Bak's work on K-theory of rings

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On the occasion of his 65 birthday,
to TONY BAK; with respect and affection

Abstract. This paper studies the work of Bak in Algebra and (lower) Algebraic K-theory and some later developments stimulated by them. We present an overview of his work in these areas, describe the setup and problems as well as the methods he introduced to attack these problems and state some of the crucial theorems. The aim is to analyse in detail some of his methods which are important and promising for further work in the subject. Among the topics covered are, unitary/general quadratic groups over form rings, structure theory and stability for such groups, quadratic K_2 and the quadratic Steinberg groups, nonstable K-theory and localisation-completion, intermediate subgroups, congruence subgroup problem, dimension theory and surgery theory.

The appendix by Max Karoubi states some periodicity theorems and conjectures in an algebraic context which are related to Bak's work.

Contents

1. Introduction

The main contributions of Anthony Bak to Algebra may be summarised as follows:

• Introduction of generalised unitary groups (aka general quadratic groups, etc.), form rings and form ideals.

• Structure of classical groups in the stable range.

• Unitary K-theory, exact sequences, K-theory of forms.

• K_1-Stability for classical groups, Λ-stable rank, generalised Witt theorems (1969–2003, final form joint with Guoping Tang, Viktor Petrov).

• Congruence subgroup problem (joint with Ulf Rehmann).

• Non-abelian K-theory, nilpotent K-theory, localisation-completion.

• Structure of classical groups in the metastable range, sandwich classification of intermediate subgroups (joint with Nikolai Vavilov, Alexei Stepanov).

• Dimension theory.

• Induction and powers of the fundamental ideal in group rings (joint with Nikolai Vavilov, Guoping Tang).

• Centrality of unitary K_2 (joint with Guoping Tang), Hasse norm theorem for K_2 (joint with Ulf Rehmann).

• Grothendieck and Witt groups of finite groups (joint with Winfred Scharlau).

• Higher degree forms and exceptional groups (joint with Nikolai Vavilov).

In the present paper we describe basic ideas and techniques underlying some of these contributions, and also some later development stimulated by them.

We agree with Dieudonné [67] that had Anthony Bak not written anything apart from his Thesis [6] and the book [14], he would already secure himself a place in the history of Mathematics of the XX century. In the first half of the paper we give a very gentle introduction to the main themes of [6] and [14]: form rings and form ideals, Bak's unitary groups, unitary K-theory, structure in the stable range, unitary stability, placing them in a broader historical context. After this very [s]low start we accelerate and become somewhat more specific in the middle of the article, where we try to explicitly state some of Bak's most influential results, and somewhat more technical towards the end, where we try to explain some of the recent results and methods.

The present paper covers roughly half of the published work by Anthony Bak, here we do not touch the topological half of his work, related to surgery theory, global actions, transformation groups and smooth actions, except the short Section 16 on surgery theory and transformation groups. We should not dwell at the point as to how much this survey reflects our personal prospective of Bak's work, the results which we understand better, which we invoked in own work, the ones on which we cooperated with Tony, his ideas and methods we subsequently used and generalised. His published work is so rich in ideas and so varied that other experts would make a completely different choice as to what Bak's most important, relevant and influential results are.

2. K-THEORY OF RINGS

In its most familiar versions algebraic K-theory consists in the study of certain functors K_i from rings to groups. These functors associate to an associative ring R a certain sequence of groups $K_i(R)$, which encode very deep arithmetic information about the ring. The functor K_0 was originally introduced in the algebro-geometric context by Alexandre Grothendieck [54] and then interpreted purely algebraically by Jean-Pierre Serre [125].

In topology, higher K-functors are obtained simply by applying K_0 to successive suspensions. This works thanks to Bott periodicity, from which it also follows that topological K-groups are periodic [4]. In the 60-ies it was not at all clear, what an appropriate algebraic analogue of suspension should be. Thus, the usual constructions of higher algebraic K-functors went in three steps. First, one associated to the ring R a certain algebraic-like group $G(R)$. Second, one associated to $G(R)$ an object $X(G(R))$ of a different nature, say a topological space, a simplicial space, or a category. Finally one recovered $K_i(R)$ from $X(G(R))$ as the values of some familiar functors, such as homotopy groups. But first one needs the group $G(R)$!

Historically the first version of algebraic K-theory was *linear* K-theory, which used the general linear group $\mathrm{GL}(n,R)$ — or rather the direct limit of these groups as n goes to infinity — as the model group. Bass used it to define K_1, Milnor to define K_2, and subsequently Quillen and others to define higher K-functors.

Let $G=\mathrm{GL}(n,R)$ be the general linear group of degree n over an associative ring R with 1. Recall that $\mathrm{GL}(n,R)$ is the group of all two-sided invertible square matrices of degree n over R, or, in other words, the multiplicative group of the full matrix ring $M(n,R)$. When one thinks of $R\mapsto \mathrm{GL}(n,R)$ as a functor from rings to groups, one writes GL_n. In the sequel for a matrix $g\in G$ we denote by g_{ij} its matrix entry in the position (i,j), so that $g=(g_{ij})$, $1\le i,j\le n$. The inverse of g will be denoted by $g^{-1}=(g'_{ij})$, $1\le i,j\le n$.

The first step in the construction of algebraic K-theory was done by Hyman Bass [45]. There is a standard embedding

$$\mathrm{GL}(n,R)\longrightarrow \mathrm{GL}(n+1,R),\qquad g\mapsto \begin{pmatrix} g & 0\\ 0 & 1\end{pmatrix},$$

called the stabilisation map, which allows us to identify $\mathrm{GL}(n,R)$ with a subgroup in $\mathrm{GL}(n+1,R)$. Now we can consider the stable general linear group

$$\mathrm{GL}(R)=\varinjlim_n \mathrm{GL}(n,R),$$

which is the direct limit (effectively the union) of the $\mathrm{GL}(n,R)$ under the stabilisation embeddings.

A crucial role is played by the elementary subgroup of $\mathrm{GL}(R)$. As usual we denote by e the identity matrix of degree n and by e_{ij} a standard matrix unit, i.e. the matrix that has 1 in the position (i,j) and zeros elsewhere. An elementary transvection $t_{ij}(\xi)$ is a matrix of the form

$$t_{ij}(\xi)=e+\xi e_{ij},\qquad \xi\in R,\quad 1\le i\ne j\le n.$$

An elementary transvection $t_{ij}(\xi)$ only differs from the identity matrix in the position (i,j), $i \neq j$, where it has ξ instead of 0. In other words, multiplication by an elementary transvection on the left/right performs what in an undergraduate linear algebra course would be called a row/column elementary transformation 'of the first kind'.

The work of Bass focused attention on the subgroup $E = E(n,R)$ of the general linear group $G = \mathrm{GL}(n,R)$, that is generated by all the elementary transvections. Thus

$$E = E(n,R) = \langle t_{ij}(\xi),\ \xi \in R,\ 1 \leq i \neq j \leq n\rangle$$

This subgroup is called the (absolute) *elementary group* of degree n over R. Since the stabilisation map sends $E(n,R)$ to $E(n+1,R)$, we can define the stable elementary group $E(R) = \varinjlim E(n,R)$. A crucial observation known as the Whitehead lemma, asserts that modulo $E(R)$ the product of two matrices in $\mathrm{GL}(n,R)$ is the same as their direct sum, and in particular, $E(R) = [\mathrm{GL}(R), \mathrm{GL}(R)]$. At this point Bass defines

$$K_1(R) = \mathrm{GL}(R)/E(R) = \mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)]$$

as the abelianisation of $\mathrm{GL}(R)$. Indeed algebraic K-theory was born as Bass observed that the functors K_0 and K_1 together with their relative versions fit into a unified theory with important applications in algebra, algebraic geometry and number theory.

Both for the development of the theory and for the sake of applications one has to extend these definitions to include relative groups. For an ideal I of R, one defines the corresponding reduction homomorphism

$$\pi_I : \mathrm{GL}(n,R) \longrightarrow \mathrm{GL}(n,R/I), \qquad (g_{ij}) \mapsto (g_{ij} + I).$$

Now the *principal congruence subgroup* $\mathrm{GL}(n,R,I)$ of level I is the kernel of reduction homomorpism π_I, while the *full congruence subgroup* $C(n,R,I)$ of level I is the inverse image of the centre of $\mathrm{GL}(n,R/I)$ with respect to this homomorphism. Clearly both are normal subgroups of $\mathrm{GL}(n,R)$.

Again, let $I \trianglelefteq R$ and let $x = t_{ij}(\xi)$ be an elementary transvection. Somewhat loosely we say that x is of level I, provided $\xi \in I$. One can consider the subgroup generated by all the elementary transvections of level I:

$$E(n,I) = \langle t_{ij}(\xi), \xi \in I, 1 \leq i \neq j \leq n\rangle.$$

This group is contained in the absolute elementary subgroup $E(n,R)$ and does not depend on the choice of an ambient ring R with 1. However, in general $E(n,I)$ has little chances to be normal in $\mathrm{GL}(n,R)$. The relative elementary subgroup $E(n,R,I)$ is defined as the normal closure of $E(n,I)$ in $E(n,R)$:

$$E(n,R,I) = \langle t_{ij}(\xi), \xi \in I, 1 \leq i \neq j \leq n\rangle^{E(n,R)}.$$

Applying the stabilisation embeddings to the families $\mathrm{GL}(n,R,I)$ and $E(n,R,I)$ generates stable versions $\mathrm{GL}(R,I)$ and $E(R,I)$, respectively, of these groups. There is no stable version of $C(n,R,I)$, though, as the stability map does not send $C(n,R,I)$ into $C(n+1,R,I)$. At this point Bass proves his famous "Whitehead lemma".

Theorem 2.1. *For any associative ring R and any ideal $I \trianglelefteq R$ one has*

$$E(R,I) = [E(R), E(R,I)] = [\mathrm{GL}(R), \mathrm{GL}(R,I)].$$

In particular $E(R,I)$ is normal in $\mathrm{GL}(R)$ and one can define the relative K_1-functor of a pair (R,I) by

$$K_1(R,I) = \mathrm{GL}(R,I)/E(R,I).$$

In the sequel we state most of the results only in the absolute case but an appropriate version of these results holds also in the relative case.

As one of important applications in algebra, Bass [45] relates the normal subgroup structure of $\mathrm{GL}(R)$ to the ideal structure of R. This leap in generality is considered as the starting point of the modern theory of linear groups.

Theorem 2.2. *Let R be an arbitrary associative ring and $H \leq \mathrm{GL}(R)$ be a subgroup normalised by the elementary group $E(R)$. Then there exists a unique ideal $I \trianglelefteq R$ such, that*

$$E(R,I) \leq H \leq \mathrm{GL}(R,I).$$

Conversely, any subgroup H satisfying these inclusions is (by Theorem 2.1) normal in $\mathrm{GL}(R)$.

Quite remarkably this result holds for arbitrary associative rings. Thus, an explicit enumeration of all normal subgroups of $\mathrm{GL}(R)$ amounts to the calculation of $K_1(R,I)$ for all ideals I in R.

The group K_1 answers essentially the question as to how far $\mathrm{GL}(n,R)$ falls short of being spanned by elementary generators. A few years later Milnor [101], [102], building on the work of Steinberg [133], [134] and Moore [105], introduced the group K_2, which measures essentially to which extent all relations among elementary generators follow from the obvious ones.

What are the obvious relations among the elementary transvections $t_{ij}(\xi)$, $\xi \in R$, $1 \leq i \neq j \leq n$? First of all, they are additive in ξ, in other words,

$$t_{ij}(\xi)t_{ij}(\zeta) = t_{ij}(\xi+\zeta) \tag{R1}$$

for any $\xi, \zeta \in R$. Secondly, they satisfy the Chevalley commutator formula, which in this case boils down to the following

$$[t_{ij}(\xi), t_{hl}(\zeta)] = \begin{cases} e, & \text{if } \ j \neq h, \ i \neq l; \\ t_{il}(\xi\zeta), & \text{if } \ j = h, \ i \neq l; \\ t_{hj}(-\zeta\xi), & \text{if } \ j \neq h, \ i = l; \end{cases} \tag{R2}$$

where $[h,g] = hgh^{-1}g^{-1}$ is the commutator of two elements $h, g \in G$. The case $j = h$, $i = l$ is excluded from the above formula, since there is no easy expression for $[t_{ij}(\xi), t_{ji}(\zeta)]$.

Let as before R be any associative ring and $n \geq 3$. Then the linear Steinberg group $\mathrm{St}(n,R)$ of degree n over R is defined as the group generated by generators $x_{ij}(\xi)$ subject to defining relations (R1) and (R2). For the case $n = 2$ relation (R2) becomes vacuous.

The usual definition of the group $\mathrm{St}(2, R)$ in [102] is only adequate for rings with many units, such as fields, but not in general.

There is a canonical homomorphism

$$\mathrm{St}(n, R) \longrightarrow E(n, R), \qquad x_{ij}(\xi) \mapsto t_{ij}(\xi).$$

By definition $K_2(n, R)$ is the kernel of this homomorphism and consequently there is an exact sequence

$$1 \longrightarrow K_2(n, R) \longrightarrow \mathrm{St}(n, R) \longrightarrow E(n, R) \longrightarrow 1.$$

As above, one has a stabilisation map $\mathrm{St}(n, R) \to \mathrm{St}(n+1, R)$, sending $x_{ij}(\xi)$ viewed as an element of the former group to $x_{ij}(\xi)$ viewed as an element of the latter. Since all the relations defining $\mathrm{St}(n, R)$ also hold in $\mathrm{St}(n+1, R)$ this extends to a homomorphism, thus a homomorphism on the level of K_2, i.e., $K_2(n, R) \to K_2(n+1, R)$. But since $\mathrm{St}(n+1, R)$ has new relations, this homomorphism does not have to be an embedding. Still, one can form the stable Steinberg group as the direct limit $\mathrm{St}(R) = \varinjlim \mathrm{St}(n, R)$. The projections $\mathrm{St}(n, R) \to E(n, R)$ extend to the projection $\mathrm{St}(R) \to E(R)$ and $K_2(R)$ is defined as its kernel. In other words, $K_2(R)$ fits into the exact sequence

$$0 \longrightarrow K_2(R) \longrightarrow \mathrm{St}(R) \longrightarrow \mathrm{GL}(R) \longrightarrow K_1(R) \longrightarrow 0.$$

Quite remarkably Kervaire [90] and Milnor [102] then proved the following crucial result.

Theorem 2.3. *The homomorphism* $\mathrm{St}(R) \to E(R)$ *is the universal central extension of* $E(R)$.

In [49], [50] Bass stressed the importance of the congruence subgroup problem [51] in the development of algebraic K-theory. He feels that interest in the new subject was substantially boosted by its application to help solving the more than half a century old problem above and by the intrinsic connections of K_1 and K_2 with deep arithmetic phenomena such as reciprocity laws of class field theory. After these initial successes — and later after the definition of higher K-functors by Quillen and others, — the whole theory exploded in two directions, which may be summarised as follows.

- Prove similar results in the *non-stable* case, that is when n is fixed.
- Prove similar results for all classical groups, including symplectic, orthogonal and unitary groups.

Both directions, known as **non-stable K-theory** — or, sometimes, for reasons that will soon become clear, non-abelian K-theory — and **unitary K-theory**, respectively, have been proved to be much more difficult than the original stable linear case. In both directions contributions of Tony Bak were extremely important, in some cases decisive. Below we try to describe them in historical prospective, starting with the second one.

3. Classical groups

We believe that in the Spenglerian sense the concept of group is the oldest concept of Mathematics, older than the concept of number itself. It is in fact inseparable from any idea of symmetry, motion, spacial or temporal pattern. From a modern viewpoint the whole classical Geometry, including that described in Euclid's "Elements", is a study of classical groups, their subgroups and actions. But it was hardly phrased that way till the late XIX century. Let us start with a *very* brief history of time before the 1960'ies.

Many results of Euler and Lagrange should be today interpreted as statements about *orthogonal* groups, and, of course, these groups become fully visible in the works of Grassmann, Hamilton, Cayley and Clifford. Similarly, *symplectic* groups were prominent in the works of Abel, Hamilton, Jacobi and Riemann. For a very good reason they were even called *Abelian groups* by Jordan, Dickson and others, until Weyl translated the word *complex* from Latin to Greek. By the same token, we are inclined to believe that Jacobi and Hermite were fully aware of *unitary* groups.

However, modern terminology could hardly have been used before the late 1820'ies when Galois started *to group permutations* ('grouper les permutations') and the early 1840'ies when Cayley officially introduced multiplication of matrices. Thus classical groups make their first official appearance in Jordan's *modest commentary* to the works of Galois [81]. In fact, in his landmark book Jordan defines and studies all classical groups over *prime* fields $\mathbb{F}_p$.

At about the same time *complex* and *real* classical groups made their triumphal appearance in the geometric and analytic context, in the works of Klein, Lie, Killing, Engel, Cartan, Picard and others. The major achievement of that period was the discovery of five exceptional groups and the classification of simple complex Lie groups by Killing, followed by classification of simple real Lie groups by Cartan. The impact of these works was such, that they effectively made all advanced mathematics into the theory of Lie groups, or, as Poincaré put it: "La théorie des groupes est, pour ainsi dire, la mathématique entière". Mathematics of the XX century was Lie theory much in the same sense as Mathematics of the XVIII century was real analysis and that of XIX century — complex analysis. Of course, the second half of the XX century interpreted Lie theory not in original analytic terms, but rather in the purely algebraic setting of algebraic groups, as in the works of Kolchin, Chevalley, Borel, Weil, Rosenlicht, Serre, Tits, Springer, Steinberg, and others.

At the brink of the XX century Dickson, who was familiar both with the book of Jordan and with Cartan's thesis, fully recognised the analogy between finite classical groups and complex Lie groups. He made his point to construct (what we now know as) *Chevalley groups* of all types over an *arbitrary* field and in fact succeeded in constructing all *classical* Chevalley groups in his 1901 paper [63]. He writes "After determining four systems of simple groups in an arbitrary domain of rationality which include the four systems of continuous groups of Lie, the writer has been led to consider the analogous problem for the five isolated simple continuous groups of 14, 52, 78, 133 and 248 parameters." Apparently, in the process of this work Dickson anticipated many crucial subsequent results pertaining to isometry groups of quadratic forms and higher degree forms [127]. However, in his book [62], also first published in 1901, he only systematically treats *classical* groups over *finite* fields. Soon thereafter he switches to other major projects and his remarkable insights concerning forms and groups of Lie type over an arbitrary field have been largely ignored and forgotten.

After that classical groups suddenly became *very* familiar to Frobenius, Burnside, Minkowski, Schur, Blichfeldt and others, who (among many other remarkable things!) proved surprising and highly non-trivial results about their finite subgroups. However, for the most part they still used Jordan's terminology, speaking of Abelian and hypo-Abelian groups, etc. The modern terminology, including the terms classical groups and symplectic group themselves, started to emerge only in the 1930'ies, mainly due to the influence of Hermann Weyl. In [176] he studied finite dimensional representations of classical groups over a field of characteristic 0. At about the same time, the algebraic theory of quadratic forms over an arbitrary field started to (re)emerge in the work of Ernst Witt [179]. In particular, Witt emphasised the crucial distinction of isotropic and anisotropic forms and groups.

Another milestone in the study of classical groups were the books of Jean Dieudonné [64], [65], [66], first published in 1948, 1951 and 1956, respectively. Therein Dieudonné systematically carried over all available structural results obtained by his predecessors to classical groups over arbitrary fields and skew-fields. In particular, he proved in this new setting analogues of all usual theorems on normal structure, generation, automorphisms, geometric properties, etc., which previously had been known only over finite or algebraically closed fields, or fields of characteristic 0.

In the first volume of his monumental work [57], Claude Chevalley gave a very readable account of the classical groups over $\mathbb{R}$ and $\mathbb{C}$ and in subsequent volumes started to develop the theory of algebraic groups. His efforts culminated in a construction of split simple groups of all types over an arbitrary field [58], which eventually led to the classification of simple algebraic groups over algebraically closed fields of arbitrary characteristic [59]. It is often said that Dieudonné's theory of classical groups is devoured by the theory of algebraic groups. There is something to it, but nevertheless we do not think this is completely true, as classical groups over infinite-dimensional skew-fields are not algebraic. In fact soon we shall see examples of classical groups over fields, which are even less algebraic than that!

In his review of [64] Weyl writes: "The term classical groups covers for the author (as it did for the reviewer) the group of linear transformations in n variables and those subgroups that leave certain nondegenerate forms invariant, namely a quadratic or a Hermitian or a skew-symmetric bilinear form." They mean the same for us, but what are quadratic and hermitian forms? In particular, everybody who has studied works of Dickson, Dieudonné and their followers is [pain]fully aware of the abysmal difference between the cases $\mathrm{char}(K) \neq 2$ and $\mathrm{char}(K) = 2$ even at the level of definitions — not to say statements and proofs!

Let us address this question first in the very classical setting. Let V be a vector space over a field K. We consider bilinear forms $B : V \times V \to K$ on V with values in K. When $B(u, v) = 0$ one says that u and v are orthogonal with respect to B and writes $u \perp v$. The form B is called *reflexive* provided the orthogonality relation defined by B is symmetric, i.e. $B(u, v) = 0$ implies $B(v, u) = 0$. In this case, the form B is also called an *inner product* on V and V itself is called an *inner product space*. One considers the isometry group of B

$$\mathrm{Isom}(V,B) = \{g \in \mathrm{GL}(V) \mid B(gu,gv) = B(u,v), \text{ for all } u,v \in V\}.$$

In other words, $\mathrm{Isom}(B)$ consists of all automorphisms of V preserving inner products. Any choice of a basis for V identifies V with K^n, B with its Gram matrix $b \in M(n,K)$, $B(u,v) = u^t bv$ and $\mathrm{GL}(V)$ with $\mathrm{GL}(n,K)$. This done, $\mathrm{Isom}(V,B)$ becomes a subgroup of $\mathrm{GL}(n,K)$

$$\mathrm{Isom}(V,B) = \{g \in \mathrm{GL}(n,K) \mid g^t bg = b\}.$$

Three classes of inner products are of particular importance. A bilinear form B is called *symmetric* if $B(u,v) = B(v,u)$ for all $u,v \in V$. Clearly, symmetric bilinear forms are inner products. A bilinear form B is called *anti-symmetric* if $B(u,v) = -B(v,u)$ for all $u,v \in V$. Clearly, anti-symmetric forms are also inner products. A form B is called *symplectic* (also *alternating*), if $B(u,u) = 0$ for all $u \in V$. Recall that a vector $u \in V$ is called *isotropic* with respect to B if $u \perp u$, or, in other words, $B(u,u) = 0$. Thus, a form is symplectic, if *all* vectors in V are isotropic with respect to this form. The class of symplectic forms *almost* coincides with that of anti-symmetric ones. Namely, a symplectic form is always anti-symmetric, and the converse is also true if $\mathrm{char}(K) \neq 2$. On the other hand, in characteristic 2, anti-symmetric is the same as symmetric. Thus any symplectic form is symmetric but it is easy to see that not every symmetric form is symplectic. For example take the symmetric form $B : K \times K \to K, (u,v) \mapsto B(u,v) = uv$. This is obviously not symplectic. This demonstrates the difference between char $\neq 2$ and char $= 2$. In the former, symplectic forms and anti-symmetric forms coincide, but are distinctly different from symmetric forms. (The only forms which are both symmetric and anti-symmetric are those which are zero on every pair of elements). In the latter, symplectic forms are a proper subset (subcategory) of anti-symmetric forms, but symmetric and anti-symmetric forms are the same.

We move on now to quadratic forms. There is exactly one classical notion of quadratic form. It is simply a map $q : V \to K$ such that for all $v \in V$, $q(v) = C(v,v)$ for some bilinear form C on V, (Except for the case $\dim(V) = 1$, a quadratic form q is defined by many different C's.) Associated to q, there is a symmetric bilinear form defined by $B(u,v) = q(u,v) - q(u) - q(v)$. The form B measures the failure of q to be additive. Setting $u = v$, we get $B(u,u) = q(2u) - 2q(u) = 2q(u)$. Suppose now that $\mathrm{char}(K) \neq 2$. Then we can recover q from B by the formula $q(u) = (1/2)B(u,u)$. The role of the condition $\mathrm{char}(K) \neq 2$ is already visible in the denominator. Conversely, starting from any symmetric bilinear form B, we can construct a quadratic form q such that $B(u,v) = q(u+v) - q(u) - q(v)$, by setting $q(u) = (1/2)B(u,u)$. This sets up a natural equivalence, when $\mathrm{char}(K) \neq 2$, between the category of quadratic forms and the category of symmetric bilinear forms, which preserves orthogonality, nonsingularity, etc.

Now this means that in characteristic $\neq 2$ we can define orthogonal groups as isometry groups of *quadratic* forms, as follows

$$O(V,q) = \mathrm{Isom}(V,q) = \{g \in \mathrm{GL}(V) \mid q(gu) = q(u)\}.$$

This suggested to Dickson to define orthogonal groups in characteristic 2 as isometry groups of *quadratic* forms, rather than bilinear forms [62]. This distinction in terminology was further emphasised by Dieudonné [66], when he treated orthogonal groups over *non-perfect* fields of characteristic 2.

The situation for unitary groups is essentially parallel. These groups are defined in terms of sesquilinear forms $B : V \times V \to K$ over a (skew-)field K with involution $^- : K \to K$. One usually rescales the form to assume it is either hermitian, $B(u, v) = \overline{B(v, u)}$, or anti-hermitian, $B(u, v) = -\overline{B(v, u)}$. A unitary group is the isometry group of an (anti-)hermitian form. Again, one usually has to distinguish the cases $\operatorname{char}(K) \neq 2$ and $\operatorname{char}(K) = 2$ even at the level of definitions, not to say proofs.

A different approach towards *algebraic* classical groups was proposed by André Weil [175]. In fact, Weil's intention was to classify classical groups as invariant points of involutions on central simple algebras over K. But again this approach runs into serious obstacles in characteristic 2. These complications were systematically removed only a posteriori in *The Book of Involutions* [94].

In most cases quadratic/hermitian forms and orthogonal/unitary groups over fields of characteristic 2 were much harder to handle, than forms and groups over fields of characteristic $\neq 2$. In some cases, however, they have been amazingly easier. One such very famous example was of course Milnor's conjecture, whose solution in characteristic 2 [87] (see also [1], [5]) came long before Voevodsky's solution in characteristic $\neq 2$ [170], and by much more elementary means.

But the two kinds of characteristic, $\neq 2$ and $= 2$, are always different. Are they really? In fact for many geometric and topological applications which started to emerge in the 1950'ies and 1960'ies one had to work over $\mathbb{Z}$, which has fields of all positive characteristic as quotients. Also, from a purely algebraic viewpoint it is preferable to work (at least) over arbitrary commutative rings.

Over $\mathbb{Z}$ there are groups which behave like symplectic groups in some primes and like orthogonal groups in other primes, the so called hybrid symplectic groups [48].

From the late 1950'ies to the early 1970'ies several attempts were made to generalise classical groups by constructing a theory which does not depend on the invertibility of 2. Most notably, one should mention Klingenberg, Wall [172] and Tits [148]. Actually special cases of their constructions were rediscovered several times; compare in particular [111] and [100].

In the classical situation of symmetric forms B or quadratic forms q, the orthogonal group was defined so as to preserve B or q, but did not in general preserved both. In 1969, Bak came up with the following two ideas, which resolved all difficulties and allowed to work characteristic free.

- A classical group should be considered as preserving a *pair* of forms (B, q),
- A quadratic form q takes its values not in R itself, but rather in its *factor-group* R/Λ, modulo a certain additive subgroup of $\Lambda \leq R$, the so called form parameter.

Both ideas are explained in some detail in the following sections.

At about the same time Klein, Mikhalev [91], [92] and Vaserstein [152] in Moscow tried to define orthogonal and unitary groups over arbitrary rings with involution. They moved in essentially the same direction as Bak, but failed to recognise the importance of form parameters and defined just the groups, corresponding to the minimal form parameter.

4. Form rings and form ideals

To describe Bak's results on unitary K-theory, structure in the stable rangle and stability we need the notions of form parameter, form ring, form ideal, general quadratic group, its elementary subgroup and corresponding relative groups. Although these notions are now classical and widely used in the literature, we briefly recall them for the convenience of the reader (for a general and comprehensive treatment, see [78] and [14]).

In his foundational paper [47] on unitary algebraic K-theory Hyman Bass writes: "This paper is intended to make propaganda for the notions of 'unitary ring' and 'unitary ideal' due to Bak. These concepts permit a refinement of the notion of quadratic hermitian form. Their inevitability, once one begins a serious study of unitary groups over rings in which 2 is not invertible, is very strikingly revealed in the results on the normal subgroups of stable unitary groups. The essential role of the notion of unitary ring in the stability theorems will also be apparent."

These notions revolutionised the whole subject, and we are tempted to reproduce another very picturesque passage from Math. Reviews N. 2033642: "All definitions of classical groups before Bak worked in terms of a bilinear *or* a quadratic map. As a result they depended on the invertibility of 2 in the ground ring. Without this restriction the theory was pestered by a swamp of technical details, with the distinctions of singular and degenerate forms, defect and non-defect orthogonal groups, etc. The essence of Bak's definition lies in the notions of form parameters and form rings."

We would like to stress that in the meantime the absolute inevitability of all these notions in the study of classical groups became even more apparent, especially that of the form ideal [6]. For example, even though symplectic groups can be defined with no reference to form rings, their subgroups over rings in which 2 is not invertible can be understood *only* using the notion of form ideals. Without this notion one is bound to stipulate restrictions such as $2 \in R^*$, or some other such condition guaranteeing that relative forms parameters corresponding to any given ideal coincide, or to introduce into the answers ad hoc technical definitions such as special pairs, quasi-ideals, Jordan ideals, etc. However, all of those are just specific manifestations of Bak's form ideals.

Let R be a ring with an *involution* denoted by $a \mapsto \overline{a}$; by definition $\overline{a+b} = \overline{a} + \overline{b}$, $\overline{ab} = \overline{b}\,\overline{a}$ and $\overline{\overline{a}} = a$ for all $a, b \in R$. Fix an element $\lambda \in \mathrm{Cent}(R)$, called the *symmetry*, such that $\lambda\overline{\lambda} = 1$. The maximal and the minimal form parameters, corresponding to this choice of involution $^-$ and symmetry λ are defined as follows:

$$\Lambda_{\min} = \{a - \lambda\overline{a} \mid a \in R\}, \qquad \Lambda_{\max} = \{a \in R \mid a = -\lambda\overline{a}\}.$$

Clearly $\Lambda_{\min}$ and $\Lambda_{\max}$ are additive subgroups of R and $\Lambda_{\min} \leq \Lambda_{\max}$. It is easy to see that they are closed with respect to the Jordan (quadratic) action of R:

$$a\Lambda_{\min}\overline{a} \leq \Lambda_{\min}, \qquad a\Lambda_{\max}\overline{a} \leq \Lambda_{\max}$$

for all $a \in A$. In general, an additive subgroup $\Lambda \leq R$ such that

(1) $\Lambda_{\min} \leq \Lambda \leq \Lambda_{\max}$

(2) $a\Lambda\overline{a} \leq \Lambda$ for all $a \in R$,

is called a *form parameter* in R. A pair (R, Λ), consisting of a ring with involution and a form parameter in it is called a *form ring*.

Let (R, Λ) and (R', Λ') be form rings relative, respectively, to the symmetries λ and λ'. A ring homomorphism $\mu : R \to R'$ such that for any $a \in R$, $\mu(\overline{a}) = \overline{\mu(a)}$, $\mu(\lambda) = \lambda'$ and $\mu(\Lambda) \leq \Lambda'$ is called a *morphism of form rings*. A morphism $\mu : (R, \Lambda) \to (R', \Lambda')$ of form rings is called *surjective* if $\mu : R \to R'$ is a surjective ring homomorphism and $\mu(\Lambda) = \Lambda'$.

These notions suffice to define unitary groups themselves, but in order to introduce later relative groups for the general quadratic group, we need also the notion of form ideal in a form ring. Let I be an ideal of R which is invariant under the involution, i.e. $\overline{I} = I$. Then the minimal and maximal relative form parameters of I are defined as follows:

$$\Gamma_{\min} = \langle x - \lambda\overline{x} \mid x \in I \rangle + \langle x\alpha\overline{x} \mid x \in I, \alpha \in \Lambda \rangle, \qquad \Gamma_{\max} = I \cap \Lambda.$$

Clearly $\Gamma_{\min} \leq \Gamma_{\max}$ depend only on the form parameter Λ and the ideal I and are closed under the quadratic action of R:

$$a\Gamma_{\min}\overline{a} \leq \Gamma_{\min}, \qquad a\Gamma_{\max}\overline{a} \leq \Gamma_{\max}$$

for all $a \in R$. In general a *relative form parameter of* I is an additive subgroup of $\Gamma \leq I$ such that

(1) $\Gamma_{\min} \leq \Gamma \leq \Gamma_{\max}$

(2) $a\Gamma\overline{a} \leq \Gamma$ for all $a \in R$.

The pair (I, Γ) is called a *form ideal* in (R, Λ).

5. Bak's unitary groups

Let (R, Λ) be a form ring, V a right R-module and let, as usual, $\mathrm{GL}(V)$ be the group of all R-linear automorphisms of V. Further, let f be a *sesquilinear* form on V, i.e., a biadditive map $f : V \times V \to R$ such that $f(ua, vb) = \overline{a}f(u, v)b$ for all $u, v \in V$ and $a, b \in R$. Define the maps $h : V \times V \to R$ and $q : V \to R/\Lambda$ by $h(u, v) = f(u, v) + \lambda\overline{f(v, u)}$ and $q(v) = f(v, v) + \Lambda$. The function q is called a Λ-*quadratic form* on V and h its associated λ-*Hermitian form*. The triple (V, h, q) is called a *quadratic module over* (R, Λ).

It is called *nonsingular*, if V is finitely generated and projective over R and the map $V \to \mathrm{Hom}_R(V, R)$, $v \mapsto h(v, -)$ is bijective, i.e. the Hermitian form h is nonsingular.

A morphism $(V, h, q) \to (V', h', q')$ of quadratic modules over (R, Λ) is an R-linear map $V \to V'$ which preserves the Hermitian and Λ-quadratic forms.

Define the *general quadratic group* $G(V,h,q)$ to be the group of all automorphisms of (V,h,q). Thus

$$G(V,h,q)=\{\sigma\in \mathrm{GL}(V)\mid h(\sigma u,\sigma v)=h(u,v),\ q(\sigma v)=q(v) \text{ for all } u,v\in V\}.$$

This group is often also called a *generalised unitary group* or *Bak's unitary group*.

Suppose that h and q are defined by the sesquilinear form f. If (I,Γ) is a form ideal in (R,Λ), define the *relative general quadratic group*

$$(1)\quad G(V,h,q,I,\Gamma)=\{\sigma\in G(V,h,q)\mid \sigma\equiv 1 \pmod I,\quad f(\sigma v,\sigma v)-f(v,v)\in\Gamma \text{ for all } v\in V\}.$$

This is what classically would be called the *principal congruence subgroup* of level (I,Γ), but now the level is a form ideal.

In his Thesis, Bak proved that, if (V,h,q) is nonsingular, then the group $G(V,h,q,I,\Gamma)$ is well defined, i.e. does not depend on the choice of f, and is normal in $G(V,h,q)$. Published proofs for the special case of the hyperbolic unitary group $G(2n,R,\Lambda)$ — which is the only case we need here — can be found in section 5.2 of the book of Hahn and O'Meara [78] or in the paper by Bak and Vavilov [41].

Much of the theory can be developed in this general context, sometimes under the assumption that h is isotropic enough, in other words, its Witt index is large enough. However, not to overcharge the present exposition with notation and precise technical details, we concentrate on the case of hyperbolic unitary group $G(2n,R,\Lambda)$ and now we recall its definition.

Let V denote a free right R-module with ordered basis $e_1,e_2,\dots,e_n,e_{-n},\dots,e_{-1}$. If $u\in V$, let $u_1,\dots,u_n,u_{-n},\dots,u_{-1}\in R$ such that $u=\sum_{i=-n}^{n}e_iu_i$. Let $f:V\times V\longrightarrow R$ denote the sesquilinear map defined by $f(u,v)=\overline{u}_1v_{-1}+\cdots+\overline{u}_nv_{-n}$. It is easy to see that if h and q are the Hermitian and Λ-quadratic forms defined by f, then

$$h(u,v)=\overline{u}_1v_{-1}+\cdots+\overline{u}_nv_{-n}+\lambda\overline{u}_{-n}v_n+\cdots+\lambda\overline{u}_{-1}v_1$$

and

$$q(u)=\Lambda+\overline{u}_1u_{-1}+\cdots+\overline{u}_nu_{-n}.$$

The ordered basis we chose here was used in [41]. As it is mentioned in [41], in all previously published works, where general quadratic groups over form rings were considered, either the ordered basis $e_1,e_{-1},\dots,e_n,e_{-n}$, or the ordered basis $e_1,\dots,e_n,e_{-1},\dots,e_{-n}$ are used. For example, this latter one is used in the book of Bak [14].

Using the basis above, we can identify $G(V,h,q)$ with a subgroup of the general linear group $\mathrm{GL}(2n,R)$ of rank $2n$. This subgroup will be denoted by $G(2n,R,\Lambda)$ and is called the *general quadratic group over* (R,Λ) *of rank* n. (In many publications, it is also called general unitary group and is denoted by $U(2n,R,\Lambda)$). Using the basis, we can identify the relative subgroup $G(V,h,q,I,\Gamma)\le G(V,h,q)$ with a subgroup of $G(2n,R,\Lambda)$ denoted by $G(2n,I,\Gamma)$. Bak describes the matrices in $G(2n,R,\Lambda)$ and $G(2n,I,\Gamma)$ in his Thesis (See also [41], [78]).

The groups introduced by Bak in his Thesis gather all *even* classical groups under one umbrella. Linear groups, symplectic groups, (even) orthogonal groups, (even) classical unitary groups, are all special cases of his construction. Not only that, Bak's construction allows to introduce a whole new range of *classical like groups*, taking into account hybridisation, defect groups, and other such phenomena in characteristic 2, which before his Thesis were considered pathological, and required separate analysis outside of the general theory. Retrospectively it becomes more and more clear that this unification has been one of the *Wendepunkte* in our understanding of classical groups, comparable in its significance to the contributions by Jordan, Dickson, Witt, Dieudonné, Tits and Wall.

To give the idea of how it works, let us illustrate how Bak's construction specialises in the case of hyperbolic groups.

• In the case when involution is trivial, $\lambda = -1$, $\Lambda = \Lambda_{\max} = R$, one gets the split symplectic group $G(2n, R, \Lambda) = \mathrm{Sp}(2n, R)$.

• In the case when involution is trivial, $\lambda = 1$, $\Lambda = \Lambda_{\min} = 0$, one gets the split even orthogonal group $G(2n, R, \Lambda) = O(2n, R)$.

• In the case when involution is non-trivial, $\lambda = -1$, $\Lambda = \Lambda_{\max}$, one gets the classical quasi-split even unitary group $G(2n, R, \Lambda) = U(2n, R)$.

• Let R^o be the ring opposite to R and $R^e = R \oplus R^o$. Define an involution of R^e by $(x, y^o) \mapsto (y, x^o)$ and set $\lambda = (1, 1^o)$. Then there is a unique form parameter $\Lambda = \{(x, -x^o) \mid x \in R\}$. The resulting unitary group

$$G(2n, R^e, \Lambda) = \{(g, g^{-t}) \mid g \in \mathrm{GL}(n, R)\}$$

may be identified with the general linear group $\mathrm{GL}(n, R)$.

Thus, in particular Bak's hyperbolic unitary groups cover Chevalley groups of types A_l, C_l and D_l. To include groups of type B_l and in general *odd* classical groups in the general theory required substantial additional effort. This was only achieved recently by Bak and Morimoto, in view of topological applications. In [27] they introduce a powerful, but rather heavy-going construction phrased in terms of two form parameters.

A simpler approach to odd unitary groups has been developed by Viktor Petrov. Being jointly supervised by Tony Bak, at the University of Bielefeld, and one of us at the Saint-Petersburg State University, he could successfully combine his expertise in Bak's unitary groups and in Chevalley groups. Departing from Bak's definition of Λ-quadratic forms and Abe's definition of twisted Chevalley groups, Petrov constructed a theory in which a quadratic form takes values not in R, as in the classical theory, and not in R/Λ as in Bak's theory, but rather in their semi-direct product $R \leftthreetimes R/\Lambda$. This allowed to fully account for non-abelian root subgroups corresponding to extra-short roots. Thus, *odd* classical groups — in particular Chevalley groups of types B_l and the twisted forms of A_l — finally obtained a natural interpretation in the general framework of Bak's theory. However, many aspects of this important and beautiful work are decisively too technical to be included in a casual presentation, since it would require a couple of pages just to state precise definitions. We refer the interested reader to Petrov's remarkable papers [113], [114] (or to his *Russian*[1] thesis [115]) which give a very clear and detailed account of this theory.

6. Elementary unitary groups

Next, we recall the definition of the elementary quadratic subgroup. For

$$i \in \Delta_n = \{1, \cdots, n, -n, \cdots, -1\},$$

let $\varepsilon(i)$ denote the sign of i, i.e., $\varepsilon(i) = 1$ if $i > 0$ and $\varepsilon(i) = -1$ if $i < 0$. Let $i, j \in \Delta_n$ be such that $i \neq j$. The *elementary transvection* $T_{ij}(a)$ is defined as follows:

$$T_{ij}(a) = \begin{cases} e + ae_{ij} - \lambda^{(\varepsilon(j)-\varepsilon(i))/2}\overline{a}e_{-j,-i} & \text{where } a \in R, \text{ if } i \neq -j \\ e + ae_{i,-i} & \text{where } a \in \lambda^{-(\varepsilon(i)+1)/2}\Lambda, \text{ if } i = -j. \end{cases}$$

Once one knows the form of the matrices in $G(2n, R, \Lambda)$, it is easy to check that $T_{ij}(a) \in G(2n, R, \Lambda)$. The symbol T_{ij}, where $i \neq -j$, is called a *short root*, whereas $T_{i,-i}$ is called a *long root*.

The subgroup generated by all elementary transvections is called the *elementary quadratic group* and is denoted by $E(2n, R, \Lambda)$. This group is the quadratic version of the elementary group in the theory of the general linear group. Note that elementary transvections corresponding to long roots are elementary matrices in $E(2n, R)$ and elementary transvections corresponding to short roots are a product of two elementary matrices in $E(2n, R)$. Let (I, Γ) be a form ideal of (R, Λ). The subgroup generated by all (I, Γ)-elementary transvections is denoted by $F(2n, I, \Gamma)$, i.e.,

$$F(2n, I, \Gamma) = \langle T_{ij}(x), T_{i,-i}(y) \mid x \in I, y \in \lambda^{-(\varepsilon(i)+1)/2}\Gamma \rangle.$$

The normal closure $F(2n, I, \Gamma)^{E(2n,R,\Lambda)}$ of $F(2n, I, \Gamma)$ in $E(2n, R, \Lambda)$ is denoted by $E(2n, I, \Gamma)$ and is called the *relative elementary quadratic subgroup of* $G(2n, R, \Lambda)$ *of level* (I, Γ). In this note we sometimes do not distinguish between short and long roots and simply write $T_{ij}(x)$, assuming that $x \in \lambda^{-(\varepsilon(i)+1)/2}\Lambda$ whenever $i = -j$.

There are standard relations among the elementary transvections, analogous to those for the elementary matrices in the general linear group. Here we follow [78] and particularly [41]. These are much simpler than the original generators and relations (up to 28 relations) in the book of Bak, which in return makes computations with them less painful (see e.g. [79],[113]). We also need these relations to define the quadratic Steinberg group. Let us list the elementary relations:

(R1) $T_{ij}(a) = T_{-j,-i}(\lambda^{(\varepsilon(j)-\varepsilon(i))/2}\overline{a})$,

(R2) $T_{ij}(a)T_{ij}(b) = T_{ij}(a+b)$,

(R3) $[T_{ij}(a), T_{hk}(b)] = 1$ where $h \neq j, -i$ and $k \neq i, -j$,

(R4) $[T_{ij}(a), T_{jh}(b)] = T_{ih}(ab)$ where $i, h \neq \pm j$ and $i \neq \pm h$,

[1]His *German* thesis is devoted to motives of flag varieties and construction of a new invariant of exceptional algebraic groups.

(R5) $[T_{ij}(a), T_{j,-i}(b)] = T_{i,-i}(ab - \lambda^{-\varepsilon(i)}\overline{b}\overline{a})$ where $i \neq \pm j$,

(R6) $[T_{i,-i}(a), T_{-i,j}(b)] = T_{ij}(ab)T_{-j,j}(-\lambda^{(\varepsilon(j)-\varepsilon(-i))/2}\overline{b}ab)$ where $i \neq \pm j$.

As the elementary subgroup has more generators and relations than in the linear case, and also more complicated relations, this — among other things! — makes computations in the setting of unitary groups more arduous, sometimes terribly much more so.

7. Unitary algebraic K-theory

There is a standard embedding

$$G(2n, R, \Lambda) \longrightarrow G(2(n+1), R, \Lambda), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & 0 & b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ c & 0 & 0 & d \end{pmatrix}$$

called the stabilisation map. In fact some other sources may give a slightly different picture of the right hand side. How the right hand side exactly looks, depends on the ordered basis we choose. With the ordered which is used in the book of Bak [14], the standard embedding has the form

$$G(2n, R, \Lambda) \longrightarrow G(2(n+1), R, \Lambda), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Define

$$G(R, \Lambda) = \varinjlim_{n} G(2n, R, \Lambda)$$

and

$$E(R, \Lambda) = \varinjlim_{n} E(2n, R, \Lambda).$$

The groups $G(I, \Gamma)$ and $E(I, \Gamma)$ are defined similarly.

One could ask, whether one can carry over Bass' results discussed in §2 to the unitary case? Bak, and in a slightly narrower situation, Vaserstein, established unitary versions of Whitehead's lemma, which in particular implies the following result.

Theorem 7.1. *Let (R, Λ) be an arbitrary form ring, and (I, Γ) be its form ideal, then*

$$E(I, \Gamma) = [E(R, \Lambda), E(I, \Gamma)] = [G(R, \Lambda), G(I, \Gamma)].$$

Now, similarly to the linear case, this allows one to introduce the *unitary K-functor*

$$K_1(I, \Gamma) = G(I, \Gamma)/E(I, \Gamma).$$

A version of unitary K-theory modeled upon Bak's unitary groups has been systematically developed by Bass in [47]. Note that, in some literature, the notation KU is used to denote the unitary K-groups. In other literature, the functor above is called a *quadratic K-functor* and the notation KQ is used. (For a lexicon of notations, see [14], §14).

The functor K_1 turned out to be very interesting in its own right and enormously important in many topological, geometrical and arithmetical applications. For example, unitary K-theory and its counterpart, hermitian K-theory, defined in terms of metabolic, rather than hyperbolic modules, turned out to be instrumental in the calculation of the L-groups of Wall, Bak-Morimoto and others. They developed into huge theories in their own right. The foundations of these theories are systematically presented in Bak's book [14], which introduces and systematically studies K-functors, both unitary KU_0, KU_1, KU_2 and hermitian KH_0, KH_1, KH_2, their properties, comparison with linear case, dependence on form parameter, exact sequences, and much more (see [88] for some applications). Here we do not develop the topic of general unitary and Hermitian K-theory any further, since it would require another survey of comparable size just to introduce necessary definitions and explain some of the basic ideas. Neither do we try to explain the relation of Bak's work in unitary and Hermitian K-theory with the work of Wall, Karoubi, Bass, Novikov, Ranicki, and others, and its relevance in topology (see [88], an Appendix by Max Karoubi to this paper, on periodicity theorems in Hermitian K-theory in an algebraic context and its relation to Bak's work).

As another piece of structure, parallel to the linear situation, let us mention the description of normal subgroups in $G(R,\Lambda)$, that holds over an arbitrary ring.

Theorem 7.2. *Let (R,Λ) be an arbitrary form ring. If $H\leq G(R,\Lambda)$ is a subgroup normalised by $E(R,\Lambda)$, then for a unique form ideal (I,Γ), one has*

$$E(I,\Gamma)\leq H\leq G(I,\Gamma)$$

Conversely, these inclusions guarantee that H is automatically normal in $G(R,\Lambda)$.

It is interesting to mention that here the course of events has been reversed as compared with the linear case. Namely the proof of this result has been written by Bass [47] (compare also §5.4D in [78]), only *after* Bak in his Thesis had already established similar results for unitary groups of finite degree, which we discuss in the next section! As Bass himself puts it, "Here we initiate a detailed study of unitary groups of hyperbolic modules over a unitary ring. We classify the normal subgroups following ideas of Bak."

Starting with the elementary relations among the generators of the elementary unitary group one can define unitary Steiberg groups StU and unitary functor KU_2. This too is done in the book of Bak [14], and we briefly describe these results in §11.

8. Structure in the stable range

Do these results carry over to groups of *finite* degree n? The much more difficult and demanding task of finding these generalisations was done in Bak's Thesis [6, 7], under some finiteness assumptions on the ground ring. In fact he proves there a non-stable version of these theorems for the quadratic case parallel to Bass' result for the linear case.

Let us recall the linear case first. These results are most conveniently stated in terms of the new type of dimension for rings, introduced by Bass, stable rank. Since later we shall discuss generalisations of this notion, we recall here its definition.

A row $(a_1, \dots, a_n) \in {}^nR$ is called *unimodular* if the elements $a_1, \dots, a_n$ generate R as a *right* ideal, i.e. $a_1R + \dots + a_nR = R$, or, what is the same, there exist $b_1, \dots, b_n \in R$ such that $a_1b_1 + \dots + a_nb_n = 1$.

A row $(a_1, \dots, a_{n+1}) \in {}^{n+1}R$ is called *stable*, if there exist $b_1, \dots, b_n \in R$ such that the *right* ideal generated by $a_1 + a_{n+1}b_1, \dots, a_n + a_{n+1}b_n$ coincides with the *right* ideal generated by $a_1, \dots, a_{n+1}$.

One says that the *stable rank* of the ring R equals n and writes $\mathrm{sr}(R) = n$ if every unimodular row of length $n+1$ is stable, but there exists a non-stable unimodular row of length n. If such n does not exist (i.e. there are non-stable unimodular rows of arbitrary length) we say that the stable rank of R is infinite.

It turned out that stable rank, on one hand, most naturally arises in the proof of results pertaining to linear groups and, on the other hand, it can be easily estimated in terms of other known dimensions of a commutative ring R, say of its Krull dimension $\dim(R)$, or its Jacobson dimension $j(R) = \dim(\mathrm{Max}(R))$. Here, $\mathrm{Max}(R)$ is the subspace of all maximal ideals of the topological space $\mathrm{Spec}(R)$, the set of all prime ideals of R, equipped with the Zariski Topology. Then $j(R)$ is the dimension of the topological space $\mathrm{Max}(R)$. Let us state a typical result in this spirit due to Bass.

Theorem 8.1. *Let R be a ring finitely generated as a module over a commutative ring A. Then $\mathrm{sr}(R) \leq \dim(\mathrm{Max}(A)) + 1$.*

In fact, usually this inequality is strict. The right hand side should be thought of as a condition expressing (a weaker form of) stability for not necessarily unimodular rows. In [68] and [86] it is shown that already $\mathrm{asr}(R) \leq \dim(\mathrm{Max}(A)) + 1$, where $\mathrm{asr}(R)$ stands for the *absolute stable rank* (see p. 22, for more discussion on this).

As a curiosity we could mention that the name `stable rank` itself and the notation sr were introduced by L.N.Vaserstein [150] as a result of mistranslation. Bass himself spoke about `stable range` rather than `stable rank`, but in Russian the word `rank` is spelled with `g`. Moreover the definition we introduced is shifted by one from Bass' original condition. The precise relationship is $\mathrm{sr}(R) \leq n$ if and only if Bass' stable range condition SR_{n+1} holds. However, our notation is now in common use, and we think that it is actually slightly more convenient both in structure theory and in stability problems than Bass' original condition.

Now we are in the position to state two of the main results of [45].

Theorem 8.2. *Assume that $n \geq max\{\mathrm{sr}(R), 3\}$ and let $I \trianglelefteq R$ be an ideal of R. Then $E(n, R, I)$ is normal in $\mathrm{GL}(n, R)$. More precisely,*

$$[\mathrm{GL}(n, R), E(n, R, I)] = [E(n, R), C(n, R, I)] = E(n, R, I)$$

Theorem 8.3. *Assume that $n \geq max\{\mathrm{sr}(R), 3\}$ and let H be a subgroup of $\mathrm{GL}(n, R)$, normalised by the elementary subgroup $E(n, R)$. Then there exists a unique ideal $I \trianglelefteq R$ such that*

$$E(n, R, I) \leq H \leq C(n, R, I).$$

The first of these theorems allows one to conclude that in the stable range the nonstable K_1-functor of degree n

$$K_1(n, R, I) = \mathrm{GL}(n, R, I)/E(n, R, I),$$

is not just a set, but in fact a group. In the next two sections we discuss both its behaviour under stabilisation maps and what happens below the stable range, again mostly limiting ourselves to the absolute case $K_1(n, R) = K_1(n, R, R)$.

In his Thesis, Bak systematically studied the structure of unitary groups at the stable level and establishes analogues of Bass results (and in his book, also of Milnor's results) in this setting. Unfortunately, many results contained in Bak's Thesis stayed unpublished for about 30 years. It is mentioned in [41] that "Unfortunately [6] was never published and was not easily available, especially in Russia and China. This did a lot of harm. In fact, many works appearing *up till the late eighties*, were proving structure theorems for classical groups over rings covered in [6], such as zero-dimensional ones.".

Of course, in the quadratic case one also has to slightly revise the upper bound group:

$$C(2n, I, \Gamma) = \{g \in G(2n, R, \Lambda) \mid [g, G(2n, R, \Lambda)] \leq G(2n, I, \Gamma)\}.$$

Note that, although the form ring is not reflected in the notation, actually the groups depend not only on (I, Γ), but also on (R, Λ). Actually, in the majority of interesting cases this subgroup coincides with the group

$$\widetilde{G}(2n, I, \Gamma) = \{g \in G(2n, R, \Lambda) \mid [g, E(2n, R, \Lambda)] \leq E(2n, I, \Gamma)\}$$

considered in [78]. The reason they coincide is that in fact, they both coincide with the larger group

$$G'(2n, I, \Gamma) = \{g \in G(2n, R, \Lambda) \mid [g, E(2n, R, \Lambda)] \leq G(2n, I, \Gamma)\}$$

which transports the smaller one of two subgroups on the left-hand side to the larger one of the two on the right-hand side. Let us quote a typical result in this style that figures as Corollary 3.4 in [6] (recall that a ring R is called almost commutative, or module finite, if it is finitely generated as a module over its centre).

Theorem 8.4. *Let R be a module finite ring. Furthermore, assume that the centre A of the ring R is Noetherian of Krull dimension d. Let $n \geq max\{d+2, 3\}$. Then for any form ideal (I, Γ) one has*

$$C(2n, I, \Gamma) = G'(2n, I, \Gamma) = \widetilde{G}(2n, I, \Gamma).$$

In [39] and [43] one can find many further instances, when these groups coincide.

With this notation two of the main results (the two implications in Theorem 1.2) established in Bak's Thesis may be stated as follows.

Theorem 8.5. *Let R be a module finite ring. Furthermore, assume that the centre A of the ring R is Noetherian of Krull dimension d. Let $n \geq max\{d+2, 3\}$. Then for any form ideal (I, Γ), $E(2n, I, \Gamma)$ is a normal subgroup of $G(2n, R, \Lambda)$ and*

$$[E(2n, R, \Lambda), C(2n, I, \Gamma)] = E(2n, I, \Gamma).$$

Theorem 8.6. *Keep the assumptions in the preceding theorem. Then for any subgroup $H \leq G(2n, R, \Lambda)$ normalised by $E(2n, R, \Lambda)$ there exists a unique form ideal (I, Γ) such that*

$$E(2n, I, \Gamma) \leq H \leq C(2n, I, \Gamma).$$

In fact Bak's original theorem imposes a slightly weaker assumption on the centre of R, namely what is now called Bass–Serre dimension (see §13 for the defintition). In the 1990'ies Bak and Vavilov [41], [40][2] and independently Vaserstein and Hong You [158] could remove the assumption on the dimension of the centre of R and prove analogues of these results for groups of degree $n \geq 3$ over classes of rings which in both cases included, in particular almost commutative rings.

One of the most important and lasting aspects of these theorems is that they emphasise the inevitablity of *form ideals* — and in particular of *relative* form parameters! — in the description of normal subgroups. This is rather non-obvious, since even now, after almost 40 years, many authors still seem not to have fully assimilated this fact. Instead, they try to introduce some ad hoc corrections to (what they believe is) the standard description in terms of ordinary ideals, with the idea to save the answer in various specific situations.

As a curiosity let us cite that even in the paper [158], which nominally acknowledges that the description should be stated in terms of form ideals, relative form parameters are omitted from the notation: "In this article we often write (J, Δ) as J". This leads to unavoidable confusion and ambiguity. On page 94 the authors define $C(2n, J, \Gamma)$ — which they alternatively denote by $O_{2n}(R, J)$ or by $O_{2n}(R, J, \Gamma)$ — as "the subgroup of matrices in $O_{2n}R$ which reduce modulo J to scalar matrices over the centre of R/J". Clearly, this group does not depend on the relative form parameter — and in fact corresponds to the *maximal* relative form parameter $\Gamma_{\max}$. But then their Theorem 1.1 (an analogue of 8.5 above) cannot be true as stated, since the last inclusion there can be a proper inclusion, rather than equality. Neither does the uniqueness of the form ideal in Theorem 1.2 (an analogue of 8.6 above) hold under this definition. That the authors indeed rely on the above (wrong) definition and do not, thereby, take into account the relative form parameter is illustrated by the naive form of level reduction on page 104: "Consider the image H' of H in $O_{2n}(R/J) \dots$ Thus H' is central in $O_{2n}(R/J)$, i.e. $H \subset O_{2n}(R, J)$". In fact, such naive level reduction gives only inclusions $E(2n, I, \Gamma_{\min}) \leq H \leq C(2n, I, \Gamma_{\max})$ and the equality $[E(2n, R, \Lambda), H] = E(2n, I, \Gamma_{\min})$ which one wants for any H in the sandwich $E(2n, I, \Gamma_{\min}) \subseteq H \subseteq C(2n, I, \Gamma_{\max})$ no longer holds, because $[E(2n, R, \Lambda), C(2n, I, \Gamma_{\max})] = E(2n, I, \Gamma_{\max})$.

Thus the results of [158], and their proofs, only stand as stated under some additional simplifying assumption which guarantees that $\Gamma_{\min} = \Gamma_{\max}$ for all ideals of R, for example, $2 \in R^*$. But both the correct definition of the group $C(2n, J, \Gamma)$ and the correct form of level reduction, which allow one to proceed without any such simplifying assumption, were introduced already in Bak's Thesis!

[2]These papers circulated in preprint form since 1994, but their authors are slow publishers.

9. Stability for unitary K-functors

Another milestone in the development of K-theory were stability theorems for K-functors. For the linear case these theorems were established by Bass himself and Vaserstein at the level of K_1, by Dennis, Vaserstein, van der Kallen, Suslin and Tulenbaev at the level of K_2 and by Suslin in general.

Let us explicitly state stability theorems for K_1 and K_2, which play a crucial role in many results. The embeddings $\mathrm{GL}(n, R) \leq \mathrm{GL}(n+1, R)$, $E(n, R) \leq E(n+1, R)$, considered in §2 induce a homomorphism

$$\psi_n : K_1(n, R) \to K_1(n+1, R).$$

It is natural to ask, under which conditions this homomorophism is an epimorphism or a monomorphism. The first one of these questions is called the problem of *surjective stability*, and the second one — the problem of *injective stability*. For classical groups it is much easier to establish surjective stability, than injective stability — for exceptional groups this is not always the case! The first stability result is due to Bass [45].

Theorem 9.1. *For any $n \geq \mathrm{sr}(R)$ the map ψ_n is surjective. In other words, one has*

$$\mathrm{GL}(n+1, R) = E(n+1, R)\,\mathrm{GL}(n, R).$$

In fact this stability result was instrumental in the proof of results we mentioned in the preceding section. On the other hand, applications to the congruence subgroup problem [51] required *injective* stability. Let us state the final version of injective stability for K_1, now known as the Bass–Vaserstein Theorem.

Theorem 9.2. *For any $n \geq \mathrm{sr}(R) + 1$ the map ψ_n is injective. In other words, one has*

$$\mathrm{GL}(n, R) \cap E(n+1, R) = E(n, R).$$

Just to give some idea how powerful this result is, let us mention that it allows sharpening standard commutator formulae as follows.

Theorem 9.3. *Let $n \geq \mathrm{sr}(R) + 1$. Then for any ideal $I \trianglelefteq R$ one has*

$$[\mathrm{GL}(n, R), \mathrm{GL}(n, R, I)] \leq E(n, R).$$

Moreover, if $n \geq 3$ and $1 \in R^* + R^*$ this inclusion is in fact an equality.

The proof of a slightly weaker result (with some extra conditions) in Bass' book [46] took about 40 pages and relied on extremely intricate matrix computations. Soon thereafter Vaserstein produced a readable 10 page proof [151]. However, to really understand what goes on and to come up with a proof that really explains things, one has to introduce additional technical tension. Plenty of such tension is introduced as one tries to generalise these results to higher K-functors and to other types of groups!

In fact, in the early 70-ies Dennis [61] and Vaserstein [153] established surjective stability for K_2 and subsequently van der Kallen [82] and Suslin–Tulenbaev [143] came up with injective stability for K_2. These theorems paved the way to the beautiful stability theorems and comparison theorems of Suslin [140], which demonstrate that the correct version of *unstable* K-theory is Volodin's K-theory.

However, let us dwell with K_1 for a while. Consider the subgroups P and Q of the group $E(n+1,R)$ defined as follows:

$$P = \left\{ \begin{pmatrix} 1 & 0 \\ u & a \end{pmatrix},\ u \in R^n,\ a \in E(n,R) \right\},$$
$$Q = \left\{ \begin{pmatrix} b & 0 \\ v & 1 \end{pmatrix},\ v \in {}^nR,\ b \in E(n,R) \right\}.$$

These subgroups should be thought of as the elementary subgroups of the parabolic subgroups P_1 and P_n in $\mathrm{GL}(n+1,R)$. Now we are all set to state Dennis–Vaserstein decomposition.

Theorem 9.4. *Let $n \geq \mathrm{sr}(R)+1$. Then every element $g \in E(n+1,R)$ can be written in the form $g = yt_{1,n+1}(\lambda)z$, where $y \in P$, $\lambda \in R$, $z \in Q$.*

This is essentially what survives over a ring of finite stable rank of several canonical forms for matrices in $\mathrm{GL}(n,K)$ over a field (such as Bruhat decomposition, Gauß decomposition, etc.)

For other classical groups the development was not as straightforward, due to the fact that it was not immediately clear what the relevant substitute of stable rank was. In fact, since for classical groups elementary transformations can modify two components of a row or a column, one is compelled to work in terms of pieces of rows and columns, which are not necessarily unimodular. As a result a host of competing stability conditions were tried as candidates over decades, with variable success. The short list starts with the obvious candidate, Jacobson dimension $j(R) = \dim(\mathrm{Max}(R))$. Another major success was an important observation by Stein that a slightly less obvious absolute stable rank $\mathrm{asr}(R)$, defined in terms of a weaker form of stability for not necessarily unimodular rows, would work here — as well as for exceptional groups! For the orthogonal group there was also Vaserstein's condition defined in terms of unimodular rows lying on a quadric, which at the time seemed quite natural. In fact, this condition became quite popular and was generalised to unitary groups by Bak's student Habdank [77], as unitary stable rank and to all Chevalley groups by Plotkin [119].

However, there was a general feeling that stability proofs for groups other than SL_n and Sp_{2l} are far from being satisfactory. Indeed, many results using diverse stability conditions had non-trivial overlaps, but would not directly imply each other. On top of that most of the published proofs, with the notable exceptions of Suslin–Tulenbaev proof in [143] and Stein's proof in [132] (and subsequent proofs by Plotkin directly modeled on it [118], [117] and references therein) were so technical and clumsy that it was not easy not only to check them in detail, but even to simply figure out what goes on there. Amazingly, this applied both to proofs phrased in terms of matrices and proofs phrased in terms of modules. Saliani's proof reproduced in [93] is generally accepted as being correct, but is by no means easy to follow either.

Andrei Suslin confessed to us that he never tried to look inside such stability proofs for classical groups and to correct all the *misprints*[3] therein, but would rather devise his own proofs. He suggested that we should proceed similarly and in [120] we made yet another attempt to understand the interrelation of different proofs and different stability conditions used therein. For example, are the Suslin–Tulenbaev proof and the Stein proof the same, or are they two different proofs? When one looks at these proofs from a caterpillar viewpoint, as matrix calculations, even the answer to such questions was not immediately obvious! First we interpreted all previous proofs in terms of the theory of algebraic groups, systematically articulating words like parabolic subgroups, unipotent radicals, Levi decomposition, etc., instead of explicit matrix formulae. As a result, we could invoke stability conditions *exactly once* in the proof of Dennis–Vaserstein decomposition, all the rest was pure structure theory of algebraic groups, essentially the Chevalley commutator formula! Thus the whole proof with all subsidiary results for *all* split classical groups = classical Chevalley groups, was condensed into 4 pages. In fact, we could fit into this space both the Suslin–Tulenbaev and Stein proofs, at the moment we discovered that they only differ in one issue, as to whether the extra zeroes needed to establish Dennis–Vaserstein decomposition are produced in the unipotent radical (according to Suslin–Tulenbaev) or in the Levi factor (according to Stein).

The decisive contribution of Bak at this point was his observation that you should not wander around in search of an appropriate ring theoretic condition, but rather should work directly from the proof and see what precisely is needed there. Consequently, in a joint paper [37], Bak and Tang boldly introduced the Λ-*stable rank* of form rings as the precise condition that is required to establish surjective stability by following the proof for unitary K_1 in [6], (4.7), and Hermitian K_1 in [37]. Namely, one says that $\Lambda\,\mathrm{sr}(R,\Lambda)\leq n$ provided $\mathrm{sr}(R)\leq n$ and for any unimodular row $(a_1,\dots,a_n,a_{-n},\dots,a_{-1})$ of length $2n$ there exists a matrix

$$y\in \mathrm{AH}(n,R,\Lambda)=\{x\in M(n,R)\mid x=-\lambda x^*, x_{ii}\in\Lambda\}$$

such that $(a_1,\dots,a_n)+(a_{-n},\dots,a_{-1})y$ is unimodular. In fact, the condition was phrased slightly differently, in the spirit of Bass' condition, its equivalence with the definition given here follows from Lemma 3.3 of [37].

It was established in [37] and [31] that all thinkable stability conditions — or, at least, all the conditions used to study unitary stability before, such as absolute stable rank, unitary stable rank, as well as conditions expressed in terms of Krull dimension, Jacobson dimension, or Bass–Serre dimension — are bounded from below by Λ-stable rank.

Even more remarkably, it turned out that this Bak's Λ-stability condition is precisely what was needed to transcribe to the unitary case the proof of [143] and [120] based on Dennis–Vaserstein decomposition! This task is beautifully accomplished in the joint work of Bak with Petrov and Tang [31]. This work *at last* provides a long desired unified version of stability results for the unitary case which generalises all previously known results! Not only that, the use of Bak's condition allows improving the bound for injective stability in

[3]We try to be as politically correct as we can and carefully control our language.

terms of Bass–Serre dimension by 1. This bound is the one predicted by that for surjective stability in Bak's Thesis [6] (4.7). As in the linear case, consider the stability map

$$\psi_n : K_1(2n, R, \Lambda) \longrightarrow K_1(2(n+1), R, \Lambda).$$

Then the main result of [31], Theorem 1.1, can be stated as follows.

Theorem 9.5. *For any $n \geq \Lambda\,\mathrm{sr}(R)$ the map ψ_n is surjective.*

Theorem 9.6. *For any $n > \Lambda\,\mathrm{sr}(R)$ the map ψ_n is injective.*

These results are direct analogues of the stability results for GL_n, *with the same bounds*. Before that work such a unified analogue was not known even for the case of surjective stability, despite the efforts of many experts! Direct analogue of the stability results for GL_n, again with the same bounds, hold also for general Hermitian groups GH_{2n} [37]. Here, stability phenomenon is concentrated in the hyperbolic part of the group and consequently the Λ-stable rank condition is needed only for the maximal form parameter $\Lambda = \Lambda_{\max}$.

10. Structure in the metastable range

In the non-stable case, as there is no "room" available for manoeuvering as in the stable case (e.g. think of the Whitehead lemma, proving $E(R)$ is normal in $\mathrm{GL}(R)$), one is forced to put some finiteness assumption on the ring. Indeed, there are counter-examples available which show that there are some rings over which the answers to the main structure problems are not standard. For any given n Gerasimov [70] produced examples of rings R for which $E(n, R)$ is as far from being normal in $\mathrm{GL}(n, R)$, as one can imagine.

However, for commutative rings and for rings satisfying appropriate commutativity conditions standard answers do hold. A major contribution in this direction is the work of Suslin [139], [149] who showed that if R is a *module finite ring* namely, a ring that is finitely generated as module over its centre, and $n \geq 3$ then $E(n, R)$ is a normal subgroup of $\mathrm{GL}(n, R)$. That Suslin's normality theorem (and the methods develop to prove it) implies the standard commutator formulae in full force was somewhat later observed independently by Borewicz–Vavilov [55] and Vaserstein [154]. Module finite rings are also called *almost commutative.*

Theorem 10.1. *Assume that R is an almost commutative ring and $n \geq 3$. Further, let $I \trianglelefteq R$ be an ideal of R. Then $E(n, R, I)$ is normal in $\mathrm{GL}(n, R)$. More precisely,*

$$[\mathrm{GL}(n, R), E(n, R, I)] = [E(n, R), C(n, R, I)] = E(n, R, I).$$

Now, it is only natural to ask, whether the second of Bass' theorems viz., the classification of subgroups of $\mathrm{GL}(n, R)$ normalised by $E(n, R)$ also holds when $n \geq 3$ and R is module finite. The following theorem was first established for commutative rings by Wilson [178], for $n \geq 4$, and independently by Golubchik [71], for $n \geq 3$.

Theorem 10.2. *Let R is an almost commutative ring, $n \geq 3$. For any subgroup $H \leq \mathrm{GL}(n, R)$ normalised by $E(n, R)$, there exists a unique ideal $I \trianglelefteq R$ such, that*

$$E(n, R, I) \leq H \leq C(n, R, I).$$

Observe that unlike the stable case now a subgroup H satisfying the conclusion of the theorem is not necessarily normal in $\mathrm{GL}(n, R)$, see §4.2D in [78], many such examples were constructed by A. Mason (see [78]).

In fact, quite remarkably Wilson and Golubchik worked in the absence of the standard commutator formulae, and thus could not directly invoke level reduction used by Bass in his proof at the stable level. As a result their proofs are considerably more complicated than subsequent proofs assuming the standard commutator formulae.

Simpler and more general proofs were devised by Borewicz–Vavilov [55] and Vaserstein [154]. In fact, these proofs are based on completely different ideas. The proof by Borewicz–Vavilov is of geometric nature and is based on reduction of rank. At the same time the proof by Vaserstein is of arithmetic nature and is based on the reduction of dimension of the ground ring (of course, for zero-dimensional rings he has to invoke reduction of rank, but for those rings it is classically known). The ultimate quarter-page proofs of these results, based on decomposition of unipotents, were proposed by Stepanov and Vavilov [137].

We do not attempt to describe many subsequent results in this style, stated in terms of various finiteness and commutativity conditions. Many variations on this theme have been published by Vaserstein and his followers. However, all these variations operate in terms of commutative localisations. A much more general approach has been taken by Golubchik and Mikhalev, who established similar results in terms of non-commutative localisations, such as Ore localisation [72]—[76]. The exact conditions are mostly far too technical to state, but their results imply the standard description for rings such as PI-rings, or weakly noetherian rings. Unfortunately, these outstanding results are mostly still not published in a form accessible to a Western reader, and the published ones do not constitute easy reading, so that their importance is largely downplayed or outright ignored.

The path to full-scale generalisation of these results to other classical groups was anything but straightforward. With many works on the structure of classical groups in various situations[4] the definitive analogues of the above results were only established in the mid-90-ies. Let us state these results for almost commutative rings, since this is the only case which is likely to be of interest to a general audience, apart from a handful of experts in classical groups.

Theorem 10.3. *Assume that R is an almost commutative ring and $n \geq 3$. Further, let $I \trianglelefteq R$ be an ideal of R. Then $E(2n, I, \Gamma)$ is normal in $G(2n, R, \Lambda)$. More precisely,*

$$[G(2n, R, \Lambda), E(2n, I, \Gamma)] = [E(2n, R, \Lambda), C(2n, I, \Gamma)] = E(2n, I, \Gamma).$$

Theorem 10.4. *Let R be an almost commutative ring, $n \geq 3$. For any subgroup $H \leq G(n, R, \Lambda)$ normalised by $E(n, R, \Lambda)$, there exists a unique form ideal $(I, \Gamma) \trianglelefteq (R, \Lambda)$ such, that*

$$E(n, I, \Gamma) \leq H \leq C(n, I, \Gamma).$$

[4]Many of which operated in terms of various finite-dimensionality conditions and thus proved special cases of results already established in [6]!

These results are established in [43], [158], [41], [40]. As far as the first one of these results is concerned, it also holds for $n = 2$ provided that $\Lambda R + R\Lambda = R$, see [43]. Nothing like that can be said in general about the second result, as can be seen from the amazing paper [60] by Costa and Keller, on the exceptional behaviour of the group $\mathrm{Sp}(4, R)$. In fact, as we observed in §8, modulo correction of errors the work [158] proves only a weaker result namely, $E(n, I, \Gamma_{\min}) \leq H \leq C(n, I, \Gamma_{\max})$, whereas the actual proof in [40] is still unpublished.

Presently Bak and Vavilov are updating their proof so that it would operate in terms of Λ-stable rank and non-commutative localisation more general than Ore localisation, rather than absolute stable rank and localisations at maximal ideals of the centre. One cannot claim, that it is the most general result possible in this direction, but at least it is the first proposal of sufficient conditions for the standard description of structural results that uniformly generalises all the previously known ones.

On a different slope, let us refer to the papers [163]—[167] by the second-named author and his students where one can find a detailed comparison of various proofs of these results for *commutative* rings, and their generalisations to exceptional groups.

One giant step forward is to characterise the subnormal subgroups of classical-like groups. This turned out to be directly related to subgroups normalised by relative elementary subgroups. The development of this line of research starts as follows.

In [16] (the original manuscript of which goes back to 1967), Bak studied the subgroups of $\mathrm{GL}(n, R)$ normalised by $E(n, R, I)$, for a ring R with the stable rank condition and obtained a sandwich classification for such subgroups. His motivation for this was to positively answer a question credited to Borel. Consider the general linear group $\mathrm{GL}(n, K)$ where K is a global field. If $n \geq 3$ and H is a noncentral subgroup of $\mathrm{GL}(n, K)$, normalised by an arithmetic subgroup of $\mathrm{GL}(n, K)$, then does H contain an arithmetic subgroup of $\mathrm{GL}(n, K)$? Bak observed that the answer to this would follow if one could establish a sandwich condition similar to the absolute case for subgroups of the special linear group $\mathrm{SL}(n, R)$ normalised by relative elementary groups where R is the ring of integers in K.

Thanks to the works of Wilson [178], Vaserstein [154, 155] and Vavilov [162], the Sandwich Theorem has been improved several times and now we have the following theorem (see [155] for a more general form, and recall that for two ideals I and J of the commutative ring R, $(I : J) = \{r \in R \mid rJ \subseteq I\}$).

Theorem 10.5. *Let R be a commutative ring, $n \geq 3$ and H a subgroup of $\mathrm{GL}(n, R)$ normalised by $E(n, R, J)$ for an ideal J. Then there exist an ideal I such that*

$$E(n, R, I) \subseteq H \subseteq C(n, R, I : J^4).$$

Theorems of the above nature are a key to classify the subnormal subgroups of $GL(n, R)$ (see proof of Theorem 1 in [155]). Namely, if

$$H = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_d = \mathrm{GL}(n, R)$$

is a subnormal subgroup of $\mathrm{GL}(n, R)$, then thanks to the above Theorem, there is an ideal J of R such that

$$E(n, R, J^4) \subseteq H \subseteq C(n, R, J).$$

In [16] Bak conjectured that his Sandwich Classification Theorem holds as well in the setting of general quadratic groups over rings with stable rank condition (in [16] Conjecture 1.3). Indeed, in the light of recent developments in the theory, one can formulate the following conjecture:

Conjecture 1. *Let (R, Λ) be a form ring with R module finite, and let (J, Γ_J) be a form ideal. Let H be a subgroup of $G(2n, R, \Lambda)$, which is normalised by $E(2n, J, \Gamma_J)$. Then there is a form ideal (I, Γ_I) and a positive integer k such that*

$$E(2n, I, \Gamma_I) \subseteq H \subseteq G(2n, I : J^k, (I : J^k) \cap \Lambda).$$

This conjecture for a commutative ring R satisfying the stable rank condition, was settled positively by Habdank [77]. Recently Zhang [180] proved the conjecture in the stable case with only the commutativity assumption on the ring (and obtained a much finer range than is predicated by the conjecture), and consequently a description of subnormal subgroups of quadratic groups in this setting followed. His refinement was to replace $(I : J^k) \cap \Lambda$ by a certain smaller relative form parameter $\Gamma_{(I:J^k)}$ and it is conjectured that the conclusion of the conjecture above holds also for this smaller relative form parameter.

11. Unitary Steinberg groups

Similarly to the linear case one can define the quadratic Steinberg group and the quadratic K_2 group. The *quadratic Steinberg group* $\mathrm{StU}(2n, R, \Lambda)$ is a group generated by $X_{ij}(a)$ where $i, j \in \Delta_n$ and $a \in R$, subject to the relations R(1) to R(6), with X_{ij} instead of T_{ij}. Here $n \geq 3$. As for $n = 1$ or 2 some of R(1) to R(6) are not valid. To be consistent with previous sections, we drop the "U" from the notation and simply write $\mathrm{St}(2n, R, \Lambda)$. Thus $K_{2,2n}(R, \Lambda) = K_2(2n, R, \Lambda)$ is defined as the kernel of the natural epimorphism $\mathrm{St}(2n, R, \Lambda) \to E(2n, R, \Lambda)$, and there is an exact sequence

$$1 \longrightarrow K_{2,2n}(R, \Lambda) \longrightarrow \mathrm{St}(2n, R, \Lambda) \longrightarrow E(2n, R, \Lambda) \longrightarrow 1.$$

To come up with a *stable* version of these groups, one should as usual consider the stablisation homomorphism

$$\mathrm{St}(2n, R, \Lambda) \longrightarrow \mathrm{St}(2(n + 1), R, \Lambda),$$

sending $X_{ij}(a)$ (in the first group) to $X_{ij}(a)$ (in the second group). However, since now the $X_{ij}(a)$'s take part in some new relations, this map is not necessarily an embedding.

Now we can set

$$\mathrm{St}(R, \Lambda) = \varinjlim_{n} \mathrm{St}(2n, R, \Lambda)$$

and define $K_2(R,\Lambda)$ as the kernel of the natural epimorphism $\mathrm{St}(R,\Lambda) \to E(R,\Lambda)$, and again, there is an exact sequence

$$1 \longrightarrow K_2(R,\Lambda) \longrightarrow \mathrm{St}(R,\Lambda) \longrightarrow E(R,\Lambda) \longrightarrow 1.$$

Similarly to the linear case one can proceed to show that the map $\mathrm{St}(R,\Lambda) \to E(R,\Lambda)$ is a central extension. In fact, in his book [14] Bak proves the following analogue of the Milnor–Kervaire theorem (see also §5.5 of [78]).

Theorem 11.1. *The homomorphism* $\mathrm{St}(R,\Lambda) \to E(R,\Lambda)$ *is a universal central extension of* $E(R,\Lambda)$.

It is natural to ask, whether this result carries over to Steinberg groups of finite degree. For the linear case the answer is given by the following remarkable theorem established by van der Kallen [83] and Tulenbaev [149].

Theorem 11.2. *Let R be a module finite ring. Then the homomorphism*

$$\mathrm{St}(n,R) \longrightarrow E(n,R)$$

is central for $n \geq 4$.

The marvelous *another presentation* proof of this theorem for commutative case given in [83] is a real masterpiece of mathematical exposition and can be explained to an undergraduate student (we've done this more than once ourselves!). The proof of the general case [149] combines van der Kallen's basic idea of "another presentation" with Suslin's "factorisation and patching" and is technically somewhat more demanding, compare also [36]. In [137] we sketch the construction of a different van der Kallen like model of the Steinberg group, which starts not with Suslin's decomposition, but rather with the decomposition of unipotents, and leads to a slightly shorter proof.

Bak and his (then) student Guoping Tang announced that by adapting a hard to read van der Kallen–Tulenbaev proof of centrality of nonstable K_2 (in the linear case) to the quadratic case, they could show the centrality of quadratic K_2. Namely, they obtained the following result.

Theorem 11.3. *Let R be a module finite ring. Then the homomorphism*

$$\mathrm{St}(n,R,\Lambda) \longrightarrow E(n,R,\Lambda)$$

is central for $n \geq 8$.

Their proof combines techniques of [41], [83], [149], but the presence of both short and long roots in the elementary quadratic subgroup, non-commutativity, non-triviality of the involution, the presence of the form parameter, among other things, make this proof *tremendously* involved. About 8 years ago we and Alexei Stepanov scrutinised most of the Bak–Tang proof and it looked quite convincing to us, but due to enormous size it is still not published.

In fact, recently one of us raised from the ground a similar proof for the case of exceptional Chevalley groups of types E_6 and E_7, [164]. Despite severe additional technical complications stemming from representation theory and more complicated root structure, we hope that, when completed, the proof in this case would be not nearly quite as complicated as the unitary proof by Bak and Tang — to get some idea, compare [79] and [80].

12. Nilpotency of K_1

Suslin's result makes it possible to define the non-stable $K_{1,n}$, when $n \geq 3$, for module finite rings. The study of these non-stable K_1's is known to be very difficult. There are examples due to van der Kallen [85] and Bak [18] which show that non-stable K_1 can be non-abelian and the natural question is how non-abelian it can be?

The breakthrough came with the brilliant work of Bak [18], who showed that this group is nilpotent by abelian (Theorem 12.2) if $n \geq 3$ and the ring satisfies some dimension condition (e.g. has a centre with finite Krull dimension).

In 1991, Bak introduced in his paper [18] his *localisation-completion* method. Using this method he was able to prove that nonstable $K_1(n, R) = \mathrm{GL}(n, R)/E(n, R)$ is a nilpotent by abelian group provided R is module finite over its centre, with finite Bass–Serre dimension and $n \geq 3$ (see §13 for definitions). Recall that, a group G is nilpotent by abelian, if there is a normal subgroup H, such that G/H is abelian and H is nilpotent. This clearly implies that G is solvable.

Theorem 12.1. *Let R be finitely generated over its centre, with finite Bass–Serre dimension. Then the group $K_1(n, R)$, $n \geq 3$, is nilpotent by abelian.*

In fact, Bak proves a much more precise result, as he explicitly constructs a descending central series in $\mathrm{SL}(n, R)$.

Theorem 12.2. *Let R be a quasi-finite A-algebra, i.e., a direct limit of module finite A-subalgebras, and $n \geq 3$. Then there is a filtration*

$$\mathrm{GL}(n, R) \geq \mathrm{SL}^0(n, R) \geq \mathrm{SL}^1(n, R) \geq \cdots \geq E(n, R),$$

where $\mathrm{GL}(n, R)/\mathrm{SL}^0(n, R)$ is abelian and $\mathrm{SL}^0(n, R) \geq \mathrm{SL}^1(n, R) \cdots$ is a descending central series. Moreover, if $i \geq \delta(R)$, where $\delta(R)$ is the Bass–Serre dimension of R, then $\mathrm{SL}^i(n, R) = E(n, R)$.

His method which consists of some "conjugation calculus" on elementary elements, plus simultaneously applying localisation-patching and completion was a source of further work in this and related areas. We analyse this important work in more detail in the Section 13. In [79] the first author adopts the same method to study nonstable K_1 of quadratic modules, establishing Theorem 12.2 for the general quadratic group $G(2n, R, \Lambda)$. In the case of general quadratic groups, as the elementary subgroups have more generators and relations, this makes, among other things, computations in this setting much more involved.

In [80] the authors apply the localisation-completion method in the setting of Chevalley groups to prove that K_1 of Chevalley groups are nilpotent by abelian. More precisely, let Φ be a reduced irreducible root system of rank at least 2 and let R be a commutative ring of finite Bass–Serre dimension or finite Krull dimension. Let $G(\Phi, R)$ be a Chevalley group of type Φ over R and let $E(\Phi, R)$ be its elementary subgroup. Then $K_1(\Phi, R) = G(\Phi, R)/E(\Phi, R)$ is nilpotent by abelian. In particular, $E(\Phi, R)$ is a characteristic subgroup of $G(\Phi, R)$. This fact plays a crucial role in A. Stepanov's recent spectacular results, partly positive, partly negative, solving the oversubgroup problem for $E(\Phi, R)$ embedded in $G(\Phi, R')$ where R' is an overring of R [136]. Also, the recent classification of subgroups of the general linear group $\mathrm{GL}(2n, R)$ containing the elementary quadratic subgroup $EU(2n, R, \Lambda)$ was made possible by employing the powerful form of Bak's "conjugation calculus" [113, 168, 169].

Putting these results together, one can write the following theorems which describe the nilpotent structure of K_1 group of classical-like groups.

Theorem 12.3. *Let $G(2n, R, \Lambda)$, $n \geq 3$, be the general quadratic group over an almost commutative form ring (R, Λ), of finite Bass–Serre dimension, and let $E(2n, R, \Lambda)$ be its elementary subgroup. Then $G(2n, R, \Lambda)/E(2n, R, \Lambda)$ is nilpotent by abelian.*

Theorem 12.4. *Let Φ be a reduced irreducible root system of rank ≥ 2, $G(\Phi, R)$ a Chevalley group of type Φ over a commutative ring R of finite Bass–Serre dimension, let $E(\Phi, R)$ its elementary subgroup. Then $G(\Phi, R)/E(\Phi, R)$ is nilpotent by abelian.*

Theorem 12.3 is proved in [79] and Theorem 12.4 in [80]. Using Bak's localisation-completion and Stein's relativisation [131] the above theorem is carried out in the relative case as well [44].

13. Localisation-completion

The methodology employed to prove Theorem 12.2, including the localisation-completion method, was the main motivation to later develop *dimension theory* in arbitrary categories (see §14), and also influenced many subsequent works. We describe here Bak's localisation-completion method. For making the method as transparent as possible, we assume the ground ring A to be commutative Noetherian[5].

We start by recalling the notion of Bass–Serre dimension for a commutative ring A. Consider the topological space $\mathrm{Spec}(A)$ of all prime ideals of A under the Zariski topology, and let, as before, $\mathrm{Max}(A)$ denote the subspace of maximal ideals of A. The Bass–Serre dimension of A, denoted by $\delta(A)$, is defined to be the smallest non-negative integer d such that $\mathrm{Max}(A) = X_1 \cup \cdots \cup X_r$ is a finite union of irreducible Noetherian subspaces X_i, with topological dimension not greater than d. It is easy to see that $\delta(A) = 0$ if and only if A is semi-local. For a module finite A-algebra R, we define $\delta(R)$ to be the dimension $\delta(A)$. Again, it is clear that if $\delta(R) = 0$, then R is a semi-local ring.

As a first step for invoking induction on $\delta(A)$ later on, Bak proves an *induction lemma.*

[5] Since any ring is a direct limit of its Noetherian subrings and the functors GL_n and E_n commute with direct limits, the proof smoothly reduces to Noetherian rings. Bak introduced his *finite completion* method to take care of this when working with general rings

Lemma 13.1. *If $s \in A$ such that for each $X_k (1 \le k \le r)$, s does not lie in some member of X_k, then $\delta(\widehat{A}_s) < \delta(A)$ where $\widehat{A}_s = \varprojlim_{p \ge 0} A/s^p A$ is the completion of A at s.*

With this lemma, Bak sets the stage for simultaneously employing *localisation-patching* to make computations in zero dimensional rings and *completion* for applying induction on $\delta(A)$. This is done as follows. For any multiplicative set S in A, let $S^{-1}A$ denote the ring of S-fractions of A. In particular, for an $s \in A$, we denote by $\langle s \rangle = \{s^i \mid i \ge 0\}$ the multiplicative set generated by s.

Let r have the same meaning, as in the definition of Bass–Serre dimension. For each k, $1 \le k \le r$, pick a maximal ideal $\mathfrak{m}_k \in X_k$. In the sequel, we take S to be the multiplicative set $S = A \setminus (\mathfrak{m}_1 \cup \cdots \cup \mathfrak{m}_r)$. For each $s \in S$, consider a diagram

$$\widehat{A}_s \longleftarrow A \longrightarrow \langle s \rangle^{-1} A$$

and the direct limit $\varinjlim_s \langle s \rangle^{-1} A = S^{-1}A$. The ring $S^{-1}A$ is semi-local and any completion over $S^{-1}A$ involving only a finite number of elements actually takes place in some $\langle s \rangle^{-1}A$. Furthermore, $\delta(\widehat{A}_s) < \delta(A)$ for any $s \in S$ by the induction lemma. Bak's strategy is to use the rings $\langle s \rangle^{-1}A$ to perform actual computations, and the rings $\widehat{A}_s$ to apply induction on the dimension of A.

To carry out this strategy, he introduces his *dimension filtration*,

$$\mathrm{SL}^d(n,R) = \bigcap_{\substack{R \longrightarrow S \\ \delta(R') \le d}} \mathrm{Ker}\Big(\mathrm{GL}(n,R) \longrightarrow \mathrm{GL}(n,R')/E(n,R')\Big),$$

where R' is a module finite A-algebra. See [79], Definition-Lemma 3.3 for technicalities.

It is clear that the following homomorphism is injective,

$$\mathrm{GL}(n,R)/\,\mathrm{SL}^0(n,R) \longrightarrow \prod_{\delta(R')=0} \mathrm{GL}(n,R')/E(n,R').$$

Since $\delta(R') = 0$, the ring R' is semi-local, and thus $\mathrm{GL}(n,R')/E(n,R')$ is abelian. It follows that $\mathrm{GL}(n,R)/\,\mathrm{SL}^0(n,R)$ is an abelian group. To show that the second half of the filtration in Theorem 12.2 is a descending central series, it suffices to show that for any $x \in \mathrm{SL}^0(n,R)$ and $y \in \mathrm{SL}^{d-1}(n,R)$, one has $[x,y] \in \mathrm{SL}^d(n,R)$. Since the filtration is functorial, we can assume $d = \delta(R) = \delta(A)$.

Here Bak uses his localisation-completion techniques. Consider the diagram

$$\mathrm{GL}(n,\widehat{R}_s) \xleftarrow{\widehat{F}_s} \mathrm{GL}(n,R) \xrightarrow{F_s} \mathrm{GL}(n, \varinjlim_s \langle s \rangle^{-1} R).$$

Since $\delta(S^{-1}R) = \delta(S^{-1}A) = 0$, the image of x in $\mathrm{GL}(n,S^{-1}R)$ lies in $E(n,S^{-1}R)$. Since $S^{-1}R = \varinjlim_s \langle s \rangle^{-1}R$, there is an $s \in S$ such that $F_s(x) \in E(n,\langle s \rangle^{-1}R)$. On the other hand, by induction on the dimension of R, $\widehat{F}_s(y) \in E(n,\widehat{R}_s)$.

It remains only to show that

$$\Big[{F_s}^{-1}\big(E(n,\langle s \rangle^{-1}R)\big), \widehat{F}_s^{-1}\big(E(n,\widehat{R}_s)\big)\Big] \subseteq E(n,R). \tag{2}$$

Let $x \in F_s{}^{-1}(E(n, \langle s\rangle^{-1}R))$ and $y \in \widehat{F}_s^{-1}(E(n, \widehat{R}_s))$. For any subset $T \subset \langle s\rangle^{-1}R$, we denote by $E^K(T)$ the subset of $E(n, \langle s\rangle^{-1}R)$, that consists of all products of at most K elementary matrices of level T. It is clear that replacing s by its sufficiently large power s^i, we can assume that $x \in F^{-1}(E^K(\frac{1}{s}R))$ for some natural K. Since $E(n, R)$ is dense in $E(n, \widehat{R}_s)$ in the s-adic topology, it is clear that $y \in E(n, R)\,\mathrm{GL}(n, s^kR)$ for any $k \geq 0$.

Thus, in order to prove the inclusion (2), it suffices to show, that for any $0 \neq s \in S$, and any natural K, there exists a k such that

$$\left[F_s^{-1}\left(E^K\left(\frac{1}{s}R\right)\right), \mathrm{GL}(n, s^kR)\right] \subseteq E(n, R). \tag{3}$$

As we noticed, $y = uz$, for some $u \in E(n, R)$ and $z \in \mathrm{GL}(n, s^kR)$. But since u does not influence the inclusion, it suffices to verify Equation 3.

Now, since R is Noetherian, there is a positive integer i, such that the map $F_s : s^iR \longrightarrow \langle s\rangle^{-1}R$ is injective, see [18], Lemma 4.10. This is one of the crucial points of the whole proof, as it guarantees that $F_s|_{\mathrm{GL}(n,s^iR)}$ is injective. Bak then shows that there exists a natural $k_i \geq i$ such that

$$\left[E^K\left(\frac{1}{s}R\right), F_s(\mathrm{GL}(n, s^{k_i}R))\right] \subseteq F_s(E(n, s^iR)) \tag{4}$$

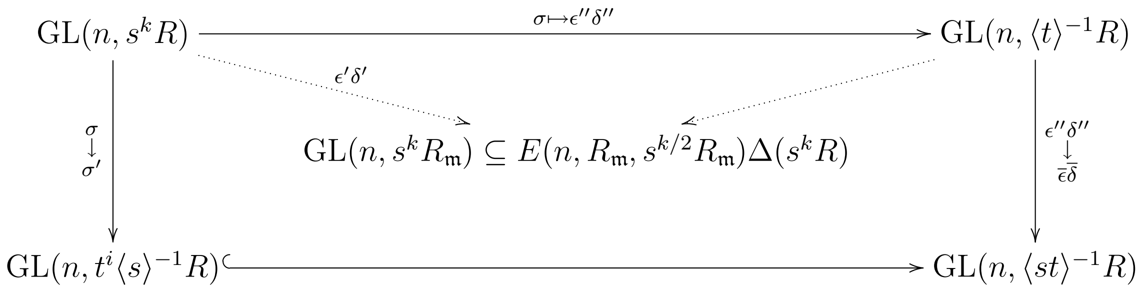
The inclusion (3) is a trivial consequence of (4) and the injectivity of F_s on $\mathrm{GL}(n, s^iR)$.

Observe, that this is a very powerful result. In particular, Suslin's normality theorem, we discussed in §10, is an immediate corollary of this result, corresponding to the case, when $s = 1$.

Finally, we outline the main idea of the proof of inclusion (4). It suffices to consider the case $K = 1$. The general case easily follows by induction on K.

Let $\epsilon(a/s) \in E^1\big(\frac{1}{s}R\big)$ and $\sigma' \in F_s(\mathrm{GL}(n, s^kR))$. If we show that for any maximal ideal $\mathfrak{m}$ of A, there is an element $t_\mathfrak{m} \in A \setminus \mathfrak{m}$, and an integer $l_\mathfrak{m}$ such that $[\epsilon(t_\mathfrak{m}^{l_\mathfrak{m}}a/s), \sigma'] \in F_s(E(n, s^qR))$ for suitable q, then since a finite number of $t_\mathfrak{m}$ generate A, it can be seen that $[\epsilon(a/s), \sigma'] \in F_s(E(n, s^pR))$. But to show this we have to use two localisations at the same time (see the diagram below). Suppose $\sigma \in \mathrm{GL}(n, s^kR)$, such that $F_s(\sigma) = \sigma'$. Since localising R at $\mathfrak{m}$, i.e., $R_\mathfrak{m}$, is a semi-local ring, the image of σ in $R_\mathfrak{m}$ can be written as $\epsilon'\delta'$ where δ' is a diagonal matrix and $\epsilon' \in E(s^{k/2}R_\mathfrak{m})$

By a direct limit argument, there is a $t \in A \setminus \mathfrak{m}$, such that the image of σ in $\langle t\rangle^{-1}R$ is $\epsilon''\delta''$ where δ'' is a diagonal matrix and $\epsilon'' \in E(s^{k/2}tR)$. On the other hand, since A is Noetherian, one can see that for a suitable positive integer i the map $t^i\langle s\rangle^{-1}R \longrightarrow \langle st\rangle^{-1}R$ is injective and thus $\mathrm{GL}(n, t^i\langle s\rangle^{-1}R) \longrightarrow \mathrm{GL}(n, \langle st\rangle^{-1}R)$ is injective. Since $\mathrm{GL}(n, t^i\langle s\rangle^{-1}R)$ is a normal subgroup of $\mathrm{GL}(n, \langle s\rangle^{-1}R)$, it follows that for any integer $l \geq i$, $[\epsilon(t^la/s), \sigma'] \in \mathrm{GL}(n, t^i\langle s\rangle^{-1}R)$. Consider the following diagram:



Now the images of σ' and $\epsilon''\delta''$ in the ring $\langle st\rangle^{-1}R$, are images of the same element σ, calculated in two different ways, and thus must coincide:

$$\left[F_t\left(\epsilon\left(\frac{t^l a}{s}\right)\right), F_t(\sigma')\right] = \left[F_t\left(\epsilon\left(\frac{t^l a}{s}\right)\right), F_s\left(E\left(\frac{s^{k/2}}{t}a'\right)\right)\overline{\delta}\right].$$

It is easy to see that $\overline{\delta}$ disappears and it only remains to verify that for any p and q there exist suitable integers l and k such that

$$\left[E\left(\frac{t^l}{s}R\right), E\left(\frac{s^k}{t}R\right)\right] \subseteq E(s^p t^q R).$$

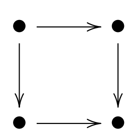
Bak achieves this by a clever application of the "conjugation calculus" of elementary matrices, see Lemmas 4.6 to 4.8 in [18].

14. Bak's dimension theory

In 1995, Bak gave a lecture series in Buenos Aires, sketching a general theory of group-valued functors on arbitrary categories with structure and dimension [19].

An arbitrary category $\mathcal{C}$ is structured by fixing a class of commutative diagrams in $\mathcal{C}$, called structure diagrams and a class of functors taking values in $\mathcal{C}$, called infrastructure functors. A function $d : \mathcal{O}bj(\mathcal{C}) \longrightarrow$ (ordinal numbers) is called a dimension function if it satisfies a certain property, called reduction, relating it to the structure on $\mathcal{C}$.

The structure diagram and infrastructure functors one takes depend on the landscape being modelled and the results being sought. We shall describe fundamental concepts and results of the theory in terms of structure diagrams of the kind $\bullet \longleftarrow \bullet \longrightarrow \bullet$ (which are automatically commutative) and dimension function taking values in $\mathbb{Z}^{\geq 0} \cup \{\infty\}$. A short description of the theory in terms of structure diagrams of the kind



as well as some of its applications, may be found in [36].

We begin by defining the notion of a category with structure, using structure diagrams of the kind $\bullet \longleftarrow \bullet \longrightarrow \bullet$.

Definition 14.1. A *category with structure* is a category $\mathcal{C}$ together with a class $\mathcal{S}(\mathcal{C})$ of diagrams $C \longleftarrow A \longrightarrow B$ in $\mathcal{C}$ called *structure diagrams* and a class $\mathcal{I}(\mathcal{C})$ of functors $F : I \longrightarrow \mathcal{C}$ from directed quasi-ordered sets I to $\mathcal{C}$ called *infrastructure functors*, satisfying the following conditions.

(1) $\mathcal{S}(\mathcal{C})$ is closed under isomorphism of diagrams.

(2) For each object A of $\mathcal{C}$, the *trivial* diagram i.e., $A \longleftarrow A \longrightarrow A$ is in $\mathcal{S}(\mathcal{C})$.

(3) $\mathcal{I}(\mathcal{C})$ is closed under isomorphism of functors.

(4) For each object A of $\mathcal{C}$, the *trivial* functor $F_A : \{*\} \longrightarrow \mathcal{C}, * \mapsto A$, is in $\mathcal{I}(\mathcal{C})$, where $\{*\}$ denotes the directed quasi-ordered set with precisely one element $*$.

(5) For each $F : I \longrightarrow \mathcal{C}$ in $\mathcal{I}(\mathcal{C})$, the direct limit $\varinjlim_I F$ exists in $\mathcal{C}$.

Next, a category with dimension is defined. To do this, we need first the notion of reduction.

Definition 14.2. Let $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}))$ be a category with structure. Let $d : \mathcal{O}bj(\mathcal{C}) \longrightarrow \mathbb{Z}^{\geq 0} \cup \infty$ be a function which is constant on isomorphism classes of objects. Let $A \in \mathcal{O}bj(\mathcal{C})$ such that $0 < d(A) < \infty$. A *d-reduction* of A is a set

$$C_i \longleftarrow A \longrightarrow B_i \ (i \in I)$$

of structure diagrams where I is a directed quasi-ordered set and $B : I \longrightarrow \mathcal{C}, i \mapsto B_i$, is an infrastructure functor such that the following holds.

(1) If $i \leq j \in I$ then the triangle

$$\begin{array}{ccc} A & & \\ \downarrow & \searrow & \\ B_i & \longrightarrow & B_j \end{array}$$

commutes.

(2) $d(\varinjlim_I B_i) = 0$.

(3) $d(C_i) < d(A)$ for all $i \in I$.

A function d is called a *dimension function* on $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}))$, if any object A of $\mathcal{C}$, such that $0 < d(A) < \infty$, has a d-reduction. In this case, the quadruple $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}), d)$ is called a *category with dimension*.

Remark 14.3. Bak has a more comprehensive theory, where he also introduces *virtual isomorphisms* in the category and the notion of *type* and an object can have d-reductions of a specific type [19].

Even in this early stage of the theory, one can prove that for any dimension function, there exist a universal one, as the following theorem shows.

Theorem 14.4. *Let $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}))$ be a category with structure and $\mathcal{C}^0$ a nonempty class of objects of $\mathcal{C}$, closed under isomorphism. Then there is a dimension function δ on $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}))$ called the universal dimension function for $\mathcal{C}^0$, such that*

- *$\mathcal{C}^0$ is the class of 0-dimensional objects of δ,*
- *if d is any other dimension function on $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}))$, whose 0-dimensional objects are contained in $\mathcal{C}^0$, then $\delta \le d$.*

For the rest of this section $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}), d)$ will denote a category with dimension and

$$\mathcal{G}, \mathcal{E} : \mathcal{C} \longrightarrow \mathcal{G}roup$$

will be a pair of group valued functors on $\mathcal{C}$ such that $\mathcal{E} \subseteq \mathcal{G}$.

Bak then defines the *dimension filtration*

$$\mathcal{G} \supseteq \mathcal{G}^0 \supseteq \mathcal{G}^1 \supseteq \cdots \supseteq \mathcal{E}$$

of $\mathcal{G}$ with respect to $\mathcal{E}$ by

$$\mathcal{G}^n(A) = \bigcap_{\substack{A \longrightarrow B \\ d(B) \le n}} \operatorname{Ker}\Big(\mathcal{G}(A) \longrightarrow \mathcal{G}(B)/\mathcal{E}(B)\Big).$$

This filtration generalises the filtration constructed for $\mathrm{GL}(n, R)$ in Section 13 .

Definition 14.5. A pair $\mathcal{G}, \mathcal{E}$ of group valued functors on $\mathcal{C}$ is called *good* if the following holds.

(1) $\mathcal{E}$ and $\mathcal{G}$ preserve direct limits of infrastructure functors.
(2) For any A of $\mathcal{C}$, $\mathcal{E}(A)$ is a perfect group.
(3) For any zero dimensional object A, $K_1(A) := \mathcal{G}(A)/\mathcal{E}(A)$ is an abelian group.
(4) For any structure diagram

$$C \longleftarrow A \longrightarrow B$$

let $H = \operatorname{Ker}(\mathcal{G}(A) \longrightarrow \mathcal{G}(B)/\mathcal{E}(B))$ and $L = \operatorname{Ker}(\mathcal{G}(A) \longrightarrow \mathcal{G}(C)/\mathcal{E}(C))$. Then one has the following inclusion $[H, L] \subseteq \mathcal{E}(A)$.

The following theorem is a central result in Bak's theory of group valued functors on categories with dimension.

Theorem 14.6. *Let $\mathcal{C} = (\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}), d)$ be a category with dimension and $(\mathcal{G}, \mathcal{E})$ be a good pair of group valued functors on $\mathcal{C}$. Then the dimension filtration*

$$\mathcal{G} \supseteq \mathcal{G}^0 \supseteq \mathcal{G}^1 \supseteq \cdots$$

of $\mathcal{G}$ with respect to $\mathcal{E}$ is a normal filtration of $\mathcal{G}$ such that the quotient functor $\mathcal{G}/\mathcal{G}^0$ takes its values in abelian groups and the filtration $\mathcal{G}^0 \supseteq \mathcal{G}^1 \supseteq \cdots$ is a descending central series such that if $d(A)$ is finite then $\mathcal{G}^n(A) = \mathcal{E}(A)$, whenever $n \ge d(A)$.

In particular, if $d(A)$ is finite, then $\mathcal{E}(A)$ is a characteristic subgroup of $\mathcal{G}(A)$.

In applications thus far of the Theorem 14.6, the structure diagram $\bullet \longleftarrow \bullet \longrightarrow \bullet$ are localisation-completion diagrams $\widehat{A}_s \longleftarrow A \longrightarrow \langle s\rangle^{-1}A$ where A is a ring, form ring, etc. On the resulting category with structure, dimension functions d are established by proving an induction lemma similar to Lemma 13.1. However, one can and should replace using Theorem 14.4, the dimension function d by the universal dimension function δ for the zero dimensional objects of d, or even better, by all objects D such that $\mathcal{G}(D)/\mathcal{E}(D)$ is abelian, and call these objects the zero dimensional ones. This δ can be considerably smaller than d. The crucial property (4) in Definition 14.5 of a good pair $(\mathcal{G}, \mathcal{E})$ of group valued functors is obtained in practice by choosing the right definition of completion $\widehat{A}_s$, so that the square

$$\begin{array}{ccc} A & \longrightarrow & \langle s\rangle^{-1}A \\ \downarrow & & \downarrow \\ \widehat{A}_s & \longrightarrow & \langle s\rangle^{-1}\widehat{A}_s \end{array}$$

is a pullback diagram. In all applications thus far, this is the notion of finite completion. Note that squares

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

instead of diagrams $\bullet \longleftarrow \bullet \longrightarrow \bullet$ have been necessary in practice when verifying that condition (4) of Definition 14.5 is satisfied. In fact, there is a version of good pair, in which squares are already required in the definition. We remark on this next.

Remark 14.7. Bak also has an alternative version of the above theorem, in which structure diagrams are commutative squares

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

instead of diagrams $\bullet \longleftarrow \bullet \longrightarrow \bullet$ and a good pair $(\mathcal{G}, \mathcal{E})$ is replaced by a natural transformation $\mathcal{S} \longrightarrow \mathcal{G}$ of group valued functors such that the following holds

(1) $\mathcal{S}$ and $\mathcal{G}$ preserve direct limits of infrastructure functors.
(2) $\mathcal{S}(A)$ is perfect for any A.
(3) $\mathcal{G}(A)/\text{image}(\mathcal{S}(A) \longrightarrow \mathcal{G}(A))$ is abelian for any zero dimensional object A.
(4) $\text{Ker}(\mathcal{S}(A) \longrightarrow \mathcal{G}(A)) \subseteq \text{Cent}(\mathcal{S}(A))$ for any finite dimensional object A.
(5) The extension $\mathcal{S} \longrightarrow \mathcal{G}$ satisfies excision on any structure square.

The conclusion of the alternative version is the same as that above, see [36].

In the light of Section 13, one can see that the localisation-completion method can serve as a way to structure the category of rings, possibly with some additional algebraic structure like form parameters. The resulting category with structure together with Bass–Serre dimension forms a category with dimension. One can then readily see that functors such as GL_n and E_n, satisfy conditions 1, 2 and 3 of Definition 14.5 and the functors GL_n and St_n, satisfy conditions 1, 2 and 3 of Remark 14.7. Thus, in order to obtain Theorem 14.6, one needs only to check that the remaining conditions hold for the functors in question.

In [108] Bak's student Mundkur uses the functors GL_n and St_n and the alternative version of "good" functors described in Remark 14.7 to recover Bak's theorem by the machinery of dimension theory. But he had to pay a price by assuming that $n \geq 4$, since centrality of the extension $\mathrm{St}(n, R) \longrightarrow E(n, R)$ for commutative rings is only known when $n \geq 4$.

On the other hand, in the case of quadratic modules the results concerning centrality of the extension $\mathrm{St}(2n, R, \Lambda) \longrightarrow E(2n, R, \Lambda)$ are not yet published. We discussed the status of this problem in Section 11. Thus, in [79], the first author had to adopt the notion of good pair in Definition 14.5 and check condition 4, which because of the presence of both short and long roots in the elementary quadratic subgroup, needs extra effort.

Bak and Stepanov use in [36] the alternative version of good functors described in Remark 14.7 to study the nonstable K-theory of *net* general linear groups introduced by Borewicz and Vavilov [55].

15. Congruence subgroup problem

As we mentioned in Section 2, the early development of lower algebraic K-theory was strongly motivated by and closely related to the congruence subgroup problem. In this section we outline the works by Bak and Rehmann, concerning the congruence subgroup problem for classical groups over central simple algebras over a global field. Here we take a very limited view and do not attempt to discuss the subsequent remarkable results by many authors. To get a broader prospective, the reader should consult the book by Platonov and Rapinchuk [116] and the recent papers by Prasad, Raghunathan [121] and Rapinchuk.

Let D be a division ring with centre K of finite index (i.e., D, as a vector space over F, has a finite bases). Let $\mathcal{O}$ be a subring of D, $\mathrm{Nrd} : \mathrm{GL}(n, D) \longrightarrow K^*$ denotes the reduced norm map for any n and let $\mathrm{SL}_n(\mathcal{O}) = \mathrm{Ker}\,\mathrm{Nrd}\,|_{\mathrm{GL}(n,\mathcal{O})}$. For any two sided ideal $\mathfrak{a}$ of $\mathcal{O}$, recall the group $\mathrm{GL}(n, \mathcal{O}, \mathfrak{a})$ from Section 2 and set $\mathrm{SL}_n(\mathcal{O}, \mathfrak{a}) = \mathrm{SL}_n(\mathcal{O}) \cap \mathrm{GL}(n, \mathcal{O}, \mathfrak{a})$. The groups $\mathrm{GL}(n, \mathcal{O}, \mathfrak{a})$ and $\mathrm{SL}_n(\mathcal{O}, \mathfrak{a})$ are called *congruence subgroups of level* $\mathfrak{a}$ of $\mathrm{GL}(n, \mathcal{O})$ and $\mathrm{SL}_n(\mathcal{O})$, respectively.

For the moment suppose $D = \mathbb{Q}$ is the field of rational numbers and $\mathcal{O} = \mathbb{Z}$ is the ring of integers. For any ideal $\mathfrak{a}$ of $\mathbb{Z}$, the quotient $\mathbb{Z}/\mathfrak{a}$ is finite. It follows that $\mathrm{GL}(n, \mathcal{O})/\,\mathrm{GL}(n, \mathcal{O}, \mathfrak{a})$ is finite and this in turn implies that $\mathrm{SL}(n, \mathcal{O})/\,\mathrm{SL}(n, \mathcal{O}, \mathfrak{a})$ is finite. Thus $\mathrm{SL}(n, \mathcal{O}, \mathfrak{a})$ has finite index in $\mathrm{SL}(n, \mathcal{O})$. In 1964, Mennicke and independtly Bass, Lazard and Serre proved that if $n \geq 3$ then the converse is also valid, namely any finite index subgroup of $\mathrm{SL}(n, \mathcal{O})$ contains a congruence subgroup. For almost a hundred years before this theorem was proved, it was known to Klein that $\mathrm{SL}(2, \mathbb{Z})$ does not follow this pattern!

Now, let D denote a finite central division algebra over a global field K. Let Σ denote a non-empty finite set of nonequivalent valuations of K which contains all archimedean valuations of K. The ring $R = \bigcap_{v \notin \Sigma} R_v$, where R_v is the valuation ring of v, is called the *ring of* Σ*-integers* of K. Let $\mathcal{O}$ be a maximal R-order on D. If $\mathfrak{a}$ is a two sided ideal of $\mathcal{O}$ then $\mathcal{O}/\mathfrak{a}$ is finite and thus as above the congruence subgroup $\mathrm{SL}(n, \mathcal{O}, \mathfrak{a})$ has finite index in $\mathrm{SL}(n, \mathcal{O})$. The *congruence subgroup problem* asks whether the converse is true, as in the above case $D = \mathbb{Q}$ and $\mathcal{O} = \mathbb{Z}$. In other words, does any finite index subgroup of $\mathrm{SL}(n, \mathcal{O})$ contain a congruence subgroup?

Serre formulates the congruence subgroup problem in terms of computing a certain group defined as follows. Let $\overline{\mathrm{SL}(n, \mathcal{O})}$, respectively, $\widehat{\mathrm{SL}(n, \mathcal{O})}$ denote the completion of $\mathrm{SL}(n, \mathcal{O})$ with respect to the topology defined by the family of congruence subgroups $\mathrm{SL}(n, \mathcal{O}, \mathfrak{a})$, respectively, family of subgroups of finite index. There is a canonical surjective homomorphism $\widehat{\mathrm{SL}(n, \mathcal{O})} \longrightarrow \overline{\mathrm{SL}(n, \mathcal{O})}$. Let $C(\Sigma, \mathrm{SL}(n, \mathcal{O}))$ be the kernel of this map. It is called the *congruence kernel*. One can check easily that $C(\Sigma, \mathrm{SL}(n, \mathcal{O}))$ is trivial if and only if the congruence subgroup problem has a positive answer. Serre's formulation of the problem is as follows.

Congruence subgroup problem according to Serre. If $n \geq 3$ or if $n = 2$ and Σ has at least two elements, is $C(\Sigma, \mathrm{SL}(n, \mathcal{O}))$ finite? When is $C(\Sigma, \mathrm{SL}(n, \mathcal{O}))$ trivial?

In 1967 Bass, Milnor and Serre solved the congruence subgroup problem for SL_n, where $n \geq 3$ and for Sp_{2n}, where $n \geq 2$ in the case $D = K$ is a global field and computed the groups $C(\Sigma, \mathrm{SL}(n, \mathcal{O}))$ and $C(\Sigma, \mathrm{Sp}_{2n}(\mathcal{O}))$.

In 1970 Serre considered the group SL_2 when Σ has at least two elements and computed the group $C(\Sigma, \mathrm{SL}_2(\mathcal{O}))$.

In the above cases the congruence kernel $C(\Sigma, G)$, where G is one of SL_n or Sp_{2n}, can be described as follows:

$$C(\Sigma, G) = \begin{cases} 1, & \text{if } \Sigma \text{ contains a noncomplex valuation,} \\ \mu(K), & \text{if } \Sigma \text{ is totally complex,} \end{cases}$$

where $\mu(K)$ is the group of roots of unity of K.

Serre also showed that if $G = \mathrm{SL}_2$ and S has only one element, then $C(S, G)$ is an infinite group, thus uniformly explaining the classical counter-examples.

In 1981, Bak and Rehmann studied the congruence subgroup problem in the general situation of a central division algebra D over a global field K [33]. They had discovered already by 1979 (see [32]) that the computation of the congruence kernel $C(\Sigma, \mathrm{SL}(n, \mathcal{O}))$ depends on $\Sigma \setminus \mathrm{Pl}_{D/K}(\Sigma)$, where

$$\mathrm{Pl}_{D/K}(\Sigma) = \{v \in \Sigma \mid K_v^* / \mathrm{Nrd}(D_v^*) \neq 1\},$$

rather than Σ itself. This influenced others working on the problem for other groups to reformulate the result they were looking for. Now we can state the precise form of the theorem established by Bak and Rehmann.

Theorem 15.1. *Let $n \geq 2$. If $n = 2$, suppose that $|\Sigma| > 2$. In addition if $D \neq K$, suppose that $\mathrm{SL}_1(\mathcal{O})$ is infinite. Then*

$$C(\Sigma, \mathrm{SL}(n, \mathcal{O})) = \begin{cases} 1, & \text{if } \Sigma \smallsetminus Pl_{D/K}(\Sigma) \text{ contains a noncomplex valuation} \\ \mu(K), & \text{if } \Sigma \smallsetminus Pl_{D/K}(\Sigma) \text{ is empty or totally complex,} \end{cases}$$

unless $2|[D : K]$ and for every 2-power root of unity $\xi \neq \pm 1$, one has $\xi - \xi^{-1} \notin K$. In this last case $C(\Sigma, SL_n(\mathcal{O}))$ is either $\mu(K)$ or $\mu(K)/\{\pm 1\}$.

It is still an open conjecture that the case $\mu(K)/\{\pm 1\}$ never occurs and $C(\Sigma, SL_n(\mathcal{O}))$ is either trivial or $\mu(K)$.

Sketch of the Proof. One shows first that

$$C(\Sigma, \mathrm{SL}(n, \mathcal{O})) \cong \varprojlim_{a \neq 0} \mathrm{SL}_n(\mathcal{O}, \mathfrak{a})/E(n, (\mathcal{O}, \mathfrak{a})).$$

By the Bass stability for the functor K_1, we know that

$$\mathrm{SL}(n, \mathcal{O}, \mathfrak{a})/E(n, (\mathcal{O}, \mathfrak{a}) \cong \mathrm{SK}_1(\mathcal{O}, \mathfrak{a}) := \mathrm{SL}(\mathcal{O}, \mathfrak{a})/E(\mathcal{O}, \mathfrak{a}).$$

Thus

$$C(\Sigma, \mathrm{SL}(n, \mathcal{O})) \cong \varprojlim_{a \neq 0} \mathrm{SK}_1(\mathcal{O}, \mathfrak{a}).$$

To compute $SK_1(\mathcal{O}, \mathfrak{a})$ one uses the following exact sequence of Bak [14], 7.36,

$$\begin{aligned} (5) \quad K_2(D) \longrightarrow \coprod_{v \notin \Sigma} \mathrm{coker}\Big(K_2(\mathcal{O}_v, \mathfrak{a}_v) \longrightarrow K_2(D_v)\Big) \longrightarrow \mathrm{SK}_1(\mathcal{O}, \mathfrak{a}) \\ \longrightarrow \prod_{v \notin \Sigma} \mathrm{SK}_1(\mathcal{O}_v, \mathfrak{a}_v) \oplus \mathrm{SK}_1(D) \longrightarrow \prod_{v \notin \Sigma} (\mathrm{SK}_1(D_v), \mathrm{SK}_1(\mathcal{O}_v)), \end{aligned}$$

where $\prod$ denotes the restricted direct product, [14],§7E. Observe that, this exact sequence holds not only for general linear groups, but for all classical groups GQ (see §2), and not only for rings of integers in division rings, but for a much larger class of rings. It contains the basic strategy for computing the answer to the congruence subgroup and metaplectic problems for all classical groups [15] and also for computing surgery groups in differential topology. Examples of surgery computations are provided in [9] and [10]. The paper [9] is of special importance, since it shows that odd dimensional surgery groups of finite odd order groups vanish. This means that there is no obstruction to performing surgery on an odd dimensional smooth compact manifold whose fundamental group is finite of odd order. Bak and Rehmann then show that for almost all v's, the group $SK_1(\mathcal{O}_v)$ is trivial.

On the other hand, since the group SK_1 is trivial for division algebras over local and global fields (by theorems of Nakayama–Matsushima and Wang, respectively), one has $\prod_{v \notin \Sigma} (SK_1(D_v), SK_1(\mathcal{O}_v)) = 1$, and the above exact sequence reduces to

$$K_2(D) \xrightarrow{\phi} \coprod_{v \notin \Sigma} K_2(D_v)/K_2(\mathcal{O}_v, \mathfrak{a}_v) \longrightarrow \mathrm{SK}_1(\mathcal{O}, \mathfrak{a}) \longrightarrow \prod_{v \notin \Sigma} SK_1(\mathcal{O}_v, \mathfrak{a}_v) \longrightarrow 1.$$

Since we are interested in the inverse limit of the $\mathrm{SK}_1(\mathcal{O},\mathfrak{a})$ over all nonzero $\mathfrak{a}$, we can restrict to the case of "small" nonzero two sided ideals $\mathfrak{a}$. Bak and Rehmann show that for such ideals $\mathrm{SK}_1(\mathcal{O}_v,\mathfrak{a}_v)=1$, [33], Lemma 6.1, and that for inclusions $\mathfrak{a}'\subseteq\mathfrak{a}''$ of such ideals, the canonical map $K_2(\mathcal{O},\mathfrak{a}')\longrightarrow K_2(\mathcal{O},\mathfrak{a}'')$ is an isomorphism. For any small ideal $\mathfrak{a}$, set $k_2(\mathcal{O}_v)=K_2(\mathcal{O}_v,\mathfrak{a}_v)$. Then the exact sequence above takes the form

$$K_2(D)\xrightarrow{\phi}\coprod_{v\notin\Sigma}K_2(D_v)/k_2(\mathcal{O}_v)\longrightarrow\mathrm{SK}_1(\mathcal{O},\mathfrak{a})\longrightarrow 1$$

for small nonzero ideals $\mathfrak{a}$.

Let $R_v=\mathcal{O}\cap K_v$, π_v the uniformising parameter of R_v, e_v the ramification index of R_v, $p_v=\mathrm{char}(R_v/R_v\pi_v)$, $\mu(K_v)_{p_v}$ p_v-th roots of unity in K_v, and finally

$$\mu(R_v\pi_v^k)=(\mu(K_v)_{p_v})^{p_v^{[k/e_v-1/(p_v-1)]}},$$

where $[k/e_v-1/(p_v-1)]$ denotes the largest integer smaller or equal to $\mid k/e_v-1/(p_v-1)\mid$. Note that if $\mathrm{char}(K_v)\neq 0$ then $\mu(K_v)_{p_v}=1$.

It remains to determine the cokernel of the map ϕ above. To do so Bak and Rehmann embed ϕ in a commutative diagram

$$\begin{array}{ccc}
K_2(D) & \xrightarrow{\quad\phi\quad} & \coprod_{v\notin\Sigma}K_2(D_v)/k_2(\mathfrak{a}_v)\\
\uparrow{\scriptstyle\psi} & & \uparrow{\scriptstyle\coprod_{v\notin\Sigma}\psi_v}\\
K^*\otimes\mathrm{Nrd}(D^*) & \xrightarrow{\phi'} & \coprod_{v\notin\Sigma}\mu(K_v)/\mu(K_v\cap\mathfrak{a}_v)
\end{array}$$

whose other maps $\psi,\coprod_{v\notin\Sigma}\psi_v$ and ϕ' are defined as follows: $\psi(a\otimes b)=(a,\beta)$, where $b=\mathrm{Nrd}(\beta)$. To define ψ_v, note that for $v\notin\Sigma$, $\mathrm{Nrd}(D_v^*)=K^*$. Consider the map $K_v^*\otimes K_v^*\longrightarrow K_2(D_v)$, $(a\otimes b)\mapsto(a,\beta)$ where $b=\mathrm{Nrd}(\beta)$. It induces a map $K_2(K_v)\longrightarrow K_2(D_v)$. This induces for any $\mathfrak{a}$ a map

$$K_2(K_v)/K_2(R_v,R_v\cap\mathfrak{a}_v)\longrightarrow K_2(D_v)/K_2(\mathcal{O}_v,\mathfrak{a}_v).$$

Identifying $K_2(K_v)/K_2(R_v,R_v\cap\mathfrak{a}_v)$ with $\mu(K_v)/\mu(R_v\cap\mathfrak{a}_v)$ via the norm residue symbol at v, one obtains a map

$$\mu(K_v)/\mu(R_v\cap\mathfrak{a}_v)\longrightarrow K_2(D_v)/K_2(\mathcal{O}_v,\mathfrak{a}_v).$$

For a small $\mathfrak{a}$, it is obvious that $\mu(R_v\cap\mathfrak{a}_v)=1$ and $K_2(\mathcal{O}_v,\mathfrak{a}_v)=k_2(\mathcal{O}_v)$, and the resulting map $\psi_v:\mu(K_v)\longrightarrow K_2(D_v)/k_2(\mathcal{O}_v)$ is the required one. The map ϕ' is induced by the norm residue homomorphisms at each v, see [33], §3 and §4 for explicit constructions. They show that the maps ψ_v are surjective and conclude trivially that the canonical map $\mathrm{coker}\phi'\longrightarrow\mathrm{coker}\phi$ is surjective. They then establish a generalisation of Moore's reciprocity law, [33], Theorem 3.2, from which it easily follows that

$$\mathrm{coker}\phi'=\begin{cases}1, & \text{if }\Sigma\setminus\mathrm{Pl}_{D/K}(\Sigma)\text{ contains a noncomplex valuation,}\\ \mu(K), & \text{if }\Sigma\setminus\mathrm{Pl}_{D/K}(\Sigma)\text{ is empty or totally complex.}\end{cases}$$

Since $C(\Sigma, \mathrm{SL}(n, \mathcal{O})) = \mathrm{coker}\phi$ and $\mathrm{coker}\phi$ is a quotient of $\mathrm{coker}\phi'$, it follows that

$$C(\Sigma, \mathrm{SL}(n, \mathcal{O})) = \begin{cases} 1, & \text{if } \Sigma \smallsetminus \mathrm{Pl}_{D/K}(\Sigma) \text{ contains a noncomplex valuation,} \\ \text{quotient of } \mu(K), & \text{if } \Sigma \smallsetminus \mathrm{Pl}_{D/K}(\Sigma) \text{ is empty or totally complex.} \end{cases}$$

The more precise computation of $C(\Sigma, SL_n(\mathcal{O}))$ in the theorem above is obtained by refining technically the argument above, in particular showing that $\ker(\psi_v) \subseteq \{\pm 1\}$ and that in many cases $\ker(\psi_v) = 1$.

We will finish this section by mentioning another result of Bak and Rehmann on K_2 of global fields. Recall that, since the norm map of finite field extensions coincides with the transfer map on K_1, the *Hasse norm theorem* can be stated in terms of K_1 functors as follows:

Theorem 15.2. *If L is a cyclic extension of a global field K, then an element of $K_1(K)$ lies in the image of transfer map $N_{L/K} : K_1(L) \longrightarrow K_1(K)$ if and only if its image in each $K_1(K_v)$ lies in the image of the transfer map*

$$N_{L_w/K_v} : K_1(L_w) \longrightarrow K_1(K_v).$$

Note that here K_v and L_w are the completions of K and L with respect to v and w and w is an extension of v to L. A natural question is whether the result is true for higher K-groups.

In [34] Bak and Rehmann succeeded in proving the Hasse norm theorem for K_2. In fact, they prove it holds not only for cyclic extensions of global fields, but for all finite extensions.

Theorem 15.3. *If L is a finite field extension of a global field K, then an element of $K_2(K)$ lies in the image of the transfer map $N_{L/K} : K_2(L) \longrightarrow K_2(K)$ if and only if its image in each $K_2(K_v)$ lies in the image of the transfer map*

$$\prod_{w/v} N_{L_w/K_v} : \prod_{w/v} K_2(L_w) \longrightarrow K_2(K_v).$$

Note, that here one considers all extensions w of v to L. This does not change Hasse's original theorem, since if L is a Galois extension of K, then

$$\mathrm{image}(N_{L_w/K_v}) = \mathrm{image}(\prod_{w/v} N_{L_w/K_v}).$$

To prove their theorem, Bak and Rehmann observe that it is equivalent to the exactness of the sequence

$$K_2(L) \xrightarrow{N_{L/K}} K_2(K) \xrightarrow{\coprod \lambda_v} \coprod_{v \in \Sigma_{L/K}} \mu(K_v) \longrightarrow 1,$$

where λ_v is the composition of the canonical map $K_2(K) \longrightarrow K_2(K_v)$ with the norm residue map $K_2(K_v) \longrightarrow \mu(K_v)$ and $\Sigma_{L/K}$ is the set of all real v such that any extension of v to L is complex. They then prove the exactness of the sequence.

It is worth mentioning that, Bak and Rehmann prove their theorem without using any Galois cohomology machinery and Tate's result connecting K_2 with Galois cohomology. Similarly Bak and Rehmann show that the Hasse–Schilling norm theorem for K_2 is equivalent to the exactness of the sequence

$$K_2(D) \xrightarrow{\text{Nrd}} K_2(K) \xrightarrow{\coprod \lambda_v} \coprod_{v \in \Sigma_{D/K}} \mu(K_v)$$

and prove the result in this form. The reader is encouraged to compare their Theorem 3 in [34] with Suslin's theorem 26.7 in [141].

16. Surgery and transformation groups

Surgery theory like K-theory was born in the 1950's and underwent its early development in the late 1950's and 60's. Its origins lie in work of Milnor [103] on exotic spheres and in the work of Kervaire and Milnor [89] on the classification of n-dimensional smooth manifolds, $n > 4$, which are homotopically equivalent to the standard n-sphere S^n. Whereas the solution to the generalized Poincare Conjecture by Smale [129] shows that every smooth closed simply connected manifold of dimension $n > 4$ which is homotopically equivalent to the standard n-sphere is homeomorphic to the standard n-sphere, the work of Kervaire and Milnor demonstrates that the diffeomorphism classes of such manifolds (for a fixed n) form a finite group under the operation of connected sum. A smooth manifold which is homeomorphic to a standard sphere, but not diffeomorphic, is called an *exotic sphere*. Their technique and methodology were extended by Browder [56] and Novikov [109] to all simply connected compact smooth manifolds of dimension $n > 4$. They classified such manifolds of a fixed dimension $n > 4$ in a given homotopy class, in terms of a surgery exact sequence and the homotopy theory of a certain classifying space G/O. This work was quickly extended in Wall [173, 174] to the case of smooth compact manifolds with finite fundamental group. The classification Wall gets is more of a program how to classify manifolds, rather than a transparent classification in terms of, for example, well understood invariants. One of the steps in the program is performing surgery on a smooth normal map $f : N \longrightarrow M$ between manifolds of the same dimension n, in order to convert f to a homotopy equivalence. This is not always possible. There is an obstruction called the surgery obstruction $\sigma(f)$, which lies in a group $L_n(G)$ called the *surgery obstruction group*, where G denotes the fundamental group of N. The fundamental theorem of surgery says that a normal map f can be converted by surgery to a normal map $f' : M' \longrightarrow N$ which is a homotopy equivalence if and only if its obstruction $\sigma(f)$ vanishes. The surgery obstruction group should actually be decorated further depending on what one means by homotopy equivalence, namely simple homotopy equivalence, arbitrary homotopy equivalence, or something in between. Thus one writes $L^s_n(G)$ if simple homotopy equivalence is meant and $L^h_n(G)$ if arbitrary homotopy equivalence is meant. However, regardless of the situation, the L-group L_n is always periodic of period 4, by construction, and for n even, (resp. n odd) L_n is a subquotient of a K_0-group (resp. K_1-group) of some category, depending on the situation, of nonsingular forms defined over the integral group ring $\mathbb{Z}G$ supplied with an appropriate involution. This means, in particular, that K-theory methods can be used to compute surgery obstruction groups and eventually surgery obstructions.

Bak did this, beginning in 1975 in [8], by showing first that for n odd and G of odd order, the Wall groups $L_n^s(G)$ and $L_n^h(G)$ vanish, which has the consequence that for smooth, compact n-manifolds, $n > 4$, whose fundamental group is finite of odd order, there is no obstruction to converting via surgery a normal map to a simple homotopy or homotopy equivalence. To make the computation above, Bak applied the unitary K-theory exact sequence 7.36b of [15], which is the analog of the exact sequence 7.36a, for ordinary K-groups. We saw in the previous section, that the latter was used several years later by Bak and Rehmann as the basis for computing the congruence kernel in the congruence subgroup problem for SL of division rings. In the current situation, Bak computes the terms surrounding $KQ_1(\mathbb{Z}G)$ in the unitary exact sequence to arrive at the answer, instead of computing, as in the later paper [33] with Rehmann, the terms surrounding $K_1(O)$ or $K_1(O, \mathfrak{a})$ in the exact sequence for ordinary K-groups. The unitary exact sequence is also the basis, as we said earlier, for Bak's solution to the congruence subgroup problem for classical groups [14]. In a further paper [9] appearing in 1976, Bak uses the $K_2 - K_1$ unitary exact sequence above and its unitary $K_1 - K_0$ version to compute the surgery groups $L_n^s(G)$ and $L_n^h(G)$ for all n and all finite groups G with abelian 2-hyperelementary subgroups. It is worth pointing out at this stage that all the exact sequences are based on localisation-completion methods, starting from localisation-completion squares of rings and form rings. Bak returned some 15 years later to ideas here in order to establish his program, which we reviewed in Sections 12, 13, and 14, concerning group valued functors, categories with dimension, and nonabelain (nonstable) K_1.

We move on now to equivariant surgery theory. After the development of non-simply connected surgery, the idea of equivariant surgery was in the air (see §14a of [174]) and several mathematicians contributed to its early development. If M is a smooth manifold then its fundamental group, say G, acts smoothly on the simply connected covering (manifold) $\widetilde{M}$ of M and $\widetilde{M}/G$ is diffeomorphic to M. It turns out that much of what was discussed above can be carried out if M is replaced by $\widetilde{M}$ supplied with the smooth action of the fundamental group of M. This is the basis of equivariant surgery theory. In this theory, one starts with a simply connected compact smooth n-dimensional manifold supplied with a smooth action of a finite group G. One makes the assumption that the fixed point manifold of any nontrivial subgroup of G has dimension less than or equal to $n/2$. The assumption is imposed so that one can perform surgery without disturbing the fixed point sets of nontrivial subgroups of G. (If M is not simply connected, but has finite fundamental group then one simply replaces M by its simply connected cover supplied with the smooth action of the semidirect product of G with the fundamental group of M. If the fixed point manifold in M of each nontrivial subgroup of G has dimension $\leq n/2$ then the same is true for the action of the semidirect product on the simply connected covering.)

T. Petrie applied in [112] an equivariant surgery theory, which he had developed, to an old problem of Montgomery and Samelson: Which spheres have a smooth one fixed point action of some finite group G? Petrie's procedure produced a G-invariant normal map $f : M \to S^n$ where M is a smooth G-manifold of dimension n with precisely one G-fixed point and G is acting smoothly on S^n. The next step was to modify f by (equivariant) G-surgery to a homotopy equivalence. It would then follow from Smale's solution to the generalized Poincare conjecture that M is an exotic sphere. But it is not difficult to see that one can convert such a sphere, without altering its fixed point sets (see Proposition 2.1 of [96]), to a standard sphere by taking the connected sum with its Kervaire-Milnor inverse [89]. Petrie was successful in carrying out his program for certain finite odd order groups G [112]. Here there was no surgery obstruction thanks to Bak's theorem on the vanishing of $L^h_n(G)$. On the other hand Petrie had set up his equivariant surgery theory under the so-called *gap hypothesis*, namely that the fixed point manifold of any nontrivial subgroup of G had dimension strictly less than $n/2$. For about a decade, it was believed that under this assumption the equivariant surgery groups were identical with the Wall surgery obstruction groups $L^h_n(G)$. This is in fact the case for groups G of odd order, which was Petrie's assumption in [112], but does not hold for arbitrary finite groups. For arbitrary finite groups it is necessary to assume that the fixed point manifold of any element of order 2 in G has dimension strictly less than $n/2 - 1$. This stronger hypothesis is now called the *strong gap hypothesis*. However if there are elements of order 2 in G whose fixed point manifold has dimension $k - 1$ where k is the largest integer less than or equal to $n/2$, then one gets a G-equivariant surgery obstruction group $L^{eq}_n(G)$ which is different than $L^h_n(G)$. Whereas Wall groups use the minimal form parameter, the G-equivariant surgery groups $L^{eq}_n(G)$ are constructed with the (nonminimal) form parameter generated by all elements g in G of order 2 such that the fixed point manifold of g has dimension $(k - 1)$. This discovery was made by Morimoto [106, 107], who introduced the correct G-equivaraint surgery obstruction groups with form parameters, christening them *Bak surgery obstruction groups*. Subsequently, Bak and Morimoto made several calculations of the new surgery groups, including some surprising vanishing theorems for odd dimensional Bak groups of even order groups G and then made various applications of these results to spheres and homotopy lens spaces, see [25].

One of the main applications of [106, 107] is to the problem of Montgomery and Samelson, regarding which standard spheres have a one smooth fixed point action of some finite group. Petrie and others had found already a few groups which could acts smoothly on some spheres with precisely one fixed point, but unfortunately their results did not always give precise information on the dimension of the spheres. Morimoto, however, was interested in an exhaustive answer to the problem. He wanted to obtain a complete list of groups which could act smoothly with one fixed point on some standard sphere and a complete list of spheres which have a one fixed point action of some finite group. The first list was completed using the equivariant surgery theory in [106, 107], but the second list had a gap at $n = 8$. (It was known that for $n < 6$, the answer was negative and that for $n \geq 6$, the answer was positive, except possibly for the case $n = 8$.) This case remained open for about a decade and was settled first in the joint paper [28]. The main obstacle the authors had to overcome was that there was no equivariant surgery theory which allowed middle dimensional fixed point manifolds of nontrivial subgroups of G. This theory was constructed in their joint papers [27] and [29] for even dimensional manifolds, which is what they needed for the application to the 8-sphere. In these papers, there occurs a new kind of form, with 2 parameters instead of just one. There is a quadratic form taking values in the ring modulo one of the parameters, and a not necessarily even, nonsingular Hermitian form, which is related in a suitable way to the quadratic form. The Hermitian form is as usual the restriction of the intersection form on the middle dimensional homology group $H_k(M)$, to the so-called *surgery kernel* $K_k(M)$, but the restriction is not necessarily even as in all other instances of surgery. The quadratic form is constructed out of the geometric self intersection form and takes values modulo the form parameter which is generated by the elements g of order 2 in G such that $\dim(M^g) = k - 1$. The other parameter, whose symmetry is opposite to that of the form parameter above, is generated by all elements g in G of order 2 such that $\dim(M^g) = k$, and it used to relate the quadratic and hermitian forms.

The surgery theory above, and in particular its applications to transformation groups, motivated some complectly algebraic work on induction/restriction theory, which included the important result that the dual of Dress' induction theorem is also true.

Dress induction says that if inducing up on a Mackey functor is surjective then restricting down on the functor is injective and the functor is hypercomputable. The dual, proved in [20] says that if restricting down on a Mackey functor is injective, then inducing up is surjective and the functor is hypercomputable. The dual has the marvelous consequence (see Corollary 1.3 of [20]) that a Mackey subfunctor of a hypercomputable Mackey functor is again hypercomputable. If a surgery group functor is a hypercomputable Mackey functor, then one can use its hypercomputability to compute the surgery obstruction of a given normal map. This is frequently done in surgery theory and includes many of the applications discussed above. It is routine to verify that surgery group functors are Mackey functors, so the crux of the matter is to show they are hypercomputable. It turns out, however, that surgery group functors may occur as Mackey subfunctors of Mackey functors which are already known to be hypercomputable. It follows therefore from the result above that such Mackey subfunctors are themselves hypercomputable.

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