A DESCRIPTION OF BAER–SUZUKI TYPE OF THE SOLVABLE RADICAL OF A FINITE GROUP

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Abstract. We obtain the following characterization of the solvable radical $\mathfrak{R}(G)$ of any finite group G: $\mathfrak{R}(G)$ coincides with the collection of $g \in G$ such that for any 3 elements $a_1, a_2, a_3 \in G$ the subgroup generated by the elements $g, a_i g a_i^{-1}$, $i = 1, 2, 3$, is solvable. In particular, this means that a finite group G is solvable if and only if in each conjugacy class of G every 4 elements generate a solvable subgroup.

Contents

1. Introduction

1.1. **Main results.** Our goal is to prove

Theorem 1.1. *The solvable radical of a finite group G coincides with the collection of $g \in G$ satisfying the property: for any 3 elements $a, b, c \in G$ the subgroup generated by the conjugates $g, aga^{-1}, bgb^{-1}, cgc^{-1}$ is solvable.*

This statement may be viewed as a theorem of Baer–Suzuki type with respect to the solvability property, in light of

Theorem 1.2. *(Baer–Suzuki) The nilpotent radical of a finite group G coincides with the collection of $g \in G$ satisfying the property: for any $a \in G$ the subgroup generated by g, aga^{-1} is nilpotent.*

Theorem 1.1 implies

Corollary 1.3. *A finite group G is solvable if and only if in each conjugacy class of G every four elements generate a solvable subgroup.*

Remark 1.4. These characterizations are the best possible: in the symmetric groups S_n ($n \geq 5$) any triple of transpositions generates a solvable subgroup.

Definition 1.5. Let $k \geq 2$ be an integer. We say that $g \in G$ is a *k-radical element* if for any $a_1, \ldots, a_k \in G$ the subgroup $H = \langle a_1 g {a_1}^{-1}, \ldots, a_k g {a_k}^{-1} \rangle$ is solvable.

Recall that a finite group G is called almost simple if it contains a unique normal simple group L such that $L \leq G \leq \mathrm{Aut}\,(L)$.

The main step in our proof of Theorem 1.1 is

Theorem 1.6. *Let G be a* **finite** *almost simple group. Then G does not contain nontrivial 4-radical elements.*

The statement of Theorem 1.1 remains true for arbitrary linear groups.

Theorem 1.7. *The solvable radical of a linear group G coincides with the collection of $g \in G$ satisfying the property: for any 3 elements $a, b, c \in G$ the subgroup generated by the conjugates $g, aga^{-1}, bgb^{-1}, cgc^{-1}$ is solvable.*

Proof. The passage from Theorem 1.1 to Theorem 1.7 is quite standard, cf. [GKPS]. For the sake of completeness we give it below.

First of all, every element of the radical $\mathfrak{R}(G)$ is a k-radical element for any k since $\mathfrak{R}(G)$ is a characteristic subgroup of G.

We shall prove the opposite inclusion, i.e. the set $S(G)$ of all 4-radical elements is contained in $\mathfrak{R}(G)$. Let H be the subgroup generated by $S(G)$. It is enough to show that H is solvable. Take a finitely generated subgroup $H_n = \langle a_1, \dots, a_n \rangle$, where $a_i \in S(G)$, $i = 1, \dots, n$. It is well known that any finitely generated subgroup of a linear group is residually finite [Mal]. Therefore H_n can be embedded into a cartesian product D of finite groups G_j, each of those is generated by 4-radical elements and is thus solvable by Theorem 1.1. Moreover, the solvability class of G_j is bounded by the rank of the linear group G. Since the class of solvable groups of fixed solvability class is closed under cartesian products, we conclude that D is solvable, hence so is H_n. We now observe that every finitely generated subgroup of H lies in some H_n and is thus solvable. This means that H is locally solvable. It remains to apply a theorem of Zassenhaus [Za] saying that any locally solvable linear group is solvable. □

Our main results can be restated is follows.

Definition 1.8. Let G be a finite nonsolvable group, and let $x \in G \setminus \mathfrak{R}(G)$. We define $\beta_G(x)$ as the smallest integer ℓ such that the conjugacy class of x contains ℓ elements generating a nonsolvable subgroup of G.

We shall often drop the subscript G.

Definition 1.9. Let G be a finite nonsolvable group. We define

$$BS(G) := \max_{x \in G \setminus \mathfrak{R}(G)} \beta(x).$$

We call this number *the Baer–Suzuki width* of G.

With this terminology, our Theorem 1.6 says that the Baer–Suzuki width of any finite almost simple group is at most 4.

Definition 1.8 should be compared with

Definition 1.10. [GS] Let G be a finite almost simple group, with $L = F^*(G)$ the unique minimal normal subgroup of G, and let $x \in G$ be a non-identity element. Then $\alpha(x)$ is defined as the minimal number of L-conjugates of x which generate the group $\langle L, x\rangle$.

Clearly, if G is a finite almost simple group and $1 \neq x \in G$, we have $\beta(x) \leq \alpha(x)$.

Another obvious remark (which will, however, be important for induction arguments) is that if H is a subgroup of G and $x \in H$, then $\beta_G(x) \leq \beta_H(x)$.

1.2. **Historical perspective, analogues and generalizations.** The whole story goes back to a pioneering paper by R. Baer [Ba] whose influence on the present article is two-fold. First, basing on a theorem of Zorn [Zo] characterizing the class of finite nilpotent groups in terms of the Engel identities, Baer obtained a description of the nilpotent radical $\mathfrak{N}(G)$ of a finite group G as the collection of the Engel elements of G. This description gave rise to an attempt to use recent characterizations of finite solvable groups in terms of explicit identities in two variables [BGGKPP1], [BGGKPP2], [BWW] for getting a similar explicit description of the solvable radical $\mathfrak{R}(G)$ [BBGKP, Conjecture 2.12]. On the other hand, the same theorem of Baer yielded another description of the nilpotent radical which, for convenience, we reformulated above as Theorem 1.2. This assertion admits many equivalent reformulations some of which are commonly known as the Baer–Suzuki theorem (a few years after the paper [Ba] appeared, Suzuki discovered a new proof of this result [Su] which played an important role in structure theory of finite groups; a very short proof was later found in [AL]). Numerous analogues and generalizations of this result are known, both in the context of finite [GR] and infinite [As], [So], [Mam] groups. Although a direct analogue of this statement for finite solvable groups cannot hold (say, because two involutions generate a dihedral group which is solvable), Flavell proved that there is an absolute constant k with the property: $\mathfrak{R}(G)$ coincides with the collection of $y \in G$ such that any k conjugates of y generate a solvable subgroup; moreover, one can choose $k = 10$ [Fl]. (Note that his proofs do not use the classification of finite simple groups.) In [GGKP1], [GGKP2] we improved on Flavell's theorem, proving that one can choose $k = 8$, and stated a conjecture that one can choose $k = 4$ (which is certainly sharp). Our proof went through yet another description of $\mathfrak{R}(G)$ in terms of commutators and heavily relied upon the classification of finite simple groups (see the above cited papers for details). In the present paper we prove this conjecture (Theorem 1.1).[1]

Let us note another result which is more close to the original Baer–Suzuki theorem. Restrict ourselves to considering elements of prime order greater than 3. For such an element x one can prove a stronger statement:

Theorem 1.11. *Let G be a finite group. An element x of prime order $p > 3$ belongs to $\mathfrak{R}(G)$ if and only if for any $y \in G$ the subgroup $\langle x, yxy^{-1}\rangle$ is solvable.*

As above, it is enough to prove that for any element x of prime order $p > 3$ in an almost simple group G we have $\beta(x) = 2$. The proof is given in [GGKP3].

We dare formulate a stronger conjecture:

Conjecture 1.12. *Let G be a finite group. An element x of order $n \geq 4$ belongs to $\mathfrak{R}(G)$ if and only if for any $y \in G$ the subgroup $\langle x, yxy^{-1}\rangle$ is solvable.*

The case $n = 4$ looks a bit suspicious, being close to the case of involutions, but nevertheless we hope it does not break the general picture.

Let us note here another parallel between the nilpotent and the solvable cases. Namely, there is yet another description of $\mathfrak{R}(G)$ [GKPS] in the style of a theorem of J. Thompson [Th]: $\mathfrak{R}(G)$ coincides with the collection of $y \in G$ such that for every $x \in G$ the subgroup $\langle x, y\rangle$ is solvable. In such a form this statement does not admit a direct analogue in the nilpotent case. However, one can reformulate this description as follows. For any $x, y \in G$ denote by $\left\langle y^{\langle x\rangle}\right\rangle$ the minimal

[1]R. Guralnick informed us that this statement, as well as Theorem 1.11, was independently proved in his unpublished joint work with P. Flavell and S. Guest [FGG]. We shall present the proof of Theorem 1.11 in [GGKP3].

normal subgroup in $\langle x, y\rangle$ containing y. Then $\mathfrak{R}(G)$ can be described as the collection of $y \in G$ such that for every $x \in G$ the subgroup $\left\langle y^{\langle x\rangle}\right\rangle$ is solvable. In this form, a direct analogue holds in the nilpotent case:

Proposition 1.13. *Let G be a finite group. The nilpotent radical $\mathfrak{N}(G)$ of G coincides with the collection of all $y \in G$ such that for any $x \in G$ the subgroup $\left\langle y^{\langle x\rangle}\right\rangle$ is nilpotent.*

Proof. Let $y \in \mathfrak{N}(G)$. Take an arbitrary $x \in G$ and consider $H = \mathfrak{N}(G) \cap \langle x, y\rangle$. We have $H \leq \mathfrak{N}(G)$, so H is nilpotent. On the other hand, H is a normal subgroup in $\langle x, y\rangle$ and $y \in H$. Since $\left\langle y^{\langle x\rangle}\right\rangle$ is the minimal normal subgroup containing y, we have $\left\langle y^{\langle x\rangle}\right\rangle \leq H$. Since H is nilpotent, $\left\langle y^{\langle x\rangle}\right\rangle$ is nilpotent too.

Conversely, suppose that y has the property that the subgroup $\left\langle y^{\langle x\rangle}\right\rangle$ is nilpotent for any $x \in G$. Evidently, for any $x \in G$ the commutator $[x, y]$ belongs to $\left\langle y^{\langle x\rangle}\right\rangle$. Since $\left\langle y^{\langle x\rangle}\right\rangle$ is nilpotent, the Engel series $[[x, y], y, \ldots, y]$ terminates at 1. Thus y is an Engel element and therefore, according to the above mentioned theorem of Baer, belongs to $\mathfrak{N}(G)$. □

The Baer–Suzuki theorem allows one to improve this characterization in the best possible way: instead of considering the subgroup $\left\langle y^{\langle x\rangle}\right\rangle$, it is enough to consider the subgroup $\langle y, y^x\rangle$ because its nilpotency for any $x \in G$ already guarantees $y \in \mathfrak{N}(G)$.

Finally, in light of the approach in a recent paper [GPS], we dare propose a further generalization, in style of problems of Burnside type.

Recall that a class of groups $\mathcal{X}$ is called a radical class if in every group G there is a maximal normal subgroup $\mathfrak{X}(G)$ belonging to $\mathcal{X}$. One can impose various conditions on $\mathcal{X}$ which guarantee the existence of $\mathfrak{X}(G)$. For example, a class $\mathcal{X}$ of finite groups closed under homomorphic images, normal subgroups and extensions is a radical class inside the class of all finite groups.

Definition 1.14. Let $\mathcal{X}$ be a radical class of finite groups. The Baer–Suzuki width of $\mathcal{X}$ is defined as the smallest integer $n := BS(\mathcal{X})$ with the property: for every finite group $G \in \mathcal{X}$, the $\mathcal{X}$-radical $\mathfrak{X}(G)$ coincides with the set of elements $g \in G$ such that for every $x_1, \ldots, x_n \in G$ the subgroup $\langle g^{x_1}, \ldots, g^{x_n}\rangle$ belongs to $\mathcal{X}$. If such an n does not exist, we set $BS(\mathcal{X}) := \infty$.

We have $BS(\mathcal{N}) = 2$ for $\mathcal{N}$ the class of finite nilpotent groups (Baer–Suzuki) and $BS(\mathcal{S}) = 4$ for $\mathcal{S}$ the class of finite solvable groups (Theorem 1.1).

Problem 1.15. Study other radical classes of finite groups for which $BS(\mathcal{X}) < \infty$.

1.3. **Notation and conventions.** Whenever possible, we maintain the notation of [GGKP2] which mainly follows [St], [Ca1], [Ca2]. In particular, we adopt the notation of [Ca2] for twisted forms of Chevalley groups (so unitary groups are denoted by $PSU_n(q^2)$ and not by $PSU_n(q)$). However, the classification of outer automorphisms follows [GLS, p. 60], [GL, p. 78]. In order to avoid misunderstandings we recall this classification. Let us call the subdivision of automorphisms of Chevalley groups into inner, diagonal, field, and graph automorphisms in the sense of [St], [Ca1], the usual one.

In the classification of finite simple groups a slightly different subdivision of automorphisms is used. Let G be an adjoint Chevalley group, untwisted or twisted (the cases where G is a Suzuki or a Ree group are treated separately). Denote by $\mathrm{Aut}\,(G)$ the group of automorphisms of G. Then ([GLS, Definition 2.5.13]):

1. *Inner-diagonal automorphisms* coincide with usual inner-diagonal automorphisms.

2. *Field automorphisms* are as follows:

2.1. If G is untwisted, then a "field" automorphism is an $\mathrm{Aut}\,(G)$-conjugate of a usual field automorphism.

2.2. If $G = {}^dG$ is a twisted group, then a "field" automorphism is an $\mathrm{Aut}\,(G)$-conjugate of a usual field automorphism of order relatively prime to d.

2.3. If G is a Suzuki or a Ree group, then a "field" automorphism is an $\mathrm{Aut}\,(G)$-conjugate of a usual field automorphism.

3. *Graph automorphisms* are as follows:

3.1. If G is untwisted, then a "graph" automorphism is an $\mathrm{Aut}\,(G)$-conjugate of a graph-inner-diagonal usual automorphism with nontrivial graph part, except for the cases B_2, F_4, G_2 with the characteristics of the ground field $p = 2, 2, 3$, respectively, in which cases there are no "graph" automorphisms.

3.2. If $G = {}^dG$ is a twisted group, then a "graph" automorphism is an element of $\mathrm{Aut}\,(G)$ whose image modulo the group of inner-diagonal automorphisms has order divisible by d.

3.3. If G is a Suzuki or a Ree group, then there are no graph automorphisms.

4. *Graph-field automorphisms* are as follows:

4.1. If G is untwisted, then a "graph-field automorphism" is an $\mathrm{Aut}\,(G)$-conjugate of a usual graph-field automorphism where both components are nontrivial, except for the cases B_2, F_4, G_2 with the characteristics of the ground field $p = 2, 2, 3$, respectively, in which cases all conjugates of usual graph-field automorphisms with nontrivial graph part are considered as "graph-field" automorphisms.

4.2. If $G = {}^dG$ is a twisted group, then there are no graph-field automorphisms.

4.3. If G is a Suzuki or a Ree group, then there are no graph-field automorphisms.

In particular, in this sense a "graph" automorphism may be a composition of an automorphism of the Dynkin diagram with an inner-diagonal automorphism, or (in the case of a twisted form ${}^dL$ of a simple group L) a field automorphism of order divisible by d.

We also use some other conventions from [GLS, pp. 410–413] without special notice.

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2. Strategy of proof

Actually, the proof grounds on a further refinement of methods and results from [GGKP2] and [GS].

We first reduce Theorem 1.1 to Theorem 1.6, exactly in the same way as in [GGKP2, Section 2].

Although this reduction is fairly standard, we sketch its main steps below. Let $S(G)$ be the set of all 4-radical elements of the group G. Obviously, $\mathfrak{R}(G)$ lies in $S(G)$ and we have to prove the opposite inclusion. We can assume that G is semisimple (i.e., $\mathfrak{R}(G)=1$), and we shall prove that G does not contain nontrivial 4-radical elements. Assume the contrary and consider a minimal counterexample, i.e. a semisimple group of smallest order with $S(G)\neq\{1\}$.

Recall that any finite semisimple group G contains a unique maximal normal centreless completely reducible (CR) subgroup (by definition, CR means a direct product of finite non-abelian simple groups) called the CR-radical of G (see [Ro, 3.3.16]). We call a product of the isomorphic factors in the decomposition of the CR-radical an isotypic component of G. Denote the CR-radical of G by V. This is a characteristic subgroup of G.

Since G is minimal, it has only one isotypic component. Any $g\in G$ acts as an automorphism $\tilde{g}$ on $V=H_1\times\cdots\times H_n$, where all H_i, $1\le i\le n$, are isomorphic nonabelian simple groups.

Suppose that $g\neq 1$ is a 4-radical element. The next step shows that g cannot act on V as a non-identity element of the symmetric group S_n.

Denote by σ the element of S_n corresponding to $\tilde{g}$.

By definition, the subgroup $\Gamma=\langle g, x_igx_i^{-1}\rangle$, $i=1,\ldots,4$, is solvable for any $x_i\in G$. Evidently, the subgroup $\langle [g,x_1],[g,x_2]\rangle$ lies in Γ.

Suppose $\sigma\neq 1$, and so $\sigma(k)\neq k$ for some $k\le n$. Take $\bar{x}_1$ and $\bar{x}_2$ of the form $\bar{x}_i=(1,\ldots,x_i^{(k)},\ldots,1)$, where $x_i^{(k)}\neq 1$ lies in H_k $(i=1,2)$. Then we may assume $(\bar{x}_i)^\sigma=(x_i^{(k)},1,\ldots,1)$, and so $[g,\bar{x}_i]=(\bar{x}_i)^\sigma\bar{x}_i^{-1}=(x_i^{(k)},1,\ldots,(x_i^{(k)})^{-1},\ldots,1)$.

As H_k is simple, it is generated by two elements, say a and b. On setting $x_1^{(k)}=a$, $x_2^{(k)}=b$, we conclude that the group generated by $[g,\bar{x}_1]$ and $[g,\bar{x}_2]$ cannot be solvable because the first components of these elements, a and b, generate the simple group H_k. Contradiction with solvability of Γ.

So we can assume that a nontrivial 4-radical element $g \in G$ acts as an automorphism of a simple group H. Then we consider the extension of the group H with the automorphism $\tilde{g}$. Denote this almost simple group by G_1. By Theorem 1.6, G_1 contains no nontrivial 4-radical elements. Contradiction with the choice of $\tilde{g}$.

Let G be an almost simple group, $L \leq G \leq \mathrm{Aut}(L)$. If $G = L$ is simple, Theorem 1.6 is an immediate consequence of [GGKP2, Theorem 1.15]. Indeed, this Theorem states that for any $x \in L$ there exist 3 elements a, b, c such that the commutators $[x, a], [x, b], [x, c]$ generate a nonsolvable subgroup. Hence the subgroup $\langle x, x^a, x^b, x^c \rangle$ is nonsolvable too. Thus we only have to consider outer automorphisms x of L. The case where L is an alternating group is straightforward (Section 3). If L is a group of Lie type, we consider the separate cases where x is an inner-diagonal (Section 5), field (Section 6), graph, or graph-field automorphism (Section 7). The first case was treated in [GGKP2] (see the discussion at the end of Section 4 of this paper for groups of small Lie rank), so we only need to complete the induction arguments. Field, graph, and graph-field automorphisms are treated using their classification. Here we mainly follow the approach of [GS], as we do when considering the groups of small Lie rank as the base of induction in Section 4. The remaining case of sporadic groups is treated in Section 8.

3. Alternating groups

Theorem 3.1. *Let $L = A_n$ be the alternating group on n letters, $n \geq 5$, and let $L \leq G \leq Aut(L)$. Then $BS(G) \leq 4$.*

Proof. We first exclude the group $G = A_6$ since this is the only non-abelian simple alternating group for which the group of outer automorphisms $\mathrm{Out}(G)$ is equal not to $\mathbb{Z}_2$ but to $\mathbb{Z}_2 \times \mathbb{Z}_2$. In the notation of [CCNPW] we have $A_6 \leq G \leq \mathrm{Aut}(G)$ for: $G = S_6 = A_6 : 2a$, $G = PGL_2(9) = A_6 : 2b$, $G = M_{10} = A_6 : 2c$, and $G = \mathrm{Aut}(A_6) = A_6.2^2$, where a, b, c are the involutions in $\mathbb{Z}_2 \times \mathbb{Z}_2$. In all these cases the statement of the theorem is checked by a direct MAGMA computation. So we assume $n \neq 6$, and G is either A_n or $\mathrm{Aut}(A_n) = S_n$. For $G = A_n$ see [GGKP2]. If $G = S_n$ and x is an automorphism of prime order, we may assume that x is an involution. If x is a transposition, we have $\beta(x) = 4$, so the estimate in the statement of the theorem is sharp. For an arbitrary involution we proceed by induction. For $n \leq 6$ we establish the result by a direct computation. Let now $n > 6$. If x fixes at least one letter, we conclude by induction. If not, $n = 2m$ is even and x is conjugate to $y = (12)(34)(56)\dots(2m-1, 2m)$. Then we can find $a_1, \dots, a_4$, lying in the subgroup $S_6 < S_n$ fixing the last $n-6$ letters, such that the group generated by $a_i z {a_i}^{-1}$, $i = 1, \dots 4$ (where $z = (12)(34)(56)$), is not solvable. Hence the group generated by $a_i y {a_i}^{-1}$, $i = 1, \dots 4$, is nonsolvable too. □

4. Groups of Lie type of small rank

Theorem 4.1. *Let G be an almost simple group of Lie type of Lie rank at most 2. Then $BS(G) \leq 4$.*

Proof. For $x \in L$, the result immediately follows from [GGKP2, Theorem 1.11], so we only have to consider outer automorphisms. We follow very closely the arguments of [GS]. Since we do not pretend to make the estimate of $BS(G)$ sharp, in our case-by-case analysis we only have to consider those x for which the estimate $\alpha(x) \leq 4$ is not established in [GS].

Remark 4.2. For all almost simple groups of Lie type of Lie rank at most 2 over the fields with 2 or 3 elements the statement of Theorem 4.1 is checked by explicit MAGMA computations.

As usual, we may and shall assume that x is an element of prime order.

Groups of Lie rank 1.

In the case $L = PSL_2(q)$, $q \geq 4$, [GS, Lemma 3.1] shows that it is enough to consider a field automorphism x of order 2 of $PSL_2(9)$. In that case we have $\langle L, x\rangle = S_6$, and 4 conjugates of x generate S_5, so $\beta(x) = 4$. If $L = PSU_3(q^2)$, $q > 2$, the result follows from [GS, Lemma 3.3]. If L is a Suzuki or a Ree group, we have $\alpha(x) \leq 3$ by [GS, Prop. 5.8].

Groups of Lie rank 2.

The case $L = PSL_3(q)$ is established in [GS, Lemma 3.2].

Let now $L = PSp_4(q)$. Although [GS, Theorem 4.1(f)] does not provide the needed estimate, we can use the arguments *mutatis mutandis*. The cases $q = 2$ and $q = 3$ are treated by a direct computation, so assume $q > 3$. Let x be a field automorphism. Then x normalizes $SL_2(q)$. So, x is a field automorphism of $SL_2(q)$ and by [GS, Lemma 3.1] we have $\alpha(x) \leq 4$.

If x is an inner-diagonal automorphism, the proof literally follows [GGKP2] for the group $\langle L, x\rangle$, see also Section 5.

If x is an involutory graph-field automorphism, then $\alpha(x) \leq 4$ ([GS]) and we are done.

If $L = G_2(q)$, [GS, Theorem 5.1] gives only $\alpha(x) \leq 5$, so we have to analyze the arguments. The case $q = 2$ is treated directly, so assume $q > 2$. If x is a field automorphism, then again x normalizes $SL_2(q)$ and we are done.

If x is an involutory graph automorphism (which exists if $q = 3^a$ with a odd), then $\alpha(x) \leq 4$ (*ibid.*).

Let us now go over to twisted groups.

Let $L = PSU_4(q^2)$. In that case [GS, Lemma 3.4] gives the required estimate $\alpha(x) \leq 4$ for all x except for an involutory graph automorphism and a transvection for $q = 2$. The latter case is treated by a direct computation, so suppose we are in the first case.

Let first q be odd. Since the case $q = 3$ can be treated by a direct computation, assume $q > 3$. According to the classification of graph automorphisms (see [GLS, Table 4.5.1]), either x normalizes (and does not centralize) $SU_3(q^2)$ (and we can use the above considerations for the groups of Lie rank 1), or $C_L(x) = PSp_4(q)$. In the latter case the argument of [GS] yields $\alpha(x) \leq 6$, so we have to reconsider it. One can choose a conjugate of x acting on $S = SU_2(q^2) \circ SU_2(q^2)$ by interchanging the components. Let a, b denote a pair of generators of the first copy of $SU_2(q^2)$. Then the subgroup in $\langle S, x\rangle$ generated by two commutators $[x, a]$ and $[x, b]$ contains the first copy of $SU_2(q^2)$ and is thus nonsolvable. Hence the subgroup generated by x, axa^{-1}, bxb^{-1} is nonsolvable too.

If now q is even, then there are two classes of such automorphisms. In the first case x normalizes (but does not centralize) $SU_2(q^2)$, and we can use the result for groups of Lie rank 1 (because $q > 2$). In the second case, one can find a conjugate of x acting on S by interchanging the components, and the above argument works because $q > 2$.

The case $G = PSU_5(q^2)$ will be considered in Section 7, along with the groups of higher rank.

It remains to consider ${}^2F_4$ and ${}^3D_4$. In the first case, let us look at the arguments in the proof of [GS, Theorem 5.1]. Let first $L = {}^2F_4(q^2)'$. If $q = 2$, the estimate $\alpha(x) \leq 4$ is given in [GS, Prop. 5.5] (and may be confirmed by a straightforward computation), so assume $q > 2$. Since x is a field automorphism, it normalizes a parabolic subgroup P. We then arrive at the rank 1 case and can proceed as in Section 6 (or as in the beginning of the proof of [GGKP2, Theorem 7.1]).

Let now $L = {}^3D_4(q^3)$. A convenient account of its properties is presented in [FM, Section 3], see also [GL, 9-1], [Kl]. The group $L = {}^3D_4(q^3)$ possesses field and graph automorphisms. Since a field automorphism acts nontrivially on $SL_2(q^3)$, we have to consider only graph automorphisms. There are two classes of such automorphisms. Denote their representatives by g_1 and g_2, respectively [Kl]. For the first one, we have $C_L(g_1) \cong G_2(q)$, and there is a subgroup $L_1 = SL_2(q^3)$ of L on which g_1 acts as a field automorphism [LS, Lemma 5.3], so the result follows from Theorem 4.1. In the second case, $C_L(g_2) = PGL_3^{\pm}(q)$ [GLS, Table 4.7.3A], if $p \neq 3$, $q \equiv \pm 1 \pmod 3$. One can choose g_2 in the form $g_2 = tg_1$ where t is (the inner automorphism corresponding to) an element of order 3 lying in $C_T(g_1)$, T standing for a maximal torus in L [GL, p. 104]. According to [FM, Lemma 3.11(3)], we have $t \in L_1 = SL_2(q^3)$, so g_2 also normalizes and does not centralize L_1, and we are done by Theorem 4.1. If $p = 3$, then g_2 normalizes (and does not centralizes) a subgroup of type A_2. This case is considered above.

□

5. Inner-diagonal automorphisms

We shall use the same approach as in [GGKP2].

Let σ be a diagonal automorphism corresponding to the Borel subgroup $B = HU$ where H is a maximal split torus of G such that $\sigma(h) = h$ for every $h \in H$. Further, let $\tilde{H} = \langle \sigma, H \rangle$. Now replace the simple groups G with the group $\tilde{G} = \langle \sigma, G \rangle$. Note that the group $\tilde{G}$

has the "Borel subgroup" $\tilde{B} = \tilde{H}U$ with the similar properties as for the group G (for instance, the Bruhat decomposition).

Let x be an inner-diagonal automorphism of G. Then we may regard x as an element of $\tilde{G}$. One can easily check that the arguments of [GGKP2] used in the case of an inner automorphism of a simple group also hold for the case of the group $\tilde{G}$. Thus, we get our statement in the same way as in Theorem 1.11 of [GGKP2].

6. Field automorphisms

Let $|q| > 3$. Since x evidently normalizes but does not centralize a rank 2 group, the result follows from Theorem 4.1.

Let $|q| = 2$ or $|q| = 3$. We choose an appropriate rank 2 or rank 1 group normalized by x. The result follows from explicit MAGMA computations.

7. Graph and graph-field automorphisms

Theorem 7.1. *Let L be a finite simple group of Lie type, and let x be a graph or graph-field automorphism of L of prime order. Then $\beta(x) \leq 4$.*

Proof. As in Section 4, we closely follow [GS].

7.1. **Linear groups.** Let $L = PSL_n(q)$, $n \geq 4$. The graph and graph-field automorphisms of prime order were classified in [AS, §19] and [Li, 3.7]. As in [GS, p. 535], we shall use the matrix description given in [LS, pp. 285–286]. They are all of order 2. We denote by τ the map sending a matrix to its inverse-transpose. If n is odd, there is only one conjugacy class of graph automorphisms represented by τ. If n is even and q is odd, there are 3 classes represented by τJ, τJ^+, and τJ^-, where

$$J = \begin{pmatrix} 0 & -1 & & & & & \\ 1 & 0 & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & 0 & -1 \\ & & & & & 1 & 0 \end{pmatrix}, \quad J^+ = \begin{pmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{pmatrix},$$

$$J^- = \begin{pmatrix} 0 & -1 & & & & & & & \\ 1 & 0 & & & & & & & \\ & & \cdot & & & & & & \\ & & & \cdot & & & & & \\ & & & & \cdot & & & & \\ & & & & & 0 & -1 & & \\ & & & & & 1 & 0 & & \\ & & & & & & & \mu & \\ & & & & & & & & 1 \end{pmatrix} \quad \text{(where } -\mu/2 \text{ is non-square).}$$

Their centralizers are of type $PSp_n(q)$, $PSO_n^+(q)$, and $PSO_n^-(q)$, respectively [GLS, Table 4.5.1]. If n and q are even, there are two classes represented by τJ and τJu, where

$$u = \begin{pmatrix} 1 & 1 & & & & \\ 0 & 1 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & 1 \end{pmatrix}.$$

Their centralizers are of type $PSp_n(q)$ and $C_{PSp_n}(q)(t)$ (where t stands for a transvection in $PSp_n(q)$), respectively [AS, (19.9)]. If x is a graph-field automorphism, then $q = q_0^2$ is a square, and x can be represented as $x = \tau\phi$ where ϕ is a field involution [LS, *loc. cit.*].

In all cases, x leaves invariant the subgroup fixing the decomposition $V = A \oplus B$ where A has codimension 1 or 2 in V (cf. [GS, p. 535]), i.e. $SL_{n-1}(q)$ or $SL_{n-2}(q)$ (for the graph automorphisms) or $SL_{n-1}(q_0)$ (for the graph-field automorphism), therefore we can use induction. Indeed, for $q > 3$ the result immediately follows from Theorem 4.1, and for $q = 2, 3$ from straightforward computations with the groups $PSL_4(2)$, $PSL_5(2)$, $PSL_3(3)$ and $PSL_4(3)$.

7.2. **Unitary groups.** Let $L = PSU_n(q^2)$, $n \geq 5$. In this case, there are no graph-field automorphisms. As in the previous subsection, we use the classification of graph automorphisms of prime order [AS, §19], [Li, 3.7]. If n is odd, such an automorphism is unique (up to conjugation), and we can represent it by a field involution. Such an involution normalizes $SU_{n-1}(q^2)$ (cf. [LS, p. 288], [GS, p. 536]), and we proceed by induction or use Section 6. Let now $n = 2m$ be even. If q is odd, there are 3 classes of graph automorphisms, with centralizers of type $PSp_n(q)$, $PSO_n^+(q)$, and $PSO_n^-(q)$, respectively [GLS, Table 4.5.1]. We analyze these cases following [GS, pp. 536–537]. In the first case, arguing as in the proof of Theorem 4.1, we can choose a conjugate of x acting on $S = SU_m(q^2) \circ SU_m(q^2)$ by interchanging the components. Choose a pair of elements (c, d) generating the first component. Then the subgroup generated in $\langle S, x\rangle$ by two commutators $[x, c]$, $[x, d]$ contains (the first copy of) $SU_m(q)$ and thus cannot be solvable (because $m \geq 3$). Therefore the group generated by x, cxc^{-1} and dxd^{-1} is non-solvable too, and $\beta(x) = 3$. In the remaining two cases, x normalizes (but does not centralize) $SU_{n-1}(q^2)$, and we proceed by induction. If q is even, there are two conjugacy classes of graph automorphisms, one of which normalizes but does not centralize $SU_{n-1}(q^2)$, and the other acts on $S = SU_m(q^2) \circ SU_m(q^2)$ by interchanging the components, so we argue as in the odd case.

(An alternative induction argument uses the case analysis of [LS, Lemma 3.14].)

7.3. **Symplectic groups.** If $n > 2$, then there are no graph or graph-field automorphisms with the single exception $L = PSp_4(q)$. This group was already treated in Theorem 4.1.

7.4. **Orthogonal groups.** In this case a graph automorphism of order 2 is conjugate to an inner-diagonal automorphism [LS, p. 287], [LLS, p. 399]. More graph automorphisms only exist for $L = PSO_8^+(q)$. Here x is of order 3, and there are two possibilities [GS, Lemma 3.15], [GS, p. 541]: either x normalizes but does not centralize $G_2(q)$ (and we can apply Theorem 4.1), or x is conjugate to the standard triality. In the latter case it is shown in [GS, *loc. cit.*] that there exists a conjugate of $G_2(q)$ normalized but not centralized by x, and we are done.

Let now x be a graph-field automorphism, so $L = PSO_n^+(q)$, $q = q_0^2$. If x is of order 2, then there is a unique class of such an involution which normalizes but does not centralize $O_{n-1}(q_0)$ [GS, p. 541]. If x is of order 3, then $n = 4$, and we proceed exactly as for graph automorphisms.

7.5. **Exceptional groups.** Having Theorem 4.1 at our disposal, we may assume the Lie rank of L is greater than 2. If $L = F_4(q)$, then there is a unique (up to a conjugation) automorphism x of order 2, and in this case $q = 2^a$, a is odd, $C_L(x) = {}^2F_4(q^2)$. This x is conjugate to some element acting as an inner involution of ${}^2F_4(q^2)$ [GS, Prop. 5.5]. We finish by applying Theorem 4.1.

If $L = E_6(q)$ or ${}^2E_6(q)$, then x normalizes but does not centralize some subgroup of type $F_4(q)$ [GS, Prop. 5.2, 5.3], and we are reduced to the above considered case.

For all other groups, there are no graph or graph-field automorphisms.

Theorem 7.1 is proved. □

Remark 7.2. In the cases of field and graph-field automorphisms one can produce an alternative induction proof based on a recent theorem of Nikolov [Ni] which implies that any such automorphism normalizes a quasisimple subgroup of type A_n defined over some subfield of the ground field.

8. Sporadic groups

Since the simple groups were treated in [GGKP2], we only have to consider the almost simple sporadic groups. Of 26 sporadic groups, only 12 have the nontrivial automorphism group (of order 2): M_{12}, M_{22}, HS, J_2, McL, Suz, He, HN, Fi_{22}, Fi_{24}', $O'N$, J_3. Those having only one conjugacy class of outer involutions x, are very easy to treat: indeed, a simple look at the lists of maximal subgroups of L and $G = \mathrm{Aut}\,(L) = L : 2$ gives an almost simple subgroup $H < L$ normalized but not centralized by x. There are 7 such cases: 1) $L = M_{12}$, $H = PSL_2(11)$; 2) $L = He$, $H = PSp_4(4)$; 3) $L = J_2$, $H = PSU_3(3^2)$; 4) $L = McL$, $H = PSU_3(5^2)$; 5) $L = HN$, $H = A_{12}$; 6) $L = O'N$, $H = A_6$; 7) $L = J_3$, $H = PSL_2(16) : 2$.

In the cases M_{22}, HS and Suz, where there are two conjugacy classes of outer involutions, we use [GS, Proof of Lemma 7.6]: for any such involution x it is proved that $\alpha(x) \leq 4$. Hence $\beta(x) \leq 4$, as needed.

The group $G = Fi_{24}$ also has two nonconjugate outer involutions (with classes 2C and 2D in the notation of [CCNPW]). An involution from the class 2C is a 3-transposition and thus belongs to Fi_{23} (and also to $PSO_7(3)$) whereas a representative of 2D belongs to $PSO_8^+(3)$ [LW, Table 10.5], and we are done.

It remains to consider $G = Fi_{22} : 2$. This group has three conjugacy classes of outer involutions (2D, 2E, 2F in the notation of [CCNPW]). Consider a subgroup $H = G_2(3)$ in Fi_{22}. According to [Wi84, Table 4], there are 3 conjugacy classes of such subgroups, one normalized (but not centralized) by an outer automorphism and two others interchanged. Therefore for an outer automorphism normalizing H, the result follows from Theorem 4.1. According to [Mo], given one outer involution x, each of two others can be obtained from x by multiplying by an inner involution t commuting with it, so each of two other outer involutions also normalizes but does not centralize a subgroup of type $G_2(3)$ (note that $G_2(3)$ is not contained in the centralizer of any outer involution), and we are done.

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