooth.

(iii) $\Delta_a = \bigcup \Delta_D$ where $D \in \Lambda_+$ runs through a fundamental set of the W-action on Λ.

(iv) Δ_D is an irreducible component if R_D is irreducible .

(v) $\dim \Delta'_D = \dim S_a + D\cdot D + 1$ for each irreducible component Δ'_D of Δ_D .

(vi) Over a generic point of Δ'_D , the fiber of the semi-universal deformation $\mathcal{V}: \mathcal{V} \to S$ has a cone singularity of degree $d = -D\cdot D$ as its only singularity.

<u>§3. Proof of Theorem 2.</u>

The last two statements of the theorem are easy corollaries of Theorem 1. The crucial point is to define an action of W on R and to show that it is the "right" action. Note that we cannot argue as in [16] since it is not known if there exists a semi-universal deformation of strictly pseudoconvex spaces with isolated singularities. We retain the notations we introduced in the previous sections.

<u>3.1</u> Let M_A denote a strictly pseudoconvex neighborhood of the maximal RDP-configuration A on M. Then M_A is a disjoint union of strictly pseudoconvex neighborhoods, say M_i, of the RDP-configurations A_i, $1 \leq i \leq n$. The semi-universal deformation $\omega: \mathcal{M} \to R$ induces deformations of (M_A, A), respectively (M_i, A_i), which we denote by $\omega_A: \mathcal{M}_A \to R$, respectively $\omega_i: \mathcal{M}_i \to R$.

<u>3.2</u> <u>Proposition.</u> Let $\omega:\mathfrak{M}\to R$ be a sufficiently small representative of the semi-universal deformation of (M,E). Then there is a product decomposition $R = R_0\times R_1\times \,..\, \times R_n$ which satisfies following properties:

(i) $R_0\times\{0\}=\bigcap R_{A_{ij}}$ where the A_{ij}'s run through the set of irreducible components of A_i ,$1\leq i\leq n$.

(ii) The restriction of ω to $R_1\times R_2\times..\times R_n$ is a representative of a semi-universal deformation of (M_A,A), say $\eta:\mathfrak{N}\to R_1\times..\times R_n$, Further there is an isomorphism $h:\mathfrak{M}_A\to R_0\times\mathfrak{N}$ such that the diagram

$$\begin{array}{ccc} \mathfrak{M}_A & \xrightarrow{h} & R_0\times\mathfrak{N} \\ & {\scriptstyle\omega_A}\searrow \quad \swarrow{\scriptstyle id\times\eta} & \\ & R= R_0\times..\times R_n & \end{array}$$

commutes .

(iii) The restriction of η to R_i, $1\leq i\leq n$, is a representative of a semi-universal deformation of (M_i,A_i), say $\eta_i:\mathfrak{N}_i\to R_i$. Further there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{M}_i & \xrightarrow{h_i} & R_0\times..\times R_{i-1}\times\mathfrak{N}_i\times R_{i+1}\times..\times R_n \\ & {\scriptstyle\omega_i}\searrow \quad \swarrow{\scriptstyle id\times\eta_i\times id} & \\ & R= R_0\times..\times R_n & \end{array}$$

where h_i is an isomorphism.

<u>Proof.</u> The obstruction map γ_A, see 1.3, for lifting all -2 curves sits in an exact sequence

(3.2.1) $$0\to H^1(\Theta_M(\log A))\to H^1(\Theta_M)\xrightarrow{\gamma_A} H^1(\mathcal{O}_A(A))\to 0$$

For each RDP-configuration A_i, we may consider the exact sequence analogous to (3.2.1) :

(3.2.2) $$0\to H^1(\Theta_{M_i}(\log A_i))\to H^1(\Theta_{M_i})\to H^1(\mathcal{O}_{A_i}(A_i))\to 0$$

Note that $H^1(\Theta_{M_i}(\log A_i)) = 0$, since rational double points are taut. Now recall Laufer's construction of the semi-universal deformation ω ,[11]. Take a Stein cover $\mathcal{U}=\{U_s\}$,$1\leq s\leq \ell$, of M such that $\overline{U}_r\cap\overline{U}_s\cap\overline{U}_t = \emptyset$ for $r\neq s\neq t$.

Let $\{\theta_{qs}^{(1)}\},\ldots,\{\theta_{qs}^{(m)}\}$ be a set of cocycles in $Z^1(\mathcal{U};\Theta_M)$ which represent a basis of $H^1(\Theta_M)$. Take R to be a small polydisc in $\mathbb{C}^m$. Then $\mathfrak{M}$ will be obtained by patching together the sets $U_s \times R$, $1\leq s\leq \ell$. The transition functions are of type

$$(x,t) \mapsto (h_{qs}(x,t),t)$$

where $h_{qs}(x,t)$ is defined by integration along $t_1\theta_{qs}^{(1)}+\ldots+t_m\cdot\theta_{qs}^{(m)}$ for time 1. Now the point is to choose above set of cocycles in a suitable way. First we arrange it that the cover $\mathcal{U}$ satisfies following requirements:

(a) To each singular point y of E there exists a unique neighborhood $U_s \in \mathcal{U}$ with $y \in U_s$.

(b) $\mathcal{U}_i := \{U_{\ell_{i-1}+1},\ldots,U_{\ell_i}\}$, $\ell_0:=0$, is a cover of M_i, $1\leq i\leq n$, and

$\mathcal{U}^* := \{U_s \mid s>\ell_n\}$ is a cover of a strictly pseudoconvex neighborhood M^* of E^* where E^* contains precisely the irreducible components E_i of E with $E_i\cdot E_i \neq -2$.

Let $d_0<d_1<\ldots<d_n$ be an increasing sequence of integers such that

$$d_i - d_{i-1} = h^1(\Theta_{M_i}) \ ,1\leq i\leq n \ .$$

Then we may choose cocycles $\{\theta_{qs}^{(j)}\}, d_{i-1}<j\leq d_i$, which represent a basis of $H^1(\Theta_{M_i})$ and vanish on $U_q\cap U_s$ if $U_q\cap U_s \not\subset M_i$. It follows from the exact sequences in (3.2.1) and (3.2.2) that the corresponding cohomology classes in $H^1(\Theta_M)$ are linearly independent and do not sit in the kernel of the obstruction map τ_A. Finally let $\{\theta_{qs}^{(1)}\},\ldots,\{\theta_{qs}^{(d_0)}\}$ be cocycles which define a basis of $H^1(\Theta_M(\log A))$. Since $H^1(\Theta_{M_i}(\log A))= 0$, we can arrange it that these cocycles vanish on $U_q\cap U_s$ if $(U_q\cap U_s)\cap M^*= \emptyset$. It is now clear how to complete the proof of Proposition 3.2.

<u>3.3 Definition.</u> Let $g:(R,0)\to(R,0)$ be an analytic automorphism. Then $g^*\tilde{\omega}:\mathfrak{W}\times_R R \to R$ is the relative Stein-factorization of the pull back $g^*\omega:\mathfrak{M}\times_R R \to R$ of ω via g. We call g a SR-<u>automorphism</u> (SR is the abbreviation of simultaneous resolution) if $\tilde{\omega}:\mathfrak{W} \to R$ and $g^*\tilde{\omega}:\mathfrak{W}\times_R R \to R$ are representatives of isomorphic deformations of (V,p) over (R,0). By $\mathcal{G}$ we denote the group of SR-automorphisms.

3.4 Suppose g is a SR-automorphism of R. Then we have a cartesian diagram

$$\begin{array}{ccc} \mathfrak{W}\times_R R & \longrightarrow & \mathcal{V}_a \\ g^*\tilde{\omega}\downarrow & & \downarrow \vartheta_a \\ R & \xrightarrow[\Phi]{} & S_a \end{array}$$

In [1; Theorem 1], Artin has shown that the functor Res is representable, see also [7; Satz 9.16] for an analytic version which is weaker but sufficient for our purposes. From this it follows that the diagram $\begin{array}{ccc} R & \xrightarrow{g} & R \\ & \Phi\searrow \; S \; \swarrow\Phi & \end{array}$ commutes.

So Φ factorizes via the quotient $R/\mathfrak{G}$. Note, since Res is representable, a SR-automorphism is already uniquely determined by its first-order map.

3.5 Elementary transformations. Let C be a -2 curve on M. Then C lifts to a smooth hypersurface $R_C \subset R$, see Theorem 1, and the lifting $\lambda : \mathcal{C} \to R_C$ defines a trivial deformation of C. As in 3.2, ω induces a deformation $\varphi : \mathfrak{X} \to R$ of a strictly pseudoconvex neighborhood X of C. Same arguments as in the proof of Proposition 3.2 show that there is a smooth 1-dimensional subgerm (B,0) of (R,0) such that $R = B \times R_C$ and that following holds: The restriction of φ to $B := B\times\{0\}$, say $\varphi_C : \mathfrak{X}_C \to B$ represents the semi-universal deformation of (X,C). Further, we have an isomorphism

$$\begin{array}{ccc} \mathfrak{X}_C \times R_C & \xrightarrow{f} & \mathfrak{X} \\ \varphi_C\times id \searrow & & \swarrow \varphi \\ & B\times R_C & \end{array}$$

over $R = B \times R_C$.

Clearly, f induces a trivialization of $\lambda : \mathcal{C} \to R_C$. Since C does not lift to B, the normal bundle of C in $\mathfrak{X}_C$ may be identified with $\mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$. Let $\sigma_C : \mathfrak{N}_C \to \mathfrak{X}_C$ be the monoidal transformation of $\mathfrak{X}_C$ with center at C. The inverse image of C is a rational ruled surface $\Sigma_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$. The proper transform is its diagonal. By [5], see also [6], Σ_0 can be blown down to $\mathbb{P}^1 \cong C$ in two different ways. One gives nothing but $\varphi_C : \mathfrak{X}_C \to B$. Let $\varphi_C^* : \mathfrak{X}_C^* \to B$ be the deformation obtained from the other blowing down which again represents a semi-universal deformation of (X,C). Thus, because of versality and construction, there is an automorphism $\tau : (B,0) \to (B,0)$ of order two inducing φ_C^* from φ_C . It is a straightforward excercise to check that τ is actually a SR-automorphism.

Now consider the monoidal transformation $\mathfrak{z}:\mathfrak{N}\to\mathfrak{M}$ of $\mathfrak{M}$ with center at $\mathfrak{C}$. Using the trivialization f, it is obvious that the inverse image of $\mathfrak{C}$,call it $\mathfrak{C}'$, has a neighborhood U which is isomorphic to $\mathfrak{N}_C \times R_C$. Further, the corresponding isomorphism is compatible with the mappings $\omega\circ\mathfrak{z}|U$ and $(\varphi_C\times id)\circ(\partial_C\times id):\mathfrak{N}_C\times R_C \to B\times R_C$. Thus we may identify $\mathfrak{C}'$ with $\Sigma_0\times R_C$,and it makes sense to say that we blow down $\mathfrak{C}'$ in two different directions. By construction, this procedure yields a SR-automorphism $\tau_C:(R,0)\to(R,0)$ of order two which is necessarily given by $\tau_C = \tau\times id$ and is called the <u>elementary transformation</u> of (R,0) defined by C.

<u>3.6</u> <u>Corollary.</u> We retain the notations of 3.5. Let A_{ij} be an irreducible component of A_i, and let τ_{ij} be the induced elementary transformation of (R,0). Then the restriction $\tau_{ij}|R_i$ defines an elementary transformation of $(R_i,0)$ with respect to the semi-universal deformation $\omega_i:\mathfrak{M}_i\to R_i$.

We omit the easy proof.

<u>3.7</u> <u>Proposition.</u> Let W^* be the group of SR-automorphisms of (R,0) generated by the elementary transformations defined by all -2 curves on M. Then W^* is isomorphic to W, and the induced action of W on (R,0) is faithful and compatible with that one on Λ, i.e.

$$s(R_D) = R_{s(D)} \qquad \text{for } s\in W \text{ and } D\in\Lambda_+$$

where we used the convention $R_D =: R_{-D}$ for $D\in\Lambda_+$.

<u>Proof.</u> Let g be a SR-automorphism of (R,0). By 3.4 and Proposition 2.4, g induces an automorphism of $T := \bigcup R_D$,$D\in\Lambda_+$. Suppose (T',0) is an irreducible component of (T,0). Then Corollary 1.11 says that there exists a unique cycle $Y\in\Lambda_+$ such that $T'\subset R_Y$. Hence

$$g(R_D) = R_Y \quad \text{for a unique } Y\in\Lambda_+ .$$

It follows from the definition of the elementary transformation τ_{ij} that the deformations $\omega:\mathfrak{M}\to R$ and $\tau_{ij}^*\omega:\mathfrak{M}\times_R R\to R$ are isomorphic over the complement $R-R_{A_{ij}}$. Applying the arguments given in[3 ;Remark 7.8], the corresponding isomorphism induces a reflection

(3.7.1) $\quad H \to H$ given by $x \mapsto x - \langle x, A_{ij}\rangle \cdot A_{ij}$.

We observe that (3.7.1) defines a representation

(3.7.2) $\quad \lambda : W^* \to GL(H)$ with $\quad w(R_D) = R_{\bar{D}},\ \bar{D} = \lambda(w)(D)$, for $w \in W^*$.

Thus, because of Lemma 2.6, it still remains to show that

(3.7.3) $\quad \lambda$ is a faithful representation .

The point is to compare it with the representation $\rho : W^* \to GL_{\mathbb{C}}(H^1(\Theta_M))$ which is given by the linearization of the action of W^* on $(R,0)$. By 3.4 and the fact that the functor Res is "representable", it follows that ρ is a faithful representation .

The obstruction map γ_{ij} to lifting the -2 curve A_{ij} yields a direct sum decomposition $\quad H^1(\Theta_M) \cong H^1(\Theta_M(\log A_{ij})) \oplus H^1_{A_{ij}}(\Theta_M)$.

Recall that $\mathcal{L}_{A_{ij}}(\mathbb{C}_1) = H^1(\Theta_M(\log A_{ij}))$. So $\rho(\tau_{ij})$ is a reflection on $H^1(\Theta_M)$ which is -1 on the line $H^1_{A_{ij}}(\Theta_M)$ and +1 on $H^1(\Theta_M(\log A_{ij}))$. On the other hand, the direct product decomposition $R = R_0 \times .. \times R_n$ induces a direct sum decomposition

(3.7.4) $\quad H^1(\Theta_M) \cong H^1(\Theta_M(\log A)) \oplus H^1_{A_1}(\Theta_M) \oplus .. \oplus H^1_{A_n}(\Theta_M)$,

see Proposition 3.2 and [4 ; Proposition 1.10]. Further Proposition and Corollary 3.6 imply that $\rho(\tau_{ij})$ is +1 on each of above direct sum components which is not equal to $H^1_{A_i}(\Theta_M)$ and that $\rho(\tau_{ij})|H^1_{A_i}(\Theta_M)$ is a reflection which is -1 on $H^1_{A_{ij}}(\Theta_M) \subset H^1_{A_i}(\Theta_M) \cong H^1(\Theta_{M_i})$ and +1 on $H^1(\Theta_{M_i}(\log A_{ij}))$. The latter may be identified with $H^1(\Theta_M(\log A_{ij})) \cap H^1_{A_i}(\Theta_M)$. Hence the direct sum components of $H^1(\Theta_M)$ in (3.7.4) are $\rho(W^*)$ invariant subspaces. Let ρ_i , $0 \le i \le n$, denote the restriction of ρ to $H^1(\Theta_M(\log A))$, $i=0$, respectively to $H^1_{A_i}(\Theta_M)$. Then we have $\quad \rho = \rho_0 \oplus \rho_1 \oplus .. \oplus \rho_n$

Let W_i^*, $1 \le i \le n$, be the subgroup of W generated by the elementary transformations τ_{ij}, $1 \le j \le n_i$. Then $\rho_i(W_k^*) = 1$ for $k \ne i$, and $\rho_i : W_i^* \to GL_{\mathbb{C}}(H^1_{A_i}(\Theta_M))$, $1 \le i \le n$, is the linearization of the action of W_i^* on $(R_i,0)$. Hence there is a natural ismorphism

(3.7.5) $$W^* \cong \prod_{1 \le i \le n} W_i^*$$

Thus, to prove (3.7.3), it suffices to show that $\lambda : W_i^* \to GL(H)$ is a faithful representation. Since W_i^* acts on $(R_i,0)$, this is true if we knew that the induced representation $\lambda_i : W_i^* \to GL(H_i)$ is faithful.

Now recall the commutative diagram, see [17;(6.21.1)]:

$$\begin{array}{ccccc} H^1(\Theta_{M_i}) & \xrightarrow{\cong} & H^1(\Omega_{M_i}) & \xrightarrow{\cong} & H^2(M_i;\mathbb{C}) \\ \gamma_{A_i}\downarrow & & \downarrow\cong & & \downarrow\cong \\ \bigoplus_j H^1(\ _{A_{ij}}(A_{ij})) & \xrightarrow{\cong} & \bigoplus_j H^1(\Omega_{A_{ij}}) & \xrightarrow{\cong} & \bigoplus H^2(A_{ij};\mathbb{C}) \end{array}$$

where γ_{A_i} is the obstruction map to lifting all irreducible components of A_i. Let β_i denote the isomorphism $H^1(\Theta_{M_i}) \to H^2(M_i;\mathbb{C})$, and let $\lambda_i^{\mathbb{C}}$ denote the complexification of λ_i. Then straightforward computations show that β_i induces an equivalence between the representations ς_i and $\lambda_i^{\mathbb{C}}$. Since ς_i is faithful, we are done.

<u>3.8</u> <u>Proposition.</u> The blowing down map $\Phi : R \to S$ factorizes via the quotient R/W and the induced map $\Phi_W : R/W \to S$ is a local embedding at 0. Further R/W is smooth and $\mathfrak{h} \cong W$.

<u>Proof.</u> Let N be the normal strictly pseudoconvex space which one obtains from M by blowing down the maximal RDP-configuration A. Since $h^1(\Theta_N) < \infty$, it follows that there exists a formal deformation $\bar{\xi} : \bar{\mathfrak{N}} \to \bar{T}$ of N which is semi-universal for the functor Def((N,Y),-) on the category of artin $\mathbb{C}$-algebras. Here Y is the exceptional set of N. General obstruction theory shows that $\bar{T}$ is smooth. In a natural way, the deformations $\omega : \mathfrak{M} \to R$ and $\vartheta : \mathcal{V} \to S$ define formal deformations, say $\bar{\omega} : \bar{\mathfrak{M}} \to \bar{R}$ and $\bar{\vartheta} : \bar{\mathcal{V}} \to \bar{S}$, which are hulls for the corresponding deformation functors on the category of artin $\mathbb{C}$-algebras. Let $\bar{\Phi} : \bar{R} \to \bar{S}$ be the induced formal blowing down morphism. Then $\bar{\Phi}$ factorizes via $\bar{T}$, i.e. there exists a commutative diagram

$$\begin{array}{ccc} \bar{R} & \xrightarrow{\bar{\Phi}} & \bar{S} \\ & \bar{\eta}\searrow \ \bar{T} \ \nearrow\bar{\sigma} & \end{array}$$

such that $\bar{\eta}^{*}\bar{\xi}: \bar{\mathcal{N}} \times_{\bar{R}} \bar{R} \to \bar{R}$ is isomorphic to the formal deformation of N which one obtains by partially blowing down of ω relative A, see 2.2, and such that there is a cartesian diagram

$$\begin{array}{ccc} \bar{\mathcal{N}} & \longrightarrow & \bar{\mathcal{V}}_a \\ \bar{\xi}\downarrow & & \downarrow\bar{\vartheta}_a \\ \bar{T} & \xrightarrow[\bar{\varepsilon}]{} & \bar{S}_a \end{array}$$

, see [16].

Now Lipman's result in [13] implies that $\bar{\varepsilon}$ is an isomorphism. Further it can be easily checked that τ_{ij} acts on $\bar{R}$ such that $\bar{\eta} \cdot \tau_{ij} = \bar{\eta}$. Hence $\bar{\eta}$ factorizes via the quotient $\bar{R}/W$. Since the action of W is faithful, it follows from [16; Thm.1.3] and Prop.3. that $\bar{T}$ and $\bar{R}/W$ may be identified. Putting altogether it follows tha R/W is smooth and that $\dot{\Phi}_W$ necessarily defines an isomorphism between $(R/W,0)$ and $(S_a,0)$.

To finish the proof of Theorem 2, it still remains to check statement (iii). But this is an immediate consequence of Corollary 1.11.

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