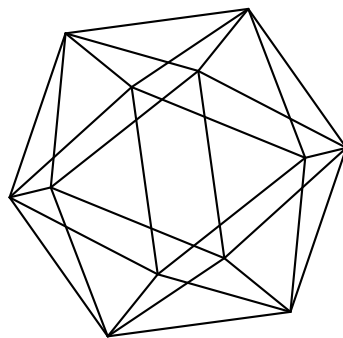


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Linear relations, monodromy and Jordan cells of a circle
valued map

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Abstract

In this paper we consider the definition of *monodromy of an angle valued map* based on linear relations as proposed in [3]. This definition provides an alternative treatment of monodromy and computationally an alternative calculation of the *Jordan cells*, topological persistence invariants of a circle valued maps introduced in [2].

We give a new geometric proof that the monodromy is actually a homotopy invariant of a pair (X, ξ) consisting of a compact ANR X and an integral cohomology class $\xi \in H^1(X; \mathbb{Z})$, without any reference to the infinite cyclic cover associated to ξ , as in [3], or to the graph representation associated an angle valued map defining ξ as in [2].

Most important, we describe an algorithm to calculate the monodromy for a simplicial angle valued map defined on a finite simplicial complex, providing a new algorithm for the calculation of the Jordan cells of the map shorter than the one proposed in [2].

We indicate the computational usefulness of *Jordan cells*, and in particular of the proposed algorithm, for the calculation of other basic topological invariants of (X, ξ) .

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1 Introduction

Absolute neighborhood retracts (ANRs) are topological spaces X which whenever $i : X \rightarrow Y$ is an embedding into a normal topological space Y there exists a neighborhood U of $i(X)$ in Y and a retraction of U onto $i(X)$ cf [15].

Let X be a compact ANR¹, $\xi \in H^1(X; \mathbb{Z})$ and κ a field with algebraic closure $\bar{\kappa}$. The r -monodromy, $r \in \mathbb{Z}_{\geq 0}$, is a similarity (= conjugacy) class of linear isomorphism $T^{(X, \xi)}(r) : V_r(X, \xi) \rightarrow V_r(X, \xi)$, with $V_r(X, \xi)$ finite dimensional κ -vector spaces, cf definition 2.1 below. The Jordan decomposition of a square matrix permits to assign to the linear isomorphism $T^{(X, \xi)}(r)$ the collection $\mathcal{J}_r(X; \xi)$ of pairs (λ, k) , $\lambda \in \bar{\kappa} \setminus 0, k \in \mathbb{Z}_{\geq 1}$, referred to as *Jordan cells* in dimension r . They provide a complete sets of invariants of the similarity class of $T^{(X, \xi)}(r)$. If $f : X \rightarrow \mathbb{S}^1$ is a tame map as in [2] and ξ_f the cohomology class defined by f , then the set $\mathcal{J}_r(X; \xi_f)$ coincides with the set of Jordan cells $\mathcal{J}_r(f)$ considered in [2] in relation with the topological persistence of the circle valued map f , cf [3].

Recall that topological persistence for a real or circle valued map $f : X \rightarrow \mathbb{R}$ or $f : X \rightarrow \mathbb{S}^1$ ($\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$) analyses the changes in the homology of the levels $f^{-1}(\theta), \theta \in \mathbb{R}$ or \mathbb{S}^1 . It records the *detectability* and the *death* of homology of the levels in terms of *bar codes* cf [2] or [7]. In case of a circle valued map, in addition to *death* and *detectability*, there is an additional feature of interest to be recorded, the *return of some homology classes* of $f^{-1}(\theta)$ when the angle θ increases or decreases with 2π . This feature is recorded as *Jordan cells* which were introduced in [2] and describe what the topologists refer to as the *homological monodromy* or simply the *monodromy*. In [3] we have proposed a definition for *monodromy* and implicitly for *Jordan cells* based on *linear relations*. For the purpose of this paper the needed background on *Linear Relations* is presented in section 3; for more background the reader can consult [23] or [3] section 8.

In this paper we review the definition of monodromy based on linear relations, provide a new geometric proof of its homotopy invariance (without any reference to Novikov homology used in [3]) and, more important, propose a new algorithm for the calculation of $\mathcal{J}_r(f) = \mathcal{J}_r(X; \xi_f)$, for X a finite simplicial complex and f a simplicial map. The notation $\mathcal{J}_r(X; \xi_f)$ indicates that the collection of pairs (λ, k) depend on the pair $(X, \xi \in H^1(X; \mathbb{Z}))$ rather than $f : X \rightarrow \mathbb{S}^1$ representing ξ .

In the present approach the monodromy is first defined for a continuous map $f : X \rightarrow \mathbb{S}^1$ and a *weakly regular angle* $\theta \in \mathbb{S}^1$ (see the definitions in section 3). Note that not all compact ANR's admit angle valued maps with weakly regular angles, cf [12]. Note also that for a simplicial map all angles are weakly regular. We reduce the general case of an arbitrary compact ANR and a continuous map, which might have no weakly regular angles, to the case of simplicial complexes and simplicial maps based on results on the topology of compact Hilbert cube manifolds.

¹The reader unfamiliar with the notion of ANR should always think to the main examples, spaces homeomorphic to simplicial complexes or to a CW complexes

Proposition 3.4 shows that the monodromy proposed is independent of the weakly regular angle, remains the same for maps which have weakly regular angles and are homotopic and does not change when one replaces the map by its composition with the projection $X \times K \rightarrow X$, K an acyclic compact ANR. These facts ultimately show that the monodromy can be associated to a pair $(X, \xi \in H^1(X; \mathbb{Z}))$, X any compact ANR, and the assignment is a homotopy invariant of the pair (X, ξ) , cf Theorem 3.6. All these facts are established in section 3, based on elementary linear algebra of linear relations summarized in section 2. They also follow from the definition of monodromy based on the homology of the infinite cyclic cover associated to ξ described in [3] but this is exactly what the present treatment wants to avoid.

The algorithm for calculating $\mathcal{J}_r(f)$ for f a simplicial angle valued map is discussed in section 4.

In section 5 we indicate applications of the calculation of Jordan cells to the calculation topological invariants whose standard definitions involve infinite cyclic covers, *computer unfriendly* objects. In particular one provides new ways to calculate Novikov Betti numbers, the Alexander polynomial of a knot and a few other invariants, see section 5.

In section 3 we notice that a generalization of the homological monodromy discussed in this paper can be obtained when the singular homology H_r is replaced by a vector space valued homotopy functor F which is half exact in the sense of A. Dold cf [13]. This F -monodromy is not investigated in this paper but it might deserve attention ²

Acknowledgements: The idea of describing the Jordan cells considered in [2] using linear relations belongs to Stefan Haller and was pursued in [3] not yet in print.

It is a pleasure to thank S.Ferry for help in relation with the Appendix 2. and for bringing to our attention the reference [12].

The final version of this work was done during the visits of the author to the MPIM-Bonn.

2 Linear relations

Fix a field κ and let $\tilde{\kappa}$ be its algebraic closure.

2.1 Generalities on linear relations

Recall from [23] and [3]:

– A linear relation $R : V_1 \rightsquigarrow V_2$ is a linear subspace $R \subseteq V_1 \times V_2$. One writes $v_1 R v_2$ iff $(v_1, v_2) \in R$, $v_i \in V_i$.

Examples:

1. Two linear maps $V_1 \xrightarrow{\alpha} W \xleftarrow{\beta} V_2$ provide the relation

$$R(\alpha, \beta) \subset V_1 \times V_2 := \{(v_1, v_2) \mid \alpha(v_1) = \beta(v_2)\}.$$

2. Two linear maps $V_1 \xleftarrow{a} U \xrightarrow{b} V_2$ provide the relation

$$R \langle a, b \rangle \subset V_1 \times V_2 := \{(v_1, v_2) \mid \exists u, a(u) = v_1, b(u) = v_2\}.$$

– Two linear relations $R_1 : V_1 \rightsquigarrow V_2$ and $R_2 : V_2 \rightsquigarrow V_3$ can be composed in an obvious way, $(v_1(R_2 \cdot R_1)v_3$ iff $\exists v_2$ such that $v_1 R_1 v_2$ and $v_2 R_2 v_3$. The diagonal $\Delta \subset V \times V$ is playing the role of the identity.

– Given a linear relation $R : V_1 \rightsquigarrow V_2$ denote by $R^\dagger : V_2 \rightsquigarrow V_1$ the relation defined by the property $v_2 R^\dagger v_1$ iff $v_1 R v_2$. Clearly $(R_1 \cdot R_2)^\dagger = R_2^\dagger \cdot R_1^\dagger$ and $R^{\dagger\dagger} = R$.

² A slightly more general situation, when X is equipped with a flat bundle of \mathcal{A} -modules W , \mathcal{A} a finite type von Neumann algebra, and W a finite type Hilbert module will be considered in a sequel of this paper in the special case \mathcal{A} is the von Neumann algebra $\mathcal{N}(\pi_1(M))$ and $W = l^2(\pi_1(M)) \otimes \mathbb{C}^n$. Such monodromy will be used to the description of L_2 -torsion in geometrically interesting situations.

The familiar category of finite dimensional vector spaces and linear maps can be extended to incorporate all linear relations as morphisms. The linear map $f : V_1 \rightarrow V_2$ can be interpreted as the relation $\text{graph } f \subset V_1 \times V_2$ denoted by $\boxed{R(f)} = R(f, id_{V_2})$, providing the embedding of the category of vector spaces and linear maps in the category of vector spaces and linear relations. This extended category remains abelian.

–The direct sums $R' \oplus R'' : V'_1 \oplus V''_1 \rightsquigarrow V'_2 \oplus V''_2$ of two relations $R' : V'_1 \rightsquigarrow V'_2$ and $R'' : V''_1 \rightsquigarrow V''_2$ is defined in the obvious way, $(v'_1, v''_1)(R' \oplus R'')(v'_2, v''_2)$ iff $(v'_1 R' v'_2)$ and $(v''_1 R'' v''_2)$.

–The relation with the same source and target $R' : V' \rightsquigarrow V'$ and $R'' : V'' \rightsquigarrow V''$ are similar and one writes $\boxed{R' \sim R''}$ if there exists the linear isomorphisms $\alpha : V' \rightarrow V''$ s.t. $R'' \cdot R(\alpha) = R(\alpha) \cdot R'$.

Recall that two linear endomorphisms $T : V \rightarrow V$ and $T' : V' \rightarrow V'$ are called similar if there exists a linear isomorphism $C : V \rightarrow V'$ s.t. $C^{-1} \cdot T' \cdot C = T$. One writes $T \sim T'$ if T and T' are similar and one denotes the similarity class of $T : V \rightarrow V$ by $[T]$; so $T \sim T'$ and $[T] = [T']$ mean the same thing.

As in the case of linear maps one denotes the similarity class of the relation $R : V \rightsquigarrow V$ by $[R]$. Clearly when $T : V \rightarrow V$ is a linear map both notations $[T]$ and $[R(T)]$ mean the same thing.

A linear relation $R : V \rightsquigarrow W$ gives rise to the following subspaces:

$$\begin{aligned} \text{dom}(R) &:= \{v \in V \mid \exists w \in W : vRw\} = pr_V(R) \\ \text{img}(R) &:= \{w \in W \mid \exists v \in V : vRw\} = pr_W(R) \\ \text{ker}(R) &:= \{v \in V \mid vR0\} \cong V \times 0 \cap R \\ \text{mul}(R) &:= \{w \in W \mid 0Rw\} \cong 0 \times W \cap R \end{aligned}$$

Here pr_V and pr_W denote the projections of $V \times W$ on V and W . We have

Observation 2.1

1. $\text{ker}(R) \subseteq \text{dom}(R) \subseteq V$ and $W \supseteq \text{img}(R) \supseteq \text{mul}(R)$,
2. $\text{ker}(R^\dagger) = \text{img}(R)$ and $\text{dom}(R^\dagger) = \text{img}(R)$,
3. $\dim \text{dom}(R) + \dim \text{ker}(R^\dagger) = \dim(R) = \dim(R^\dagger) = \dim \text{dom}(R^\dagger) + \dim \text{ker}(R)$.

It is immediate, in view of the above definitions and above observation that :

Lemma 2.2

1. A linear relation $R : V \rightsquigarrow W$ is of the form $R(f)$ for $f : V \rightarrow W$ linear map iff $\text{dom}R = V$ and $\text{mul}R = 0$.

2. A linear relation $R : V \rightsquigarrow V$ is of the form $R(T)$ for $T : V \rightarrow V$ a linear isomorphism iff $\text{dom}R = V$ and $\text{ker}R = 0$.

Let $R : V \rightsquigarrow V$ be a linear relation. Define

1. $D := \{v \in V \mid \exists v_i \in V, i \in \mathbb{Z}, v_i R v_{i+1}, v_0 = v\}$. The relation R restricts to a relation $R_D : D \rightsquigarrow D$.
2. $K_+ := \{v \in V \mid \exists v_i, i \in \mathbb{Z}_{\geq 0}, v_i R v_{i+1}, v_0 = v, v_i = 0, i \text{ large enough}\}$.
3. $K_- := \{v \in V \mid \exists v_i, i \in \mathbb{Z}_{\leq 0}, v_{i-1} R v_i, v_0 = v, v_{-i} = 0, i \text{ large enough}\}$.
4. $V_{reg} := \frac{D}{D \cap (K_+ + K_-)}$, $\pi : D \rightarrow \frac{D}{D \cap (K_+ + K_-)}$ the quotient map and $\iota : D \rightarrow V$ the inclusion.

Consider the composition of relations

$$R_D = R(\iota)^\dagger \cdot R \cdot R(\iota)$$

and define

$$R_{reg} := R(\pi) \cdot R_D \cdot R(\pi)^\dagger : V_{reg} \rightsquigarrow V_{reg}.$$

Proposition 2.3 (cf [3])

1. There exists a linear isomorphism $T^R : V_{reg} \rightarrow V_{reg}$ such that $R_{reg} = R(T^R)$.
2. If $R : V \rightsquigarrow V$ and $R' : V' \rightsquigarrow V'$ are similar relations, i.e. there exists an isomorphism of vector spaces $\omega : V \rightarrow V'$ such that $R' = R(\omega) \cdot R \cdot R(\omega^{-1})$, then T^R and $T^{R'}$ are similar linear isomorphisms (i.e. $T^{R'} = \underline{\omega} \cdot T^R \cdot \underline{\omega}^{-1}$ for some isomorphism $\underline{\omega}$).
3. $R_{reg}^{-1} = (R^\dagger)_{reg}$.
4. $(R' \oplus R'')_{reg} = R'_{reg} \oplus R''_{reg}$.
5. Suppose $R_i : V_i \rightsquigarrow V_{i+1}$, $i = 1, 2, \dots, k$ with $V_1 = V_{k+1}$ then $(R_i \cdots R_{i-1} \cdots R_1 \cdot R_k \cdot R_{k-1} \cdots R_{i+1})_{reg} \sim (R_k \cdot R_{k-1} \cdots R_2 \cdot R_1)_{reg}$.

In view of the definition of R_{reg} it is immediate that :

Observation 2.4

1. If $\alpha, \beta : V \rightarrow W$ are two isomorphisms then $T^{R(\alpha, \beta)} = \beta^{-1} \cdot \alpha$.
2. If $f : V \rightarrow V$ is a linear map and V_0 is the generalized eigen-space of the eigenvalue 0 then:
 $f(V_0) \subset V_0$,
 f induces $\hat{f} : V/V_0 \rightarrow V/V_0$ and
 $T^{R(f)} \sim \hat{f} : V/V_0 \rightarrow V/V_0$.

The following technical Proposition will be used in section 4.3, where an algorithm for the calculation of $R(a, b)_{reg}$, part of an algorithm for the calculation of the r -monodromy will be presented.

Proposition 2.5

1. Consider the diagram

$$\begin{array}{ccccc} V & \xrightarrow{\alpha} & W & \xleftarrow{\beta} & V \\ \uparrow \subseteq & & \uparrow \subseteq & & \uparrow \subseteq \\ V' & \xrightarrow{\alpha'} & W' & \xleftarrow{\beta'} & V' \end{array} \quad (1)$$

and suppose that:

$$W' \supseteq \text{img} \alpha \cap \text{img} \beta$$

$$V' = \alpha^{-1}(W') \cap \beta^{-1}(W') \text{ and}$$

α' and β' the restriction of α and β .

$$\text{Then } R(\alpha, \beta)_{reg} = R(\alpha', \beta')_{reg}.$$

2. Consider the diagram

$$\begin{array}{ccccc}
 V & \xrightarrow{\alpha} & W & \xleftarrow{\beta} & V \\
 \downarrow p' & & \downarrow p & & \downarrow p' \\
 V' & \xrightarrow{\alpha'} & W' & \xleftarrow{\beta'} & V'
 \end{array} \tag{2}$$

with both α and β surjective. Define :

$$V' = V/\ker \alpha, W' = W/\beta(\ker \alpha),$$

$p : W \rightarrow W', p' : V \rightarrow V'$ the canonical quotient maps,

$$\bar{\alpha} : V' \rightarrow W \text{ induced from } \alpha, \alpha' = p \cdot \bar{\alpha},$$

β' induced by passing to quotient from β .

$$\text{Then } R(\alpha, \beta)_{reg} = R(\alpha', \beta')_{reg}.$$

For the reader's convenience the proofs of Propositions 2.3 and 2.5 are included in Appendix 1.

2.2 Jordan cells, characteristic polynomial and the characteristic divisors

Recall that a Jordan matrix $T(\lambda, k)$ is determined by a pair (λ, k) , $\lambda \in \bar{\kappa}$ and k a positive integer. When $\lambda \neq 0$ the pair (λ, k) is called in [2] *Jordan cell*.

$$T(\lambda; k) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \lambda & 1 \\ 0 & \cdots & 0 & 0 & \lambda \end{pmatrix}.$$

Any invertible square $n \times n$ - matrix is conjugated with a direct sum of Jordan cells (by Jordan decomposition theorem, cf [14]) with λ eigenvalue of the matrix. In different words any conjugacy class of linear isomorphisms $T : V \rightarrow V$, denoted by $[T]$, is determined by a unique collection of pairs, the Jordan cells $\mathcal{J}([T])$ or $\mathcal{J}(T)$. Note that Any such collection determines and is determined by the collection of monic polynomials

$$P^T(z)|P_1^T(z)|P_2^T(z)|\cdots|P_{n-1}^T(z)$$

where $P^T(z) = \det(zI - T)$ and $P_i^T(z)$ is the greatest common divisor of all $(n-i) \times (n-i)$ - minors of $zI - T$, cf [14]. The polynomials $P^T(z)|P_1^T(z)|P_2^T(z)|\cdots|P_{n-1}^T(z)$ ³ do not involve the algebraic closure $\bar{\kappa}$. The precise relation between them and the elements of $\mathcal{J}([T])$ is given in [14].

Definition 2.6 *The Jordan cells of the linear relation $R: V \rightsquigarrow V$ is the collection $\mathcal{J}([T^{R_{reg}}])$.*

³ $P(z)|Q(z)$ means that there exists a polynomial $R(z)$ such that $P(z) = Q(z)R(z)$.

3 Monodromy

In this section the homology of a space X is the singular homology with coefficients in a field κ fixed once for all and is denoted by $H_r(X)$, $r = 0, 1, 2, \dots$.

An *angle* is a complex number $\theta = e^{it} \in \mathbb{C}$, $t \in \mathbb{R}$ and the set of all angles is denoted by $\mathbb{S}^1 = \{\theta = e^{it} \mid t \in \mathbb{R}\}$. The space of angles identified to \mathbb{S}^1 , is equipped with the distance

$$d(\theta_1, \theta_2) = \inf\{|t_2 - t_1| \mid e^{it_1} = \theta_1, e^{it_2} = \theta_2\}.$$

In this paper **all real valued or angle valued maps $f : X \rightarrow \mathbb{R}$ or $f : X \rightarrow \mathbb{S}^1$ are proper continuous maps with X an ANR**. The properness of f forces the space X to be locally compact in the first case and compact in the second.

– A value $t \in \mathbb{R}$ or $\theta \in \mathbb{S}^1$ is *weakly regular* if $f^{-1}(t)$ or $f^{-1}(\theta)$ is an ANR, hence a compact ANR⁴.

– A map f whose set of weakly regular values is not empty is called *good* and a map with all values weakly regular is called *weakly tame*. For X a (locally finite) simplicial complex any \mathbb{R} or \mathbb{S}^1 -valued simplicial map f is weakly tame.

– An ANR X whose set of weakly tame maps is dense in the space of all maps with the C^0 -fine topology⁵ is called a *good ANR*. There exist compact ANR's (actually compact homological n -manifolds, cf [12]) with no co-dimension one subsets which are ANR's, hence compact ANR's which are not *good ANR*'s. The spaces homeomorphic to simplicial complexes, finite dimensional topological manifolds, or Hilbert cube manifolds (see Appendix 2 for definitions) are all *good ANR*'s. The first because any continuous map can be approximated by simplicial maps w.r. to a convenient subdivision, the last by more subtle reasons explained in Appendix 2.

As pointed out in introduction, the r -*monodromy*, cf Definition 3.3 below, will be first defined for good maps and will involve an angle θ , which is a weakly regular value. It will be shown that different choices of such angles lead to the same r -monodromy, and that the r -monodromy depends only on the cohomology class ξ_f associated with the map f .

Once some elementary properties will be established for good ANRs and maps⁶, using results on Hilbert cube manifolds, it will be shown that the r -monodromy can be associated to any angle valued map and is a homotopy invariants for any pair $(X, \xi \in H^1(X; \mathbb{Z}))$, X any compact ANR.

The following observations will be useful.

Proposition 3.1

1. Two maps $f, g : X \rightarrow \mathbb{S}^1$ with $D(f, g) = \sup_{x \in X} d(f(x), g(x)) < \pi$ are homotopic by a canonical homotopy, the "convex combination" homotopy.
2. Suppose X is a good ANR, $f, g : X \rightarrow \mathbb{S}^1$ two homotopic angle valued maps and $\epsilon > 0$. Then there exists a finite collection of maps $f_0, f_1, \dots, f_k, f_{k+1}$, such that:
 - a) $f_0 = f, f_{k+1} = g$,
 - b) f_i are weakly tame maps for $i = 1, 2, \dots, k$,
 - c) $D(f_i, f_{i+1}) < \epsilon$.

⁴A compact ANR has the homotopy type of finite simplicial complex.

⁵For this paper the concepts of *good map*, *tame map* and *good ANR* will be considered under the hypothesis that the space is compact, in which case C^0 -fine topology is the same as the familiar compact open topology

⁶actually it suffices to established them for simplicial complexes and simplicial maps

Indeed if f and g are viewed as maps with values in \mathbb{C} then the map $h_t(x) = \frac{tg(x)+(1-t)f(x)}{|tg(x)+(1-t)f(x)|}$, $0 \leq t \leq 1$, provides the desired homotopy stated in item 1. The condition $D(f(x), g(x)) < \pi$ insures that $|tg(x) + (1-t)f(x)| \neq 0$.

Item 2. follows from the local contractibility of the space of maps when equipped with the distance D .

3.1 Real valued maps

For $f : X \rightarrow \mathbb{R}$ a real valued map and $a \in \mathbb{R}$ denote by:

X_a^f , the sub-level $X_a^f := f^{-1}((-\infty, a])$; if a is weakly regular value then $X_a^f := f^{-1}((-\infty, a])$ is an ANR,

X_f^a , the super-level $X_f^a := f^{-1}([a, \infty)$; if a is weakly regular value then $X_f^a := f^{-1}([a, \infty)$ is an ANR.

For $f : X \rightarrow \mathbb{R}$ and $g : X \rightarrow \mathbb{R}$ maps $a < b$ s.t. $f^{-1}(a) \subset g^{-1}(-\infty, b)$ denote by

$$X_{a,b}^{f,g} := X_b^g \cap X_f^a.$$

If b is a weakly regular value for g and a is weakly regular value for f then $X_{a,b}^{f,g}$ is a compact ANR. This insures that $H_r(g^{-1}(a))$, $H_r(f^{-1}(b))$ and $H_r(X_{a,b}^{f,g})$ have finite dimension.

Denote by $R_{a,b}^{f,g}(r)$ the linear relation defined by the linear maps $i_1(r)$ and $i_2(r)$ induced by the inclusions $f^{-1}(a) \subset X_{a,b}^{f,g}$ and $g^{-1}(b) \subset X_{a,b}^{f,g}$.

$$H_r(f^{-1}(a)) \xrightarrow{i_1(r)} H_r(X_{a,b}^{f,g}) \xleftarrow{i_2(r)} H_r(g^{-1}(b)).$$

Proposition 3.2 *Let $t_1 < t_2 < t_3$. Suppose that t_1 is weakly regular for f , t_2 is weakly regular for g and $g^{-1}(t_2) \subset f^{-1}((t_1, t_3))$. Then one has*

$$R_{t_2,t_3}^{g,f}(r) \cdot R_{t_1,t_2}^{f,g} = R_{t_1,t_3}^{f,f}(r).$$

Proof: The verification is a consequence of the exactness of the following piece of Meyer–Vietoris sequence

$$H_r(g^{-1}(t_2)) \xrightarrow{i'_1 \oplus i'_2} H_r(X_{t_1,t_2}^{f,g}) \oplus H_r(X_{t_2,t_3}^{g,f}) \xrightarrow{i_1 - i_2} H_r(X_{t_1,t_3}^{f,f}) \quad (3)$$

whose linear maps involved in the sequence (3) and in the commutative diagram below are induced by obvious inclusions ⁷.

$$\begin{array}{ccccc} & & H_r(X_{t_1,t_3}^{f,f}) & & \\ & \nearrow I_1 & & \nwarrow I_2 & \\ & & & & \\ H_r(f^{-1}(t_1)) & \xrightarrow{j_1} & H_r(X_{t_1,t_2}^{f,g}) & \xleftarrow{i'_1} & H_r(g^{-1}(t_2)) & \xrightarrow{i'_2} & H_r(X_{t_2,t_3}^{g,f}) & \xleftarrow{j_2} & H_r(f^{-1}(t_3)) \end{array} \quad (4)$$

Indeed the commutativity of the diagram (4) implies that $xR_{t_1,t_2}^{f,g}y$, for $x \in H_r(f^{-1}(t_1))$ and $y \in H_r(f^{-1}(t_3))$ iff $i_1(j_1(x)) - i_2(j_2(y)) = 0$.

By the exactness of the sequence (3) one has $i_1(j_1(x)) - i_2(j_2(y)) = 0$ iff there exists $u \in H_r(g^{-1}(t_2))$ such that $(i'_1 \oplus i'_2)(u) = (j_1(x), j_2(y))$. This happens iff $xR_{t_1,t_2}^{f,g}u$ and $uR_{t_2,t_3}^{g,f}y$. which means $xR_{t_1,t_3}^{f,f}y$. ■

⁷In order to lighten the writing, in both (3) and (4), "r" was dropped off i 's and i' 's the notations for the inclusion induced linear maps in r -homology

3.2 Angle valued maps

Let $f : X \rightarrow \mathbb{S}^1$ be an angle valued map. Let $u \in H^1(\mathbb{S}^1; \mathbb{Z}) \cong \mathbb{Z}$ be the generator defining the orientation of \mathbb{S}^1 . Here \mathbb{S}^1 is regarded as an oriented one dimensional manifold. Let $f^* : H^1(\mathbb{S}^1; \mathbb{Z}) \rightarrow H^1(X; \mathbb{Z})$ be the homomorphism induced by f in integral cohomology and $\xi_f = f^*(u) \in H^1(X; \mathbb{Z})$. It is a well known fact in homotopy theory that the assignment $f \rightsquigarrow \xi_f$ establishes a bijective correspondence between the set of homotopy classes of continuous maps from X to \mathbb{S}^1 and $H^1(X; \mathbb{Z})$.

The cut at θ (with respect to the map $f : X \rightarrow \mathbb{S}^1$)

For $\theta \in \mathbb{S}^1$, a weakly regular value for f , define **the cut at $\theta = e^{it}$** to be the space \overline{X}_θ^f , the two sided compactification of $X \setminus f^{-1}(\theta)$ with sides $f^{-1}(\theta)$. Precisely as a set \overline{X}_θ^f is a disjoint union of three parts, $\overline{X}_\theta^f = f^{-1}(\theta)(1) \sqcup f^{-1}(\mathbb{S}^1 \setminus \theta) \sqcup f^{-1}(\theta)(2)$, with $f^{-1}(\theta)(1)$ and $f^{-1}(\theta)(2)$ two copies of $f^{-1}(\theta)$.

The topology on \overline{X}_θ^f is the only topology which makes \overline{X}_θ^f compact and the map from \overline{X}_θ^f to X defined by identity on each part continuous and a homeomorphism on the image when restricted to each part. The compact space \overline{X}_θ^f is a compact ANR.

The obvious inclusions $i_1, i_2, f^{-1}(\theta) \xrightarrow{i_1} \overline{X}_\theta^f \xleftarrow{i_2} f^{-1}(\theta)$ induce in homology in dimension r the linear maps (between finite dimensional vector spaces) $i_1(r)$ and $i_2(r)$,

$$H_r(f^{-1}(\theta)) \xrightarrow{i_1(r)} \overline{H}_r(X_\theta) \xleftarrow{i_2(r)} H_r(f^{-1}(\theta)) .$$

These linear maps define the linear relation $R(i_1(r), i_2(r)) := R_\theta^f(r)$ and then, by Proposition 2.3 the linear relation $(R_\theta^f(r))_{\text{reg}}$ and the linear isomorphism $T_\theta^f(r) : V_\theta^f(r) \rightarrow V_\theta^f(r)$, s.t. $(R_\theta^f(r))_{\text{reg}} = R(T_\theta^f(r))$.

Definition 3.3 *The r -monodromy of $f : X \rightarrow \mathbb{S}^1$ at $\theta \in \mathbb{S}^1$, for θ a weakly regular value, is the similarity class of the relation $(R_\theta^f(r))_{\text{reg}}$, equivalently the similarity class of the linear isomorphism $T_\theta^f(r)$. One denotes these similarity classes by $[(R_\theta^f(r))_{\text{reg}}]$ or $[T_\theta^f(r)]$.*

For a map $f : X \rightarrow \mathbb{S}^1$ and K a compact ANR denote by

$$\overline{f}_K : X \times K \rightarrow \mathbb{S}^1$$

the composition of f with the projection of $X \times K$ on X . Note that if θ is a weakly regular value for f then it remains a weakly regular value for \overline{f}_K and $(\overline{X} \times \overline{K})_{\overline{f}_K}^{\theta} = \overline{X}_\theta^f \times K$. Therefore, in view of the Kunnetth formula (expressing the homology of the product of two spaces) one has

$$\begin{aligned} V_\theta^{\overline{f}_K}(r) &= \oplus_l V_\theta^f(r-l) \otimes H_l(K) \\ T_\theta^{\overline{f}_K}(r) &= \oplus_l T_\theta^f(r-l) \otimes Id_{H_l(K)} \end{aligned} \tag{5}$$

where $Id_{H_l(K)}$ denotes the identity map on $H_l(K)$.

In particular if K is contractible then

$$[T_\theta^{\overline{f}_K}(r)] = [T_\theta^f(r)] \tag{6}$$

and if $K = \mathbb{S}^1$ then

$$[T_\theta^{\overline{f}_K}(r)] = \begin{cases} [T_\theta^f(0)] & \text{if } r = 0 \\ [T_\theta^f(r) \oplus T_\theta^f(r-1)] & \text{if } r \geq 1. \end{cases} \tag{7}$$

Proposition 3.4

1. If θ_1 and θ_2 are two different weakly regular values for f then $[T_{\theta_1}^f(r)] = [T_{\theta_2}^f(r)]$.
2. If X is a good ANR and $f, g : X \rightarrow \mathbb{S}^1$ are two homotopic maps with θ_1 a weakly regular value for f and θ_2 a weakly regular value for g then $[T_{\theta_1}^f(r)] = [T_{\theta_2}^g(r)]$.
3. If $f : X \rightarrow \mathbb{S}^1$ and $g : Y \rightarrow \mathbb{S}^1$ are two maps with θ_1 weakly regular value for f and θ_2 weakly regular value for g then $[T_{\theta_1}^f(r)] = [T_{\theta_2}^g(r)]$ iff $[T_{\theta_1}^{\bar{f}}(r)] = [T_{\theta_2}^{\bar{g}}(r)]$.
4. If $f : X \rightarrow \mathbb{S}^1$ and $g : Y \rightarrow \mathbb{S}^1$ are two maps with θ_1 weakly regular value for f and θ_2 weakly regular value for g , and $\omega : X \rightarrow Y$ is a homeomorphisms such that $g \cdot \omega$ and f are homotopic then $[T_{\theta_1}^f(r)] = [T_{\theta_2}^g(r)]$.

Proof of 1.: For X a compact ANR and $\xi \in H^1(X; \mathbb{Z})$ consider $\pi : \tilde{X} \rightarrow X$ an infinite cyclic cover⁸ associated to ξ .

Any map $f : X \rightarrow \mathbb{S}^1$ such that $f^*(u) = \xi$, u the canonical generator of $H^1(\mathbb{S}^1)$, has lifts $\tilde{f} : \tilde{X} \rightarrow \mathbb{R}$, which make the diagram below with $p(t)$ is given by $p(t) = e^{it} \in \mathbb{S}^1$, a pull-back diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{p} & \mathbb{S}^1 \\ \uparrow \tilde{f} & & \uparrow f \\ \tilde{X} & \xrightarrow{\pi} & X. \end{array} \quad (8)$$

Consider $\theta_1 = e^{it_1}, \theta_2 = e^{it_2} \in \mathbb{S}^1$ two different weakly regular values for f (i.e. with $t_2 - t_1 \leq \pi$ hence $t_1 < t_2 < t_1 + 2\pi < t_2 + 2\pi$). We apply the discussion in the subsection 3.1 to the real valued map $\tilde{f} : \tilde{X} \rightarrow \mathbb{R}$ and note that

$$R_{\theta_1}^f(r) = R_{t_1, t_1+2\pi}^{\tilde{f}, \tilde{f}}(r) = R_{t_2, t_1+2\pi}^{\tilde{f}, \tilde{f}}(r) \cdot R_{t_1, t_2}^{\tilde{f}, \tilde{f}}(r)$$

and

$$R_{\theta_2}^f(r) = R_{t_2, t_2+2\pi}^{\tilde{f}, \tilde{f}}(r) = R_{t_1+2\pi, t_2+2\pi}^{\tilde{f}, \tilde{f}}(r) \cdot R_{t_2, t_1+2\pi}^{\tilde{f}, \tilde{f}}(r).$$

Using the linear isomorphisms induced by π , the linear relations $R_{t_1, t_2}^{\tilde{f}, \tilde{f}}(r)$ and $R_{t_1+2\pi, t_2+2\pi}^{\tilde{f}, \tilde{f}}(r)$ can be identified to the linear relation $R' := R_{\theta_1}^f(r) : H_r(f^{-1}(\theta_1)) \rightsquigarrow H_r(f^{-1}(\theta_2))$ while $R_{t_2, t_1+2\pi}^{\tilde{f}, \tilde{f}}(r)$ to the linear relation $R'' = R_{\theta_2}^f(r) : H_r(f^{-1}(\theta_2)) \rightsquigarrow H_r(f^{-1}(\theta_2))$.

Therefore $R_{\theta_1}^f(r) = R'' \cdot R'$ and $R_{\theta_2}^f(r) = R' \cdot R''$, which in view of Proposition 2.3 (5) imply that $(R_{\theta_1}^f(r))_{\text{reg}} \sim (R_{\theta_2}^f(r))_{\text{reg}}$.

Proof of 2.: In view of Proposition 3.1 it suffices to prove the statement under the following additional hypotheses:

1. At least one of the maps f or g is weakly tame,
2. $D(f, g) < \pi$,

⁸ An infinite cyclic cover is a map $\pi : \tilde{X} \rightarrow X$ together with a free action $\mu : \mathbb{Z} \times \tilde{X} \rightarrow \tilde{X}$ such that $\pi(\mu(n, x)) = \pi(x)$ and the map induced by π from \tilde{X}/\mathbb{Z} to X is a homeomorphism. The above cover is called *associated to* ξ if any $\tilde{f} : \tilde{X} \rightarrow \mathbb{R}$ which satisfies $\tilde{f}(\mu(n, x)) = \tilde{f}(x) + 2\pi n$ induces a map from X to $\mathbb{R}/2\pi\mathbb{Z} = \mathbb{S}^1$ representing the cohomology class $\xi_f = \xi$. For two infinite cyclic covers $\pi_i : \tilde{X}_i \rightarrow X$ representing ξ there exists homeomorphisms $\omega : \tilde{X}_1 \rightarrow \tilde{X}_2$ which intertwine the free actions μ_1 and μ_2 and satisfy $\pi_2 \cdot \omega = \pi_1$.

3. The angles $\theta_1 = e^{it_1}$ and $\theta_2 = e^{it_2}$ satisfy $|t_2 - t_1 - \pi| < \pi/4$.

Since f and g are homotopic $\xi_f = \xi_g$. For any infinite cyclic cover $\tilde{X} \rightarrow X$ associated with $\xi = \xi_f = \xi_g$ both f and g have lifts \tilde{f} and \tilde{g} as indicated in the diagrams below

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{p} & \mathbb{S}^1 \\ \uparrow \tilde{f} & & \uparrow f \\ \tilde{X} & \xrightarrow{\pi} & X \end{array} \quad \begin{array}{ccc} \mathbb{R} & \xrightarrow{p} & \mathbb{S}^1 \\ \uparrow \tilde{g} & & \uparrow g \\ \tilde{X} & \xrightarrow{\pi} & X. \end{array} \quad (9)$$

Under the additional hypotheses one can find lifts \tilde{f} and \tilde{g} such that $g^{-1}(t_2) \subset \tilde{f}^{-1}(t_1, t_1 + 2\pi)$ and $\tilde{f}^{-1}(t_1 + 2\pi) \subset \tilde{g}^{-1}(t_2, t_2 + 2\pi)$. We apply the considerations in subsection 3.1 to the real valued maps $\tilde{f}, \tilde{g} : \tilde{X} \rightarrow \mathbb{R}$ and conclude that :

$$R_{\theta_1}^f(r) = R_{t_1, t_1 + 2\pi}^{\tilde{f}, \tilde{f}}(r) = R_{t_2, t_1 + 2\pi}^{\tilde{g}, \tilde{f}}(r) \cdot R_{t_1, t_2}^{\tilde{f}, \tilde{g}}(r)$$

and

$$R_{\theta_2}^g(r) = R_{t_2, t_2 + 2\pi}^{\tilde{g}, \tilde{g}}(r) = R_{t_1 + 2\pi, t_2 + 2\pi}^{\tilde{f}, \tilde{g}}(r) \cdot R_{t_2, t_1 + 2\pi}^{\tilde{g}, \tilde{f}}(r).$$

Let $R' := R_{t_2, t_1 + 2\pi}^{\tilde{g}, \tilde{f}}(r)$ and $R'' := R_{t_1, t_2}^{\tilde{f}, \tilde{g}}(r) = R_{t_1 + 2\pi, t_2 + 2\pi}^{\tilde{f}, \tilde{g}}(r)$. Then $R_{\theta_1}^f(r) = R'' \cdot R'$ and $R_{\theta_2}^g(r) = R' \cdot R''$ which, by Proposition 2.3 item (5), imply that $(R_{\theta_1}^f(r))_{\text{reg}} \sim (R_{\theta_2}^g(r))_{\text{reg}}$.

Proof of 3.: Recall that for a linear isomorphism $T : V \rightarrow V$ one denotes by $\mathcal{J}(T)$ the set of Jordan cells which is a similarity invariant.

First observe that if $T_1 : V_1 \rightarrow V_1$ and $T_2 : V_2 \rightarrow V_2$ are two linear isomorphism then $\mathcal{J}(T_1 \oplus T_2) = \mathcal{J}(T_1) \sqcup \mathcal{J}(T_2)$.

If so $[T_1 \oplus T_2] = [T_1' \oplus T_2']$, hence $\mathcal{J}([T_1]) \sqcup \mathcal{J}([T_2]) = \mathcal{J}([T_1']) \sqcup \mathcal{J}([T_2'])$, and $[T_1] = [T_1']$, hence $\mathcal{J}([T_1]) = \mathcal{J}([T_1'])$, imply $\mathcal{J}([T_2]) = \mathcal{J}([T_2'])$, hence $[T_2] = [T_2']$.

We apply this observation to $T_1 = T_{\theta_1}^f(r-1)$, $T_1' = T_{\theta_2}^g(r-1)$ and $T_2 = T_{\theta_1}^f(r)$, $T_2' = T_{\theta_2}^g(r)$. Then by induction on r formula (7) implies item 3..

Proof of 4.: In view of item 2. one has $[T_{\theta_2}^{g, \omega}(r)] = [T_{\theta_1}^f(r)]$. Since ω induces a homeomorphism between $\overline{X}_{\theta_2}^{g, \omega}$ and $\overline{Y}_{\theta_2}^{g, \omega}$ then $R_{\theta_2}^{g, \omega}(r) \sim R_{\theta_2}^g(r)$ which implies $[T_{\theta_2}^{g, \omega}(r)] = [T_{\theta_2}^g(r)]$ which implies $[T_{\theta_1}^f(r)] = [T_{\theta_2}^g(r)]$.
q.e.d.

In view of Proposition 3.4 (1) $[T_{\theta}^f(r)]$ is independent on θ , so for a *weakly tame map* f one can write $[T^f(r)]$ instead of $[T_{\theta}^f(r)]$. In view of Proposition 3.4 (2) if f_1 and f_2 are two good maps with $D(f_1, f_2) < \pi$ then one has $[T^{f_1}(r)] = [T^{f_2}(r)]$.

If X is a *good ANR* for a map f choose a weakly tame maps f' with $D(f, f') < \pi/2$ and in view of Proposition 3.4 (2) $[T^{f'}(r)]$ provides an unambiguous definition of the r -monodromy for the map f . Indeed for two such maps f'_1 and f'_2 one has $D(f'_1, f'_2) < \pi$ and then Proposition 3.4 (2) guaranties that $[T^{f'_1}(r)] = [T^{f'_2}(r)]$. Moreover, if f and g are homotopic then $[T^f(r)] = [T^g(r)]$. Then for X a *good ANR* and $\xi \in H^1(X; \mathbb{Z})$. one chooses f , with $\xi_f = \xi$, and one defines

$$[T^{(X; \xi)}(r)] := [T^f(r)].$$

In order to show that $[T^{(X; \xi)}(r)]$ can be extended to any compact ANR and is a homotopy invariant of the pair (X, ξ) ,⁹ one uses Proposition 3.4 (3) and (4) and the Stabilization Theorem below. This theorem

⁹This means that for (X_1, ξ_1) , and (X_2, ξ_2) pairs with $X_i, i = 1, 2$ compact ANRs, $\xi_i \in H^1(X_i; \mathbb{Z}), i = 1, 2$, the existence of a homotopy equivalence $\omega : X_1 \rightarrow X_2$ satisfying $\omega^*(\xi_2) = \xi_1$ implies $[T^{(X_1, \xi_1)}] = [T^{(X_2, \xi_2)}]$.

is a consequence remarkable topological results of Edwards and Chapman about Hilbert cube manifolds, cf [8]. An homological proof is also possible but requires a little bit of algebraic topology, cf [3].

Theorem 3.5 Stabilization theorem (*R. Edwards and T. Chapman*) *There exists a contractible compact ANR, Q , with the following properties.*

1. *For any compact ANR X the product $X \times Q$ is a good compact ANR.*
2. *Given $\omega : X \rightarrow Y$ a homotopy equivalence of compact ANR's the map $\omega \times Id_{Q \times \mathbb{S}^1} : X \times Q \times \mathbb{S}^1 \rightarrow Y \times Q \times \mathbb{S}^1$ is homotopic to a homeomorphism $\omega' : X \times Q \times \mathbb{S}^1 \rightarrow Y \times Q \times \mathbb{S}^1$.*
The compact ANR, Q , is the product of countable many copies of the segment $[0, 1]$.

The statements above are rather straightforward consequences of Edwards and Chapman results however neither 1. nor 2., as formulated above, can be found in their work or in [8]. They can be derived from the mathematics presented in [8] as explained in Appendix 2.

Extension of r -monodromy to all pairs (X, ξ)

To any pair (X, ξ) , X compact ANR and $\xi \in H^1(X; \mathbb{Z})$, and any $r \in \mathbb{Z}_{\geq 0}$ one defines the r -monodromy by

$$[T^{X, \xi}(r)] := [T^{X \times K, \bar{\xi}}(r)]$$

with $\bar{\xi}$ is the pull back of ξ by the projection of $X \times K \rightarrow X$. In view of the equality (7) if X was already a good ANR then $[T^{X, \xi}(r)] = [T^{X \times K, \bar{\xi}}(r)]$.

To verify the homotopy invariance consider $f_i : X_i \rightarrow \mathbb{S}^1$ representing the cohomology class ξ_i . Since $\omega^*(\xi_2) = \xi_1$ the composition $f_2 \cdot \omega$ and f_1 are homotopic and then in view of item 2. of Stabilization Theorem one has the homeomorphism ω' homotopic to $\omega \times id_{K \times \mathbb{S}^1}$. Hence $(\bar{f}_2)_{K \times \mathbb{S}^1} \cdot \omega'$ is homotopic to $(\bar{f}_1)_{K \times \mathbb{S}^1}$. Then by Proposition 3.4 (4) one has $[T^{(\bar{f}_2)_{K \times \mathbb{S}^1}}(r)] = [T^{(\bar{f}_1)_{K \times \mathbb{S}^1}}(r)]$. In view of Proposition 3.4 (3), $[T^{(\bar{f}_2)_K}(r)] = [T^{(\bar{f}_1)_K}(r)]$, hence $[T^{(X_1, \xi_1)}] = [T^{(X_2, \xi_2)}]$.

As a summary one has the following Theorem.

Theorem 3.6 *To any pair (X, ξ) , and $r = 0, 1, 2, \dots$, X compact ANR, and $\xi \in H^1(X; \mathbb{Z})$ one can associate the similarity class of linear isomorphisms $[T^{(X, \xi)}(r)]$ which is a homotopy invariant of the pair. When $f : X \rightarrow \mathbb{S}^1$ is a good map with $\xi_f = \xi$ this is the r -monodromy defined for a good map f and a weakly regular value.*

The collection $\mathcal{J}_r(X; \xi)$ consisting of the pairs with multiplicity, (λ, k) , $\lambda \in \bar{\kappa}, k \in \mathbb{Z}_{>0}$, which determine the similarity class $[T^{(X, \xi)}(r)] = [\oplus_{(\lambda, k) \in \mathcal{J}_r(X, \xi)} T(\lambda, k)]$ is referred to as *the Jordan cells of the r -monodromy* of (X, ξ) .

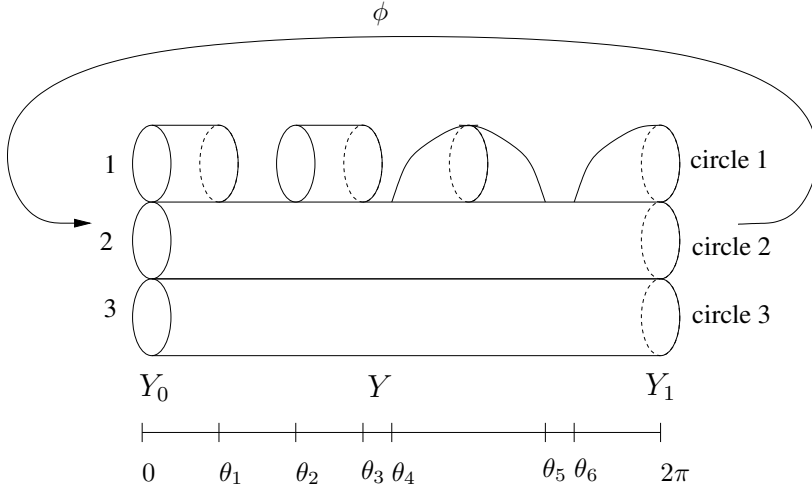
An example

The picture below is taken from [2] but with a different gluing map.

Consider the space X obtained from Y by identifying its right end Y_1 (a union of three circles) to the left end Y_0 (a union of three circles) following the map $\phi : Y_1 \rightarrow Y_0$ defined by the matrix

$$\begin{pmatrix} 3 & 3 & 0 \\ 2 & 3 & -1 \\ 1 & 2 & 3 \end{pmatrix}.$$

The meaning of this matrix as a map is the following: Circle (1) is divided in 6 parts, circle (2) in 8 parts and circle (3) in 4 parts ; the first three parts of circle (1) wrap clockwise around circle (1) to cover it three times, the next 2 wrap clockwise around circle (2) to cover it twice and around circle three to cover it



three times. Similarly circle (2) and (3) wrap over circles (1)(2) and (3) as indicated by the matrix. The first part of circle (2) wraps counterclockwise on circle (2).

The map $f: X \rightarrow S^1$ is induced by the projection of Y , on the interval $[0, 2\pi]$ which becomes S^1 when 0 and 2π are identified. This map has all values weakly regular.

In this example $\mathcal{J}_0(f) = \{(\lambda = 1, k = 1)\}$, $\mathcal{J}_1(f) = \{(\lambda = 2, k = 2)\}$ and $\mathcal{J}_2(f) = \emptyset$. The first and last calculations are obvious. The second will be derived by applying the algorithm described in Section 4.

3.3 F- monodromy

For a field κ , instead of the homology vector space $H_r(X)$, one can consider a more general functor F , a so called Dold half-exact functor cf [13]. Recall that this is a covariant functor defined from the category Top_c of compact ANR's and continuous maps (or any subcategory with the same homotopy category) to the category $\kappa - Vect$ of finite dimensional vector spaces and linear maps which satisfies the following properties:

1. $F(f) = F(g)$ for any two homotopic maps f and g ,
2. F satisfies the Meyer Vietoris property. Precisely, if A is a compact ANR with A_1 and A_2 closed subsets such that A_1, A_2 and $A_{12} = A_1 \cap A_2$ are ANR's and $A = A_1 \cup A_2$ then the sequence

$$F(A_{12}) \xrightarrow{i} F(A_1) \oplus F(A_2) \xrightarrow{j} F(A)$$

is exact. Here $i = F(i_1) \oplus F(i_2)$, $j = F(j_1) - F(j_2)$, i_1, i_2 the obvious inclusions of A_{12} in A_1 and A_2 and j_1, j_2 the obvious inclusion of A_1 and A_2 in A .

Analogues of Propositions 3.2 and 3.4 hold for F instead of H_r since they are based only on the Meyer-Vietoris property.

The same constructions applied to F instead of H_r work and one defines the $F - monodromy$ on the same lines. There are plenty of such functors and the $F - monodromy$ might be an invariant which deserve attention.

3.4 Comments

Theorem 3.6 is implicit in [3] (cf section 4 combined with with Theorem 8 .14) based on the interpretation of the monodromy as the similarity class of the linear isomorphism induced by the generator of the group

of deck transformations, on the vector space $V_r(X, \xi) = \ker(H_r(\tilde{X}) \rightarrow H_r^N(X, \xi))$. Here \tilde{X} denotes is the infinite cyclic cover of X defined by ξ and $H_r^N(X, \xi)$ denotes the Novikov homology of (X, ξ) .

In [3] it is shown that the Jordan cells $\mathcal{J}_r(f)$ defined in [2] as invariants for persistence of the circle valued map f are the same as the Jordan cell defined above. Since [3] is not yet in print, for the reader familiar with the notations in [2] section 5, we will provide a short explanations of this statement in Appendix 3.

The characteristic polynomial of $[T^{(X, \xi)}(1)]$ for the pair $(X; \xi)$, $X = S^3 \setminus K$, K an open tube around an embedded oriented circle (knot) and ξ the canonical generator of $H^1(S^3 \setminus K) = \mathbb{Z}$ is exactly the monic Alexander polynomial of the knot. as explained in section 5.

The alternating product of the characteristic polynomials $P_r(z)$ of the monodromies $[T^{X; \xi}(r)]$,

$$A(X; \xi)(z) = \prod P_r(z)^{(-1)^r},$$

known to topologists as the Alexander rational function, calculates (essentially ¹⁰) the Reidemeister torsion of X equipped with the degree one representation of $\pi_1(X)$ defined by ξ , and the complex number $z \in \mathbb{C}$, $z \neq 0$ when interpreted as an homomorphism $\pi_1(X, x) \rightarrow GL_1(\mathbb{C})$. This was pointed out first by J. Milnor and refined by V. Turaev, cf [24]. Given the need of additional background and definitions a precise formulation of this calculation will be discussed elsewhere.

4 The calculation of Jordan cells of an angle valued map

4.1 Generalities

Cell complexes

Recall that:

- A *convex k -cell* σ in an affine space \mathbb{R}^n , $n \geq k$, is the convex hull of a finite collection of points e_0, e_1, \dots, e_N called vertices, with the property that :
 1. there are subsets with $(k + 1)$ -points linearly independent ¹¹ but no subset of $(k + 2)$ -points linearly independent,
 2. no vertex lies in the topological interior of this convex hull

The topology of the cell is the one induced from the ambient affine space \mathbb{R}^n .

A k -simplex is a convex k -cell with exactly $k + 1$ vertices.

A k' -face σ' of σ , $k' < k$, is a convex k' -cell whose set of vertices is a subset of the set of vertices of σ . One indicates that σ' is a face of σ by writing $\sigma' \prec \sigma$.

A space homeomorphic to a convex k -cell is called simply a k -cell and the subset homeomorphic to a face continues to be called *face*.

- A *cell complex* Y is a space together with a locally finite collection \mathcal{Y} of compact subsets $\sigma \subset Y$, each homeomorphic with a convex cell with the following properties:

¹⁰a precise formulation requires additional data whose explanations are beyond the purpose of this paper

¹¹ In an affine space $(k + 1)$ - points are linearly independent if they lie in a k -dimensional affine subspace but not in any $(k - 1)$ -dimensional affine subspace.

1. If a k -cell σ is a member of the collection \mathcal{Y} then any of its faces $\omega \prec \sigma$ is a member of the collection \mathcal{Y} .
2. If σ and σ' are two cells members of the collection \mathcal{Y} then their intersection is a union of cells and each cell of this union is face of both σ and σ' .
3. $\bigcup_{\sigma \in \mathcal{Y}} \sigma = Y$.

The concept of sub complex $Y' \subset Y$ is obvious. Precisely Y' is the union of the cells in the subset $\mathcal{Y}' \subset \mathcal{Y}$ with the property that any face of cell in \mathcal{Y}' is in \mathcal{Y}' .

A finite *simplicial complex* is a finite cell complex with all cells simplexes.

For a cell complex Y with cells \mathcal{Y} denote by \mathcal{Y}_k the set of the k -cells in \mathcal{Y} . Clearly \mathcal{Y}_0 is the set of all vertices of the cells in \mathcal{Y} .

- An *oriented cell* is a cell $\sigma \in \mathcal{Y}$ equipped with an orientation $o(\sigma)$. This orientation induces an orientation for any codimension one face σ described by the rule : *first the induced orientation, next the normal vector pointing inside give the orientation $o(\sigma)$.*

If each cell σ of a cell complex is equipped with an orientation $o(\sigma)$ one has the incidence function $\mathbb{I} : \mathcal{Y} \times \mathcal{Y} \rightarrow \{0, +1, -1\}$ defined as follows:

$$\mathbb{I}(\sigma, \tau) := \begin{cases} \mathbb{I}(\tau, \sigma) = +1 & \text{if } \sigma \in \mathcal{Y}_k, \tau \in \mathcal{Y}_{k+1}, \sigma \prec \tau, o(\sigma)|_{\sigma'} = o(\sigma'), \\ \mathbb{I}(\tau, \sigma) = -1 & \text{if } \sigma \in \mathcal{Y}_k, \tau \in \mathcal{Y}_{k+1}, \sigma \prec \tau, o(\sigma)|_{\sigma'} \neq o(\sigma'), \\ \mathbb{I}(\tau, \sigma) = 0 & \text{if } \sigma \cap \sigma' = \emptyset \end{cases} \quad (10)$$

The incidence function determines the homology of Y with coefficients in any field.

- A *good order* of the set \mathcal{Y} of cells of Y is a total order " \leq " if:

(1) $\sigma \prec \tau$ implies $\sigma < \tau$.

In this case, if the cardinality of \mathcal{Y} is N , then the function $\mathbb{I}(\dots, \dots)$ can be regarded as $N \times N$ upper triangular matrix (all entries on and below diagonal are 0) and is referred below as the *incidence matrix* of Y .

Suppose that inside Y one has two disjoint sub complexes, $Y_1, Y_2 \subset Y$. In this case a *good order compatible with \mathcal{Y}_1 and \mathcal{Y}_2* needs, in addition to (1) above, the following requirements satisfied:

(2) If $\sigma_1 \in \mathcal{Y}_1$ and $\sigma_2 \in \mathcal{Y}_2$ then $\sigma_1 \prec \sigma_2$ and

(3) If $\sigma' \in \mathcal{Y}_i$ and $\sigma \in \mathcal{Y} \setminus \mathcal{Y}_1 \sqcup \mathcal{Y}_2$ imply $\sigma' \prec \sigma$.

Note that:

1. Given a total order of the cells in \mathcal{Y} a simple algorithm referred below as the *Ordering algorithm* permits to correct this order into a good total order. The Ordering Algorithm is based on the inspection of the n -th cell with respect to all previous cells. If the requirements (1)-(3) are not violated move to the $(n+1)$ -cell. If at least one of the three requirements is violated, change the position of this cell, and implicitly of the preceding ones if the case, by moving the cell to the left until (1), (2), or (3) are no more violated.
2. With the requirements 1, 2, 3 for the *good order* satisfied the incidence matrix of Y , $\mathbb{I}(\dots, \dots)$, has the form

$$\begin{pmatrix} A_1 & 0 & X \\ 0 & A_2 & Y \\ 0 & 0 & Z \end{pmatrix} \quad (11)$$

with $A_1 = \mathbb{I}_1$, $A_2 = \mathbb{I}_2$ the incidence matrices for Y_1 and for Y_2 .

3. The persistence algorithm [11], [25] permits to calculate from the incidence matrix \mathbb{I}

- (a) first, a base for $H_r(Y_1)$, then a base for $H_r(Y_2)$, then a base for $H_r(Y)$,
- (b) second, the $\dim H_r(Y) \times \dim H_r(Y_1)$ -matrix A and the $\dim H_r(Y) \times \dim H_r(Y_2)$ -matrix B representing the linear maps induced in the homology in dimension r by the inclusions of Y_1 and Y_2 in Y .

The cut at θ (with respect to f).

Let σ be a k -dimensional simplex with vertices e_0, e_1, \dots, e_k , i.e. a convex k -cell generated by $(k+1)$ linearly independent points located in some vector space. Let $f : \sigma \rightarrow \mathbb{R}$ be a linear map determined by the values of $f(e_i)$ by the formula

$$f\left(\sum_i t_i e_i\right) = \sum_i t_i f(e_i), \quad t_i \geq 0, \quad \sum_i t_i = 1 \quad (12)$$

and let $t \in \mathbb{R}$. Suppose that $\sup_i f(e_i) > t$ and $\inf_i f(e_i) < t$.

The map f and the number t determine two k -convex cells σ_+, σ_- and a $(k-1)$ -convex cell σ' :

$$\begin{aligned} \sigma_+ &= f^{-1}([t, \infty)) \cap \sigma \\ \sigma_- &= f^{-1}((-\infty, t]) \cap \sigma \\ \sigma' &= f^{-1}(t) \cap \sigma. \end{aligned} \quad (13)$$

An orientation $o(\sigma)$ on σ provides orientations $o(\sigma_+), o(\sigma_-)$ on σ_+, σ_- and induces an orientation $o'(\sigma')$ on σ' , precisely the unique orientation which followed by the direction provided by the vector field $\text{grad} f$ defines the orientation of $o(\sigma)$. Then $I(\sigma_{\pm}, \sigma') = \pm 1$.

Recall that the map $f : X \rightarrow \mathbb{S}^1 \subset \mathbb{C}$ is simplicial if the restriction of "ln f "¹² to any simplex σ is a linear map as considered above.

4.2 The algorithm

The algorithm we propose inputs a simplicial complex X , a simplicial map f and an angle θ different from the values of f on vertices and outputs for each r , $0 \leq r \leq \dim X$, in STEP 1 two $m \times n$ matrices A_r and B_r with m , the number of rows equal to the dimension of $H_r(\overline{X}_\theta^f)$ and n , the number of columns, equal to the dimension of $H_r(f^{-1}(\theta))$. The matrices represent the linear maps induced in homology by the two inclusions, from the left and from the right, of $f^{-1}(\theta) = Y_1 = Y_2$ in \overline{X}_θ^f . In STEP 2 one obtains from the matrices A_r and B_r the invertible square matrices A'_r and B'_r such that $(B'_r)^{-1}A'_r$ represents the r -monodromy and in STEP 3 one derives from $(B'_r)^{-1}A'_r$ the Jordan cells $\mathcal{J}_r(f)$.

STEP 1.

The simplicial set X is recorded by :

¹²in view of 1-connectivity of each simplex "ln f " has continuous univalent determination when the value on one vertex of the simplex is specified

- the set of vertices with an arbitrary chosen total order,
- a specification of the subsets which define the collection \mathcal{X} of simplices.

Implicit in this data is an orientation $o(\sigma)$ of each simplex, orientation provided by the relative ordering of the vertices of each simplex, and therefore the incidence number $\mathbb{I}(\sigma', \sigma)$ of any two simplexes σ' and σ in \mathcal{X} .

(Implicit is also a total order of the simplexes of \mathcal{X} provided by the *lexicographic order* induced from the order of the vertices.)

The simplicial map f ¹³ is indicated by

- the sequence of $N_0 =$ the number of vertices, the values of f on vertices.

The map f and the angle $\theta = e^{it}$ provide a decomposition of the set \mathcal{X} as $\mathcal{X}' \sqcup \mathcal{X}''$ with $\mathcal{X}' := \{\sigma \in \mathcal{X} \mid \sigma \cap f^{-1}(\theta) \neq \emptyset\}$ and $\mathcal{X}'' := \mathcal{X} \setminus \mathcal{X}'$.

From these data we can derive :

– first, the collections \mathcal{Y} with the sub collections $\mathcal{Y}(1)$ and $\mathcal{Y}(2)$ of the cells of the complex $Y = \overline{X_\theta^f}$ and the sub complexes $Y_1 = f^{-1}(\theta)$ and $Y_2 = f^{-1}(\theta)$,

- second, the incidence function on $\mathcal{Y} \times \mathcal{Y}$,
- third, a good order for the elements of \mathcal{Y} .

These all lead to the incidence matrix $\mathbb{I}(Y)$.

Description of the cells of Y : Each oriented simplex σ in \mathcal{X}'' provides a unique oriented cell σ in \mathcal{Y} .

Each oriented k -simplex σ in \mathcal{X}' provides two oriented k -cells σ_+ and σ_- and two oriented $(k-1)$ -cells $\sigma'(1)$ and $\sigma'(2)$, copies of the oriented cell σ' . So the cells of Y are of five types

$$\mathcal{Y}'_k(1) = \mathcal{X}'_{k+1},$$

$$\mathcal{Y}'_k(2) = \mathcal{X}'_{k+1},$$

$$\mathcal{Y}'_{k-} = \mathcal{X}'_k,$$

$$\mathcal{Y}'_{k+} = \mathcal{X}'_k,$$

$$\mathcal{Y}''_k = \mathcal{X}''_k.$$

Note that \mathcal{Y}'_{k+} and \mathcal{Y}'_{k-} are two copies of the same set \mathcal{X}'_k and $\mathcal{Y}'_k(1)$ and $\mathcal{Y}'_k(2)$ are in bijective correspondence with the set \mathcal{X}'_{k+1} .

Inside the cell complex Y we have two sub complexes Y_1 and Y_2 whose cells are $(\mathcal{Y}_1)_k = \mathcal{Y}'_k(1)$, $(\mathcal{Y}_2)_k = \mathcal{Y}'_k(2)$, two copies of the same set \mathcal{X}'_{k+1} .

Incidence of cells of \mathcal{Y} : The incidence of two cells in the same group (one of the five types) are the same as the incidence of the corresponding simplexes. The incidence of two cells one in \mathcal{Y}_1 the other in \mathcal{Y}_2 or one in the group $\mathcal{Y}'(i)$, $i = 1, 2$ the other in the group \mathcal{Y}'' is always zero. The rest of incidences are provided by the formulae (10).

The good order: Start with a good order of \mathcal{Y}_1 followed by \mathcal{Y}_2 with the same order (translated by the number of the elements of \mathcal{Y}_1) followed by the remaining elements of \mathcal{Y} . Without changing the order in the collection $\mathcal{Y}_1 \sqcup \mathcal{Y}_2$, since no violation of the requirements 1, 2, 3, appear, one can realize a good order for the entire collection \mathcal{Y} with all remaining cells being preceded by the cells of $\mathcal{Y}_1 \cup \mathcal{Y}_2$. Simply we apply the *ordering algorithm* to obtain a good order.

As a result we have the incidence matrix $\mathbb{I}(Y)$ which is of the form

$$\begin{pmatrix} \mathbb{I} & 0 & X \\ 0 & \mathbb{I} & Y \\ 0 & 0 & Z \end{pmatrix} \quad (14)$$

with \mathbb{I} the incidence matrix of Y_1 and Y_2 .

¹³for simplicity one supposes that f takes different values on different vertices

Running the persistence algorithm of [11], [25] leads to the matrices representing $A_r : H_r(Y_1) \rightarrow H_r(Y)$ and $B_r : H_r(Y_2) \rightarrow H_r(Y)$ as follows.

We run the persistence algorithm on the incidence matrix A to compute a base for of the homology of $H_r(Y_1) = H_r(Y_2)$. We continue the procedure by adding columns and rows to the matrix to obtain a base of $H_r(Y)$. It is straightforward to compute a matrix representation for the the inclusion induced linear maps $H_r(Y_i) \rightarrow H_r(Y), i = 1, 2$.

The time complexity of this step was discussed in [2] and is $O(M(n))$, the time complexity of multiplying $n \times n$ matrices, with $M(n) = O(n^\omega), \omega < 2.376$ see references in [2].

STEP 2. One uses the algebraic algorithm to pass from A_r, B_r to the invertible matrices A'_r, B'_r and then to $(B'_r)^{-1}(A'_r)_r$ described in the next subsection. This is based by reducing matrices to echelon form as described in subsection 4.3 below.

STEP 3. One uses the standard algorithms to put the matrix $(B'_r)^{-1}A'_r$ in Jordan diagonal form (i.e. as block diagonal matrix with Jordan cells on diagonal). Since the resulting matrix from Step 2 is a $k \times k$ matrix with $k < \inf\{m, n\}$ the time complexity of STEP 3 is at most $O(k^5 \log k)$, cf [21], however there are apparently algorithms with better time complexity like [10] with $O(5/4k^4)$ and I understand even $O(M(k))$.

All basic softwares which carry linear algebra packages contain sub packages which input a matrix and output its (reduced) row/column echelon form as well as the matrix C or D in Proposition 4.2 involved in Step 2; this permit an easy implementation of step 2. Most of them also contain sub packages which input a square matrices and output their Jordan form making also easy the implementation of step 3.

4.3 An algorithm for the calculation of $R(A, B)_{\text{reg}}$

The algorithm presented below (STEP 2.) above inputs two $m \times n$ matrices (A, B) defining a linear relation $R(A, B)$ and outputs two $k \times k, k \leq \inf\{m, n\}$, invertible matrices (A', B') such that $R(A, B)_{\text{reg}} \sim R(A', B')_{\text{reg}}$. It is based on three modifications T_1, T_2, T_3 described below. The simplest way to perform these modification is to use familiar procedures of bringing a matrix to row or column echelon form (REF) or (CEF) explained below, but less is actually needed as the reader can see in the presentation of the algorithm.

Modification $T_1(A, B) = (A', B')$:

Produces the invertible $m \times m$ matrix C and the invertible $n \times n$ matrix D so that

$$CAD = \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix} \text{ and } CBD = \begin{pmatrix} B_{11} & B_{12} \\ B_{2,1} & 0 \end{pmatrix}.$$

Precisely, one constructs first C which puts A in REF (reduced row echelon form) such that

$$CA = \begin{pmatrix} A_1 \\ 0 \end{pmatrix} \text{ and makes } CB = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

Second, one constructs D which puts B_2 in CEF (column echelon form). Precisely,

$$B_2D = \begin{pmatrix} B_{21} & 0 \end{pmatrix}.$$

Clearly CAD and CBD are as stated above.

Take $A' = A_{12}, B' = B_{12}$.

In view of Proposition 2.5 (1) one has $R(A, B)_{\text{reg}} = R(A', B')_{\text{reg}}$.

Modification $T_2(A, B) = (A', B')$:

Produces the invertible $m \times m$ matrix C and the invertible $n \times n$ matrix D so that

$$CAD = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{pmatrix} \text{ and } CBD = \begin{pmatrix} B_{11} & B_{12} \\ 0 & 0 \end{pmatrix}.$$

Precisely, one constructs C which puts B in REF (row echelon form) such that

$$CB = \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \text{ and makes } CA = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}.$$

Then one constructs D which puts A_2 in RCEF (column echelon form), precisely $A_2D = (A_{21} \ 0)$.

Take $A' = A_{12}, B' = B_{12}$.

Clearly CAD and CBD are as stated above.

In view of Proposition 2.5 (1) one has $R(A, B)_{\text{reg}} = R(A', B')_{\text{reg}}$.

Note that if A was surjective then A' remains surjective.

Modification $T_3(A, B) = (A', B')$:

Produces the invertible $n \times n$ matrix D and the $m \times m$ invertible matrix C so that

$$CAD = \begin{pmatrix} A_{11} & 0 \\ A_{21} & 0 \end{pmatrix} \text{ and } CBD = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & 0 \end{pmatrix}.$$

Precisely, one constructs D which puts A in CEF (reduced row echelon form) i.e.

$$AD = (A_1 \ 0) \text{ and makes } BD = (B_1 \ B_2).$$

Then one constructs C to put B_2 in REF precisely,

$$CB_2 = \begin{pmatrix} B_{21} \\ 0 \end{pmatrix}.$$

Take $A' = A_{21}, B' = B_{21}$.

Clearly CAD and CBD are as stated above.

In view of Proposition 2.5 (2) one has $R(A, B)_{\text{reg}} = R(A', B')_{\text{reg}}$.

Note that if both A and B were surjective then A' and B' remain surjective.

Here is how the algorithm works.

- (I) Inspect A

if surjective move to (II)

else:

- apply T_1 and obtain A' and B' .
- make $A = A'$ and $B = B'$ and
- go to (I)

- (II) Inspect B

if surjective move to (III)

else :

- apply T_2 and obtain A' and B' .
- make $A = A'$ and $B = B'$ and
- go to (II)

(Note that if A was surjective by applying T_2 , A' remains surjective.)

- (III) Inspect A

if injective go to (IV).

else

- apply T_3 and obtain A' and B' .
- make $A = A'$ and $B = B'$ and
- go to (III)

- (IV) Calculate $B^{-1} \cdot A$.

(Note that if A and B were surjective by applying T_3 , A' remains surjective.)

Echelon form for $n \times m$ matrices

Let κ be a field.

Let M be a $m \times n$ matrix with coefficient in the field κ .

$$M = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} \end{pmatrix}.$$

Recall that:

a row or column is zero-row or zero-column if all entries are zero,

the leading entry in a row or a column is the first nonzero entry w.r.to the index which varies.

Definition 4.1

1. The matrix M is in row echelon form, REF, if the following hold:

(a) All zero rows are below nonzero ones.

(b) For each row the leading entry is to the right of the leading entry of the previous row.

As example the matrix M below is in row echelon form

$$M = \begin{pmatrix} 0 & 0 & \alpha & x & x & x & 0 & x \\ 0 & 0 & 0 & \beta & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

with $\alpha, \beta, \gamma \neq 0$ and x various (possibly zero) elements in κ .

2. The matrix M is in column echelon form, CEF, iff the transposed matrix M^t is in REF i.e. the following hold:

(a) All zero columns succeed nonzero ones.

(b) For each column the leading entry is below of the leading entry of the previous column.

As example the matrix below is in reduced row echelon form

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 \\ x & \beta & 0 & 0 & 0 \\ x & x & 0 & 0 & 0 \\ x & x & 0 & 0 & 0 \\ x & x & \gamma & 0 & 0 \end{pmatrix}.$$

with $\alpha, \beta, \gamma \neq 0$ and x various (possibly zero) elements in κ .

Proposition 4.2

1. For any $(m \times n)$ matrix M one can produce an invertible $n \times n$ matrix C such that the composition CM is in REF.
2. For any $(m \times n)$ matrix matrix M one can produce an invertible $m \times m$ matrix D such that the composition MD in in CEF.

The construction of C is based on "Gauss elimination" procedure consisting in operation of "permuting rows , multiplying rows with a nonzero element in κ and replacing a row by itself plus a multiple of an other row, each such operation is realizable by left multiplication by elementary matrix or permutation matrix cf [14].

The construction of D is done by : transpose, then apply the construction of C , then transpose again.

The complexity of applying any of the modifications T_1, T_2 , or T_3 , construction the matrices which perform is reduction to echelon form is $O(k^\omega)$, for $k = \sup\{n, m\}$ cf [16].

4.4 An example

We illustrate Step 2 of the algorithm with $A = A_1$ and $B = B_1$ representing the inclusion induced linear maps in homology in dimension one derived from the example in Section 3. We take the cut at the angle $\theta = 0$, i.e. the level corresponding to the complex number $1 \in \mathbb{S}^1$. One has $H_1(f^{-1}(\theta)) = \kappa^3$, $H_1(\overline{X}_\theta^f) = \kappa^4$. It is immediate from the description of f that the matrices A_1 and B_1 are given by

$$A_1 = A = \begin{pmatrix} 3 & 3 & 0 \\ 2 & 3 & -1 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix} \quad B_1 = B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \quad (15)$$

Proceed according to the algorithm:

Inspect A , since not surjective apply T_1 and find $C = Id$ and $D = Id$. Then

$$A' = \begin{pmatrix} 3 & 3 & 0 \\ 2 & 3 & -1 \\ 1 & 2 & 3 \end{pmatrix} \quad B' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (16)$$

Update

$$A = \begin{pmatrix} 3 & 3 & 0 \\ 2 & 3 & -1 \\ 1 & 2 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (17)$$

Since A is surjective inspect B . Since B is not surjective apply T_2 and find

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and then } D = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\text{Then } CAD = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 3 \\ 3 & 0 & 0 \end{pmatrix} \quad \text{and } CBD = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \quad B' = \begin{pmatrix} 1 & 0 \\ 0 & 1. \end{pmatrix} \quad (18)$$

Since both A' and B' are invertible, consider

$$B^{-1} \cdot A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \text{ invertible matrix.}$$

According to Step 3. $\mathcal{J}([R(A, B)_{\text{reg}}]) = \{(2, 2)\}$.

5 A few computational applications

In view of the previous section we view the Jordan cells as *computer friendly* invariants (since they are computable by implementable effective algorithms). In this section we indicate how they can be used, possibly complemented by other computer friendly invariants like the standard Betti numbers to derive relevant topological invariants of interest even outside topology. More details and additional computations (applications) will be treated in future work.

5.1 Novikov Betti numbers and L_2 -Betti numbers

To a pair $(X, \xi \in H^1(X; \mathbb{Z}))$, X a compact ANR, and κ a field, in addition to the familiar Betti numbers $\beta_r(X; \kappa)$, one can associate the Novikov–Betti numbers $\beta_r^N(X, \xi; \kappa)$. They are interesting numerical invariants of geometric and topological relevance¹⁴. For $\kappa = \mathbb{C}$ the L_2 -Betti numbers $\beta_r^{L_2}(\tilde{X})$, $\tilde{X} \rightarrow X$ the infinite cover determined by ξ , are the same as $\beta_r^N(X, \xi; \kappa)$.

Recall from [19] or [20] that The Novikov–Betti numbers $\beta_r^N(X, \xi; \kappa)$ are defined using the infinite cyclic cover $\tilde{X} \rightarrow X$ associated to ξ by the equality

$$\beta_r^N(X, \xi; \kappa) = \dim_{\kappa[t^{-1}, t]} H_r(\tilde{X}) \otimes_{\kappa[t^{-1}, t]} \kappa[t^{-1}, t].$$

Here $\kappa[t^{-1}, t]$ denotes the ring of Laurent polynomials in t with coefficients in κ , $\kappa[[t^{-1}, t]]$ denotes the field of Laurent power series in t with coefficients in κ and $H_r(\tilde{X}; \kappa)$ is viewed as a $\kappa[[t^{-1}, t]]$ -module whose $\kappa[t^{-1}, t]$ -module structure determined by the action of \mathbb{Z} on \tilde{X} as the group of deck transformations.

Note that when X is a finite simplicial complex the standard Betti numbers are *computer friendly* which means computable by effective algorithms (for instance the persistence algorithm [6], [25], [11]) and so are the Jordan cells in view of the algorithm presented in section 4 or of the algorithm described in [2]. In view of the definition above, even when X is a finite simplicial complex, \tilde{X} is an infinite simplicial complex, hence the calculation of the Novikov Betti numbers via their definition is not computer friendly.

The formula (19) established in [3] Theorem 1.1, permits however to express the Novikov–Betti numbers in terms of *computer friendly* quantities and provides an alternative definition of them not based on infinite coverings.

$$\beta_r^N(\xi; \kappa) = \beta_r(X; \kappa) - \#\mathcal{J}_r(\xi)(1) - \#\mathcal{J}_{r-1}(\xi)(1) \quad (19)$$

In this formula $\mathcal{J}_r(\xi)(1) = \{J = (\lambda, n) \in \mathcal{J}_r(\xi) \mid \lambda = 1\}$ and $\#$ denotes cardinality.

5.2 Other type of Betti numbers

The calculation of the sets $\mathcal{J}_r(X; \xi)$ lead to the calculation of the dimension of the homology with coefficients in the local system (of one dimensional κ -vector spaces) $\hat{u} \cdot \xi$, $u \in \kappa \setminus 0$, defined by the composition

$$H_1(X; \mathbb{Z}) \xrightarrow{\xi} \mathbb{Z} \xrightarrow{\hat{u}} \kappa \setminus 0 \text{ with } \hat{u} : \mathbb{Z} \rightarrow \kappa \setminus 0 \text{ given by } \hat{u}(n) = u^n.$$

¹⁴for example, for X is a closed Riemannian manifold and $f : X \rightarrow \mathbb{S}^1$ a Morse angle valued function (i.e. all critical points non degenerated) one has the same relation between the numbers of critical points and the Novikov Betti numbers (Novikov inequalities) as the familiar relations between the critical points of a real valued Morse map and the standard Betti numbers (Morse inequalities)

Precisely one has the following formula established in [4], Theorem 7.1,

$$\dim H_r(X; \hat{u}\xi) = \beta_r^N(X, \xi; \kappa) + \#\mathcal{J}_r(1/u) + \#\mathcal{J}_{r-1}(u). \quad (20)$$

For $\omega \in \bar{\kappa} \setminus 0$, $\mathcal{J}_r(\omega)$ denotes the set $\{J = (\lambda, n) \in \mathcal{J}_r(\xi) \mid \lambda = \omega\}$.

5.3 The Betti number of the homotopy theoretic fiber of ξ .

For $\beta_r^N(X, \xi; \kappa) = 0$, in view of the [3], the $\kappa[t^{-1}, t]$ -module $H_r(\tilde{X})$ is torsion module, hence equal to $V_r(X; \xi)$, hence

$$\dim H_r(\tilde{X}) = \sum_{J=(\lambda_J, n_J) \mid J \in \mathcal{J}_r(\xi)} n_J. \quad (21)$$

If ξ is represented by a fibration $f : X \rightarrow \mathbb{S}^1$ with compact fiber $f^{-1}(\theta)$ then $f^{-1}(\theta)$ and \tilde{X} have the same homology, hence $\beta_r(f^{-1}(\theta); \kappa) = \dim H_r(\tilde{X})$. By Theorem 3.6 $\mathcal{J}_r(\xi) = \mathcal{J}_r(g)$ for $g : X \rightarrow \mathbb{S}^1$ any map, not necessary a fibration, in the homotopy class representing by ξ . These sets can be computed using the algorithm described in section 4¹⁵.

5.4 Alexander polynomial of a knot and generalizations

A knot $K \subset S^3$, K a simple closed curve (i.e. homeomorphic to the oriented circle \mathbb{S}^1), is a locally flat embedding in the three dimensional sphere S^3 . Consider $X = S^3 \setminus N$ where N is a open tubular neighborhood of K in S^3 . Clearly X is homotopy equivalent to $S^3 \setminus K$. The Alexander dual of the generator $u \in H_1(K; \mathbb{Z})$ is an integral cohomology class $\xi \in H^1(X; \mathbb{Z})$. The Alexander polynomial of the knot, a fundamental invariant of the knot, is a polynomial with integral coefficients

$$a_r z^r + \cdots a_1 z + a_0$$

with $a_0 \neq 0$ and $a_n \geq 0$ is defined as the only generator of the principal ideal $(P(t))$ defined by the isomorphism $H_r(\tilde{X}; \mathbb{Z}) \cong \mathbb{Z}[t^{-1}, t]/(P(t))$ where $\mathbb{Z}[t^{-1}, t]$ denotes the ring of Laurent polynomial with coefficients in \mathbb{Z}

For detailed definitions and examples one recommends ([22])¹⁶. As established first by Milnor, cf [18], the monic polynomial $1/a_n \cdot P(t)$ can be calculated as the characteristic polynomial of the 1- monodromy of (X, ξ) , and is exactly

$$\prod_{J=(\lambda_J, n_J) \mid J \in \mathcal{J}_1(\xi)} (z - \lambda_J)^{n_J}. \quad (22)$$

The algorithm described in Section 4 provides a new algorithm to calculate the monic Alexander polynomial¹⁷.

¹⁵This formula was used by the author to calculate the Betti numbers of the Milnor fiber of some isolated singularities fiber for some isolated singularities and will be described in subsequent work.

¹⁶ For example for the familiar figure eight knot $P(t) = t^2 - 3t + 1$, cf [22] page 166 and the torus knot $(4, 7)$ $P(t) = t^{18} - t^{17} + t^{14} - t^{13} + t^{11} - t^9 + t^7 - t^5 + t^4 - t + 1$ cf [22] page 178

¹⁷ a more in detailed discussion for the calculation of Alexanders polynomials of knots and links including knots in higher dimension and the role of the algorithm provided in section 4 is in preparation. In a similar vein important cases of the Milnor-Turaev torsion of (M, ξ) , a rational function defined on $GL(n, \mathbb{C})$, when regarded as the variety of rank n complex representations of Z can be calculated by implementable algorithms containing as part the calculation of the Jordan cell for monodromies.

6 Appendices

6.1 Appendix 1

For the proof of Propositions 2.3 and 2.5 one needs the following observation.

Observation 6.1

i). $x \in D$ iff there exists $x_i \in V$, $i \in \mathbb{Z}$ with $x_i R x_{i+1}$, $x_0 = x$.

ii). $y \in K_+ + K_-$ iff there exists a nonnegative integer k , the sequences $x_1^+, x_2^+, \dots, x_k^+$ all elements in V , and the sequence $x_1^-, x_0^-, x_{-1}^-, \dots, x_{-k}^-$ all elements in V , such that:

1. $y = x_1^+ + x_1^-$,
2. $x_1^+ R x_2^+ R \dots R x_k^+ R 0$,
3. $0 R x_{-k}^- R x_{-(k-1)}^- R \dots R x_0^- R x_1^-$.

Proof of Proposition 2.3 (cf [3])

To establish item 1. one uses Lemma 2.2 (2) applied to the relation R_{reg} . Clearly in view of the surjectivity of $\pi : D \rightarrow V_{reg}$ and Observation 6.1 (i) one has $\text{dom} R_{reg} = V_{reg}$, so it remains to check that $\ker(R_{reg}) = 0$.

To verify this we start with $x \in D$ s.t xRy , $y \in (K_+ + K_-)$ and want to check that $x \in D \cap (K_+ + K_-)$.

One produces the elements $x_i \in V$, $i \in \mathbb{Z}$, $x_1^+, x_2^+, \dots, x_k^+ \in V$ and the elements $x_1^-, x_0^-, x_{-1}^-, \dots, x_{-k}^- \in V$ as stated in Observation 6.1 (ii) and with the properties stated. One observes that:

1. $x_0^- \in K_-$,
2. $(x - x_0^-) \in D \cap K_+$ since

$$\dots R x_{-(k-1)}^- R (x_{-k} - x_{-k}^-) R \dots R (x_0 - x_0^-) R ((y - x_1^-) = x_1^+) R x_2^+ \dots R x_k^+ R 0$$

and therefore $x_0^- = -x + x - x_0^- \in D$, hence

3. $x_0^- \in D \cap K_-$,

Combining (2.) and (3.) above one obtains $x = x - x_0 + x_0 \in (D \cap K_+) + (D \cap K_-) \subseteq D \cap (K_+ + K_-)$.

Items 2. 3. and 4. (in Proposition 2.3) are straightforward.

To verify item 5. it suffices to check the equality for $k = 2$. which can be concluded in view of Observation 6.1 (i).

Proof of Proposition 2.5

Item 1. follows by observing that D and $D \cap (K^+ + K^-)$ for both $R(\alpha, \beta)$ and $R(\alpha', \beta')$ are actually the same.

To check this consider the sequences

$$\begin{array}{ccccccccccc} \dots & v_{-1} & \xrightarrow{\alpha} & w_0 & \xleftarrow{\beta} & v_0 = (v_0^- + v_0^+) & \longrightarrow & w_1 & \longleftarrow & v_1 & \longrightarrow & w_2 \dots \\ & & & & & & & & & & & & \\ & v_0^+ & \longrightarrow & w_1^+ & \longleftarrow & v_1^+ & \longrightarrow & w_2^+ & \longleftarrow & \dots & w_{k+1}^+ & \longleftarrow & 0 \\ & & & & & & & & & & & & \\ & 0 & \longrightarrow & w_{-(k)}^- & \longleftarrow & v_{-k}^- & \longrightarrow & w_{-(k-1)}^- & \longleftarrow & \dots & v_{-1} & \longrightarrow & w_0^- \end{array}$$

Indeed, by Observation 6.1 (i), $v_0 \in D$ implies the existence of the first sequence, which implies that $v_i \in V'$ and $w_i \in W'$, which guarantee that $D = D'$.

If $v_0 \in D \cap (K_+ + K_-)$ all three sequences above exist, which imply that $v_0 - v_0^- = v_0^+ \in D \cap K_+^+ \subseteq D' \cap (K_+^+ K_+^+)$. Similarly $v_0 - v_0^+ = v_0^- \in D' \cap K_-^+ \subseteq D' \cap (K_+^+ + K_-^+)$, and therefore $v_0 = v - v_0^- + v - v_0^+ = v_0 \in D' \cap ((K_+^+ + K_-^+))$.

To check item 2. observe that the diagram (2) (in Section 2) induces the linear map $\pi : D/D \cap (K_+ + K_-) \rightarrow D'/D' \cap (K_-^+ + K_+^+)$. This map is obviously surjective since both pairs α, β and α', β' being surjective make $V = D$ and $V' = D'$. To check that is injective we will verify that $p'^{-1}(K_\pm^+) \subset K_\pm$.

For this purpose consider diagram (2) with α' and β' as specified by hypotheses.

Lemma 6.2 *If $w \in W, w' \in W', v' \in V'$ such that $p(w) = w'$ and $\beta'(v') = w'$ then there exists $v \in V$ such that $\beta(v) = w$ and $p'(v) = v'$.*

Proof: We first choose \underline{v} with the property $p'(\underline{v}) = v'$, observe that $p(w - \beta(\underline{v})) = 0$, hence in view of the definition of the diagram (2) $w - \beta(\underline{v}) = \beta(u), u \in \ker \alpha$ and correct \bar{v} to v by taking $v = \underline{v} - u$.

q.e.d

With Lemma 6.2 established observe that given a sequence $v'_0, v'_1, \dots, v'_k \in V'$ and $v_0 \in V$ with the property that

$$\begin{aligned} \alpha'(v'_{i-1}) &= \beta'(v'_i), \quad 1 \leq i \leq k \\ p(v_0) &= v'_0 \end{aligned} \tag{23}$$

one can produce $v_1, v_2, \dots, v_k \in V$ such that

$$\begin{aligned} \alpha(v_{i-1}) &= \beta(v_i) \\ p(v_i) &= v'_i. \end{aligned} \tag{24}$$

Indeed suppose inductively that $v_1, v_2, \dots, v_i, i \leq r$ satisfying properties (24) are produced. Apply Lemma 6.2 to $w = \alpha(v_i), w' = \alpha'(v'_i)$ and $v' = v'_{r+1}$ and obtain v_{r+1} .

To conclude $p'^{-1}(K_+^+) \subset K_+$ one chooses the sequence $\{v'_i\}$ to have (for some k) $\alpha(v'_k) = 0$, which means that $v'_0 \in K_+^+$. Then v_k constructed as above is in $\ker \alpha$ which means that $v_0 \in K_+$.

To conclude $p'^{-1}(K_-^+) \subset K_-$ one chooses a sequence $\{v'_i\}$ to have (for some k) $v' = v'_k \in K_-^+$ and $v'_0 = 0$. Then for $v_0 = 0$ one construct the sequence $v_1, v_2, \dots, v_k \in V$. Then $v_k \in K_-$, hence $p'^{-1}(K_-^+) \subset K_-$.

q.e.d.

6.2 Appendix 2.

Recall that:

The Hilbert cube Q is the infinite product $Q = \prod_{i \in \mathbb{Z}_{\geq 0}} I_i = I^\infty$ with $I_i = I = [0, 1]$. The topology of Q is given by the metric $d(u, v) = \sum_i |u_i - v_i|/2^i$ with $u = \{u_i \in I, i \in \mathbb{Z}_{\geq 0}\}$ and $v = \{v_i \in I, i \in \mathbb{Z}_{\geq 0}\}$.

A compact Hilbert cube manifold is a compact Hausdorff space locally homeomorphic to the Hilbert cube.

The space Q is a compact ANR and so is $X \times Q$ when X is a compact ANR.

For any n , positive integer, write $Q = I^n \times Q'_n$ and denote by $\pi_n : Q \rightarrow I^n$ the first factor projection and by $\pi_n^X : X \times Q \rightarrow X \times I^n$ the product $\pi_n^X = id_X \times \pi_n$.

For $F : X \times Q \rightarrow \mathbb{R}$ let F_n be the restriction of F to $X \times I^n$ and \bar{F}_n the composition $\bar{F}_n := F_n \cdot \pi_n^X$.

For $f : X \rightarrow \mathbb{R}$ denote by $\bar{f} := f \cdot \pi_X$ where $\pi_X : X \times Q \rightarrow X$ is the canonical projection on X . Note that

Observation 6.3

1. If $f : X \rightarrow \mathbb{R}$ is a tame map so is \bar{f} .
2. The sequence of maps \bar{F}_n is uniformly convergent to the map F .

The following are two results about Hilbert cube manifolds whose proof can be found in [8].

Theorem 6.4

1. (R Edwards) If X is a compact ANR then $X \times Q$ is a Hilbert cube manifold.
2. (T Chapman) If $\omega : X \rightarrow Y$ is a homotopy of equivalence between two finite simplicial complexes with Whitehead torsion $\tau(\omega) = 0$ then there exists a homeomorphism $\omega' : X \times Q \rightarrow Y \times Q$ such that ω' and $\omega \times id_Q$ are homotopic.

Recall (for the non expert reader) that for a homotopy equivalence $f : X \rightarrow Y$ between two finite simplicial complexes one can associate an element $\tau(f) \in Wh(\pi_1(X, x))$ which measures the obstruction to f to be a "simple homotopy equivalence" in the sense of J.H. Whitehead. Here $Wh(\Gamma)$ denotes the Whitehead group of Γ , which is an abelian group associated with a discrete group Γ , cf [17]. It is also known [17] that if K is a finite cell complex (actually a compact ANR) with $\chi(K) = 0$ then $\tau(f \times Id_K) = 0$ in particular $\tau(f \times Id_{S^1}) = 0$. In view of theorem above Chapman has extended $\tau(f)$ to a homotopy equivalence of compact Hilbert manifolds.

Proof of Stabilization theorem:

Items 1. and 2. in Stabilization theorem follow from item 1. respectively item 2. combined with item 3. in Theorem 6.4.

One has also the following result whose proof was provided by S. Ferry:

Proposition 6.5 *A compact Hilbert cube manifold is a " good ANR ".*

Proof: Let M be a Hilbert cube manifold and $F : M \rightarrow \mathbb{R}$ a continuous map. We want to show that for $\epsilon > 0$ one can produce a tame map $P : M \rightarrow \mathbb{R}$ such that $|F(u) - P(u)| < \epsilon$ for any $u \in M$. For this purpose write $M = K \times Q$, K a finite simplicial complex, cf [8] section 11.

It suffices to produce an n and a simplicial map $p : K \times I^n \rightarrow \mathbb{R}$ such that $|F - p \cdot \pi_n^K| < \epsilon$.

The continuity of F and the compactness of M insure the existence of $\delta > 0$ such that $|u - v| < \delta$ implies $|F(u) - F(v)| < \epsilon/2$, $u, v \in M = K \times Q$.

Choose n such that $|u - (\pi_n^K(u), 0)| < \delta$, $u \in K \times Q$ (here $(\pi_n^K(u), 0) \in (K \times I^n) \times Q'_n = K \times Q$).

Choose $p : K \times I^n \rightarrow \mathbb{R}$ a simplicial map (with respect to a convenient subdivision) with $|p(x) - F_n(x)| < \epsilon/2$, $x \in K \times I^n$, and take $P = p \cdot \pi_n^K$. Since p is tame so is P .

Then $|F(u) - p \cdot \pi_n^K(u)| \leq |F(u) - F_n \cdot \pi_n^K(u)| + |F_n \cdot \pi_n^K(u) - p \cdot \pi_n^K(u)| < \epsilon$. q.e.d.

6.3 Appendix 3

Recall from [2] sections 4 and 5 the following notations:

– The oriented graph G_{2m} has vertices x_1, x_2, \dots, x_{2m} and the oriented edges $a_i : x_{2i-1} \rightarrow x_{2i}$, $b_i : x_{2i+1} \rightarrow x_{2i}$ with $x_{2m+1} = x_1, i = 1, \dots, m$.

– A G_{2m} -representation ρ is given by a collection of linear maps $\alpha_i : V_{2i-1} \rightarrow V_{2i}, \beta_i : V_{2i+1} \rightarrow V_{2i}$ with V_i vector space corresponding to the vertex x_i , and the linear map α_i resp. β_i corresponding to the arrow a_i resp. b_i .

–For $f : X \rightarrow \mathbb{S}^1$ a tame map in the sense of [2] with m critical angles $0 < s_1 < s_2 < \dots < s_m \leq 2\pi$ and t_1, t_2, \dots, t_m regular values such that $0 < t_1 < s_1 < t_2 < \dots < s_{m-1} < t_m < s_m$ one associates the G_{2m} –representation ρ_r with $V_{2i-1} = H_r(\tilde{f}^{-1}(t_i))$, $V_{2i} = H_r(f^{-1}(s_i))$ and α_i^r, β_i^r the linear maps induced in homology by the continuous maps a_i and b_i , considered in [2] Section 5.

Let $\tilde{f} : \tilde{X} \rightarrow \mathbb{R}$ be the canonical infinite cyclic cover of the tame map $f : X \rightarrow \mathbb{S}^1$. Put $t_{m+1} = t_1 + 2\pi$ and observe that $V_{2i} = H_r(\tilde{f}^{-1}(t_t, t_{i+1}))$ and the relation $R_{t_i, t_{i+1}}^{\tilde{f}, \tilde{f}}(r)$ is exactly

$$R_{t_i, t_{i+1}}^{\tilde{f}, \tilde{f}}(r) = R(\alpha_i^r, \beta_i^r) = R(\beta_i^r)^\dagger \cdot R(\alpha_i^r).$$

Clearly the composition

$$R_{t_m, t_{m+1}}^{\tilde{f}, \tilde{f}}(r) \cdot R_{t_{m-1}, t_m}^{\tilde{f}, \tilde{f}}(r) \cdots R_{t_2, t_3}^{\tilde{f}, \tilde{f}}(r) \cdot R_{t_1, t_2}^{\tilde{f}, \tilde{f}}(r)$$

identifies to $R_{t_1}^f(r)$.

To a G_{2m} –representation ρ one associates the linear relation $R(\rho) : V_1 \rightsquigarrow V_1 = R^\dagger(\beta_m) \cdot R(\alpha_m) \cdots R^\dagger(\beta_1) \cdot R(\alpha_1)$ and one denotes by $\mathbb{J}(\rho) := \mathcal{J}([R(\rho)_{\text{reg}}])$. Clearly one has:

Observation 6.6

1. $R(\rho \oplus \rho') = R(\rho) \oplus R(\rho')$ and therefore $\mathbb{J}(\rho) \sqcup \mathbb{J}(\rho')$,
2. $\mathbb{J}(\rho^I) = \emptyset$,
3. $\mathbb{J}(\rho^I(\lambda, k)) = \{(\lambda, k)\}$.

The Jordan cells $(R_{t_1}^f)_{\text{reg}}(r)$ are the Jordan cells of the r –monodromy $T^{(X, \xi_f)}(r)$ and then, by Observation 6.6, they are the Jordan cells of the representation ρ_r defined in [2].

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