Selberg zeta functions associated with theta multiplier systems of $SL_2(\mathbb{Z})$ and Jacobi forms

by

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Tsuneo Arakawa

§ 0. *Introduction.*

0.1. This is a continuation to our previous paper [Ar]. The purpose of the present paper is to define certain Selberg zeta functions associated with a theta multiplier system of $SL_2(\mathbb{Z})$ and to obtain the analytic continuations and certain functional equations for them. For that purpose we describe Selberg trace formula for certain spaces of automorphic forms of $SL_2(\mathbb{R})$ associated with the theta multiplier system. Those spaces of automorphic forms correspond to the spaces of Jacobi forms in some cases. As an application we can derive certain relations between the dimensions of the spaces of Jacobi forms of lower weights and the orders of the zeros at $s=3/4$ of our Selberg zeta functions.

0.2. To be more precise let m be a fixed positive integer. For each $r \in R=\mathbb{Z}/2m\mathbb{Z}$, denote by $\theta_r(\tau,z)$ the classical theta series given by (1.1). Let $\chi(M)$ ($M \in SL_2(\mathbb{Z})$) be the theta multiplier system given by (1.3), which plays a key role to describe the theta transformation formula (1.2) for $\theta_r(\tau,z)$. We define two Selberg zeta functions $Z_{\Gamma,m,+}(s)$, $Z_{\Gamma,m,-}(s)$ associated with the theta multiplier system χ

(for the precise definition see (2.3)). The zeta functions $Z_{\Gamma,m,\pm}(s)$ are a kind of Selberg zeta functions studied by Hejahl [He] and Fischer [Fi]. Our main result is that the Selberg zeta functions $Z_{\Gamma,m,\pm}(s)$ are analytically continued to meromorphic functions in the whole s-plane which satisfy certain functional equations under the substitution $s \longrightarrow 1-s$ (THEOREM 4.3). To obtain this result we describe the resolvent Selberg trace formula (THEOREM 3.1) involving the logarithmic derivatives of $Z_{\Gamma,m,\pm}(s)$ for certain spaces of automorphic forms associated with χ. For a positive integer ℓ, denote by $J_{\ell,m}$ (resp. $J^{*}_{\ell,m}$) the space of holomorphic Jacobi forms of Eichler -Zagier [E-Z] (resp. the space of skew-holomorphic Jacobi forms of Skoruppa [Sk2]) of index m and weight ℓ. Denote by $J^{cusp}_{\ell,m}$ and $J^{*cusp}_{\ell,m}$ the subspaces of cusp forms of $J_{\ell,m}$ and $J^{*}_{\ell,m}$, respectively. As an application of the trace formula we obtain the following relation between the dimensions of the spaces of Jacobi forms of lower weights and the order of the zero at s=3/4 of $Z_{\Gamma,m,\pm}(s)$:

$$
(0.1)\qquad
\begin{aligned}
\dim_{\mathbb{C}} J^{cusp}_{2,m} &= \mathrm{Ord}_{s=3/4}(Z_{\Gamma,m,+}(s)) + \lambda(m,2)\\
\dim_{\mathbb{C}} J^{*}_{1,m} &= \mathrm{Ord}_{s=3/4}(Z_{\Gamma,m,+}(s))\\
\dim_{\mathbb{C}} J^{*cusp}_{2,m} &= \mathrm{Ord}_{s=3/4}(Z_{\Gamma,m,-}(s)) + \mu(m,-1)\\
\dim_{\mathbb{C}} J_{1,m} &= \mathrm{Ord}_{s=3/4}(Z_{\Gamma,m,-}(s)),
\end{aligned}
$$

where $\lambda(m,k)$, $\mu(m,k)$ $(k \in \mathbb{Z})$ are the numbers given by (5.4).

REMARK. The dimension of the space $J^{*}_{1,m}$ has been calculated by Skoruppa-Zagier [S-Z1, § 2] in an explicit form. By virtue of their result we discover that

$$\mathrm{Ord}_{s=3/4}(Z_{\Gamma,m,+}(s)) = \frac{1}{2}\{\sum_{d|m\ d>0} 1 + \delta(m=\square)\},$$

where the symbol $\delta(m=\square)$ takes the values 1 or 0 according as m is a square of some integer or not. Consequently, the Selberg zeta function $Z_{\Gamma,m,+}(s)$ does not satisfy the Riemann hypothesis on the real interval $(1/2,1]$.

For the proof we employ a new formulation of the Selberg trace formula for $SL_2(\mathbb{R})$ due to Fischer [Fi] and some results of [Ar].

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§ 1. *Automorphic forms associated with a theta multiplier system*

First we recall basic facts on certain spaces of automorphic forms associated with the theta multiplier system arising from classical theta series.

Let m be a positive integer and fix it once and for all. Let R denote the $\mathbb{Z}$-module $\mathbb{Z}/2m\mathbb{Z}$ of residue classes mod 2m. Denote by $V=\mathbb{C}^{2m}$ the $\mathbb{C}$-vector space of column vectors $(x_r)_{r\in R}$ $(x_r \in \mathbb{C})$ indexed by the set R. A positive definite hermitian scalar product on V×V is given by

$$(x,y) = \sum_{r\in R} x_r\bar{y}_r \qquad (x=(x_r)_{r\in R},\quad y=(y_r)_{r\in R} \in V).$$

We use the symbol $e^m(\alpha)$ (resp. $e(\alpha)$) as an abbreviation for $\exp(2\pi i m\alpha)$ (resp. $\exp(2\pi i\alpha)$). Let $\mathfrak{H}$ be the upper half plane. We write, for $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $z \in \mathfrak{H}$,

$$J(M,\tau)= c\tau+d \quad \text{and} \quad d\omega(\tau)= \eta^{-2}d\xi d\eta, \quad \xi=\mathrm{Re}(\tau), \quad \eta=\mathrm{Im}(\tau).$$

We take the branch of z^{α} ($z\neq 0$, $\alpha \in \mathbb{R}$) with $-\pi<\arg z\leq\pi$. For a real number μ, let $\sigma_{\mu}(A,B)$ (A, $B \in SL_2(\mathbb{R})$) be the cocycle given by

$$\sigma_{\mu}(A,B)= e(\mu w(A,B)) \quad \text{with}$$
$$2\pi w(A,B)= \arg J(A,B\tau)+\arg J(B,\tau)-\arg J(AB,\tau),$$

where $\arg z$ ($z\neq 0$) is chosen so that $-\pi<\arg z\leq\pi$. For each $r \in R$, we define the theta series $\theta_r(\tau,z)$ to be the sum

$$(1.1) \qquad \sum_{q \in \mathbb{Z}} e^{m}\left(\tau\left(q+ \frac{r}{2m}\right)^2+ 2z\left(q+ \frac{r}{2m}\right)\right) \qquad (\tau \in \mathfrak{H}, \quad z \in \mathbb{C}).$$

The collection

$$\Theta(\tau,z)= (\theta_r(\tau,z))_{r\in R}$$

of the theta series $\theta_r(\tau,z)$ as a column vector satisfies the theta transformation formula

$$(1.2) \quad \Theta\left(M\tau, \frac{z}{c\tau+d}\right)= e^{m}\left(\frac{cz^2}{c\tau+d}\right)(c\tau+d)^{1/2}U(M)\Theta(\tau,z) \quad \left(M=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})\right)$$

with a certain unitary matrix $U(M)$ of size $2m$ with respect to the scalar product $(\ ,\ )$. We set

$$(1.3) \qquad \chi(M)= \overline{U(M)}, \qquad \text{the complex conjugate of } U(M).$$

Let k be a fixed rational integer and set

$$\kappa= \frac{1}{2}\left(k- \frac{1}{2}\right).$$

By the formula (1.2), χ satisfies the following identity

$$(1.4) \qquad \chi(M_1M_2)= \sigma_{2\kappa}(M_1,M_2)\chi(M_1)\chi(M_2) \qquad (M_1,\ M_2 \in SL_2(\mathbb{Z}))$$

as a unitary multiplier system (see also [Ar, (2.1)]). We set

$$j_M(\tau,\kappa)= \exp(2i\kappa\cdot \arg J(M,\tau)) \qquad (M \in SL_2(\mathbb{R}),\ \tau \in \mathfrak{H}).$$

We write $j_M(\tau)$ for $j_M(\tau,\kappa)$ if there is no fear of confusion. This automorphic factor has the property

$$(1.5) \qquad j_A(B\tau)j_B(\tau)= \sigma_{2\kappa}(A,B)j_{AB}(\tau) \qquad (A,\ B \in SL_2(\mathbb{R})).$$

We assume for simplicity that $\Gamma=SL_2(\mathbb{Z})$. Let $\mathcal{H}_\kappa$ be the space of measurable functions $f\colon \mathfrak{H} \longrightarrow V$ satisfying

1) $f(M\tau)= \chi(M)j_M(\tau)f(\tau) \qquad$ for all $M \in \Gamma$,

2) $\displaystyle\int_{\mathcal{F}} (f(\tau),f(\tau))d\omega(\tau) < +\infty,$

where $\mathcal{F}$ is a fundamental domain of Γ in $\mathfrak{H}$. Then, $\mathcal{H}_\kappa$ forms a Hilbert space via the Petersson scalar product (f,g) on $\mathcal{H}_\kappa\times\mathcal{H}_\kappa$:

$$(f,g)= \int_{\mathcal{F}} (f(\tau),g(\tau))d\omega(\tau) \qquad (f,\ g \in \mathcal{H}_\kappa).$$

We set

$$\Delta_\kappa= \eta^2\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} \right) - 2i\kappa\eta \frac{\partial}{\partial\xi} .$$

Denote by $\mathcal{D}_\kappa$ the set of all twice continuously differentiable functions $f \in \mathcal{H}_\kappa$ with the property $\Delta_\kappa f \in \mathcal{H}_\kappa$. Roelcke showed that

$$(-\Delta_\kappa f,g)= (f,-\Delta_\kappa g) \qquad \text{for } f,\ g \in \mathcal{D}_\kappa \qquad \text{([Ro1, Satz 3.1])}$$

and moreover that the linear operator $-\Delta_\kappa\colon \mathcal{D}_\kappa \longrightarrow \mathcal{H}_\kappa$ has the unique self-adjoint extension $-\Delta_\kappa^\sim\colon \mathcal{D}_\kappa^\sim \longrightarrow \mathcal{H}_\kappa$ which is a closed operator ([Ro1, Satz 3.2]).

We recall the definition and some properties of real analytic Eisenstein series on $\mathfrak{H}$ associated with $\Gamma=SL_2(\mathbb{Z})$ and the theta multiplier system χ ([Ar, § 2]). For each $r \in R$ denote by e_r the column vector of V whose ℓ-th component is one or zero according as $\ell=r$ or not; $e_r=(\delta_{r\ell})_{\ell\in R}$, $\delta_{r\ell}$ being the Kronecker symbol. We define the subsets $R^{(2)}$, R^{null} of $R=\mathbb{Z}/2m\mathbb{Z}$ as follows:

$$R^{(2)}= \{r \in R \mid r\equiv -r \bmod 2m\} \quad \text{and} \quad R^{null}=\{r \in R \mid r^2\equiv 0 \bmod 4m\}.$$

Set, for each $r \in R$,

$$w_r= \begin{cases} (e_r+(-1)^k e_{-r})/2 & \ldots \ r \in R^{(2)} \\ (e_r+(-1)^k e_{-r})/\sqrt{2} & \ldots \ r \in R-R^{(2)}. \end{cases}$$

Denote by 1_n the identity matrix of size n. Let Γ_∞ be the subgroup of Γ generated by the elements $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and -1_2. Set, for each $r \in R^{null}$,

$$E_{k,m,r}(\tau,s)= \sum_{M \in \Gamma_\infty\backslash\Gamma} j_M(\tau)^{-1}(\mathrm{Im}(M\tau))^s\chi(M)^{-1}w_r.$$

As is seen in [Ar, § 2], the V-valued function $E_{k,m,r}(\tau,s)$ is well-defined and absolutely convergent for $\mathrm{Re}(s)>1$. We omit the indices k, m and write $E_r(\tau,s)$ in place of $E_{k,m,r}(\tau,s)$ for simplicity. Since $E_r(\tau,s)=(-1)^k E_{-r}(\tau,s)$ ([Ar, (2.6)]), it suffices to consider a half of the elements of R^{null}. There exists a subset $R^\sim$ of R such that

$$R= R^{(2)}\cup R^\sim \cup \{-R^\sim\} \qquad \text{(disjoint union)},$$

where $-R^\sim=\{-r \mid r \in R^\sim\}$. We choose such a set $R^\sim$ and fix it throughout the paper. We define the subsets R_k, R_k^{null} by

$$(1.6) \qquad R_k= \begin{cases} R^\sim \cup R^{(2)} & \ldots \text{ if } k \text{ is even,} \\ R^\sim & \ldots \text{ if } k \text{ is odd,} \end{cases}$$

and

$$R_k^{null}= R_k \cap R^{null}.$$

Denote by t_∞ the cardinality of the set R_k^{null}. Set, for each $r \in R$, (1.7) $\beta_r = \langle -r^2/4m \rangle$,

where $\langle x \rangle$ $(x \in \mathbb{R})$ denotes the real number satisfying

$$x - \langle x \rangle \in \mathbb{Z} \quad \text{and} \quad 0 \le \langle x \rangle < 1.$$

As we see in [Ar, Proposition 2.2], the Eisenstein series $E_r(\tau,s)$ for each $r \in R_k^{null}$ has the Fourier expansion of the form:

$$E_r(\tau,s) = \sum_{p \in R_k^{null}} (\delta_{rp}\eta^s + \varphi_{rp}(s)\eta^{1-s}) w_p + \sum_{p \in R_k} q_{rp}(\tau,s) w_p$$

$(\xi = \mathrm{Re}(\tau),\ \eta = \mathrm{Im}(\tau),\ \mathrm{Re}(s) > 1)$,

where each $\varphi_{rp}(s)$ is a holomorphic function in the region $\mathrm{Re}(s) > 1$, and where

$$q_{rp}(\tau,s) = \begin{cases} \sum_{n=-\infty}^{\infty} q_{rp,n}(\eta,s) e((n+\beta_p)\xi) & \cdots \quad p \in R_k - R_k^{null} \\ \sum_{n=-\infty,\ n \neq 0}^{\infty} q_{rp,n}(\eta,s) e(n\xi) & \cdots \quad p \in R_k^{null}. \end{cases}$$

Let $\Phi(s)$ denote the matrix of size t_∞ whose (r,p)-component $(r,\ p \in R_k^{null})$ is given by $\varphi_{rp}(s)$. Moreover, set

$$\mathbb{E}(\tau,s) = (\ldots, E_r(\tau,s), \ldots)_{r \in R_k^{null}} \cdot$$

which is a $2m \times t_\infty$ matrix. It has been observed by [Ar, Proposition 2.5] that $E_r(\tau,s)$ are analytically continued to meromorphic functions of s in the region $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \ge 1/2\}$ except on the interval $(1/2, 1]$ and moreover that $\mathbb{E}(\tau,s)$ and $\Phi(s)$ satisfy the functional equations

$$\mathbb{E}(\tau,1-s) = \mathbb{E}(\tau,s)\,{}^t\Phi(1-s) \quad \text{and} \quad \Phi(s)\Phi(1-s) = 1_{t_\infty}.$$

Now we recall briefly a preliminary version of Fischer's resolvent trace formula ([Fi, § 2, Theorem 2.1.2]) in a form suitable to our situation (the fact that our theta multiplier system χ does not satisfy the condition (a) of (1.3.4) in [Fi] causes a little difference). Set, for τ, $w \in \mathfrak{H}$,

$$H(\tau,w)= \left(\frac{w-\bar{\tau}}{\tau-\bar{w}} \right)^{\kappa} \quad \text{and} \quad \sigma(\tau,w)= \frac{|\tau-\bar{w}|^2}{4\mathrm{Im}(\tau)\mathrm{Im}(w)} .$$

Recall that $\kappa=(k-1/2)/2$. Set, for $s \in \mathbb{C}$ and $\sigma>1$,

$$k_s(\sigma)= \sigma^{-s}\cdot\frac{\Gamma(s+\kappa)\Gamma(s-\kappa)}{4\pi\Gamma(2s)}\cdot F(s+\kappa,s-\kappa,2s;1/\sigma),$$

where $F(a,b,c;z)$ is the hypergeometric function:

$$F(a,b,c;z)= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\cdot\frac{z^n}{n!} \qquad \text{with} \quad (a)_n= \frac{\Gamma(a+n)}{\Gamma(a)} .$$

For τ, $w \in \mathfrak{H}$, we write $\tau\equiv w \bmod \Gamma$, if there is some element γ of Γ with $\tau=\gamma w$, and $\tau\not\equiv w \bmod \Gamma$, otherwise. Denote by $\mathbb{N}$ the set of positive integers and set $\mathbb{N}_0= \mathbb{N} \cup\{0\}$. For $s \in \mathbb{C}-\{|\kappa|-\ell| \ \ell \in \mathbb{N}_0\}$, $\mathrm{Re}(s)>1$, $\lambda=s(1-s)$, and τ, $w \in \mathfrak{H}$, set

$$(1.8) \qquad G_{\kappa,\lambda}(\tau,w)= \frac{1}{2} \sum_{M \in \Gamma} \chi(M)j_M(w)H(\tau,Mw)k_s(\sigma(\tau,Mw)).$$

It is known by Elstrodt [El, p.318], Fischer [Fi, Proposition 1.4.8] that the infinite series on the right side of (1.8) converges normally in variables (z,w,s) with z, $w \in \mathfrak{H}$, $z\not\equiv w \bmod \Gamma$, $s \in \mathbb{C}-\{|\kappa|-\ell| \ \ell \in \mathbb{N}_0\}$, $\mathrm{Re}(s)>1$ (for the notion of normal convergence, see [El, p.302]). Denote by $\psi(s)$ the logarithmic derivative of the gamma function: $\psi(s)=\Gamma'(s)/\Gamma(s)$. Let $\lambda=s(1-s)$, $\mu=a(1-a)$ with s, $a \in \mathbb{C}-\{|\kappa|-\ell| \ \ell \in \mathbb{N}_0\}$, $\mathrm{Re}(s)$, $\mathrm{Re}(a)>1$ and τ, $\tau' \in \mathfrak{H}$. In a manner similar to that in [Fi, 2.1], we get, instead of [Fi, p.45, (2.1.2)],

$$\lim_{\tau'\longrightarrow\tau} (G_{\kappa\lambda}(\tau,\tau')-G_{\kappa\mu}(\tau,\tau'))=$$

$$-\frac{1}{8\pi}(\psi(s+\kappa)+\psi(s-\kappa)-\psi(a+\kappa)-\psi(a-\kappa))'\cdot(1_{2m}+\chi(-1_2)j_{-1_2}(\tau))$$

$$+\lim_{\tau'\longrightarrow\tau}\ \frac{1}{2}\sum_{M\in\Gamma-\{\pm 1_2\}}\chi(M)j_M(\tau')H(\tau,M\tau')(k_s(\sigma(\tau,M\tau')-k_a(\sigma(\tau,M\tau')).$$

In view of the results of Roelcke ([Ro1, 2, Satz 5.7, Satz 7.2]), there exists a complete set of the eigen values $\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots$ counted with multiplicities of the self-adjoint operator $-\Delta_\kappa^\sim$: $\mathfrak{D}_\kappa^\sim \longrightarrow \mathcal{H}_\kappa$. Then one can write

(1.9) $\quad \lambda_n = \frac{1}{4} + r_n^2 \quad$ with $\quad r_n \in \{r|\ r\geq 0\} \cup \{ir|\ r>0\} \quad (n \in \mathbb{N}_0)$.

Let P denote the discrete subset $\{1/2\pm ir_n|\ n \in \mathbb{N}_0\}$ of $\mathbb{C}$. As is shown in [Fi, Theorem 1.6.5], the infinite series

(1.10) $$S_{\Gamma,k,m}(s,a) = \sum_{n=0}^{\infty}\left(\frac{1}{(s-1/2)^2+r_n^2} - \frac{1}{(a-1/2)^2+r_n^2}\right)$$

is absolutely convergent for any s, a $\in \mathbb{C}$-P, and furthermore, a being fixed, indicates a holomorphic functiuon of s in the domain $\mathbb{C}$-P.

Let L be a matrix of size 2m (or a linear transformation of V) characterized by

(1.11) $\quad Le_r = e_{-r} \quad$ for any $\quad r \in R$.

We note here that

(1.12) $\quad \chi(-1_2) = e^{\pi i/2}L \qquad$ ([Ar, (1.3)])

and hence that

$$\mathrm{tr}\chi(-1_2)j_{-1_2}(\tau) = (-1)^k \sum_{r\in R}(Le_r, e_r) = 2(-1)^k.$$

Therefore by the same argument as in [Fi, 2.1], Fischer's theorem [Fi, Theorem 2.1.2] is replaced by the following in our case.

THEOREM 1.1 (Fischer). *Assume* $\Gamma=SL_2(\mathbb{Z})$. *Let* $s, a \in \mathbb{C}$, $Re(s)$, $Re(a)>1$ *and* $|\kappa|-s$, $|\kappa|-a \notin \mathbb{N}_0$. *Set* $\lambda=s(1-s)$, $\mu=a(1-a)$. *Then,*

(1.13) $$S_{\Gamma,k,m}(s,a)= - \frac{1}{12}(m+(-1)^k)(\psi(s+\kappa)+\psi(s-\kappa)-\psi(a+\kappa)-\psi(a-\kappa))$$

$$+ \int_{\mathcal{F}} \Big[\frac{1}{2} \sum_{M \in \Gamma-\{\pm 1_2\}} tr(\chi(M) j_M(\tau))\{k_s(\sigma(\tau,M\tau))-k_a(\sigma(\tau,M\tau))\} -$$

$$\sum_{q \in R_k^{null}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \Big(\frac{1}{(s-1/2)^2+t^2} - \frac{1}{(a-1/2)^2+t^2}\Big) |E_q(\tau,\frac{1}{2}+it)|^2 dt\Big] d\omega(\tau),$$

where the integral on the right hand side of the equality is absolutely convergent.

Our purpose from now on is to simplify the integral on the right hand side of (1.13) with the use of arithmetic or analytic quantities related to Γ and χ.

§ 2. *Selberg zeta functions*

In this paragraph we assume that Γ is a subgroup of $SL_2(\mathbb{Z})$ with finite index having the element -1_2. We define two Selberg zeta functions associated with Γ and the theta multiplier system χ. Since the matrix L given by (1.11) has the eigen value 1 (resp. -1) with multiplicity m+1 (resp. m-1), there exists an element Q of $GL_{2m}(\mathbb{C})$ with

(2.1) $$L= Q\begin{pmatrix} 1_{m+1} & 0 \\ 0 & -1_{m-1} \end{pmatrix} Q^{-1}.$$

One may choose as Q a unitary matrix with respect to the scalar product (,). Every $\chi(M)$ ($M \in SL_2(\mathbb{Z})$) commutes with L via the relations (1.12), (1.4). Therefore, $\chi(M)$ has an expression of the form

(2.2) $$\chi(M)= Q\begin{pmatrix} \chi_+(M) & 0 \\ 0 & \chi_-(M) \end{pmatrix} Q^{-1}$$

with unitary matrices $\chi_+(M)$, $\chi_-(M)$ of size $m+1$, $m-1$, respectively. Every hyperbolic element P of Γ has an expression

$$P= \pm A\begin{pmatrix} N(P)^{1/2} & 0 \\ 0 & N(P)^{-1/2} \end{pmatrix} A^{-1} \quad \text{with } N(P)>1 \text{ and some } A \in SL_2(\mathbb{R}),$$

where the uniquely determined number N(P) is called the norm of P. Denote by $\{P\}_\Gamma^{hyp}$ (resp. $\{P_0\}_\Gamma^{prim}$) the Γ-conjugacy classes of hyperbolic (resp. primitive hyperbolic) elements of Γ. We define the Selberg zeta functions $Z_{\Gamma,m,+}(s)$, $Z_{\Gamma,m,-}(s)$ as follows:

(2.3) $$Z_{\Gamma,m,\pm}(s)= \prod_{\{P_0\}_\Gamma^{prim},\ trP_0>2} \ \prod_{n=0}^{\infty} \det(1_{m\pm 1}-\chi_\pm(P_0)N(P_0)^{-s-n}),$$

where the signs $\pm$ are taken in the corresponding manner and the first product indicates that P_0 runs over the Γ-conjugacy classes of primitive hyperbolic elements of Γ with $trP_0>2$. The infinite products on the right hand side of (2.3) are absolutely and uniformly convergent on any compact subset in the half plane $\{s \in \mathbb{C} \mid Re(s)>1\}$ (for instance Lemma 2.2.2 and Corollary 2.2.6 in [Fi] ensure this assertion). Therefore the zeta functions $Z_{\Gamma,m,\pm}(s)$ indicate holomorphic functions in the same half plane. It shoud be noted that $Z_{\Gamma,m,\pm}(s)$ are independent of the choice of the matrix Q satisfying

(2.1). Set, for $\lambda=s(1-s)$, $Re(s)>1$, $|\kappa|-s \notin \mathbb{N}_0$,

$$G_{\kappa\lambda,hyp}(\tau)= \frac{1}{2} \sum_{M} tr\chi(M) j_M(\tau)\cdot H(\tau,M\tau) k_s(\sigma(\tau,M\tau)) \qquad (\tau \in \mathfrak{H}),$$

where M runs over all hyperbolic elements of Γ. It is known that $G_{\kappa\lambda,hyp}(\tau)$, if restricted to a fundamental domain $\Gamma\backslash\mathfrak{H}$ of Γ in $\mathfrak{H}$, defines an L^1-function on $\Gamma\backslash\mathfrak{H}$ ([Fi, 2.2.5]). Then it follows by the same argument as in [Fi, Proposition 2.2.5] that

$$\int_{\Gamma\backslash\mathfrak{H}} G_{\kappa\lambda,hyp}(\tau)d\omega(\tau)= \frac{1}{2(2s-1)} \sum_{\{P\}_\Gamma^{hyp},\ trP>2} \{tr\chi(P)+(-1)^k tr(L\chi(P))\} \log N(P_0)\cdot \frac{N(P\ )^{-s}}{1-N(P)^{-1}} ,$$

where P_0 is the primitive hyperbolic element associated to P. We note that χ also satisfies [Fi, a) of Corollary 1.3.9], i.e,

$$\chi(P^n)= \chi(P)^n \quad \text{for any hyperbolic element P of } \Gamma \text{ with } trP>2.$$

Taking (2.1), (2.2) into account, we thus have

(2.4)

$$\int_{\Gamma\backslash\mathfrak{H}} G_{\kappa\lambda,hyp}(\tau)d\omega(\tau)= \frac{1}{2s-1} \sum_{\substack{\{P_0\}_\Gamma^{prim} \\ trP_0>2}} tr(\chi_\pm(P_0)^n)\log N(P_0)\cdot \frac{N(P_0)^{-ns}}{1-N(P_0)^{-n}}$$

$$= \frac{1}{2s-1}(Z'_{\Gamma,m,\pm}/Z_{\Gamma,m,\pm}(s) \qquad (Re(s)>1,\ |\kappa|-s \notin \mathbb{N}_0),$$

where the signs +, - are chosen according as k is even or odd.

§ 3. *Description of Selberg trace formula*

To describe Selberg trace formula for the space $\mathcal{H}_\kappa$ given in § 1 we first calculate the contributions from the elliptic or parabolic conjugacy classes of $\Gamma=SL_2(\mathbb{Z})$ to the preliminary resolvent trace formula (1.13).

In [Sk-Za2, §4], Skoruppa and Zagier gave a method of calculating the trace of a certain linear operator $U_m(\xi)$ ($\xi \in \Gamma^J$, for the Jacobi group Γ^J see § 5) of the $\mathbb{C}$-linear space spanned by the theta series θ_r ($r \in R$). If $M \in SL_2(\mathbb{Z})$, the trace of the operator $U_m(M)$ coincides with the trace of our $U(M)$ given by (1.2). For $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ denote by $Q_M(\lambda,\mu)$ the binary quadratic form $b\lambda^2+(d-a)\lambda\mu-c\mu^2$ and moreover set

$$\varepsilon(M)=\begin{cases} -1 \;\ldots\; \text{if } c<0 \text{ and } \mathrm{tr}M<2 \\ +1 \;\ldots\; \text{otherwise.} \end{cases}$$

Hence Theorem 2 of [Sk-Za1] implies that

$$\text{(3.1)} \qquad \mathrm{tr}U(M)=\frac{\varepsilon(M)\,\mathrm{sign}(t-2)(t-2)^{1/2}}{|\det(1_2-M)|}\sum_{x\,\in\,\mathbb{Z}^2/\mathbb{Z}^2(1_2-M)} e^m\left(\frac{1}{t-2}\,Q_M(x)\right),$$

where $t=\mathrm{tr}M$ and $\mathbb{Z}^2$ is the lattice of row vectors of size two of rational integers. The fact that $Q_M(x+y)\equiv Q_M(x) \bmod (t-2)$ for any $y \in \mathbb{Z}^2(1_2-M)$ and $x \in \mathbb{Z}^2$ ensures the well-definedness of the summation on the right hand side of (3.1).

Set, for $s \in \mathbb{C}$, $|\kappa|-s \notin \mathbb{N}_0$, $\lambda=s(1-s)$,

$$G_{\kappa\lambda,\mathrm{ell}}(\tau)=\frac{1}{2}\sum_{M}\ \mathrm{tr}\chi(M)\cdot j_M(\tau)H(\tau,M\tau)k_s(\sigma(\tau,M\tau)),$$

where M runs over all elliptic elements of Γ. For an elliptic element R of Γ, let $2\nu(R)$ be the order of the centralizer $Z_\Gamma(R)$ of R in Γ. Denote by $\{R\}_\Gamma$ the Γ-conjugacy classes of elliptic elements of Γ. It has been proved in [Fi, Proposition 2.3.4] that $G_{\kappa\lambda,ell}(\tau)$, if it is restricted to $\mathcal{F}=\Gamma\backslash\mathfrak{H}$, defines an L^1-function on $\mathcal{F}$. Then we easily have, similarly as in [Fi, pp. 62-64],

$$(3.2) \qquad \int_{\mathcal{F}} G_{\kappa\lambda,ell}(\tau)d\omega(\tau)= \frac{1}{2}\sum_{\{R\}_\Gamma} \frac{\mathrm{tr}\chi(R)}{\nu(R)}\cdot I(R),$$

where

$$I(R)= \int_{\mathfrak{H}} j_R(\tau)H(\tau,R\tau)k_s(\sigma(\tau,R\tau))d\omega(\tau).$$

For $M=\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in SL_2(\mathbb{R})$, we write $c(M)$ for the (2,1)-component c. Assume that an elliptic element R of Γ is $SL_2(\mathbb{R})$-conjugate to some $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ with $0<\theta<2\pi$, where θ is uniquely determined by R. We often write $\theta(R)$ for θ. If $0<\theta(R)<\pi$, the integral $I(R)$ has been explicitly evaluated in [Fi, pp. 65, 66]:

$$(3.3) \quad I(R)= \frac{1}{2s-1}\cdot\frac{ie^{2i\kappa\theta}}{2\nu\sin\theta}\sum_{j=0}^{\nu-1}\left(e^{i\theta(2j+1)}\psi\left(\frac{s+\kappa+j}{\nu}\right) - e^{-i\theta(2j+1)}\psi\left(\frac{s-\kappa+j}{\nu}\right)\right)$$

$$(0<\theta=\theta(R)<\pi),$$

where $\nu=\nu(R)$, half the order of the centralizer $Z_\Gamma(R)$. If $\pi<\theta=\theta(R)<2\pi$, we write $\theta=\pi+\theta'$ with $\theta<\theta'<\pi$. Set $R'=-R$. Then R' is $SL_2(\mathbb{R})$-conjugate to $\begin{pmatrix}\cos\theta' & -\sin\theta'\\ \sin\theta' & \cos\theta'\end{pmatrix}$. In this case $c(R)=-c(R')<0$. Moreover,

$$j_R(\tau)= j_{-R'}(\tau)= \sigma_{2\kappa}(-1_2,R')^{-1}j_{-1_2}(R'\tau)j_{R'}(\tau)$$

$$= \mathrm{sign}(-c(R'))e^{2i\kappa\pi}j_{R'}(\tau)= -e^{2i\kappa\pi}j_{R'}(\tau).$$

Thus $I(R)= -e^{2i\kappa\pi}I(R')$, which together with (3.3) implies that

$$(3.4) \quad I(R)= \frac{-1}{2s-1}\cdot\frac{ie^{2i\kappa\theta}}{2\nu\sin\theta}\sum_{j=0}^{\nu-1}\left(e^{i\theta(2j+1)}\psi\left(\frac{s+\kappa+j}{\nu}\right) - e^{-i\theta(2j+1)}\psi\left(\frac{s-\kappa+j}{\nu}\right)\right)$$

$$(\pi<\theta=\theta(R)<2\pi).$$

Set

$$J=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \quad \text{and} \quad W=\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}.$$

We may take J, J^3, W, W^2, W^4, W^5 as a complete set of representatives of $\{R\}_\Gamma$. Taking the relation $\chi(M)=\overline{U(M)}$ into account, then by an easy calculation with the use of (3.1), we have

$$(3.5)\quad \begin{aligned}&\mathrm{tr}\chi(J)= iG_2(m), \qquad \mathrm{tr}\chi(J^3)= -iG_2(m)\\ &\mathrm{tr}\chi(W)=i, \quad \mathrm{tr}\chi(W^2)=i\overline{G_3(m)}, \quad \mathrm{tr}\chi(W^4)=-iG_3(m), \quad \mathrm{tr}\chi(W^5)=-i,\end{aligned}$$

where, for any positive integer ℓ,

$$G_\ell(m)= \frac{1}{\sqrt{\ell}} \sum_{\lambda \bmod \ell} e^m(\lambda^2/\ell).$$

Then the right hand side of (3.2) equals

$$C_J + C_W^{I} + C_W^{II}$$

with

$$C_J= \frac{1}{4}\{\mathrm{tr}\chi(J)I(J)+\mathrm{tr}\chi(J^3)I(J^3)\},$$

$$C_W^{I}= \frac{1}{6}\{\mathrm{tr}\chi(W)I(W)+\mathrm{tr}\chi(W^5)I(W^5)\}, \quad C_W^{II}= \frac{1}{6}\{\mathrm{tr}\chi(W^2)I(W^2)+\mathrm{tr}\chi(W^4)I(W^4)\}.$$

Thus we get, by an elementary but tedious calculation,

$$C_J= \frac{1}{8(2s-1)} G_2(m)\sin(\pi(k-1/2)/2)\cdot\left(\psi\left(\frac{s+\kappa}{2}\right)+\psi\left(\frac{s-\kappa}{2}\right)-\psi\left(\frac{s+\kappa+1}{2}\right)-\psi\left(\frac{s-\kappa+1}{2}\right)\right),$$

$$C_W^{I}= \frac{1}{9(2s-1)} \sum_{j=0}^{2} \frac{\sin(\pi(k-2j)/3)}{2\sin(\pi/3)}\cdot\left(\psi\left(\frac{s-\kappa+j}{3}\right)-\psi\left(\frac{s+\kappa+2-j}{3}\right)\right),$$

$$C_W^{II}= \frac{1}{9(2s-1)} \sum_{j=0}^{2} \sum_{\lambda \bmod 3} \frac{\sin(2\pi(k-3/4-m\lambda^2-2j)/3)}{3}\cdot\left(\psi\left(\frac{s-\kappa+j}{3}\right)-\psi\left(\frac{s+\kappa+2-j}{3}\right)\right).$$

Consequently we have calculated the integral on the left hand side of

(3.2) in an explicit form:

$$(3.6)\quad \int_{\mathcal{F}} G_{\kappa\lambda,\mathrm{ell}}(\tau)d\omega(\tau)=\frac{\epsilon_2(k,m)}{8(2s-1)}\cdot\left(\psi\left(\frac{s+\kappa}{2}\right)+\psi\left(\frac{s-\kappa}{2}\right)-\psi\left(\frac{s+\kappa+1}{2}\right)-\psi\left(\frac{s-\kappa+1}{2}\right)\right)$$

$$+\frac{1}{9(2s-1)}\sum_{j=0}^{2}\epsilon_3(k-2j,m)\left(\psi\left(\frac{s-\kappa+j}{3}\right)-\psi\left(\frac{s+\kappa+2-j}{3}\right)\right),$$

where

$$\epsilon_2(k,m)=\frac{(1+e(m/2))}{\sqrt{2}}\sin(\pi(k-1/2)/2),$$

(3.7)

$$\epsilon_3(k,m)=\frac{\sin(\pi k/3)}{\sqrt{3}}+\frac{1}{3}\sum_{\lambda \bmod 3}\sin(2\pi(k-3/4-m\lambda^2)/3).$$

It should be noted that the symbols $\epsilon_2(k,m)$, $\epsilon_3(k,m)$ take only the values 0, ±1, and moreover that $\epsilon_2(k,m)$ (resp. $\epsilon_3(k,m)$) depends on m mod 2 and k mod 4 (resp. m mod 3 and k mod 6).

Next we calculate the contribution from the parabolic elements of Γ. By a standard argument as in [Fi, 2.4, p.69], we get

$$(3.8)\quad \frac{1}{2}\sum_{M\in\Gamma,\ \text{parabolic}} \mathrm{tr}\chi(M)j_M(\tau)H(\tau,M\tau)\mathbf{k}_s(\sigma(\tau,M\tau))$$

$$=\frac{1}{2}\sum_{M\in\Gamma,\ \mathrm{tr}M=2}\left(\mathrm{tr}\chi(M)j_M(\tau)+\mathrm{tr}\chi(-M)j_{-M}(\tau)\right)H(\tau,M\tau)\mathbf{k}_s(\sigma(\tau,M\tau))$$

$$=\frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(\mathrm{tr}\chi(U^n)+\mathrm{tr}\chi(-U^n)j_{-U^n}(\gamma\tau)\right)H(\gamma\tau,U^n\gamma\tau)\mathbf{k}_s(\sigma(\gamma\tau,U^n\gamma\tau)),$$

where $U=\begin{pmatrix}1&1\\0&1\end{pmatrix}$. Since $\chi(U^n)e_r=e(n\beta_r)e_r$ for any $r\in R$, $n\in\mathbb{Z}$ (see [Ar, (2.2)]), we immediately have

$$\mathrm{tr}\chi(U^n)=\sum_{r\in R}e(n\beta_r)\qquad(\text{for } \beta_r,\ \text{see (1.7)}).$$

Using the properties (1.4), (1.12) of χ and (1.5) of j_M, we see that

$$\mathrm{tr}\chi(-U^n)j_{-U^n}(\tau) = (-1)^k \sum_{r\in R} e(n\beta_r)(e_{-r},e_r)$$

$$= (-1)^k \sum_{r\in R^{(2)}} e(n\beta_r) \qquad (n \in \mathbb{Z}).$$

Taking the last equality in (3.8) into account, we define the regular parabolic part of $G_{\kappa\lambda}(\tau)$ as follows:

$$G_{\kappa\lambda,\mathrm{par},\mathrm{reg}}(\tau) = \frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\ \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}$$

$$\Big\{\sum_{\substack{r\in R\\ r^2\not\equiv 0 \bmod 4m}} e(n\beta_r) + \sum_{\substack{r\in R^{(2)}\\ r^2\not\equiv 0 \bmod 4m}} (-1)^k e(n\beta_r)\Big\}\cdot H(\gamma\tau,U^n\gamma\tau)\mathbf{k}_s(\sigma(\gamma\tau,U^n\gamma\tau))$$

$$(\mathrm{Re}(s)>\mathrm{Max}(1,|\kappa|),\quad \lambda=s(1-s)).$$

Set, for simplicity,

$$(3.9)\qquad R^* = \{r\in R \mid r^2\not\equiv 0 \bmod 4m\} \quad\text{and}\quad R_k^* = R_k \cap R^*$$

(for the set R_k, see (1.6)).

By [Fi, Lemma 2.4.1], the restriction of $G_{\kappa\lambda,\mathrm{par},\mathrm{reg}}(\tau)$ to $\mathcal{F}=\Gamma\backslash\mathfrak{H}$ defines an L^1-function on $\mathcal{F}$ with respect to $d\omega(\tau)$, and moreover,

$$\int_{\mathcal{F}} G_{\kappa\lambda,\mathrm{par},\mathrm{reg}}(\tau)d\omega(\tau) = \sum_{r\in R_k^*} I(\beta_r;s),$$

where we put

$$I(\beta;s) = \int_{\Gamma_\infty\backslash\mathfrak{H}} \sum_{n\in\mathbb{Z}-\{0\}} e(n\beta)H(\tau,U^n\tau)k_s(\sigma(\tau,U^n\tau))d\omega(\tau)$$

$$(0<\beta<1,\quad \mathrm{Re}(s)>\mathrm{Max}(1,|\kappa|)).$$

It has been shown in [Fi, pp.72-77] that the integral $I(\beta,s)$ is absolutely convergent and that

$$I(\beta;s) = \frac{1}{2s-1}\{-\log 2-\log\sin(\pi\beta)+(\psi(s+\kappa)-\psi(s-\kappa))(1/2-\beta)\}.$$

Therefore,

$$(3.10)\quad \int_{\mathcal{F}} G_{\kappa\lambda,\mathrm{par},\mathrm{reg}}(\tau)d\omega(\tau)= \frac{1}{2s-1}\Big(\sum_{r \in R_k^*} \{-\log 2-\log\sin(\pi\beta_r) + (\psi(s+\kappa)-\psi(s-\kappa))(1/2-\beta_r)\}\Big) \qquad (\mathrm{Re}(s)>\mathrm{Max}(1,|\kappa|)).$$

As a final step in this paragraph we consider the non-regular part of the contribution from the parabolic elements of Γ. Set, for $\mathrm{Re}(s)$, $\mathrm{Re}(a)>1$, $|\kappa|-s$, $|\kappa|-a \notin \mathbb{N}_0$,

$$k_{s,a}(\sigma)= k_s(\sigma)-k_a(\sigma),$$

$$f(t;s,a)= \frac{1}{(s-1/2)^2+t^2} - \frac{1}{(a-1/2)^2+t^2},$$

and

$$K_{s,a}(\tau)= \sum_{n \in \mathbb{Z}-\{0\}} H(\tau,U^n\tau)k_{s,a}(\sigma(\tau,U^n\tau)).$$

The integral we have to consider is the following:

$$(3.11)\qquad I= \int_{\mathcal{F}} \Big(\sum_{\gamma \in \Gamma_\infty\backslash\Gamma} t_\infty K_{s,a}(\gamma\tau) - \sum_{q \in R_k^{\mathrm{null}}} \frac{1}{4\pi} \int_{-\infty}^{\infty} f(t;s,a)|E_q(\tau,1/2+it)|^2dt\Big)d\omega(\tau)$$

$$(\mathrm{Re}(s),\ \mathrm{Re}(a)>\mathrm{Max}(1,|\kappa|)),$$

where we recall that t_∞ is the cardinality of the set R_k^{null}. Now we take as $\mathcal{F}$ the usual fundamental domain of $\Gamma=SL_2(\mathbb{Z})$ in $\mathfrak{H}$:

$$\mathcal{F}= \{\tau \in \mathfrak{H} \mid -1/2<\mathrm{Re}(\tau)\le 1/2,\ |\tau|\ge 1 \text{ and } \mathrm{Re}(\tau)\ge 0 \text{ on } |\tau|=1\}.$$

Thanks to Lemma 2.4.11 of [Fi], the integral I is modified into the form:

(3.12)

$$I=\lim_{Y\longrightarrow+\infty}\Big[\frac{t_\infty}{2s-1}\Big(\log Y-\log 2+\frac{1}{2}\psi(s+\kappa)+\frac{1}{2}\psi(s-\kappa)-\psi(s)-\psi(s+1/2)+\frac{1}{2s-1}\Big)$$

$$-\frac{t_\infty}{2a-1}\Big(\log Y-\log 2+\frac{1}{2}\psi(a+\kappa)+\frac{1}{2}\psi(a-\kappa)-\psi(a)-\psi(a+1/2)+\frac{1}{2a-1}\Big)$$

$$-\int_{\mathcal{F}_Y}\sum_{q\in R_k^{null}}\frac{1}{4\pi}\int_{-\infty}^{\infty}f(t;s,a)|E_q(\tau,1/2+it)|^2dtd\omega(\tau)\Big],$$

where

$$\mathcal{F}_Y=\{\tau\in\mathcal{F}\mid \operatorname{Im}(\tau)\le Y\}.$$

For each $r\in R_k^{null}$, let $E_r^Y(\tau,s)$, with $Y>0$ being chosen sufficiently large, denote the compact form of $E_r(\tau,s)$:

$$E_r^Y(\tau,s)=\begin{cases}E_r(\tau,s)-u_r(\operatorname{Im}(\tau),s) & \ldots \text{ if } \tau\in\mathcal{F}-\mathcal{F}_Y\\ E_r(\tau,s) & \ldots \text{ if } \tau\in\mathcal{F}_Y,\end{cases}$$

where $u_r(\eta,s)$ ($\eta=\operatorname{Im}(\tau)>0$) is the constant term of the Fourier expansion of $E_r(\tau,s)$ given by

$$u_r(\eta,s)=\eta^s w_r+\eta^{1-s}\sum_{p\in R_k^{null}}\varphi_{rp}(s)w_p \qquad \text{([Ar, Proposition 2.2]).}$$

Set

$$\varphi(s)=\det\Phi(s).$$

The Maass-Selberg relation for $E_r(\tau,s)$ (see [Ar, Proposition 2.4] or in more detail [Ro, Lemma 11.2]) implies that, if $\sigma>1/2$ and $t\neq 0$,

(3.13)
$$\int_{\mathcal{F}}\sum_{q\in R_k^{null}}|E_q^Y(\tau,\sigma+it)|^2d\omega(\tau)=$$

$$\frac{t_\infty Y^{2\sigma-1}-\operatorname{tr}(\Phi(\sigma+it)\Phi(\sigma-it))Y^{1-2\sigma}}{2\sigma-1}+\frac{\operatorname{tr}\Phi(\sigma-it)Y^{2it}-\operatorname{tr}\Phi(\sigma+it)Y^{-2it}}{2it}.$$

On the both hand sides of (3.13), one can take the limit of $\sigma\longrightarrow 1/2$

and get

$$(3.14)\quad \int_{\mathcal{F}} \sum_{q \in R_k^{null}} |E_q^Y(\tau,1/2+it)|^2 d\omega(\tau)= 2t_\infty \log Y - \frac{\varphi'}{\varphi}(1/2+it)$$

$$+ 2\mathrm{Re}\left(\overline{\mathrm{tr}\Phi(1/2+it)}\cdot\frac{Y^{2it}}{2it}\right),$$

where we have used the identities

$$\frac{\varphi'}{\varphi}(s)= \frac{d}{ds}\log\det\Phi(s)= \mathrm{tr}(\Phi'(s)\Phi(s)^{-1})= \mathrm{tr}(\Phi'(s)\Phi(1-s)).$$

Thus we have, by [Fi, Lemmas 2.4.18, 2.4.19] together with (3.14),

$$(3.15)\quad \lim_{Y\longrightarrow+\infty}\Big[\int_{\mathcal{F}_Y}\sum_{q \in R_k^{null}} \frac{1}{4\pi}\int_{-\infty}^{\infty} f(t;s,a)|E_q(\tau,1/2+it)|^2 dt d\omega(\tau)$$

$$-t_\infty\left(\frac{1}{2s-1} - \frac{1}{2a-1}\right)\log Y\ \Big]$$

$$= -\frac{1}{4\pi}\int_{-\infty}^{\infty} f(t;s,a)\frac{\varphi'}{\varphi}(1/2+it)dt + \frac{1}{4}\mathrm{tr}\Phi(1/2)\left(\frac{1}{(s-1/2)^2} - \frac{1}{(a-1/2)^2}\right).$$

Consequently it follows from (3.12), (3.15) that

$$(3.16)\qquad I= \frac{1}{2s-1}\Big[-t_\infty\log 2+ \frac{t_\infty}{2}(\psi(s+\kappa)+\psi(s-\kappa)-2\psi(s)-2\psi(s+1/2))+$$

$$\frac{1}{2s-1}\mathrm{tr}(1_{t_\infty}-\Phi(1/2))\Big] - \frac{1}{2a-1}\Big[-t_\infty\log 2+ \frac{t_\infty}{2}(\psi(a+\kappa)+\psi(a-\kappa)-2\psi(a)-2\psi(a+1/2))$$

$$+ \frac{1}{2a-1}\mathrm{tr}(1_{t_\infty}-\Phi(1/2))\Big] + \frac{1}{4\pi}\int_{-\infty}^{\infty} f(t;s,a)\frac{\varphi'}{\varphi}(1/2+it)dt$$

$(\mathrm{Re}(s),\ \mathrm{Re}(a)>\mathrm{Max}(1,|\kappa|))$.

Now all the contributions of the elements of Γ to the integral on the right side of (1.13) have been calculated. Therefore with the help of (1.13), (2.4), (3.6), (3.10), (3.11) and (3.16), the resolvent trace formula of Fischer ([Fi, Theorem 2.5.1]) can be formulated in our case as the following.

THEOREM 3.1. *Let* $\Gamma=SL_2(\mathbb{Z})$. *Let* k *be an integer and* m *a positive integer. Set* $\kappa=(k-1/2)/2$. *Assume* $Re(s), Re(a)>Max(1,|\kappa|)$. *Then,*

$$S_{\Gamma,k,m}(s,a)= -\frac{1}{12}(m+(-1)^k)(\psi(s+\kappa)+\psi(s-\kappa)) + \frac{1}{2s-1}(Z'_{\Gamma,m,\pm}/Z_{\Gamma,m,\pm})(s)$$

$$+\frac{1}{2s-1}\Big[\sum_{j=0}^{1}\frac{\epsilon_2(k-2j,m)}{8}\Big(\psi\Big(\frac{s-\kappa+j}{2}\Big)-\psi\Big(\frac{s+\kappa+1-j}{2}\Big)\Big)$$

$$+\sum_{j=0}^{2}\frac{\epsilon_3(k-2j,m)}{9}\Big(\psi\Big(\frac{s-\kappa+j}{3}\Big)-\psi\Big(\frac{s+\kappa+2-j}{3}\Big)\Big)\Big]$$

$$+\frac{1}{2s-1}\Big[-\log 2\cdot(m+(-1)^k)-\log\Big(\prod_{r\in R_k^*}\sin(\pi\beta_r)\Big)+t_\infty(\psi(s-\kappa)-\psi(s)-\psi(s+1/2))$$

$$+(\psi(s+\kappa)-\psi(s-\kappa))\Big(\frac{m+(-1)^k}{2}-\sum_{r\in R_k^*}\beta_r\Big)+\frac{1}{2s-1}tr(1_{t_\infty}-\Phi(1/2))\Big]$$

$$+\frac{1}{4\pi}\int_{-\infty}^{\infty}\Big(\frac{1}{(s-1/2)^2+t^2}-\frac{1}{1/4+t^2}\Big)\frac{\varphi'}{\varphi}(1/2+it)dt$$

+ {*the same expression with* s *being replaced by* a},

where $S_{\Gamma,k,m}(s,a)$ *is given by* (1.10) *and the signs* +, - *are chosen according as* k *is even or odd.*

§ 4. *Analytic continuation and functional equation*

Follwing [Fi, 3.1], we explain briefly how the Selberg zeta functions $Z_{\Gamma,m,\pm}(s)$ are analytically continued to meromorphic functions in the whole s-plane and satisfy certain functional equations (basic ideas are similar to those of Hejhal [He, Ch.10]).

Let the notation be the same as in the previous paragraphs. Let $G(z)$ denote the Barnes G-function given by

$$G(z+1)= (2\pi)^{z/2}\exp(-z(z+1)/2-\gamma z^2/2)\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^n\exp\left(-z+\frac{z^2}{2n}\right),$$

where γ is the Euler constant. Then, $G(z)$ is an entire function whose zeros are located at $z=1-n$ ($n \in \mathbb{N}$) with order n. We define the functions $\Xi_I(s)$, $\Xi_{hyp}(s)$, $\Xi_{ell}(s)$ and $\Xi_{par}(s)$, following Fischer [Fi]. Set

$$\Xi_I(s)= \exp\Big(\frac{m+(-1)^k}{6}\{s\log(2\pi)+s(1-s)+(1/2+\kappa)\log\Gamma(s+\kappa)+$$

$$(1/2-\kappa)\log\Gamma(s-\kappa)-\log G(s+\kappa+1)-\log G(s-\kappa+1)\}\Big),$$

which is a zero-free holomorphic function in $\mathbb{C}-(-\infty,|\kappa|)$. It follows from [Fi, Remark 3.1.3] that

$$\frac{\Xi_I'}{\Xi_I}(s)= -(2s-1)\frac{m+(-1)^k}{12}(\psi(s+\kappa)+\psi(s-\kappa)).$$

Set

$$\Xi_{hyp}(s)= \begin{cases} Z_{\Gamma,m,+}(s) & \ldots \text{ if } k \text{ is even} \\ Z_{\Gamma,m,-}(s) & \ldots \text{ if } k \text{ is odd.} \end{cases}$$

Then, $\Xi_{hyp}(s)$ is a holomorphic function in the half-plane $Re(s)>1$. Define a holomorphic function $\Xi_{ell}: \mathbb{C}-(-\infty,|\kappa|) \longrightarrow \mathbb{C}$ by

$$\Xi_{ell}(s)= \prod_{j=0}^{1}\left(\frac{\Gamma\left(\frac{s-\kappa+j}{2}\right)}{\Gamma\left(\frac{s+\kappa+1-j}{2}\right)}\right)^{\epsilon_2(k-2j,m)/4}\cdot\prod_{j=0}^{2}\left(\frac{\Gamma\left(\frac{s-\kappa+j}{3}\right)}{\Gamma\left(\frac{s+\kappa+2-j}{3}\right)}\right)^{\epsilon_3(k-2j,m)/3}.$$

Moreover, set

$$\Xi_{par}(s)= 2^{-(m+(-1)^k)s}\cdot\left(\frac{\Gamma(s+\kappa)}{\Gamma(s-\kappa)}\right)^{(m+(-1)^k)/2-\beta*}\cdot$$

$$\prod_{r\in R_k^*}(\sin(\pi\beta_r))^{-s}\cdot\left(\frac{\Gamma(s-\kappa)}{\Gamma(s)\Gamma(s+1/2)}\right)^{t_\infty},$$

where

(4.1) $\beta^*=\sum_{r\in R_k^*}\beta_r,\quad \beta_r=\langle -r^2/4m\rangle$ (for the set R_k^*, see (3.9)).

The function $\Xi_{par}(s)$ defines a holomorphic function in $\mathbb{C}-(-\infty,\mathrm{Max}(1,|\kappa|))$. We remark that the form of our $\Xi_{par}(s)$ is a little different from that in [Fi, Corollary 2.4.22]. Recall that $\varphi(s)=\det\Phi(s)$. We set, for $\mathrm{Re}(s)>1$,

(4.2) $$\xi_{par,\Phi}(s)= \frac{1}{2s-1}\mathrm{tr}(1_{t_\infty}-\Phi(1/2))+$$

$$\frac{2s-1}{4\pi}\int_{-\infty}^{\infty}\left(\frac{1}{(s-1/2)^2+t^2}-\frac{1}{1/4+t^2}\right)\cdot\frac{\varphi'}{\varphi}(1/2+it)dt$$

It has been proved in [Fi, Lemma 2.4.19 and p.103, (2.4.6)] that the integral on the right hand side of (4.2) is absolutely convergent for $\mathrm{Re}(s)>1/2$ and that $\xi_{par,\Phi}(s)$ is analytically continued to a meromorphic function in the whole s-plane which has at most only simple poles whose residues are all rational integers. In particular $\xi_{par,\Phi}(s)$ is holomorphic in the half plane $\{s\in\mathbb{C}|\ \mathrm{Re}(s)>1/2\}$ and has a simple pole at $s=1/2$ with the residue $(t_\infty-\mathrm{tr}\Phi(1/2))/2\in\mathbb{Z}$ (note that $\Phi(1/2)$ is a diagonalizable matrix with eigen values ±1, since $\Phi(1/2)$ is a hermitian matrix and $\Phi(1/2)^2=1_{t_\infty}$ ([Ar, Propositions 2.3, 2.5])). Moreover with the help of [He, p.440, Proposition 2.17], one can prove the functional equation

(4.3) $$\xi_{par,\Phi}(s) + \xi_{par,\Phi}(1-s) = \frac{\varphi'}{\varphi}(s).$$

(One can also prove (4.3) by using [Fi, Lemma 2.4.16 and (2.4.6)]). Now the resolvent trace formula (Theorem 3.1) can be reformulated as

PROPOSITION 4.1. *Assume that* Re(s), Re(a)>Max(1, $|\kappa|$). *Then,*

$$S_{\Gamma,k,m}(s,a) = \frac{1}{2s-1}\left(\frac{\Xi'_I}{\Xi_I}(s) + \frac{\Xi'_{hyp}}{\Xi_{hyp}}(s) + \frac{\Xi'_{ell}}{\Xi_{ell}}(s) + \frac{\Xi'_{par}}{\Xi_{par}}(s) + \xi_{par,\Phi}(s)\right)$$

$$- \frac{1}{2a-1}\left(\frac{\Xi'_I}{\Xi_I}(a) + \frac{\Xi'_{hyp}}{\Xi_{hyp}}(a) + \frac{\Xi'_{ell}}{\Xi_{ell}}(a) + \frac{\Xi'_{par}}{\Xi_{par}}(a) + \xi_{par,\Phi}(a)\right).$$

Let $\mathcal{H}_\kappa$, Δ_κ, $\mathcal{D}_\kappa$, $\Delta^\sim_\kappa$, $\mathcal{D}^\sim_\kappa$ be the same as in § 1. For $s \in \mathbb{C}$, denote by $\mathcal{H}_\kappa(s)$ the subspace consisting of $f \in \mathcal{D}^\sim_\kappa$ with $-\Delta^\sim_\kappa f = s(1-s)f$. It follows from [Ro1, Satz 5.6, 5.7] that

(4.4) $$\mathcal{H}_\kappa(s) = \{f \in \mathcal{D}_\kappa \mid -\Delta_\kappa f = s(1-s)f\}.$$

Let $d_{k,m}(s)$ denote the multiplicity of the eigen value $s(1-s)$ of the self-adjoint operator $-\Delta^\sim_\kappa : \mathcal{D}^\sim_\kappa \longrightarrow \mathcal{H}_\kappa$. Obviously,

$$d_{k,m}(s) = \dim_{\mathbb{C}}\mathcal{H}_\kappa(s).$$

By the definition of the numbers r_n, $\mathcal{H}_\kappa(s) \neq \{0\}$ if and only if $s = 1/2 \pm ir_n$ $(n \in \mathbb{N}_0)$.

We set

$$\Xi^*(s) = \Xi_I(s)\Xi_{hyp}(s)\Xi_{ell}(s)\Xi_{par}(s).$$

Via the formula in Proposition 4.1 and (4.3), $(\Xi^{*\prime}/\Xi^*)(s)$ can be analytically continued to a meromorphic function in the whole s-plane satisfying the functional equation

$$(4.5) \qquad \frac{\Xi^{*\prime}}{\Xi^{*}}(s) + \frac{\Xi^{*\prime}}{\Xi^{*}}(1-s) + \frac{\varphi'}{\varphi}(s)=0.$$

The function $(2s-1)S_{\Gamma,k,m}(s,a)$, with a being fixed, indicates a meromorphic function in the whole s-plane that has at most simple poles at $s=1/2\pm ir_n$ ($n \in \mathbb{N}_0$) with the residue $d_{k,m}(1/2+ir_n)$ (resp. $2d_{k,m}(1/2)$) if $r_n \neq 0$ (resp. $r_n=0$). Therefore by Proposition 4.1, the singularities of the function $(\Xi^{*\prime}/\Xi^{*})(s)$ are located at $s=1/2\pm ir_n$ ($n \in \mathbb{N}_0$) and at any poles of $\xi_{par,\Phi}(s)$ (they are all simple poles). As we have seen, the residues at those simple poles of $(\Xi^{*\prime}/\Xi^{*})(s)$ are all rational integers. Consequently, $\Xi^{*}(s)$ itself is analytically continued to a meromorphic function in the whole s-plane. We note that

$$(4.6) \qquad \varphi(1/2)= (-1)^{(t_\infty-\mathrm{tr}\Phi(1/2))/2}$$

and that $(\Xi^{*\prime}/\Xi^{*})(s)$ has a simple pole at $s=1/2$ with the residue $2d_{k,m}(1/2)-(t_\infty-\mathrm{tr}\Phi(1/2))/2$ ($\in \mathbb{Z}$) according to Proposition 4.1 and (4.2). Hence the identities (4.5), (4.6) imply that $\Xi^{*}(s)$ itself satisfies the functional equation

$$(4.7) \qquad \Xi^{*}(1-s)= \varphi(s)\Xi^{*}(s).$$

Furthermore all the poles of the function

$$(4.8) \qquad \frac{\Xi'_I}{\Xi_I}(s)+ \frac{\Xi'_{ell}}{\Xi_{ell}}(s)+ \frac{\Xi'_{par}}{\Xi_{par}}(s)$$

are located at $s=\pm\kappa-\ell$, $s=-\ell$, $s=-1/2-\ell$ ($\ell \in \mathbb{N}_0$) (they are all simple poles). It is immediate to see that the residues at the poles $s=-\ell$, $-1/2-\ell$ ($\ell \in \mathbb{N}_0$) are rational integers. We set, for $\ell \in \mathbb{N}_0$,

$$(4.9) \qquad \lambda(m,k;\ell)= \frac{m+(-1)^k}{12}(2\kappa-2\ell+5)-\frac{\epsilon_2(k-2\ell,m)}{4}-\frac{\epsilon_3(k-2\ell,m)}{3}-\beta^*-t_\infty,$$

$$(4.10) \qquad \mu(m,k;\ell)= \frac{m+(-1)^k}{12}(-2\kappa-2\ell-7)+\frac{\epsilon_2(k+2\ell-2,m)}{4}+\frac{\epsilon_3(k+2\ell-4,m)}{3}+\beta^*$$

(for $\epsilon_2(k,m)$, $\epsilon_3(k,m)$, see (3.7) and for $^*\beta$ see (4.1)).

It is easy to see from the definition of $\Xi_I(s)$, $\Xi_{ell}(s)$, $\Xi_{par}(s)$ that the residue of the function (4.8) at each simple pole $s=\kappa-\ell$ (resp. $s=-\kappa-\ell$) with $\ell \in \mathbb{N}_0$ equals the number $\lambda(m,k;\ell)$ (resp. $\mu(m,k;\ell)$).

LEMMA 4.2. *The residues of the function* (4.8) *at the simple poles are all rational integers.*

Proof. It suffices to prove that $\lambda(m,k;\ell)$, $\mu(m,k;\ell)$ with any $\ell \in \mathbb{N}_0$ are rational integers. We discuss only the case of $\lambda(m,k;\ell)$. The assertion that $\lambda(m,k;\ell) \in \mathbb{Z}$ depends on k mod 12 and ℓ mod 6. Therefore we may assume that k is a positive integer sufficiently large relative to ℓ. Taking the residues at $s=\kappa-\ell$ (>1) of the both hand sides of the formula in Proposition 4.1, we have

$$\lambda(m,k;\ell)= d_{k,m}(\kappa-\ell) \in \mathbb{N}_0.$$

q.e.d.

According to Lemma 4.2, the function

$$\Xi_I(s)\Xi_{ell}(s)\Xi_{par}(s)$$

is meromorphically continued over the whole s-plane. Therefore, $\Xi_{hyp}(s)$ can be analytically continued to a meromorphic function in the whole s-plane via the relation

$$\Xi_{hyp}(s)=\Xi^{*}(s)\cdot(\Xi_{I}(s)\Xi_{ell}(s)\Xi_{par}(s))^{-1}.$$

Thus we obtain the following theorem.

THEOREM 4.3. *The Selberg zeta functions* $Z_{\Gamma,m,\pm}(s)$ *are analytically continued to meromorphic functions in the whole* s-*plane which satisfy the functional equation* (4.7).

§ 5. *The spaces of Jacobi forms*

We recall the notion of (holomorphic) Jacobi forms and skew-holomorphic Jacobi forms following Eichler-Zagier [E-Z] and Skoruppa [Sk2].

Denote by G^{J} the real Jacobi group of degree one. As a set,

$$G^{J}=\{(M,(\lambda,\mu),\rho)\mid M\in SL_{2}(\mathbb{R}),\ \lambda,\ \mu,\ \rho\in\mathbb{R}\}.$$

For two elements $g_{j}=(M_{j},(\lambda_{j},\mu_{j}),\rho_{j})\in G^{J}$ $(j=1,2)$, the composition $g_{1}g_{2}$ is defined by

$$g_{1}g_{2}=(M_{1}M_{2},(\lambda_{1},\mu_{1})M_{2}+(\lambda_{2},\mu_{2}),\rho_{1}+\rho_{2}+(\lambda_{1},\mu_{1})M_{2}\begin{pmatrix}\mu_{2}\\-\lambda_{2}\end{pmatrix}).$$

The group G^{J} acts on the product space $\mathfrak{H}\times\mathbb{C}$ in the manner:

$$g(\tau,z)=(M\tau,\ \frac{z+\lambda\tau+\mu}{c\tau+d})\quad\text{for } g=(M,(\lambda,\mu),\rho)\in G^{J},\ M=\begin{pmatrix}a&b\\c&d\end{pmatrix},\ (\tau,z)\in\mathfrak{H}\times\mathbb{C}.$$

Let m be a fixed positive integer. Set

$$J_{m}(g,(\tau,z))=e^{m}\left(-\lambda^{2}\tau-2\lambda z-\lambda\mu-\rho+\frac{c(z+\lambda\tau+\mu)^{2}}{c\tau+d}\right),$$

where g and (τ,z) be the same as above. Then, $J_{m}(g,(\tau,z))$ satisfies the property as an automorphic factor:

$$J_m(g_1g_2,(\tau,z))= J_m(g_1,g_2(\tau,z))J_m(g_2,(\tau,z)) \quad (g_1,\ g_2 \in G^J).$$

Let ℓ be a non-negative integer. For simplicity we discuss only the case of $\Gamma=SL_2(\mathbb{Z})$. Set

$$\Gamma^J= \{(M,(\lambda,\mu),\rho) \mid M \in SL_2(\mathbb{Z}),\ \lambda,\ \mu,\ \rho \in \mathbb{Z}\},$$

which forms a discrete subgroup of G^J. For $\tau \in \mathfrak{H}$, we write $\eta=\mathrm{Im}(\tau)$. Then the space $J_{\ell,m}$ (resp. $J^*_{\ell,m}$) of holomorphic Jacobi forms (resp. skew-holomorphic Jacobi forms) of index m and weight ℓ is defined to be the space consisting of all functions ϕ: $\mathfrak{H}\times\mathbb{C} \longrightarrow \mathbb{C}$ which satisfy the following conditions:

(i) $\phi(\tau,z)$ is a holomorphic function in $\tau \in \mathfrak{H}$ and $z \in \mathbb{C}$ (resp. $\phi(\tau,z)$ is a smooth function in $\tau \in \mathfrak{H}$ and holomorphic in z),

(ii) $\phi(\tau,z)$ satisfies the functional equation

$$\phi(\gamma(\tau,z))= J_m(\gamma,(\tau,z))J(M,\tau)^{\ell}\phi(\tau,z)$$

(resp. $\phi(\gamma(\tau,z))= J_m(\gamma,(\tau,z))\overline{J(M,\tau)}^{\ell-1}|J(M,\tau)|\phi(\tau,z))$ for all $\gamma \in \Gamma^J$.

(iii) $\phi(\tau,z)$ has the Fourier-Jacobi expansion of the form:

$$\phi(\tau,z)= \sum_{n,r \in \mathbb{Z},\ 4mn\geq r^2} c(n,r)e(n\tau+rz)$$

(resp. $\phi(\tau,z)= \sum_{n,r \in \mathbb{Z},\ 4mn\leq r^2} c(n,r)e\left(n\bar{\tau}+ \frac{1}{2m}ir^2\eta+rz\right)$).

Denote by $J^{cusp}_{\ell,m}$ (resp. $J^{*cusp}_{\ell,m}$) the subspace of cusp forms of $J_{\ell,m}$ (resp. $J^*_{\ell,m}$). Namely, $J^{cusp}_{\ell,m}$ (resp. $J^{*cusp}_{\ell,m}$) consists of all Jacobi forms $\phi \in J_{\ell,m}$ (resp. all skew-holomorphic Jacobi forms $\phi \in J^*_{\ell,m}$) whose Fourier coefficients c(n,r) in the above (iii) vanish if $4mn=r^2$.

Now let k be a rational integer and $\kappa=(k-1/2)/2$ as before. For s $\in$

$\mathbb{C}$, let $\mathcal{H}_\kappa(s)$ denote the subspace of $\mathcal{H}_\kappa$ given in (4.4). We are much concerned with the special subspsce $\mathcal{H}_\kappa(\kappa)$ (resp. $\mathcal{H}_\kappa(-\kappa)$) if $\kappa>0$ (resp. $\kappa<0$). As is easily seen from the results of [Ro, §2-§5], one has

$$(5.1)\qquad \begin{aligned} &\mathcal{H}_\kappa(\kappa) = \{f \in \mathcal{H}_\kappa \mid \eta^{-\kappa} f(\tau) \text{ is holomorphic in } \tau \in \mathfrak{H}\} \quad \text{if } \kappa>0, \\ &\mathcal{H}_\kappa(-\kappa) = \{f \in \mathcal{H}_\kappa \mid \eta^{\kappa} f(\tau) \text{ is anti-holomorphic in } \tau \in \mathfrak{H}\} \quad \text{if } \kappa<0. \end{aligned}$$

Assume that $\kappa>0$ (i.e., $k\geq 1$). Any $\phi \in J_{k,m}$ has an expression as a linear combination of theta series:

$$(5.2)\qquad \phi(\tau,z) = \sum_{r\in R} \eta^{-\kappa} f_r(\tau)\theta_r(\tau,z).$$

If $k\geq 2$ and $\phi \in J_{k,m}^{cusp}$, then the collection $f(\tau)=(f_r(\tau))_{r\in R}$ as a column vector gives rise to an element of $\mathcal{H}_\kappa(\kappa)$. If $k=1$ ($\kappa=1/4$), then, for any $\phi \in J_{1,m}$, $f(\tau)=(f_r(\tau))_{r\in R}$ becomes an element of $\mathcal{H}_\kappa(\kappa)$, since it is easy to see from the Fourier-Jacobi expansion of ϕ that

$$\int_{\mathcal{F}} |f(\tau)|^2 d\omega(\tau) < +\infty.$$

Next let $\kappa<0$ (i.e., $k\leq 0$). In this case any $\phi \in J^*_{1-k,m}$ has an expression of the form

$$(5.3)\qquad \phi(\tau,z) = \sum_{r\in R} \eta^{\kappa} g_r(\tau)\theta_r(\tau,z).$$

If $k\leq -1$ and $\phi \in J^{*cusp}_{1-k,m}$, then as is easily seen, the collection $g(\tau)= (g_r(\tau))_{r\in R}$ as a column vector becomes an element of $\mathcal{H}_\kappa(-\kappa)$. If $k=0$ ($\kappa=-1/4$), then, for any $\phi \in J^*_{1,m}$, $g(\tau)=(g_r(\tau))_{r\in R}$ is also an element of $\mathcal{H}_\kappa(-\kappa)$.

RROPOSITION 5.1. *Let* k *be a rational integer.*

(i) *If* $k\geq 2$ (*resp.* $k=1$), *the space* $J_{k,m}^{cusp}$ (*resp.* $J_{1,m}$) *is isomorphic to*

$\mathfrak{K}_\kappa(\kappa)$ *via the correspondence* $\phi \longrightarrow (f_r)_{r\in R}$ *in* (5.2) *as* $\mathbb{C}$-*linear spaces.*

(ii) *If* $k\leq -1$ (*resp.* $k=0$), *the space* $J^{*cusp}_{1-k,m}$ (*resp.* $J^*_{1,m}$) *is isomorphic to* $\mathfrak{K}_\kappa(-\kappa)$ *via the correspondence* $\phi \longrightarrow (g_r)_{r\in R}$ *in* (5.3) *as* $\mathbb{C}$-*linear spaces.*

The proof is immediate from (5.1), so we omit it.

Now we employ the resolvent trace formula to obtain some information on the dimensions of the spaces of Jacobi forms.

Denote by $\lambda(m,k)$ (resp. $\mu(m,k)$) the number $\lambda(m,k;0)$ (resp. $\mu(m,k;0)$) given by (4.9) (resp. (4.10)) for $\ell=0$. Namely,

$$\lambda(m,k)=\frac{m+(-1)^k}{24}(2k+9)-\frac{\epsilon_2(k,m)}{4}-\frac{\epsilon_3(k,m)}{3}-\beta^*-t_\infty$$

(5.4)

$$\mu(m,k)=\frac{m+(-1)^k}{24}(-2k-13)+\frac{\epsilon_2(k-2,m)}{4}+\frac{\epsilon_3(k-4,m)}{3}+\beta^*.$$

For a meromorphic function $f(z)$ having a pole at $z=\alpha$, denote by $\mathrm{Res}_{z=\alpha}f(z)$ the residue at the pole $z=\alpha$ of $f(z)$.

THEOREM 5.2. *Let* m *be a positive integer and* k *an integer.*

(i) *If* $k\geq 3$, *then,* $\dim_{\mathbb{C}} J^{cusp}_{k,m} = \lambda(m,k)$.

(ii) *If* $k=2$, *then,* $\dim_{\mathbb{C}} J^{cusp}_{2,m} = \mathrm{Res}_{s=3/4}\left((Z'_{\Gamma,m,+}/Z_{\Gamma,m,+})(s)\right)+\lambda(m,2)$.

(iii) *If* $k=1$, *then,* $\dim_{\mathbb{C}} J_{1,m} = \mathrm{Res}_{s=3/4}\left((Z'_{\Gamma,m,-}/Z_{\Gamma,m,-})(s)\right)$.

(iv) *If* $k=0$, *then,* $\dim_{\mathbb{C}} J^*_{1,m} = \mathrm{Res}_{s=3/4}\left((Z'_{\Gamma,m,+}/Z_{\Gamma,m,+})(s)\right)$.

(v) *If* $k=-1$, *then,* $\dim_{\mathbb{C}} J^{*cusp}_{2,m} = \mathrm{Res}_{s=3/4}\left((Z'_{\Gamma,m,-}/Z_{\Gamma,m,-})(s)\right)+\mu(m,-1)$.

(vi) *If* $k\leq -2$, *then*, $\dim_{\mathbb{C}} J^{*cusp}_{1-k,m} = \mu(m,k)$.

Proof. According to Proposition 4.1, we have

$$(5.5)\quad (2s-1)S_{\Gamma,k,m}(s,a) = \frac{\Xi'_{hyp}}{\Xi_{hyp}}(s) + \frac{\Xi'_{I}}{\Xi_{I}}(s) + \frac{\Xi'_{ell}}{\Xi_{ell}}(s) + \frac{\Xi'_{par}}{\Xi_{par}}(s) + \xi_{par,\Phi}(s)$$

$$- \frac{2s-1}{2a-1}\left(\frac{\Xi'_{hyp}}{\Xi_{hyp}}(a) + \frac{\Xi'_{I}}{\Xi_{I}}(a) + \frac{\Xi'_{ell}}{\Xi_{ell}}(a) + \frac{\Xi'_{par}}{\Xi_{par}}(a) + \xi_{par,\Phi}(a)\right)$$

Let a be fixed ($\mathrm{Re}(a) > \mathrm{Max}(1, |\kappa|)$). The functions of s on the both hand sides of (5.5) are meromorphically continued over the whole s-plane and $\xi_{par,\Phi}(s)$ is holomorphic for $\mathrm{Re}(s) > 1/2$. First assume that $k\geq 2$ (then, $\kappa > 1/2$). Comparing the residues at the simple pole $s=\kappa$ of the functions on the both hand sides of (5.5), we have

$$d_{k,m}(\kappa) = \lambda(m,k) \qquad \ldots \quad \text{if } k>2$$

$$d_{2,m}(\kappa) = \mathrm{Res}_{s=3/4}\left((Z'_{\Gamma,m,+}/Z_{\Gamma,m,+})(s)\right) + \lambda(m,2) \quad \ldots \text{ if } k=2.$$

Thus with the help of (i) of Propositin 5.1, we immediately obtain the assertions (i), (ii). Next assume $k\leq -1$ (then, $\kappa < -1/2$). In this case we calculate the residues of the function $(2s-1)S_{\Gamma,k,m}(s,a)$ at the simple pole $s=-\kappa$ in two manners via (5.5). Thus together with Proposition 5.1, (ii), we get the assertions (v), (vi). If $k=1$, then, $\kappa=1/4$. We consider the residues of the functions on the both hand sides of (5.5) at the simple pole $s=1-\kappa=3/4$. Then the assertion (iii) follows in a manner similar to the proofs of (i), (ii). The assertion (iv) is similarly verified. q.e.d.

The identities (0.1) in the introduction are a direct consequence of Theorem 5.2. Moreover as an immediate corollary to it, we obtain the following.

COROLLARY 5.3. (i) $\dim_{\mathbb{C}} J_{2,m}^{cusp} = \dim_{\mathbb{C}} J_{1,m}^{*} + \lambda(m,2)$.
(ii) $\dim_{\mathbb{C}} J_{2,m}^{*cusp} = \dim_{\mathbb{C}} J_{1,m} + \mu(m,-1)$.

REMARK. Skoruppa-Zagier [S-Z1, 2] obtained a general trace formula for Hecke operators acting on the space of Jacobi forms. Corollary 5.3, (i) is a part of their trace formula for k=2 ([S-Z2, § 3, (11)].

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