DOUBLE AFFINE HECKE ALGEBRAS AND MACDONALD'S CONJECTURES

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Dedicated to I.M. Gelfand on the occasion of his 80 birthday

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0. Introduction. In this paper we prove the main results about the structure of double affine Hecke algebras announced in [C1, C2]. The technique is based on the realization of these algebras in terms of Demazure - Lusztig operators [BGG, D, L2, LS, Ch3] and rather standard facts from the theory of affine Weyl groups. In particular, it completes the proof (partially published in [C2]) of the Macdonald scalar product conjecture (see [M1],(12,6')) , including the famous Macdonald constant term conjecture (the q, t-case).

We mainly follow Opdam paper [O] where the Macdonald-Mehta conjectures in the degenerate (differential) case were deduced from certain properties of the Heckman-Opdam operators [HO] and the existence of the shift operators. Heckman's interpretation of these operators via the so-called Dunkl operators (see [He] and also [Ch5]) was important to our approach.

We note that the HO operators are closely related to the so-called quantum many-body problem (Calogero, Sutherland, Moser, Olshanetsky, Perelomov), the conformal field theory (Knizhnik- Zamolodchikov equations), the harmonic analysis on symmetric spaces (Harish-Chandra, Helgason etc.), and (last but not the least) the classic theory of the hypergeometric functions.

Establishing the connection between the difference counterparts of Heckman-Opdam operators introduced in [Ch4] and the Macdonald theory [M1,M2] including the construction of the difference shift operators is the main result of this paper. Once this is done it is not very difficult to calculate the scalar squares of the Macdonald polynomials and prove the constant term conjecture from his fundamental paper [M3].

To simplify the exposition, we consider the reduced root systems only and impose the relation $q = t^k$ for $k \in \mathbf{Z}_+$ (to avoid infinite products in the definition of Macdonald's pairing). The purpose of this work is to present a concrete application of the new technique. Arbitrary q, t can be handled in a similar way and will be considered in the subsequent paper(s). The passage to non-reduced systems is straightforward. It is worth mentioning that the final problem in this line is to

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calculate the scalar products of the eigenfunctions of arbitrary difference Heckman-Opdam operators for any eigenvalues.

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Let $R = \{\alpha\} \subset \mathbf{R}^n$ be a root system of type $A, B, ..., F, G$ with respect to a euclidean form (z, z') on $\mathbf{R}^n \ni z, z'$. We fix the set R_+ of positive roots ($R_- = -R_+$), the corresponding simple roots $\alpha_1, ..., \alpha_n$, and their dual counterparts $a_1, ..., a_n, a_i = \alpha_i^\vee$, where $\alpha^\vee = 2\alpha/(\alpha,\alpha)$. The fundamental weights $\beta_1, ..., \beta_n$ and the dual fundamental weights $b_1, ..., b_n$ are determined from the relations $(\beta_i, a_j) = \delta_i^j = (\alpha_i, b_j)$ for the Kronecker delta. We will also introduce the lattices

$$Q = \oplus_{i=1}^n \mathbf{Z}\alpha_i \subset P = \oplus_{i=1}^n \mathbf{Z}\beta_i, \ A = \oplus_{i=1}^n \mathbf{Z}a_i \subset B = \oplus_{i=1}^n \mathbf{Z}b_i,$$

and $Q_\pm, P_\pm, A_\pm, B_\pm$ for $\mathbf{Z}_\pm = \{m \in \mathbf{Z}, \pm m \geq 0\}$ instead of $\mathbf{Z}$. (In the standard notations, $B = P^\vee, P_+ = P^{++}, \beta_i = \omega_i$ etc.) Later on,

$$\begin{aligned} \nu_\alpha &= (\alpha,\alpha), \ \nu_i = \nu_{\alpha_i}, \ \nu_R = \{\nu_\alpha, \alpha \in R\}, \\ \rho_\nu &= (1/2)\sum_{\nu_\alpha=\nu} \alpha = \sum_{\nu_i=\nu} \beta_i, \ \text{ for } \ \alpha \in R_+. \end{aligned} \tag{0.1}$$

We set $\kappa_\nu = |\{\alpha \in R_+, \ \nu_\alpha = \nu\}|, l_\alpha^\nu = (\alpha^\vee, \rho_\nu)$, where $|\ |$ means the number of elements, and introduce $e_{\nu'} = \{\delta_\nu^{\nu'}, \nu \in \nu_R\}$. We will consider

$$e_\alpha \stackrel{def}{=} e_{\nu_\alpha}, \ \rho \stackrel{def}{=} \{\rho_\nu\}, \ \kappa \stackrel{def}{=} \{\kappa_\nu\}, \ l_\alpha \stackrel{def}{=} \{l_\alpha^\nu\}, \ \nu \in \nu_R, \tag{0.2}$$

as vectors and use the dot product $(\{x_\nu\} \cdot \{y_\nu\} = \sum_{\nu \in \nu_R} x_\nu y_\nu)$.

Let us put formally $x_i = exp(\beta_i), \ x_\beta = exp(\beta) = \prod_{i=1}^n x_i^{k_i}$ for $\beta = \sum_{i=1}^n k_i\beta_i$, and introduce the algebra $\mathbf{C}_t[x]$ of polynomials in terms of $x_i^{\pm 1}$ with the coefficients belonging to the field $\mathbf{C}_t = \mathbf{C}(t)$ of rational functions in terms of an indefinite complex parameter t. The coefficient of $x^0 = 1$ (*the constant term*) will be denoted by $\langle\ \rangle$. We will also involve $x_\beta^{1/2} = exp(\beta/2) = \prod_{i=1}^n x_i^{k_i/2}$, belonging to a proper extension of $\mathbf{C}_t[x]$.

Given $k = \{k_\nu\}, k_\nu \in \mathbf{Z}_+^2$, let $k_\alpha = k_{\nu_\alpha}$,

$$\begin{aligned} \mu_k = \prod_{\alpha \in R_+} &\{(t^{k_\alpha - 1} x_\alpha^{1/2} - t^{1-k_\alpha} x_\alpha^{-1/2}) \cdots \\ &\cdots (t^{1-k_\alpha} x_\alpha^{1/2} - t^{k_\alpha - 1} x_\alpha^{-1/2}) \ (t^{-k_\alpha} x_\alpha^{1/2} - t^{k_\alpha} x_\alpha^{-1/2})\}. \end{aligned} \tag{0.3}$$

THEOREM 0.1.

$$(-1)^{k\cdot\kappa}\langle\mu_k\rangle \;=\; \prod_{\alpha\in R_+}\prod_{i=0}^{k_\alpha-1}\{(t^{k\cdot l_\alpha+i+1}-t^{-k\cdot l_\alpha-i-1})/(t^{k\cdot l_\alpha-i}-t^{-k\cdot l_\alpha+i})\}. \qquad (0.4)$$

□

Note that μ_k belongs to $\mathbf{C}_{t^2}[x]$. We use t to make the notations more symmetric. Formula (0.4) for coinciding $\{k_\nu\}$ is equivalent to Conjecture (3.4) (see also (3.2)) from [M3] (Macdonald's q equals t^{-2}, his t is our q^{-2}).

In the case of A_n, the proof of the (more general) Andrews conjecture can be found in [BZ] (see also [St]) . The BC_1 was considered in [A]. Paper [K] (containing the most elaborate theory based on the Aomoto-Selberg integrals) is devoted to the proof of the Macdonald- Morris conjecture for BC_n. It was also proved for G_2 (see [H]) and F_4 ([GG]). See [M2,K,O] for futher information about related results and special cases.

A generalization (based on the cohomology of certain Lie algebras) was found in [Ha1]. The proof of the so-called strong Macdonald conjecture was done in [Ha2] in the case of A_n. A certain approach to this conjecture was suggested by Feigin.

We will obtain Theorem 0.1 as a particular case of the so-called Macdonald scalar product conjecture.

1. Affine root systems. In the above notations, the vectors $\tilde\alpha = [\alpha, k] \in \mathbf{R}^n \times \mathbf{R} \subset \mathbf{R}^{n+1}$ for $\alpha \in R, k \in \mathbf{Z}$ form the *affine root system* $R^a \supset R$ ($z \in \mathbf{R}^n$ are identified with $[z,0]$). We add $\alpha_0 \stackrel{def}{=} [-\theta, 1]$ to the simple roots for the *maximal root* $\theta \in R$. The corresponding set R^a_+ of positive roots coincides with $R_+ \cup \{[\alpha,k],\ \alpha \in R,\ k>0\}$.

We will use the Dynkin diagram Γ and its affine completion Γ^a with $\{\alpha_j, 0 \le j \le n\}$ as the vertices ($m_{ij} = 2,3,4,6$ if α_i and α_j are joined by 0,1,2,3 laces respectively). The set of the indices of the images of α_0 by all the automorphisms of Γ^a will be denoted by O ($O = \{0\}$ for E_8, F_4, G_2). Let $O^* = r \in O, r \neq 0$.

Without going into detail, we mention that $(\theta^\vee, \alpha) \le 1$ for $\theta \neq \alpha \in R_+$. More precisely, $\theta = \sum_i \beta_i$, where $m_{i0} > 2$. The multiplicity (b_r, α) of the roots α_r in arbitrary $\alpha \in R_+$ is also not more than 1 for $r \in O^*$ (i.e. b_r are minuscule co-weights). For instance, $(b_r, \theta) = 1$ (see [B,V,C4]).

Given $\tilde\alpha = [\alpha,k] \in R^a,\ b \in B$, let

$$s_{\tilde\alpha}(\tilde z) \;=\; \tilde z - (z,\alpha^\vee)\tilde\alpha,\quad b'(\tilde z) \;=\; [z, \zeta - (z,b)] \ \text{ for } \ \tilde z = [z,\zeta] \in \mathbf{R}^{n+1}. \quad (1.1)$$

The *affine Weyl group* W^a is the span $< s_{\tilde{\alpha}} >$. It is generated by the simple reflections $s_j = s_{\alpha_j}, 0 \leq j \leq n$, and can be represented as the semi-direct product $W \ltimes A'$ of its subgroups $W =< s_\alpha, \alpha \in R_+ >$ and $A' = \{a', a \in A\}$, where

$$a' = s_\alpha s_{[\alpha,1]} = s_{[-\alpha,1]} s_\alpha \text{ for } a = \alpha^\vee. \tag{1.2}$$

The *extended Weyl group* W^b generated by W and B' (instead of A') is isomorphic to $W \ltimes B'$:

$$(wb')([z,\zeta]) = [w(z), \zeta - (z,b)] \text{ for } w \in W, b \in B. \tag{1.3}$$

DEFINITION 1.1.
i) Given $b_+ \in B_+$, *let*

$$\omega_{b_+} = w_0 w_0^+ \in W, \ \pi_{b_+} = b'_+ (\omega_{b_+})^{-1} \in W^b, \ \omega_i = \omega_{b_i}, \pi_i = \pi_{b_i}, \tag{1.4}$$

where w_0 *(respectively,* w_0^+*) is the longest element in* W *(respectively, in* W_{b_+} *generated by* s_i *preserving* b_+*) relative to the set of generators* $\{s_i\}$ *for* $i > 0$.
ii) If b *is arbitrary then there exist unique elements* $w \in W$, $b_+ \in B_+$ *such that* $b = w(b_+)$ *and* $(\alpha, b_+) \neq 0$ *if* $(-\alpha) \in R_+ \ni w(\alpha)$. *We set*

$$\omega_b = \omega_{b_+} w^{-1}, \ \pi_b = w \pi_{b_+}. \tag{1.5}$$

□

We will discuss general properties of $\{\omega_b, \pi_b\}$ later. Now we only note that the elements $\pi_r, r \in O$, leave Γ^a invariant and form a group denoted by Π, which is isomorphic to B/A by the natural projection $\{b_r \to \pi_r\}$. As to $\{\omega_r\}$, they preserve the set $\{-\theta, \alpha_i, i > 0\}$. The relations $\pi_r(\alpha_0) = \alpha_r = (\omega_r)^{-1}(-\theta)$ distinguish the indices $r \in O^*$. These elements are important because:

$$W^b = \Pi \ltimes W^a, \text{ where } \pi_r s_i \pi_r^{-1} = s_j \text{ if } \pi_r(\alpha_i) = \alpha_j. \tag{1.6}$$

Basically, this property is due to [B]. In this very form, (1.6) was established in [V] (1.2), as well as an equivalent version of (1.9) below and some other facts of this kind. Then similar results appeared in different places (say in [KW] and [C4]). I'd like to thank Kumar for the reference to [V] (which still seems one of the most complete papers on this stuff).

To go further we need the notion of length and its geometric interpretation. Given $\nu \in \nu_R$, $r \in O^*$, $\tilde{w} \in W^a$, and a reduced decomposition $\tilde{w} = s_{j_l} \ldots s_{j_2} s_{j_1}$

with respect to $\{s_j, 0 \le j \le n\}$, we call $l = l(\hat{w})$ the *length* of $\hat{w} = \pi_r \tilde{w} \in W^b$ and introduce the sets

$$\lambda(\hat{w}) = \{\tilde{\alpha}^1 = \alpha_{j_1},\ \tilde{\alpha}^2 = s_{j_1}(\alpha_{j_2}),\ \tilde{\alpha}^3 = s_{j_1}s_{j_2}(\alpha_{j_3}), \ldots, \tilde{\alpha}^l = \tilde{w}^{-1}s_{j_l}(\alpha_{j_l})\},$$
$$\lambda_\nu(\hat{w}) = \{\tilde{\alpha}^m,\ \nu(\tilde{\alpha}^m) = \nu(\tilde{\alpha}_{j_m}) = \nu\} \ \text{ for } \ \nu([\alpha, k]) \stackrel{def}{=} \nu_\alpha,\ 1 \le m \le l. \qquad (1.7)$$

One has: $l = \sum_\nu l_\nu$, where $l_\nu = l_\nu(\hat{w}) = |\lambda_\nu(\hat{w})|$ denotes the corresponding number of elements.

To see that these sets do not depend on the choice of the reduced decomposition we will use the following (affine) action of W^b on $z \in \mathbf{R}^n$:

$$(wb')\langle z\rangle = w(b+z),\ w \in W, b \in B,$$
$$s_{\tilde{\alpha}}\langle z\rangle = z - ((z,\alpha)+k)\alpha^\vee,\ \tilde{\alpha} = [\alpha, k] \in R^a, \qquad (1.8)$$

and the affine Weyl chamber:

$$C^a = \bigcap_{j=0}^{n} L_{\alpha_j},\ L_{\tilde{\alpha}} = \{z \in \mathbf{R}^n,\ (z,\alpha)+k > 0\}. \qquad (1.9)$$

PROPOSITION 1.2.

$$\lambda_\nu(\hat{w}) = \{\tilde{\alpha} \in R^a,\ \hat{w}^{-1}\langle C^a\rangle \not\subset L_{\tilde{\alpha}},\ \nu(\tilde{\alpha}) = \nu\}$$
$$= \{\tilde{\alpha} \in R^a,\ l_\nu(\hat{w}s_{\tilde{\alpha}}) < l_\nu(\hat{w})\}. \qquad (1.9)$$

□

As to the latter condition, direct calculation shows that

$$l(\ \hat{w}s_{\tilde{\alpha}\{1\}}\ldots s_{\tilde{\alpha}\{p\}}\ ) > l(\ \hat{w}s_{\tilde{\alpha}\{1\}}\ldots s_{\tilde{\alpha}\{p+1\}}\ ),\ \text{if}$$
$$\tilde{\alpha}\{q\} \stackrel{def}{=} \tilde{\alpha}^{m_q},\ l \ge m_1 > m_2 > \ldots > m_p > m_{p+1} \ge 1. \qquad (1.10)$$

Vice versa, an arbitrary sequence of positive roots $\tilde{\alpha}\{1\}, \tilde{\alpha}\{2\}, \ldots$ satisfying the consequent conditions (1.10) for $p = 0, 1, \ldots$ can be obtained by the above construction (i.e. belongs to $\lambda_\nu(\hat{w})$ and corresponds to a certain reduced decomposition of $\tilde{w}$). We will not use this fact and only mention that it results from the following rather standard proposition.

PROPOSITION 1.3. *(see e.g. [C4], Proposition 1.4).*

Each of the following conditions for $x, y \in W^b$ is equivalent to the relation $l_\nu(xy) = l_\nu(x) + l_\nu(y)$:

$$a)\ \lambda_\nu(xy) = \lambda_\nu(y) \cup y^{-1}(\lambda_\nu(x)),\ b)\ y^{-1}(\lambda_\nu(x)) \subset R^a_+$$
$$c)\ \lambda_\nu(y) \subset \lambda_\nu(xy),\ d)\ y^{-1}(\lambda_\nu(x)) \subset \lambda_\nu(xy). \qquad (1.11)$$

□

Now everything is prepared to motivate the construction of $\{\pi_b\}$.

THEOREM 1.4.
i) In the above notations,

$$\lambda(b') = \{\tilde\alpha, \alpha\in R_+, (b,\alpha) > k \geq 0\} \cup \{\tilde\alpha, \alpha\in R_-, (b,\alpha)\geq k > 0\}, \quad (1.12)$$
$$\lambda(\pi_b^{-1}) = \{\tilde\alpha, -(b,\alpha) > k \geq 0\}, \quad \textit{where } \tilde\alpha = [\alpha, k] \in R_+^a, b\in B. \quad (1.13)$$

ii) If $\hat w \in b'W$ *(i.e.* $\hat w\langle 0\rangle = b$*) then* $\hat w = \pi_b w$ *for* $w \in W$ *such that* $l(\hat w) = l(\pi_b) + l(w)$ *. Given* $b \in B$*, this property (valid for any* $\hat w$ *taking* 0 *to* b*) determines* π_b *uniquely.*

Proof. Formula (1.12) is verified directly (see Proposition 1.6, b) from [C4]). By the way, it gives the useful formulas (cf. [L1], 1.4) :

$$l_\nu(b') = \sum_\alpha |(b,\alpha)|, \quad \text{where } |\;| = \text{abs. value}, \alpha\in R_+, \nu_\alpha = \nu \in \nu_R,$$
$$l_\nu(b'_+) = 2(b,\rho_\nu), \quad \text{when } b\in B_+. \quad (1.14)$$

One can follow the same proposition (assertion a)) to check that

$$\lambda(\omega_{b'_+}) = \{\alpha\in R_+, (b_+,\alpha) > 0\} \quad \text{for } b_+\in B_+. \quad (1.15)$$

It proves (1.13) for B_+, due to Proposition 1.3, a) and the relation $\lambda(\hat w^{-1}) = -\hat w\langle\lambda(\hat w)\rangle$ (resulting from Proposition 1.2).

Let $b = w(b_+)$ for positive b_+ and $w \in W$. We can multiply w on the right by elements preserving b_+ (i.e. belonging to W_{b_+}). If the length of w is the least possible, then $\lambda(w)$ does not contain roots $\alpha \in R_+$ orthogonal to b_+ (Proposition 1.2) and w is defined uniquely. This condition is from Definition 1.1, ii).

Setting $b = \pi\omega$ for $\omega \in W$, where $\pi \in W$ has the least possible length $l(\pi)$, we are going to calculate $\lambda(\omega)$ and $\lambda(\pi^{-1})$.

The set $\lambda(\pi)$ containes only roots $\tilde\alpha = [\alpha, k]$ with $k > 0$. Otherwise we could find in this set a root from R_+ and apply the second formula from (1.9) to reduce π by the corresponding reflection from W. Hence, $w^{-1}(\lambda(\pi)) \subset R_+^a$ and the decomposition $b = \pi\omega$ satisfies condition (1.11). Moreover, $w^{-1}\langle\lambda(\pi)\rangle$ contains all the elements from $\lambda(b)$ with $k > 0$ (since $w \in W$ – use (1.11) again). It is enough to calculate $\lambda(\omega)$ because $\lambda(b)$ is already known. We will arrive at the same formula (1.15) (but now for ω and $b \in B$). Applying (1.11) after the passage to $-b$, we obtain precisely (1.13) for $\lambda(\pi^{-1})$.

Let us calculate $\lambda(\omega_b)$ and $\lambda(\pi_b^{-1})$. Thanks to formula (1.15) for b_+ and the properties of w (see above) we have the embedding $\lambda(w) \subset \lambda(\omega_{b_+})$. Hence the decomposition $\omega_{b_+} = \omega_b w$ satisfies conditions (1.11) and

$$\lambda(\omega_b) = w(\lambda(\omega_{b_+}) \setminus \lambda(w)) = w(\lambda(\omega_{b_+})) \cap R_+$$
$$= w(\{\alpha\in R, (\alpha, b_+) > 0\}) \cap R_+ \;=\; \{\alpha'\in R_+, (\alpha', b) > 0, \}.$$

Here one can use Proposition 1.3 with the relation $\lambda(w) = \{\alpha \in R_+, w(\alpha) \in R_-\}$ resulting directly from (1.9). We see that (abstact) ω defined above and ω_b from (1.5) coincide (they have the same λ-sets). It gives the coincidence of π and π_b, formula (1.13), and statement ii). As for the latter, if $\hat{w}\langle 0\rangle = b$, then $\hat{w} = \pi_b w', w' \in W$. However we know that $l(\pi_b w') = l(\pi_b) + l(w')$ for any $w' \in W$. □

We set

$$c \preceq b, b \succeq c \ \text{ for } \ b, c \in B \quad \text{if} \quad b - c \in A_+, \tag{1.16}$$

and use $\prec, \succ$ respectively if $b \neq c$. Given $b \in B$, let $b_+ = w_+^{-1}(b) \in B_+$ for w_+ from Definition 1.1. The sets

$$\begin{aligned} \sigma^\vee(b) &\stackrel{def}{=} \{g \in B, w(c) \preceq b_+ \ \text{ for any } w \in W\}, \\ \sigma_0^\vee(b) &\stackrel{def}{=} \{c \in B, w(c) \prec b_+ \ \text{ for any } w \in W\} \end{aligned} \tag{1.17}$$

are W-invariant (which is evident) and convex. The latter means that if $c, c^* = c + r\alpha^\vee \in \sigma^\vee(b) (\in \sigma_0^\vee(b))$ for $\alpha \in R, r \in \mathbf{Z}_+$, then

$$\{c,\ c+\alpha^\vee, ..., c+(r-1)\alpha^\vee,\ c^*\} \subset \sigma^\vee(b)(\subset \sigma_0^\vee(b)). \tag{1.18}$$

Really, $w(c + r'\alpha^\vee)$ for $0 < r' < r$ is always between $w(c), w(c^*)$ for any w with respect to the ordering '$\prec$' and therefore belongs to (1.17) because $w(c), w(c^*)$ do.

For the sake of completeness, we will check another well known property of $\sigma^\vee(b)$. It contains the orbit $W(b)$. If $w(b) \preceq b_+$ and $l(ws_\alpha) > l(w)$ for $\alpha \in R_+$, then $w(\alpha) \in R_+$ and $ws_\alpha(b_+) = w(b_+ - (b_+, \alpha)\alpha^\vee) \preceq b_+$. Hence we can argue by induction.

PROPOSITION 1.5.

i) Given $\hat{w} \in W^b, \tilde{\alpha} \in \lambda(\hat{w})$, let $b = \hat{w}\langle 0\rangle, \hat{w}_* = \hat{w}s_{\tilde{\alpha}}, b_* = \hat{w}_*\langle 0\rangle$. Then $b_* \in \sigma^\vee(b)$. If $b \in B_+$ and $b_* \neq b$, then $b_* \in \sigma_0^\vee(b)$.

ii) In the above hypotheses, $\ell(\hat{w}) > \ell(b'_+)$ if $b_+ \neq b$, and

$$\ell(\hat{w}_*) < \ell(\hat{w}) \ \text{ if } \ b_* \neq b, \quad \text{where} \quad \ell(\hat{w}) = \ell(b') \stackrel{def}{=} l(\pi_b). \tag{1.19}$$

iii) Let $\hat{w}_* = s_{\tilde{\alpha}\{p\}} ... s_{\tilde{\alpha}\{1\}}\hat{w}$,where we take any sequence (1.10) for $\hat{w}^{-1}$ (instead of $\hat{w}$) such that $\ell(s_{\tilde{\alpha}\{1\}}\hat{w}) < \ell(\hat{w})$. Then $\ell(\hat{w}_*) < \ell(\hat{w})$ and $\hat{w}_*\langle 0\rangle \neq b$. If $b = b_+$ then $\{b_*\} = \sigma_0^\vee(b)$.

Proof. One has: $\lambda(\tilde{w}^{-1}) \subset \{\tilde{\alpha} = [\alpha, k] \in R_+^a, -(b, \alpha) \geq k \geq 0\}$ (use (1.9)). Hence,

$$b_* = s_{\tilde{\alpha}}\langle b\rangle = b - ((b, \alpha) + k)\alpha^\vee$$

is between b and $s_\alpha(b)$ with respect to the odering '$\preceq$'. If $b \in B_+$ (i.e. $b = b_+$) and $b_* \neq b$, then $\alpha \in R_-, k > 0$, and $b \prec b_* \prec s_\alpha(b)$. It completes i). Assertions ii) and iii) follow directly from the definitions of π_b and $\ell(\ )$. □

2. Double affine Hecke algebras.

The construction depends on $\delta \in \mathbf{C}^*$ and $\{q_\nu \in \mathbf{C}^*,\ \nu \in \nu_R\}$ which will be regarded as formal parameters; $\mathbf{C}_{\delta,q}$ means the field of rational functions in $\{\delta, q_\nu\}$. In the hypotheses of the previous sections, we denote the least common order of the elements of Π by m ($m = 2$ for D_{2k}, otherwise $m = |\Pi|$) and set

$$\Delta \ = \ \delta^m\ ,\ q_{\tilde\alpha} = q_{\nu(\tilde\alpha)},\ q_j = q_{\alpha_j},\ \text{ where } \tilde\alpha \in R^a, 0 \le j \le n. \tag{2.1}$$

We remind that $x_i = exp(\beta_i),\ x_\beta = exp(\beta) = \prod_{i=1}^n x_i^{k_i}$ for $\beta = \sum_{i=1}^n k_i\beta_i$, $\mathbf{C}_\delta[x] = \mathbf{C}_\delta[x_\beta]$ means the algebra of polynomials in terms of $x_i^{\pm 1}$ with the coefficients depending on δ rationally. We will also use

$$X_{\tilde\beta} \ = \ \prod_{i=1}^n X_i^{k_i}\delta^{mk} \ \text{ if } \ \tilde\beta = [\beta, k],\ \beta = \sum_{i=1}^n k_i\beta_i \in P,\ mk \in \mathbf{Z}, \tag{2.2}$$

where $\{X_i\}$ are independent variables which act in $\mathbf{C}_\delta[x]$ naturally:

$$X_{\tilde\beta}(p(x)) \ = \ x_{\tilde\beta}p(x), \ \text{ where } \ x_{\tilde\beta} \stackrel{def}{=} x_\beta\delta^{mk},\ p(x) \in \mathbf{C}_\delta[x]. \tag{2.3}$$

The elements $\tilde w \in W^b$ act in $\mathbf{C}_\delta[x]$, $\mathbf{C}_\delta[X] = \mathbf{C}_\delta[X_\beta]$ by the formulas:

$$\tilde w(x_{\tilde\beta}) \ = \ x_{\tilde w(\tilde\beta)},\ \ \tilde w X_{\tilde\beta}\tilde w^{-1} \ = \ X_{\tilde w(\tilde\beta)}. \tag{2.4}$$

In particular (we will use this in the sequel):

$$\pi_r(x_\beta) \ = \ x_{\omega_r^{-1}(\beta)}\delta^{m(\beta, b_{r^*})} \ \text{ for } \ \alpha_{r^*} \stackrel{def}{=} \pi_r^{-1}(\alpha_0),\ r \in O^*. \tag{2.5}$$

DEFINITION 2.1. *(see [C1,C2])*

The double affine Hecke algebra $\mathfrak{H}$ *is generated over the field* $\mathbf{C}_{\delta,q}$ *by the elements* $T_j,\ 0 \le j \le n\}$, *pairwise commutative* $\{X_\beta,\ \beta \in P\}$, *and the group* Π, *satisfying the following relations:*

(o) $(T_j - q_j)(T_j + q_j^{-1}) \ = \ 0,\ 0 \ \le \ j \ \le \ n;$
(i) $T_iT_jT_i... \ = \ T_jT_iT_j...$, m_{ij} *factors on each side;*
(ii) $\pi_r T_i \pi_r^{-1} \ = \ T_j$ *if* $\pi_r(\alpha_i) = \alpha_j;$
(iii) $T_iX_\beta T_i \ = \ X_\beta X_{\alpha_i}^{-1}$ *if* $(\beta, a_i) = 1,\ 1 \le i \le n;$
(iv) $T_0^{-1}X_\beta T_0^{-1} \ = \ X_{s_0(\beta)} \ = \ X_\beta X_\theta^{-1}\Delta$ *if* $(\beta, \theta^\vee) = 1;$

(v) $T_i X_\beta = X_\beta T_i$ *if* $(\beta, a_i) = 0$, *where* $a_0 = \theta^\vee$;
(vi) $\pi_r X_\beta \pi_r^{-1} = X_{\pi_r(\beta)} = X_{\omega_r^{-1}(\beta)}\delta^{m(b_{r^*},\beta)}$, $r \in O^*$.

□

Given $\tilde{w} \in W^a, r \in O$, the product

$$T_{\pi_r\tilde{w}} \stackrel{def}{=} \pi_r \prod_{k=1}^{l} T_{i_k}, \quad \text{where} \quad \tilde{w} = \prod_{k=1}^{l} s_{i_k}, l = l(\tilde{w}), \tag{2.6}$$

does not depend on the choice of the reduced decomposition (because $\{T\}$ satisfy the same "braid" relations as $\{s\}$ do). Moreover,

$$T_{\tilde{v}} T_{\tilde{w}} = T_{\tilde{v}\tilde{w}} \text{ whenever } l(\tilde{v}\tilde{w}) = l(\tilde{v}) + l(\tilde{w}) \text{ for } \tilde{v}, \tilde{w} \in W^b, \tag{2.7}$$

which follows from (2.6) and relations (ii). In particular, we arrive at the pairwise commutative operators (use (2.7) and (1.14)):

$$Y_b = \prod_{i=1}^{n} Y_i^{k_i} \text{ if } b = \sum_{i=1}^{n} k_i b_i \in B, \text{ where } Y_i \stackrel{def}{=} T_{b_i'}. \tag{2.8}$$

PROPOSITION 2.2.

$$\begin{aligned} T_i^{-1} Y_b T_i^{-1} &= Y_b Y_{a_i}^{-1} \ \textit{if} \ (b, \alpha_i) = 1, \\ T_i Y_b &= Y_b T_i \ \textit{if} \ (b, \alpha_i) = 0, \ 1 \le i \le n. \end{aligned} \tag{2.9}$$

Proof(cf. [L1], 2.7). We will deduce these relations from (i)-(ii). It sufices to check that

$$T_i^{-1} Y_i T_i^{-1} = Y_i Y_{a_i}^{-1}, \ T_i Y_j = Y_j T_i \text{ for } 1 \le i \ne j \le n. \tag{2.10}$$

Applying (1.15) to $\tilde{b} = s_i(b_i) = b_i - a_i$, we see that $l(\tilde{b}') = \sum_{\alpha \in R_+} |(b_i, s_i(\alpha))| = l(b_i') - 2$, since $s_i(\alpha) \in R_+$ for $\alpha \in R_+ \setminus \{\alpha_i\}$. Hence formula (2.7) works for the triple decomposition $b_i' = s_i \tilde{b} s_i$. If $j \ne i$, then $\alpha_j \notin \lambda(b_i')$ (see (1.12)) and $l(b_i' s_j) = l(b_i') + 1$. Now we only have to use the commutativity of b_i and s_j. □

Vice versa, imposing (i), one can deduce formally (ii) from (2.9) (and the commutativity of Y). When $m \ne 1$ there is rather straightforward proof using $\{\pi_r\}$. Here we will obtain it as an application of the following construction.

The *Demazure-Lusztig operators* (see [KL, KK, C1], and [C5] for more detail)

$$\hat{T}_j = q_j s_j + (q_j - q_j^{-1})(X_{\alpha_j} - 1)^{-1}(s_j - 1), \ 0 \le j \le n, \tag{2.11}$$

act in $\mathbf{C}_{\delta,q}[x]$. It is worth mentioning that

$$\hat{T}_0 = q_0 s_0 + (q_0 - q_0^{-1})(\Delta X_\theta^{-1} - 1)^{-1}(s_0 - 1),$$
$$\text{where } s_0(X_i) = X_i X_\theta^{-(\beta_i,\theta^\vee)} \Delta^{(\beta_i,\theta^\vee)}. \quad (2.12)$$

THEOREM 2.3.

i) The map $\hat{\zeta}(T_j) = \hat{T}_j$, $\hat{\zeta}(X_\beta) = X_\beta$ (see (2.3)), $\hat{\zeta}(\pi_r) = \pi_r$ (see (2.5)) induces a $\mathbf{C}_{\delta,q}$-linear homomorphism from $\mathfrak{H}$ to the algebra of linear endomorphisms of $\mathbf{C}_{\delta,q}[x]$, well-defined for arbitrary values $\{\delta \in \mathbf{C}^* \ni q\}$.

ii) This representation is faithful if the value of δ is not a root of unity and q are arbitrary (from $\mathbf{C}^*$). In this case, any given element $H \in \mathfrak{H}$ has the unique "normal form":

$$H = \sum_{b\in B, w\in W} Y_b h_{b,w} T_w, \; h_{b,w} \in \mathbf{C}_{\delta,q}[X]. \quad (2.13)$$

Proof. Relations (ii),(vi) (involving π) results directly from (1.6). As to the relations with X ((iii),(iv),(v)), they follow from the formulas $s_j X_{\beta_j} = X_{s_j(\beta_j)} s_j$ (in the non-affine case they are due to Lusztig). One may check (o) and (i) only for $j > 0$ (moreover, there is a reduction to the case of rank 2), when these relations are from [L2] (see also [LS] about roots of type A).

Let as assume that δ is not a root of unity. Arbitrary element $\hat{H} = \hat{\zeta}(H)$, $H \in \mathfrak{H}$, can be uniquely represented as follows:

$$\hat{H} = \sum_{b\in B, w\in W} b' g_{b,w} w, \quad (2.14)$$

where $g_{b,w}$ are rational functions in $\{X_1, ..., X_n\}$.

LEMMA 2.4.

Given $b \in B$ and $\hat{w} = \pi_b \omega, \omega \in W$,

$$\hat{T}_{\hat{w}} = \xi(\hat{T}_{\hat{w}}) + \sum_{b_*, w\in W} g_{b_*,w}\, b'_* w, \;\; \text{where } b_* \in \sigma^\vee(b), \ell(b'_*) < \ell(b') \quad (2.15)$$

$$\text{for } \xi(\hat{T}_{\hat{w}}) = \prod_{\tilde{\alpha}\in\lambda(\pi_b^{-1})} \frac{q_{\tilde{\alpha}} X_{\tilde{\alpha}} - q_{\tilde{\alpha}}^{-1}}{X_{\tilde{\alpha}} - 1} b'(\omega_b^{-1} \hat{T}_\omega), \quad (2.16)$$

that is invertible. Moreover, if $b \in B_+$ then $b_* \in \sigma_0^\vee(b)$ (in this case $T_{\hat{w}} = Y_b T_{\omega_b}^{-1} T_\omega$) .

Proof. Following [C4], let

$$G_{\tilde{\alpha};q} = G_{\tilde{\alpha}} = q_{\tilde{\alpha}} + (q_{\tilde{\alpha}} - q_{\tilde{\alpha}}^{-1})(X_{\tilde{\alpha}}^{-1} - 1)^{-1}(1 - s_{\tilde{\alpha}}), \; \tilde{\alpha} \in R^a. \quad (2.17)$$

Given a reduced decomposition $\hat{w} = \pi_r s_{j_l} \cdots s_{j_1}$, $l = l(\hat{w}), r \in O$, one has (see (1.7)):

$$\hat{T}_{\hat{w}} = \hat{w} G_{\tilde{\alpha}^l} \cdots G_{\tilde{\alpha}^1}, \quad \text{for} \quad \tilde{\alpha}^1 = \alpha_{j_1}, \tilde{\alpha}^2 = s_{j_1}(\alpha_{j_2}), \tilde{\alpha}^3 = s_{j_1} s_{j_2}(\alpha_{j_3}), \ldots \quad (2.18)$$

We may assume that $\omega = s_{j_e} \cdots s_{j_1}$, $e = l(\omega)$. If we take only the terms without $s_{\tilde{\alpha}}$ from the binomials $G_{\tilde{\alpha}^p}$, where $\tilde{\alpha} = \tilde{\alpha}^p$, $p > e$, then the corresponding product coincides exactly with (2.16). Apply (1.13) to check it. Any other terms contribute to the elements $g_{b_*,w}\, b'_* w$ with $b'_* \neq b$. (see Proposition 1.5). □

We come back to the theorem. First of all, the existence of (2.13) follows from Definition 2.1 and Proposition 2.2.

If there is a nontrivial expression in the form (2.13) vanishing identically and involving $\{Y\}$, one can make all b positive (multiplying on the left by a proper Y). Then take $\hat{\zeta}$ and rewrite $\hat{T}_{b'} = \hat{Y}_b, b \in B_+$, due to the lemma. Using the ℓ-length, we arrive at a nontrivial sum of type (2.14) without Y and representing zero . A contradiction with [L1],[C3].

As a by-product, we proved that $\hat{\zeta}$ is a faithful representation. □

3. Difference Heckman-Opdam operators. Let $\mathcal{H}_X$, $\mathcal{H}_Y$ be the *affine Hecke algebras* generated over $\mathbf{C}$ by abstract $\{T_i, 1 \leq i \leq n\}$ and pairwise commutative $\{X_i\}$, $\{Y_i\}$ satisfying relations (o,i,iii,v) from Definition 2.1 (for $1 \leq i, j \leq n$) and (2.10). We assume that δ is not a root of unity and regard them as subalgebras of $\mathfrak{H}$ (which is possible thanks to Theorem 2.3, ii)). From now on $\mathfrak{H}$ is identified with its image with respect to $\hat{\zeta}$. We write T, Y instead of $\hat{T}, \hat{Y}$.

The algebra of W-invariant elements in the $\mathbf{C}[x]$ is denoted by $\mathbf{C}[x]^W$. Here (and in similar cases) x will be replaced by X and other letters without more comment. We will often use that $\mathbf{C}[X]^W$ is the center of $\mathcal{H}_X$. The same of course holds for $\mathbf{C}[Y]^W$ and $\mathcal{H}_Y$. This property is due to Bernstein (see e.g. [L1], [C3]).

Theorem 2.3 gives that an arbitrary element $H \in \mathcal{H}_Y$ can be uniquely represented in the form :

$$H = \sum_{w \in W, b \in B} h_{b,w}\, Y_b T_w = \sum_{w \in W, b \in B} b' g_{b,w} w, \quad (3.1)$$

where $h_{b,w} \in \mathbf{C}$, $g_{b,w}$ are rational functions in $\{X_1, \ldots, X_n\}$. Let us check that they are regular at the points

$$\diamond \stackrel{def}{=} (X_1 = \ldots = X_n = 0), \ \bowtie \stackrel{def}{=} (X_1 = \ldots = X_n = \infty).$$

Really, $\{(X_{\tilde{\alpha}}^{\pm} - 1)^{-1}\}$ (from (2.11) and (2.18)) are well-defined at these points either for positive or for negative $\tilde{\alpha} \in R^a$.

Let us define the *difference Harish-Chandra homomorphism*:

$$\chi(\sum_{w\in W, b\in B} b' g_{b,w} w) \;=\; \sum_{w,b}(\prod_{\nu\in\nu_R} q_\nu^{-2(b,\rho_\nu)}) g_{b,w}(\diamondsuit) y_b \;\in \mathbf{C}_{\delta,q}[y], \qquad (3.2)$$

where $\{y_b\}$ is one more set of variables introduced for independent $y_1, ..., y_n$ in the same way as $\{x_b\}$ were. We mention that one can switch g and b' in (3.2) because the point $\diamondsuit$ is B'-invariant.

PROPOSITION 3.1.

$$\chi(\sum_{w\in W, b\in B} h_{b,w}\, Y_b T_w\,) \;=\; \sum_{w\in W, b\in B}(\prod_\nu q_\nu^{l_\nu(w)} h_{b,w}\, y_b. \qquad (3.3)$$

Proof. Firstly, one can consider $\{Y_b\}$ only. Let us deduce from formula (2.16), that the χ-value of the leading term $\xi(\hat{T}_{b'})$ gives exactly y_b. It is clear, that $\chi(\omega_b^{-1}\hat{T}_{\omega_b}) = \prod_\nu q_\nu^{l_\nu(\omega_b)}$. As to the product before b', let us look at (1.13).

Given $\alpha \in R_+$, if $-(b,\alpha) > 0$ then the number of roots $\tilde{\alpha} = [\alpha, k]$ in the product equals $-(b,\alpha)$. Otherwise (if $(b,\alpha) > 0$), the number of roots $\tilde{\alpha} = [-\alpha, k]$ in the product is (b,α). In the first case, $X_{\tilde{\alpha}}(\diamondsuit) = 0$. The second leads to ∞. The corresponding ratio in the considered product is either $q_{\nu_\alpha}^{-1}$ or q_{ν_α} respectively. Together with the ω_b-part calculated above (the roots from $\lambda(\omega_b)$ are all non-affine and positive), we arrive at the required statement.

Here b was arbitrary. We can say more for positive b. Any other terms contribute to the coefficients $g_{b_*,w}$ with $b_* \in \sigma_0^\vee(b)$ and come from the s-parts of the subproducts (cf. (1.10)) :

$$b' G_{\tilde{\alpha}\{1\}} \cdots G_{\tilde{\alpha}\{p\}}, \quad \text{where } \tilde{\alpha}\{1\} = \tilde{\alpha}^{m_1}, ..., \; l \geq m_1 > ... > m_p \geq 1.$$

Moreover, $m_1 > e$ for the first G, which gives the factor $(X_{[\alpha,k]}^{-1} - 1)^{-1}$ for $\tilde{\alpha}\{1\} = [\alpha, k], \alpha \in R_+$. Its value at $\diamondsuit$ is 0 and will remain unchangable after transforming and taking χ . Thus $\chi(Y_b) = y_b$ and

$$g_{b_*,w}(\diamondsuit) = 0 \;\text{ for }\; b \in B_+, w \in W\; b_* \in \sigma_0^\vee(b). \qquad (3.4)$$

Let us consider now $b \in B_-$. Theorem 1.4 gives, that $\pi_b = b'$, $\ell(b') = l(b') > \ell(b'_*)$ for any $b \neq b_* \in \sigma_0^\vee(b)$.

LEMMA 3.2.

Given $b \in B_-$,

$$Y_b = \xi(Y_b) + \sum_{b_*,w\in W} g_{b_*,w}\, b'_* w,\; b \neq b_* \in \sigma^\vee(b),$$

$$\xi(Y_b) \;=\; \prod_{\tilde{\alpha}\in\lambda(-b')} \frac{q_{\tilde{\alpha}} X_{\tilde{\alpha}} - q_{\tilde{\alpha}}^{-1}}{X_{\tilde{\alpha}} - 1} b',\; g_{b_*,w}(\diamondsuit) = 0 \textit{ for any } b_*, w. \qquad (3.5)$$

Proof. Since $(G_{\tilde{\alpha};q})^{-1} = G_{-\tilde{\alpha};q^{-1}}$ (see (2.17), [C4]), Y_{-b}^{-1} can be obtained from $T_{b'}$ by the following substitution : $q_\nu \to q_\nu^{-1}, \delta \to \delta^{-1}$, $X_i \to X_i^{-1}$ for all $\{\nu, 1 \leq i \leq n\}$. Thus we may use Lemma 2.4. The last relation from (3.5) holds because $\lambda(-b')$ contains only $[\alpha, k]$ with positive α and we have to replace X_α by X_α^{-1} for all α afterwards. □

Turning to arbitrary $b \in B$, let $b = b_+ + b_-$, where $b_\pm \in B_\pm$. Then (see (2.8)), $Y_b = Y_{b_+} Y_{b_-}$. Since we have conditions (3.5) for both $Y_{b_\pm}$ it completes the proof. □

Given $f \in \mathbf{C}[y]$, let

$$\mathcal{L}_f = f(Y_1, ..., Y_n) = \sum_{w \in W, b \in B} b' g_{b,w} w, \; L_f \stackrel{def}{=} (\mathcal{L}_f)_{red} = \sum_{w \in W, b \in B} b' g_{b,w}. \quad (3.6)$$

We notice that the restrictions of $\mathcal{L}_f$ and L_f on $\mathbf{C}_{\delta,q}[x]^W$ coincide.

THEOREM 3.3.

The operators $\{L_f,\ f \in \mathbf{C}[y]^W\}$ are pairwise commutative, W-invariant (i.e $w L_f w^{-1} = L_f$ for all $w \in W$) and preserve $\mathbf{C}_{\delta,q}[x]^W$.

Proof. First, the operators $\{\mathcal{L}_f\}$ are pairwise commutative and preserve $\mathbf{C}_{\delta,q}[x]$ (because $\{T_j, 0 \leq j \leq n\}$ and $\{\pi_r\}$ do). Then $T_i \mathcal{L}_f = \mathcal{L}_f T_i$ for all $i \geq 1$ and f, since $f(Y)$ are central in $\mathcal{H}_Y$ (due to Bernstein). It results in the relations $T_i(\mathcal{L}_f(p(x))) = q_i \mathcal{L}_f(p(x))$ for any $p(x) \in \mathbf{C}_{\delta,q}[x]^W$. We see that $\{\mathcal{L}_f\}$ and $\{L_f\}$ leave $\mathbf{C}_{\delta,q}[x]^W$ invariant. Hence the commutativity of $\{\mathcal{L}\}$ gives the commutativity of $\{L\}$ upon the restriction to $\mathbf{C}_{\delta,q}[x]^W$, which leads to the required commutativity. Cf. also [Ch4], Theorem 3.6 (the rational case). □

Proposition 3.1 supplies us with the χ-values of $\{\mathcal{L}_f\}$ (which will be necessary to prove the Macdonald conjecture). Moreover we can calculate the main terms of these operators.

PROPOSITION 3.4.

i) Given $b \in B_+$, let $n_b = \sum_{w \in W/W_b} y_{w(b)}$, where W_b is the stabilizer of b in W. Then

$$N_b \stackrel{def}{=} L_{n_b} \;=\; \xi(N_b) + \sum_{b_*} g_{b_*} b'_*, \quad \text{where } b_* \in \sigma_0^\vee(b),$$

$$\xi(N_b) \;=\; \sum_{w \in W/W_b} \prod_{\tilde{\alpha} \in \lambda(b)} \frac{q_{\tilde{\alpha}} X_{w(\tilde{\alpha})} - q_{\tilde{\alpha}}^{-1}}{X_{w(\tilde{\alpha})} - 1} (w(-b))' \quad (3.7)$$

ii) If $r \in O^*$ then $N_{b_r} \;=\; \xi(N_{b_r})$: Moreover, $N_{\theta^\vee} - \xi(N_{\theta^\vee})$ is a scalar function.

Proof. In the operator $\mathcal{N}_b = \mathcal{L}_{n_b}$, the term with $-b'$ can come only from Y_{-b}. It results directly from (2.15) and (3.5). The latter contains the formula for this term. The W-invariance of $N_b = (\mathcal{N}_b)_{red}$ completes this reasoning. □

Assertion ii) generalizes Theorem A.3. from [C4] (about a construction of Macdonald's operators for A_n via affine Hecke algebras). The operators $N_{b_r}, N_{\theta^\vee}$ are exactly the operators corresponding to (minuscule) $\{b_r\}$ and (quasi-minuscule) $\theta^\vee$ from [M2]. Actually, this observation alone is enough to establish the connection of our construction and the Macdonald theory. In the next section, we will discuss this issue in more detail involving the Macdonald pairing.

Let us fix $\epsilon = \{\epsilon_\nu, \nu \in \nu_R\}$ (cf. (0.2)). We introduce the *shift operator* by the formula $\mathcal{G}_\epsilon = \mathcal{X}_\epsilon^{-1}\mathcal{Y}_\epsilon$, where

$$\mathcal{X}_\epsilon = \prod_{e_\alpha \le \epsilon} (q_\alpha X_\alpha^{1/2} - q_\alpha^{-1} X_\alpha^{-1/2}),\ \mathcal{Y}_\epsilon = \prod_{e_\alpha \le \epsilon} (q_\alpha^{-1} Y_{\alpha^\vee}^{1/2} - q_\alpha Y_{\alpha^\vee}^{-1/2}). \tag{3.8}$$

Here $\alpha \in R_+, e_\alpha$ are from (0.2), $e_\alpha \le \epsilon$ means that the components of $\epsilon - e_\alpha$ are from $\mathbf{Z}_+$, and $X^{1/2}, Y^{1/2}$ are understood in the same way as $x^{1/2}$ were (Sec.0). Elements $\mathcal{X}_\epsilon, \mathcal{Y}_\epsilon$ belong to $\mathbf{C}_q[X], \mathbf{C}_q[Y]$ respectively (i.e. are polynomials in terms of X_β, Y_b for $\beta \in P$, $b \in B$, with the coefficients in $\mathbf{C}_q$).

PROPOSITION 3.5.
If $e_i \stackrel{def}{=} e_{\alpha_i} \le \epsilon$, $1 \le i \le n$, then

$$\begin{aligned}
&(T_i + q_i^{-1})\mathcal{X}_\epsilon = (q_i X_{\alpha_i}^{-1/2} - q_i^{-1} X_{\alpha_i}^{1/2})(q_i X_{\alpha_i}^{1/2} - q_i^{-1} X_{\alpha_i}^{-1/2})^{-1} \mathcal{X}_\epsilon (T_i - q_i),\\
&(T_i + q_i^{-1})\mathcal{Y}_\epsilon = (q_i Y_{a_i}^{1/2} - q_i^{-1} Y_{a_i}^{-1/2})(q_i Y_{a_i}^{-1/2} - q_i^{-1} Y_{a_i}^{1/2})^{-1} \mathcal{Y}_\epsilon (T_i - q_i),\\
&(T_i - q_i)\mathcal{X}_\epsilon^\iota = (q_i X_{\alpha_i}^{1/2} - q_i^{-1} X_{\alpha_i}^{-1/2})(q_i X_{\alpha_i}^{-1/2} - q_i^{-1} X_{\alpha_i}^{1/2})^{-1} \mathcal{X}_\epsilon^\iota (T_i + q_i^{-1}),\\
&\text{for } \mathcal{X}_\epsilon^\iota \stackrel{def}{=} \prod_{e_\alpha \le \epsilon} (q_\alpha^{-1} X_\alpha^{1/2} - q_\alpha X_\alpha^{-1/2}).
\end{aligned} \tag{3.9}$$

Otherwise (if $e_i \not\le \epsilon$,), $T_i\mathcal{X}_\epsilon = \mathcal{X}_\epsilon T_i$, $T_i\mathcal{Y}_\epsilon = \mathcal{Y}_\epsilon T_i$. The operators $\mathcal{G}_\epsilon$, $\mathcal{F}_\epsilon \stackrel{def}{=} \mathcal{X}_\epsilon^\iota \mathcal{Y}_\epsilon$, and $G_\epsilon \stackrel{def}{=} \mathcal{X}_\epsilon^{-1}(\mathcal{Y}_\epsilon)_{red}$ preserve $\mathbf{C}_{\delta,q}[x]^W$. Moreover, G_ϵ is W-invariant.

Proof. We set $\mathcal{X}_\epsilon = D_i \mathcal{D}_i$ for $e_i \le \epsilon$, where $D_i = (q_i X_{\alpha_i}^{1/2} - q_i^{-1} X_{\alpha_i}^{-1/2})$. Here $D, \mathcal{D}$ are considered as operators acting in $\mathbf{C}_{\delta,t}[x_1^{\pm 1/2}, \ldots, x_n^{\pm 1/2}]$. Then $T_i X_{\alpha_i}^{1/2} T_i = X_{\alpha_i}^{-1/2}$ and

$$\begin{aligned}
&(T_i + q_i^{-1})D_i =\\
&(q_i X_{\alpha_i}^{-1/2} - q_i^{-1} X_{\alpha_i}^{1/2})(q_i X_{\alpha_i}^{1/2} - q_i^{-1} X_{\alpha_i}^{-1/2})^{-1} D_i (T_i - q_i)
\end{aligned} \tag{3.10}$$

The relation $s_i \mathcal{D}_i = \mathcal{D}_i s_i$ results in $T_i \mathcal{D}_i = \mathcal{D}_i T_i$ and in (3.9) for $\mathcal{X}_\epsilon$. One can put $D_i = 1$ in this reasoning to include the case $e_i \not\le \epsilon$,. As for $\mathcal{Y}_\epsilon$, use the statement which has been already checked and the substitution $\{Y_i^\vee = X_i^{-1}\}$ identifying $\mathcal{H}_X$

with the algebra $\mathcal{H}_Y^\vee$ for the dual root system. The automorphism $\{q_i \to -q_i^{-1}\}$ in $\mathcal{H}_X$ takes $\mathcal{X}_\epsilon$ to $\pm\mathcal{X}_\epsilon^\iota$, which proves the remaining relation.

Given $p \in \mathbf{C}_{\delta,q}[x]^W$, formula (3.9) shows that the polynomial $p' = \mathcal{Y}_\epsilon(p)$ satisfies the relations $T_i(p') = -q_i^{-1}p'$ if $e_i \not\leq \epsilon$ and $T_i(p') = q_i p'$ otherwise. Hence $\mathcal{X}_\epsilon^{-1}(p')$ is W-invariant. It is also a polynomial, which can be deduced from the corresponding statements from [B] (in the case $q = 1$) by standard deformation reasoning. We see that the operators $\mathcal{G}_\epsilon$ and G_ϵ preserve $\mathbf{C}_{\delta,q}[x]^W$. Therefore G_ϵ is invariant. As for $\mathcal{F}_\epsilon$, it is a product of $\mathcal{G}_\epsilon$ by a central element from $\mathcal{H}_X$ and leaves $\mathbf{C}_{\delta,q}[x]^W$ invariant as well (one can use (3.9) directly to see this). □

We follow (1.16):

$$\gamma \succeq \beta, \beta \preceq \gamma \ \text{ for } \ \beta,\gamma \in P \quad \text{if} \quad \gamma - \beta \in Q_+,$$
$$\sigma(\beta) \stackrel{def}{=} \{\gamma \in P, w(\gamma) \succeq \beta \ \text{ for any } w \in W\},\ \beta \in P_-,$$
$$\sigma_0(\beta) \stackrel{def}{=} \{\gamma \in P, w(\gamma) \succ \beta \ \text{ for any } w \in W\},\ \beta \in P_-. \tag{3.11}$$

These sets are W-invariant and convex. The first contains $W(\beta)$.

PROPOSITION 3.6.
Operators $\{T_j, 0 \leq j \leq n\}$, $\{Y_i, 1 \leq i \leq n\}$, $\{\mathcal{L}_f, f \in \mathbf{C}[y]\}$ preserve $\Sigma(\beta) \stackrel{def}{=} \oplus_{\gamma\in\sigma(\beta)}\mathbf{C}_{\delta,q}x_\gamma$ and the $\Sigma_0(\beta)$ (defined for $\sigma_0(\beta)$ in the same way) for arbitrary $\beta \in P_-$ (cf. [H], Prop.3.5).

Proof. It suffices to check the statement for (one of) $\{T_j\}$. Given $\gamma \in \sigma(\beta)$, its image $T_j(x_\beta)$ is a linear combination of $\{x_\gamma\}$ such that $\{\gamma, \gamma - \alpha, \ldots, \gamma - r\alpha\} \subset \sigma(\beta)$ for $s_j(\gamma) = \gamma - r\alpha$, where α is either $\alpha_i, i > 0$, or $-\theta$. □

4. Macdonald's polynomials. We set $m_\beta = \sum_{\gamma\in W(\beta)} x_\gamma$ for $\beta \in P_-$. These *monomial symmetric functions* form a base of $\mathbf{C}[x]^W$. Let us introduce the involution ι: $\delta \to \delta^{-1}$, $q^\nu \to (q^\nu)^{-1}$ on $\mathbf{C}_{\delta,q}$, and the Macdonald conjugation $\bar{x}_\beta \stackrel{def}{=} x_{-\beta}$, leaving δ, q invariant. From now on we impose the conditions

$$q_\nu \ = \ t^{k_\nu} \ = \delta^{mk_\nu/2} \ \text{ for } \ k = \{k_\nu, \nu \in \nu_R\},\ k_\nu \in \mathbf{Z}_+, \tag{4.1}$$

though much holds good without this restriction. One has: $q_{\tilde\alpha} = t^{k_{\tilde\alpha}}$, where $k_{\tilde\alpha} \stackrel{def}{=} k_{\nu_\alpha}$ for $\tilde\alpha = [\alpha, \cdot]$, $q_j = t^{k_j}$ for $k_j = k_{\alpha_j}$. We set

$$\langle f, g\rangle_k \ = \langle \mu_k f\ \bar{g}^\iota\rangle \ \text{ for } \ f, g \in \mathbf{C}_\delta[x] \ \text{ and } \ \mu_k \text{ from (0.3)}. \tag{4.2}$$

One can check that this *skew Macdonald pairing* (with ι) is non-degenerate over $\mathbf{C}_\delta$ (cf. [M1,M2]). Indeed, it is definite if $\delta = 1$ and the coefficients of $f \neq 0$ are real:

$$\mu_k f\bar{f}^\iota = (-1)^{k\cdot\kappa} F\bar{F}^\iota \text{ for } F = f\prod_{\alpha\in R_+}(x_\alpha^{1/2} - x_\alpha^{-1/2})^{k_\alpha},\ \langle F\bar{F}^\iota\rangle > 0. \quad (4.3)$$

Hence, given a finite-dimensional $V \subset \mathbf{C}[x]$, one makes $\langle f, f\rangle_k \neq 0$ for any $0 \neq f \in V$ if $\delta \in \mathbf{C}$ is rather close to 1.

Thus $\langle f, f\rangle_k \neq 0 \in \mathbf{C}_\delta$ for any f, and we may introduce the *Macdonald polynomials* $p_\beta(x) = p_\beta^{(k)}(x)$, $\beta \in P_-$, by means of the conditions

$$p_\beta - m_\beta \in \oplus_{0\succeq\gamma\succ\beta}\mathbf{C}_\delta m_\gamma,\ \langle p_\beta, m_\gamma\rangle_k = 0 \text{ for } \gamma \succ \beta, \gamma \in P_-. \quad (4.4)$$

They can be determined by the Gram -Schmidt process and form a base in $\mathbf{C}_\delta[x]^W$.

We mention that $\bar{\mu}_k^\iota = \mu_k$ (μ_k is a product of an even number of terms satisfying relations $(\bar{\ })^\iota = -(\ )$), and $p_\beta^\iota = p_\beta$ for any $\beta \in P_-$ because p_β^ι satisfy (4.4) as well. It makes our results compatible with the Macdonald original definitions. We need ι because of the following theorem.

Let us introduce two anti-involutions on the operators from Sec.3 (acting in $\mathbf{C}_\delta[x]$):

$$H^+ = \sum w^{-1}\bar{g}(X)\iota(-b)',\ H^* = \mu_k^{-1}H^+\mu_k \text{ for } H = \sum b'g(X)w, \quad (4.5)$$

and rational functions $g(x)$, $w \in W, b \in B$ (b' is defined in (1.1)). The second involution serves the skew Macdonald pairing: $\langle Hf, g\rangle_k = \langle f, H^*g\rangle_k$.

Here we regarded μ_k as an operator substituting X for x (as it was done for g). We will do it permanently for this and some other functions without any comment.

THEOREM 4.1.

$$(\mathcal{L}_f)^* = \mathcal{L}_{\bar{f}} \text{ for } f \in \mathbf{C}[y],\ \bar{y}_b = y_{-b} \quad (4.6)$$

Proof. First of all, let us rewrite Y_b in terms of b' and

$$G_{\tilde{\alpha}} = q_{\tilde{\alpha}} + (q_{\tilde{\alpha}} - q_{\tilde{\alpha}}^{-1})(X_{\tilde{\alpha}}^{-1} - 1)^{-1}(1 - s_{\tilde{\alpha}}),\ \tilde{\alpha} \in R^a. \quad (4.7)$$

We follow (2.17). Given $b' = \pi_r\tilde{w} \in B'_+$ (positive b are enough to conside) and a reduced decomposition $\tilde{w} = s_{j_l}\cdots s_{j_1}$, $l = l(\tilde{w})$, one has:

$$Y_b = b'G_{\alpha^l}\cdots G_{\alpha^1}, \text{ where } \alpha^1 = \alpha_{j_1}, \alpha^2 = s_{j_1}(\alpha_{j_2}), \alpha^3 = s_{j_1}s_{j_2}(\alpha_{j_3}), \ldots \quad (4.8)$$

We have to check that $\mu_k Y_b^{-1} = Y_b^+\mu_k$, which can be rewritten as $\mu_k G_{\alpha^1}^{-1}\cdots(b')^{-1} = G_{\alpha^1}^{+}\cdots(b')^{-1}\mu_k$. A straightforward calculation for A_1 gives that $\mu_k G_{\alpha^1}^{-1} = G_{\alpha^1}^{+}\mu_k'$ for $\mu_k' = s_{j_1}(\mu_k)$. Hence we can continue replacing $\mu_k' G_{\alpha^2}^{-1}$ by $G_{\alpha^2}^{+}\mu_k''$, where $\mu_k'' = s_{j_1}s_{j_2}(\mu_k)$, and so on. Finally (after $G_{\alpha_l}^{(l)}$) we arrive at $\mu_k^{(l)} = \tilde{w}^{-1}(\mu_k)$. To make the proof complete, we need the relation $\mu_k^{(l)}(b')^{-1} = (b')^{-1}\mu_k$, which directly follows from (0.3) $((b'\tilde{w}^{-1})(\mu_k) = \pi_r(\mu_k) = \mu_k)$. □

COROLLARY 4.2.

$$\phi_k^{-1} L_f^{+} \phi_k \;=\; L_{\bar f} \;\; \text{for} \;\; f \in \mathbf{C}[y]^W, \;\; \text{where} \;\; \phi_k = \eta_k^{-1}\mu_k,$$
$$\eta_k \stackrel{def}{=} \prod_{\alpha\in R_+} \{(t^{-k_\alpha} x_\alpha^{1/2} - t^{k_\alpha} x_\alpha^{-1/2})/(x_\alpha^{1/2} - x_\alpha^{-1/2})\} \;\in \mathbf{C}_t[x]. \tag{4.9}$$

Proof. We omit some details because this statement will not be applied to the Macdonald conjectures. The relations $T_i\mathcal{L}_f = \mathcal{L}_f T_i$ for $1 \le i \le n$ and the formulas

$$T_i \;=\; -q_i^{-1} + (s_i+1)(q_i^{-1}X_i^{1/2} - q_i X_i^{-1/2})/(X_i^{1/2} - X_i^{-1/2}), \tag{4.10}$$

result in $\mathcal{L}_f\mathcal{P} = \mathcal{P}\eta_k\mathcal{L}_f\eta_k^{-1}$ for $\mathcal{P} \stackrel{def}{=} |W|^{-1}\sum_{w\in W} w$. Use the W-invariance of ϕ_k and the relations $\mathcal{L}_f\mathcal{P} = L_f\mathcal{P} = \mathcal{P}L_f$ to check that

$$\phi_k L_f \phi_k^{-1}\mathcal{P} = \phi_k(\mathcal{L}_f\mathcal{P})\phi_k^{-1} = \mathcal{P}\mu_k\mathcal{L}_f\mu_k^{-1} =$$
$$\mathcal{P}\mathcal{L}_{\bar f}^{+} = (\mathcal{L}_{\bar f}\mathcal{P})^{+} = L_{\bar f}^{+}\mathcal{P}. \tag{4.11}$$

□

We note that ϕ_k when used instead of μ_k does not alter the Macdonald polynomials (see [M2]). Moreover, the following statements hold true.

PROPOSITION 4.3.

The orthogonality condition in (4.4) is equivalent to the requirement:

$$\langle \psi_k^\epsilon p_\beta x_{-\gamma'}\rangle = 0 \;\; \text{for} \;\; \gamma' \in \sigma_0(\beta - \rho_\epsilon) \;\; \text{for} \;\; \psi_k^\epsilon = \phi_k\xi_\epsilon^{-1}, \tag{4.12}$$

where $\xi_\epsilon \stackrel{def}{=} \prod_{e_\alpha\le\epsilon}(x_\alpha^{1/2} - x_\alpha^{-1/2})$, $\alpha \in R_+$, $\epsilon = \{\epsilon_\nu, \nu \in \nu_R\}$ *is from (3.9),* $\rho_\epsilon = \epsilon\cdot\rho = \sum_\nu \epsilon_\nu\rho_\nu$.

Proof. One has:

$$\langle \psi_k^\epsilon p_\beta x_{-\gamma'}\rangle = \langle \psi_k^\epsilon p_\beta \mathcal{P}_-^\epsilon(x_{-\gamma'})\rangle \;\; \text{for}$$
$$\mathcal{P}_-^\epsilon \stackrel{def}{=} |W|^{-1}\sum_{w\in W}(-1)^{l_\epsilon(w)}w,\;\; l_\epsilon(w) = \epsilon\cdot\{l_\nu(w)\}.$$

Hence, we may check the above conditions for $\xi_\epsilon m_{-\gamma}$ instead of $x_{-\gamma'}$, where $\gamma \succ \beta, \gamma \in P_-$ (since $\mathcal{P}_-^\alpha(x_{-\gamma'})$ is divisible by ξ_α). Thus (4.12) can be replaced by $\langle \phi_k p_\beta m_{-\gamma}\rangle = 0$, which is equivalent to the relation $\langle \mu_k p_\beta m_{-\gamma}\rangle = 0$ because of the following lemma.

LEMMA 4.4.

Let $W(\beta) = \{w(\beta), w \in W\}$. Then $\langle \eta_k f\rangle = d_k\langle f\rangle$ for $f \in \mathbf{C}[x]^W$,

$$d_k \stackrel{def}{=} |W(k\cdot\rho)|^{-1} \prod_{\alpha\in R_+}^{k_\alpha\neq 0} (t^{k\cdot l_\alpha + k_\alpha} - t^{-k\cdot l_\alpha - k_\alpha})/(t^{k\cdot l_\alpha} - t^{-k\cdot l_\alpha}). \quad (4.13)$$

Proof.

$$d_k = |W|^{-1}(\sum_{w\in W} w(\eta_k))\eta_k^{-1} = $$
$$|W|^{-1}\sum_{w\in W} t^{k\cdot\{l_\nu(w_0) - 2l_\nu(w)\}}.$$

The formula for the latter is known (see [M2], Sec.12). ◫

MAIN THEOREM 4.5.

We take $\epsilon = \{\epsilon_\nu\}$ such that $\epsilon_\nu = 1$ if $k_\nu > 0$ for $\nu \in \nu_R$. Given $f(y_1, ..., y_n) \in \mathbf{C}[y]^W$ and $\beta \in P_-$,

$$L_f(p_\beta^{(k)}) = f(t^{k\cdot l_{\alpha_1} - (\alpha_1^\vee,\beta)}, \ldots, t^{k\cdot l_{\alpha_n} - (\alpha_n^\vee,\beta)})p_\beta^{(k)}, \quad (4.14)$$

$$G_\epsilon(p_\beta^{(k)}) =$$
$$(-1)^{\epsilon\cdot\kappa} t^{k\cdot\kappa} \prod_{e_\alpha \leq \epsilon} (t^{k\cdot l_\alpha - k_\alpha - (\alpha^\vee,\beta)} - t^{-k\cdot l_\alpha + k_\alpha + (\alpha^\vee,\beta)})p_{\beta+\rho_\epsilon}^{(k+\epsilon)}, \quad (4.15)$$

where $\alpha \in R_+$, $p_\gamma = 0$ if $\gamma \notin P_-$.

Proof. Proposition 3.6 gives that $\mathcal{L}_f$ preserve $\Sigma(\beta)$. Hence L_f preserve $\oplus_{0\succeq\gamma\succeq\beta}\mathbf{C}_\delta m_\gamma$ for arbitrary f, β. The standard arguments (due to Macdonald) show that Theorem 4.1 ensures the proportionality of $L_f(p_\beta^{(k)})$ and $p_\beta^{(k)}$. The corresponding coefficient is determined by Proposition 3.1. As for G_ϵ, it takes $p_\beta^{(k)}$ to a polynomial from $\oplus_{0\succeq\gamma\succeq\beta+\rho_\epsilon}\mathbf{C}_\delta m_\gamma$ which (we are going to check it now) is orthogonal to $\oplus_{0\succeq\gamma\succ\beta+\rho_\epsilon}\mathbf{C}_\delta m_\gamma$ relative to $\langle\ ,\ \rangle_{k+\epsilon}$. Again it means the proportionality (cf. the proof of Theorem 3.15 from [H]) and we can calculate the coefficient using Proposition 3.1.

The relations $\mathcal{Y}_\epsilon^* = \pm\mathcal{Y}_\epsilon$ (see Theorem 4.1), $\mathcal{X}_\epsilon^* = \pm\mathcal{X}_\epsilon$, and $\mathcal{X}_\epsilon = \eta_k\xi_\epsilon$ lead to the formulas

$$\mathcal{F}_\epsilon^+ = \phi_k\eta_k(\mathcal{Y}_\epsilon\mathcal{X}_\epsilon^\iota)\eta_k^{-1}\phi_k^{-1} = \phi_k\xi_\epsilon^{-1}\mathcal{X}_\epsilon^\iota\mathcal{Y}_\epsilon\xi_\epsilon\phi_k^{-1} = \psi_k^\epsilon\mathcal{F}_\epsilon\{\psi_k^\epsilon\}^{-1}, \quad (4.16)$$

$$\langle\psi_{k+\epsilon}^\epsilon G_\epsilon(p_\beta^{(k)})x_{-\gamma'}\rangle = \langle\psi_k^\epsilon\mathcal{F}_\epsilon(p_\beta^{(k)})x_{-\gamma'}\rangle = \langle\psi_k^\epsilon p_\beta^{(k)}\{\mathcal{X}_\epsilon^\iota\mathcal{Y}_\epsilon(x_{\gamma'})\}^{-\iota}\rangle, \quad (4.17)$$

where $(\ )^{-\iota} = (^-)^{\iota}$. If $\gamma' \in \sigma_0(\beta)$ then $\mathcal{X}_\epsilon^\iota \mathcal{Y}_\epsilon(x_{\gamma'}) \in \Sigma_0(\beta - \rho_\epsilon)$ (Proposition 3.6) and the r.h.s. of (4.17) equals zero (Proposition 4.3). However the l.h.s. is straight from (4.12) for $G_\epsilon(p_\beta^{(k)}), \beta + \rho_\epsilon$, and $k + \epsilon$ instead of $p_\beta^{(k)}, \beta$, and k.

□

5. The Macdonald conjecture. We preserve the notations from Sec.4 and put $\mathcal{L}^{(k)}, L^{(k)}$ and so on to emphasize the dependence of k.

THEOREM 5.1.

Given arbitrary $\beta, \gamma \in P_-$, *one has:* $\langle p_\beta^{(k)},\ p_\gamma^{(k)}\rangle_k = 0$ *if* $\beta \neq \gamma$ *and*

$$\langle p_\beta^{(k)},\ p_\beta^{(k)}\rangle_k = (-1)^{k \cdot \kappa} |W(k\cdot\rho - \beta)|\ |W(k\cdot\rho)|^{-1}$$
$$\prod_{\alpha\in R_+}^{k_\alpha \neq 0} (t^{k\cdot l_\alpha + k_\alpha} - t^{-k\cdot l_\alpha - k_\alpha})/(t^{k\cdot l_\alpha} - t^{-k\cdot l_\alpha})$$
$$\prod_{\alpha\in R_+} \prod_{i=1}^{k_\alpha - 1} \frac{t^{k\cdot l_\alpha - (\alpha^\vee,\beta) + i} - t^{-k\cdot l_\alpha + (\alpha^\vee,\beta) - i}}{t^{k\cdot l_\alpha - (\alpha^\vee,\beta) - i} - t^{-k\cdot l_\alpha + (\alpha^\vee,\beta) + i}}. \tag{5.1}$$

Proof. The orthogonality of $p_\beta^{(k)},\ p_\gamma^{(k)}$ for $\beta \neq \gamma$ results from (4.6) and (4.14):

$$\langle L_f^{(k)}(p_\beta^{(k)}),\ p_\gamma^{(k)}\rangle_k = \langle p_\beta^{(k)}, L_{\bar f}^{(k)}(p_\gamma^{(k)})\rangle_k =$$
$$f(t^{k\cdot l_{\alpha_1} - (\alpha_1^\vee,\beta)}, \ldots)\langle p_\beta^{(k)},\ p_\gamma^{(k)}\rangle_k = f(t^{k\cdot l_{\alpha_1} - (\alpha_1^\vee,\gamma)}, \ldots)\langle p_\beta^{(k)},\ p_\gamma^{(k)}\rangle_k. \tag{5.2}$$

The corresponding eigenvalues distinguish $\beta \neq \gamma$ for a proper W-invariant polynomial f. It gives the desired statement. It was established by Macdonald by means of the simplest self-adjoint difference operators satisfying (we introduced them in assertion of Proposition 3.4).

Actually one operator is enough to split $\{p_\beta\}$, and therefore to check the orthogonality. However we need the comlplete set $\{L_f\}$ is important to establish (5.1) (we need rather complicated operator $\hat{\mathcal{Y}}_\epsilon \mathcal{Y}_\epsilon$ - see (5.8)) and to some other applications. By the way, it gives a uniform proof of the ortogonality for all root systems.

The remaining (main) part is based on the following chain of the shift operators that will be applied to $p_{\beta - k\cdot\rho}^{(0)} = m_{\beta - k\cdot\rho}$ one after one:

$$G_{\{1,1\}}^{(k-\{1,1\})} G_{\{1,1\}}^{(k-\{2,2\})} \cdots G_{\{1,1\}}^{(k-\{s,s\})} G_\epsilon^{(k-\{s,s\}-\epsilon)} \cdots G_\epsilon^{(0)}, \tag{5.3}$$

where $k = s\{1,1\} + r\epsilon$ for $\epsilon = \{\epsilon_\nu\}$ such that $\prod_\nu \epsilon_\nu = 0$. We use the visual but not quite correct notation $\{1,1\}$ for the vector with unit components. Let

$k(i), \epsilon(i)$, $0 \le i \le r+s-1$, be the corresponding indices of $G = G(i)$ beginning from $G(0) = G_\epsilon^{(0)}$.

Firstly, we have to consider the product of the G for the indices $0, 1, ..., r-1$. If $0 \le i < r$, then $k(i) = i\epsilon$, $\epsilon(i) = \epsilon$. Our aim is to express M_{i+1} in terms of

$$M_i = \langle p^{(i\epsilon)}_{\beta'+i\rho_\epsilon},\ p^{(i\epsilon)}_{\beta'+i\rho_\epsilon}\rangle_{i\epsilon} \ \text{ for } \ \beta' \stackrel{def}{=} \beta - k\cdot\rho. \tag{5.4}$$

Formula (4.15) results in the relation

$$\begin{aligned} &M_{i+1} = (g_i g_i^\iota)^{-1}\langle G_\epsilon^{(i\epsilon)}(p^{(i\epsilon)}_{\beta'+i\rho_\epsilon}),\ G_\epsilon^{(i\epsilon)}(p^{(i\epsilon)}_{\beta'+i\rho_\epsilon})\rangle_{(i+1)\epsilon},\\ &\quad\text{for } \ g_i = \prod_{e_\alpha\le\epsilon}(t^{i\epsilon_\alpha - i\epsilon\cdot l_\alpha + (\alpha^\vee,\beta'+i\rho_\epsilon)} - t^{-i\epsilon_\alpha + i\epsilon\cdot l_\alpha - (\alpha^\vee,\beta'+i\rho_\epsilon)}) = \\ &\prod_{e_\alpha\le\epsilon}(t^{i\epsilon_\alpha + (\alpha^\vee,\beta')} - t^{-i\epsilon_\alpha - (\alpha^\vee,\beta')}). \end{aligned} \tag{5.5}$$

On the other hand, $G_\epsilon^{(i\epsilon)} = (\mathcal{X}_\epsilon^{(i\epsilon)})^{-1}\mathcal{Y}_\epsilon^{(i\epsilon)}$ and

$$\begin{aligned} &N_i \stackrel{def}{=} \langle \mathcal{Y}_\epsilon^{(i\epsilon)}(p^{(i\epsilon)}_{\beta'+i\rho_\epsilon}),\ \mathcal{Y}_\epsilon^{(i\epsilon)}(p^{(i\epsilon)}_{\beta'+i\rho_\epsilon})\rangle_{i\epsilon} \ = \\ &\langle \zeta_i G_\epsilon^{(i\epsilon)} p^{(i\epsilon)}_{\beta'+i\rho_\epsilon},\ G_\epsilon^{(i\epsilon)} p^{(i\epsilon)}_{\beta'+i\rho_\epsilon}\rangle_{(i+1)\epsilon} \ \text{ for } \ \zeta_i \ = \\ &\prod_{e_\alpha\le\epsilon}(t^{i\epsilon_\alpha}x_\alpha^{1/2} - t^{-i\epsilon_\alpha}x_\alpha^{-1/2})(t^{-(i+1)\epsilon_\alpha}x_\alpha^{1/2} - t^{(i+1)\epsilon_\alpha}x_\alpha^{-1/2})^{-1}. \end{aligned} \tag{5.6}$$

Applying Lemma 4.4 twice, we arrive at the relation

$$N_i \ = \ (-1)^{\epsilon\cdot\kappa}(d_{i\epsilon}/d_{(i+1)\epsilon})(g_i g_i^\iota)M_{i+1}. \tag{5.7}$$

Then one may use Theorem 4.1 and (4.14) to calculate N_i:

$$\begin{aligned} N_i \ &= \ \langle \hat{\mathcal{Y}}_\epsilon^{(i\epsilon)}\mathcal{Y}_\epsilon^{(i\epsilon)}(p^{(i\epsilon)}_{\beta'+i\rho_\epsilon}),\ p^{(i\epsilon)}_{\beta'+i\rho_\epsilon}\rangle_{i\epsilon} \ = \ g_i\hat{g}_i M_i, \ \text{ where} \\ \hat{\mathcal{Y}}_\epsilon^{(i\epsilon)} \ &= \ \prod_{e_\alpha\le\epsilon}(t^{-i\epsilon_\alpha}Y_{\alpha^\vee}^{-1/2} - t^{i\epsilon_\alpha}Y_{\alpha^\vee}^{1/2}), \\ \hat{g}_i \ &= \ \prod_{e_\alpha\le\epsilon}(t^{i\epsilon_\alpha - (\alpha^\vee,\beta')} - t^{-i\epsilon_\alpha + (\alpha^\vee,\beta')}). \end{aligned} \tag{5.8}$$

Here we replaced $\mathcal{Y}^*$ by $\hat{\mathcal{Y}}$. This changes N_i by a term $\langle \mu_{i\epsilon}\mathcal{X}_\epsilon^{(i\epsilon)}(f - (-1)^{\epsilon\cdot\kappa}w_0(f))\rangle$ for the longest element $w_0 \in W$ and a proper $f \in \mathbf{C}_t[x]$, that is zero because of the W-invariance of $\langle\ \rangle$. Then we made use of the fact that the product $\hat{\mathcal{Y}}_\epsilon^{(i\epsilon)}\mathcal{Y}_\epsilon^{(i\epsilon)}$ corresponds to an element from $\mathbf{C}_t[y]^W$.

Finally,

$$M_{i+1} = (-1)^{\epsilon\cdot\kappa}(d_{(i+1)\epsilon}/d_{i\epsilon})\hat{g}_i(g_i^t)^{-1}M_i. \tag{5.9}$$

In the case $i \geq r$ the formula will be just the same with $k(i) = (i-r)\{1,1\} + r\epsilon$, $\epsilon(i) = \{1,1\}$, $d_{k(i)}$ instead of $i\epsilon$, ϵ, $d_{i\epsilon}$.

To complete the proof let us put together the relations for $i = 0, \ldots, r+s-1$:

$$M_{r+s} = (-1)^{k\cdot\kappa}d_k \prod_{i=0}^{r+s-1} \hat{g}_i(g_i^t)^{-1}M_0 = \text{r.h.s. of } (5.1). \tag{5.10}$$

Here we regrouped the terms with respect to α and used the equality $d_0 = 1$. □

COROLLARY 5.2.

$$(-1)^{k\cdot\kappa}\langle\mu_k\rangle = \prod_{\alpha\in R_+}^{k_\alpha\neq 0} \{(t^{k\cdot l_\alpha+k_\alpha} - t^{-k\cdot l_\alpha-k_\alpha})/(t^{k\cdot l_\alpha} - t^{-k\cdot l_\alpha})\}$$

$$\prod_{\alpha\in R_+}\prod_{i=1}^{k_\alpha-1}\{(t^{k\cdot l_\alpha+k_\alpha-i} - t^{-k\cdot l_\alpha-k_\alpha+i})/(t^{k\cdot l_\alpha-i} - t^{-k\cdot l_\alpha+i})\}. \tag{5.11}$$

□

This formula coincides with (0.4) and could be somewhat simplified by means of formula (3.2) from [M3].

COROLLARY 5.3.
Let $\tilde{p}_\beta^{(k)} = h_\beta^{(k)}p_\beta^{(k)}$, $\beta' = \beta - k\cdot\rho$ *(see (5.4), where*

$$h_\beta^{(k)} = \prod_{i,\ \alpha\in R_+}^{k_*>i\geq(k_*-k_\alpha)} (t^{i+(\alpha^\vee,\beta')} - t^{-i-(\alpha^\vee,\beta')}), \quad \textit{for } k_* = \max\{k_\nu\}. \tag{5.12}$$

Then the coefficients of $\tilde{p}_\beta^{(k)}$ *expressed in terms of* $\{m_\gamma\}$ *are Laurent polynomials in* t^2 *over* $\mathbf{Z}$ *(belong to* $\mathbf{Z}[t^2, t^{-2}]$*).*

The proof is based on the representation of $p_\beta^{(k)}$ by means of the chain from (5.3). The operators G act over $\mathbf{Z}[t, t^{-1}]$ (which follows from the the same property of $\{T\}$). Hence we can apply (4.15) again (and use the functions g from (5.5)). The resulting statement is connected with (6.3) (and the corresponding Conjecture) from [M1]. □

We mention that (5.1) is equivalent to Conjecture 1 from the end of [M1] in the case of coinciding $\{k_\nu\}$ and Conjecture (12.6') from [M2] when $\prod_\nu k_\nu \neq 0$.

The latter was checked in [AI] (for A_1), [AW] (the case of BC_1), and proved by Macdonald for A_n (unpublished).

By the way, formulas (5.1), (5.11) were checked by computer in the case of the root systems $B_{2,3}$ for quite a few k, β (in the range of 30 MB). It seems that the $\{p_\beta\}$ have (many) other rather remarkable algebraic properties. Some of them are connected with those from papers [S,M1], where the so-called Jack polynomials and their q,t-counterparts were considered (in the case of A_n).

There are quite a few works about orthogonal polynomials. We mentioned here only (a small part of the) papers directly connected with the Macdonald q-conjecture. We would like to add that a certain generalization of the q-Jacobi-Askey-Wilson- Macdonald polynomials of type BC_n can be found in [Ko]. The coincidence of the polynomials of type A for $q = t^2$, $q = t^{1/2}$ with q-spherical functions of the symmetric spaces $GL(n)/SO(n)$, $GL(2n)/Sp(2n)$ was established in [N].

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