On certain finite group related to cubic theta function

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ABSTRACT. Given the Kubota-Patterson cubic theta function, one can attach to it a family of 27 shifted theta functions. Then one can naturally define some group of permutations of shifted theta functions, which occurs to be isomorphic to a subgroup of the famous group of permutations of 27 lines on cubic surface.

0 Introduction. These notes give rise from some observation on the Kubota–Patterson cubic theta function. Let $\Gamma(3)$ be the principal congruence subgroup mod 3 in $\mathrm{SL}(2,\mathbf{Z}[\omega])$, where $\omega = \exp(2\pi i/3)$, and let $\widetilde{\Gamma}(3)$ be the subgroup of $\mathrm{SL}(2,\mathbf{Z}[\omega])$ generated by the groups $\Gamma(3)$ and $\mathrm{SL}(2,\mathbf{Z})$. Let us recall that the cubic theta function we mean, denoted $\Theta_{\text{K-P}}\colon \mathbf{H} \to \mathbf{C}$, is defined on the 3-dimensional hyperbolic space $\mathbf{H} \simeq \mathrm{SL}(2,\mathbf{C})/\mathrm{SU}(2)$ and satisfies the transformation low

$$\Theta_{\text{K-P}}(\gamma w) = \kappa(\gamma)\Theta_{\text{K-P}}(w) \quad \text{for all} \quad \gamma \in \widetilde{\Gamma}(3), \quad w \in \mathbf{H},$$

where $\kappa\colon \widetilde{\Gamma}(3) \to \mathbf{C}^*$ is the cubic Kubota homomorphism; we shall explain this in more details in sections 3 and 4.

One may notice that $\widetilde{\Gamma}(3)$ and $\Gamma(3)$ are subgroups of indexes 27 and 648 in $\mathrm{SL}(2,\mathbf{Z}[\omega])$, and that the numbers 27 and 648 give rise also in connection with configuration of lines on a smooth cubic surface. Given a smooth cubic surface, say $\mathcal{X}$, in the 3-dimensional projective space over algebraically closed field, there are 27 lines $l_1, l_2, \ldots, l_{27}$ contained in $\mathcal{X}$. In fact not only is the number of lines independent of particular choice of surface, but so is the configuration of these lines. By that we mean that the group $\mathcal{P}$ of permutations of the lines $l_1, l_2, \ldots, l_{27}$ leaving fixed the incidence relations does not depend (up to isomorphism) of particular choice of $\mathcal{X}$. The $\mathcal{P}$ is the group of order $51840 = 2^7 \cdot 3^4 \cdot 5$ isomorphic to the Weyl group $W(E_6)$ of the root system E_6. We can refer to [11], [9], [17], [8], [14], and to many other sources. It was observed by A. Cayley one and half century ago that some finite number of lines must lie on the surface; then G. Salmon showed that this number is 27. Let us restrict ourselves by this remark, and to refer to [9] for further history.

Our aim is to define some group of order $648 = 2^3 \cdot 3^4$ in connection with the cubic theta function $\Theta_{\text{K-P}}$, and to show that this group is isomorphic to a subgroup of $\mathcal{P}$. It is worth to say that the group we define has a geometrical interpretation. Actually, several different realizations are known as for $\mathcal{P}$ itself as for its commutator subgroup, say $\mathcal{P}'$, which is the simple group of order $25920 = 2^6 \cdot 3^4 \cdot 5$. In particular, $\mathcal{P}'$ is isomorphic to the projective symplectic group of rank 2 over the finite field $\mathbf{F}_3$. Let us mention also that one can find these groups in the classical theory of theta functions of genus two with characteristics.

These notes are organized into several sections. To attach a finite group to the cubic theta function we adopt the well known general principle explained in Section 1. Then, in Section 2, we state our results. Basic properties of the Kubota's κ and of the $\Theta_{\text{K-P}}$ are summarized in Section 3 and Section 4. In Section 5 we give the necessary definitions and explanations concerning the $W(E_6)$ and the symplectic groups. Then, in sections $6, \ldots, 10$ we prove all the statements given in Section 2.

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1 Basic construction. Let Δ' be a subgroup of finite index, say n, in a group Δ. Assume that Δ acts on some space, say $\mathcal{S}$, and that $\lambda\colon \Delta' \to \mathbf{C}^*$ is a group homomorphism. Given any function $f\colon \mathcal{S} \to \mathbf{C}$ satisfying

$$f(\gamma w) = \lambda(\gamma) f(w) \quad \text{for all} \quad \gamma \in \Delta' \quad \text{and} \quad w \in \mathcal{S},$$

we can define function $F\colon \mathcal{S} \to \mathbf{C}^n$ and a group homomorphism

$$\psi:\ \Delta\ \to \left\{ \begin{array}{c} \text{monomial} \\ \text{matrices } n \times n \end{array} \right\} \hookrightarrow\ \mathrm{GL}(n, \mathbf{C})$$

in such a way that

$$F(\gamma w) = \psi(\gamma) F(w) \quad \text{for all} \quad \gamma \in \Delta \quad \text{and} \quad w \in \mathcal{S}.$$

To this end, we take $\sigma_1, \sigma_2, \ldots, \sigma_n \in \Delta$ such that

$$\Delta = \bigcup_{j=1,2,\ldots,n} \Delta' \sigma_j \tag{1.1}$$

is a disjoint decomposition, and then we set

$$F(w) = \begin{pmatrix} f(\sigma_1 w) \\ f(\sigma_2 w) \\ . \\ . \\ f(\sigma_n w) \end{pmatrix} \quad \text{for all} \quad w \in \mathcal{S}.$$

Given any $\gamma \in \Delta$ and $j = 1, 2, \ldots, n$ we have $\sigma_j \gamma \in \Delta$, and there is a unique k such that $\sigma_j \gamma \in \Delta' \sigma_k$. With this k, we have $f(\sigma_j \gamma w) = f(\sigma_j \gamma \sigma_k^{-1} \sigma_k w) = \lambda(\sigma_j \gamma \sigma_k^{-1}) f(\sigma_k w)$. Now we can define $\psi(\gamma)$ to be the monomial matrix[†] $n \times n$ with jk-th entry $\lambda(\sigma_j \gamma \sigma_k^{-1})$. Clearly, ψ occurs to be a group homomorphism with desired properties.

Now we can attach to our date Δ', Δ, λ, $\sigma_1, \sigma_2, \ldots, \sigma_n$, and f the group

$$\mathcal{E} = \{\psi(\gamma) \mid \gamma \in \Delta\},$$

which is a subgroup of $\mathrm{GL}(n, \mathbf{C})$. We will interested also in the subgroup $\mathcal{Z}$ of all diagonal matrices in $\mathcal{E}$, in the subgroup $\mathcal{C}$ of all scalar matrices in $\mathcal{E}$, and especially in the factor group $\mathcal{E}/\mathcal{Z}$. (We should notice that $\mathcal{Z}$ is a normal subgroup of $\mathcal{E}$ just because $\mathcal{E}$ consists of monomial matrices only.)

Up to isomorphism, and even up to conjugation in $\mathrm{GL}(n, \mathbf{C})$, the groups $\mathcal{C}$, $\mathcal{Z}$, $\mathcal{E}$, and $\mathcal{E}/\mathcal{Z}$ do not depend on the particular choice of the cosets representatives σ_j in (1.1). More precisely, if the groups, say $\hat{\mathcal{C}}$, $\hat{\mathcal{Z}}$, $\hat{\mathcal{E}}$, are attached as above to Δ', Δ, λ, f and to some $\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_n$ instead of $\sigma_1, \sigma_2, \ldots, \sigma_n$, then there is a monomial matrix $h \in \mathrm{GL}(n, \mathbf{C})$ such that $\hat{\mathcal{C}} = h^{-1}\mathcal{C}h$, $\hat{\mathcal{Z}} = h^{-1}\mathcal{Z}h$, $\hat{\mathcal{E}} = h^{-1}\mathcal{E}h$.

2 Main theorem. Let us apply the construction described in Section 1 with the following date

$$\begin{aligned}
&\Delta = \mathrm{B}(\mathcal{O}) = \text{ the subgroup of } \mathrm{SL}(2, \mathbf{C}) \text{ generated by } \mathrm{SL}(2, \mathcal{O}) \text{ and } \iota;\\
&\Delta' = \mathrm{B}(3) = \text{ the subgroup of } \mathrm{SL}(2, \mathbf{C}) \text{ generated by } \widetilde{\Gamma}(3) \text{ and } \iota;\\
&\mathcal{S} = \mathbf{H} \text{ that is the 3-dimensional hyperbolic space;}\\
&f = \Theta_{\text{K-P}}, \text{ that is the Kubota-Patterson cubic theta function;}\\
&\lambda = \kappa, \text{ that is the extended Kubota homomorphism } \mathrm{B}(3) \to \mathbf{C}^*.
\end{aligned} \tag{2.1}$$

Here and in what follows we use the following notations.

$\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{C}$ are the ring of rational integers, the field of rational and the field of complex numbers;

$\mathbf{C}^*$ is the multiplicative group of $\mathbf{C}$;

$\mathcal{O} = \mathbf{Z}[\omega]$ is the integers ring of the field $\mathbf{Q}(\sqrt{-3})$;

$\omega = \exp(2\pi i/3) = (-1 + \sqrt{-3})/2$;

$$\iota = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad i = \sqrt{-1}; \tag{2.2}$$

$\mathrm{diag}(c_1, c_2, \ldots, c_n)$ is the $n \times n$ diagonal matrix with c_l at the intersection of l^{th} row and l^{th} column;

I_n is the identity matrix $n \times n$.

Also, if q is either an ideal of $\mathcal{O}$ or a non-zero element of $\mathcal{O}$, then

[†] By a monomial matrix we mean a matrix with unique non-zero entry in every row and in every column.

$\Gamma(q)$ is the principal congruence subgroup $\bmod q$ in $\mathrm{SL}(2,\mathcal{O})$;

$\pm\Gamma(q)$ is the subgroup of $\mathrm{SL}(2,\mathcal{O})$ generated by $\Gamma(q)$ and $-I_2$;

$\widetilde{\Gamma}(q)$ is the subgroup of $\mathrm{SL}(2,\mathcal{O})$ generated by $\Gamma(q)$ and $\mathrm{SL}(2,\mathbf{Z})$;

$\mathrm{B}(q)$ is the subgroup of $\mathrm{SL}(2,\mathbf{C})$ generated by $\widetilde{\Gamma}(q)$ and ι.

In particular, $\mathrm{B}(\mathcal{O})$, i.e. the subgroup of $\mathrm{SL}(2,\mathbf{C})$ generated by $\mathrm{SL}(2,\mathcal{O})$ and ι, is known as the Bianchi extension of $\mathrm{SL}(2,\mathcal{O})$. The Bianchi group $\mathrm{B}(\mathcal{O})$ is a maximal discrete subgroup of $\mathrm{SL}(2,\mathbf{C})$, while $\mathrm{SL}(2,\mathcal{O})$ is its subgroup of index 2; see Elstrodt, Grunewald, Mennicke [7], 7.4, and the references therein.

Theorem. *Let* $\psi\colon \mathrm{B}(\mathcal{O}) \to \mathrm{GL}(n,\mathbf{C})$ *be the homomorphism attached to the date listed in* (2.1) *in accordance with the procedure described in Section* 1. *Let* $\mathcal{E}$ *be the image of* ψ, *and let* $\mathcal{Z}$ *be the subgroup of all the diagonal matrices in* $\mathcal{E}$. *Then* $n = 27$, $\mathcal{Z}$ *is a normal subgroup of* $\mathcal{E}$, *and the factor group* $\mathcal{E}/\mathcal{Z}$ *is isomorphic to some subgroup of order* 648 *in the Weyl group* $W(E_6)$ *of the root system* E_6.

Some additional information on the groups involved and on the homomorphism ψ is given in the three propositions below.

Proposition 1. *With the notations above one has*

(a) $\begin{array}{l}[\mathrm{SL}(2,\mathcal{O}):\Gamma(3)] = 648\\ [\widetilde{\Gamma}(3):\Gamma(3)] = 24\end{array}$ *and* $[\mathrm{SL}(2,\mathcal{O}):\widetilde{\Gamma}(3)] = 27$;

(b) $[\mathrm{B}(\mathcal{O}):\mathrm{SL}(2,\mathcal{O})] = [\mathrm{B}(3):\widetilde{\Gamma}(3)] = 2$;

(c) $\begin{array}{l}[\mathrm{B}(\mathcal{O}):\pm\Gamma(3)] = 648\\ [\mathrm{B}(3):\pm\Gamma(3)] = 24\end{array}$ *and* $[\mathrm{B}(\mathcal{O}):\mathrm{B}(3)] = 27$.

Proposition 2. *Let* $\sigma_1,\sigma_2,\ldots,\sigma_{27}$ *represent cosets in* $\widetilde{\Gamma}(3)\backslash\mathrm{SL}(2,\mathcal{O})$, *so that*

$$\mathrm{SL}(2,\mathcal{O}) = \bigcup_{j=1,2,\ldots,27} \widetilde{\Gamma}(3)\,\sigma_j \tag{2.3}$$

is a disjoint decomposition. Then the same matrices $\sigma_1,\sigma_2,\ldots,\sigma_{27}$ *represent cosets in* $\mathrm{B}(3)\backslash\mathrm{B}(\mathcal{O})$, *so that*

$$\mathrm{B}(\mathcal{O}) = \bigcup_{j=1,2,\ldots,27} \mathrm{B}(3)\,\sigma_j \tag{2.4}$$

is a disjoint decomposition too.

Proposition 3. *Under the assumption of Theorem, for the homomorphism* $\psi\colon \mathrm{B}(\mathcal{O}) \to \mathrm{GL}(27,\mathbf{C})$ *one has*

(a) $\psi(\gamma)$ *is the identity matrix for* $\gamma = \pm I_2$;

(b) $\psi(\gamma)$ *is a scalar matrix for* $\gamma \in \pm\Gamma(9)$;

(c) $\psi(\gamma)$ *is a diagonal matrix for* $\gamma \in \pm\Gamma(3)$;

(d) $\psi(\gamma)$ *is not a diagonal matrix for* $\gamma \notin \pm\Gamma(3)$.

3 Kubota homomorhpism. Let $\left(\frac{\cdot}{\cdot}\right)$ be the cubic residue symbol in $\mathcal{O}$. Given

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(3), \quad \text{we set} \quad \kappa(\gamma) = \begin{cases} \left(\frac{c}{d}\right)^{-1} & \text{if} \quad c \neq 0, \\ 1 & \text{if} \quad c = 0. \end{cases} \tag{3.1}$$

According to Kubota [12], the so defined $\kappa\colon \Gamma(3) \to \mathbf{C}^*$ is a group homomorphism. It is known, see [15], [16], it can be extended to homomorphism $\widetilde{\Gamma}(3) \to \mathbf{C}^*$ by setting $\kappa(\gamma) = 1$ for $\gamma \in \mathrm{SL}(2,\mathbf{Z})$; recall that the group $\widetilde{\Gamma}(3)$ is generated by $\Gamma(3)$ and $\mathrm{SL}(2,\mathbf{Z})$. We would like to notice here that

the Kubota homomorphism κ can be extended to homomorphism
$\mathrm{B}(3) \to \mathbf{C}^*$ *(which we still denote by κ) just by setting $\kappa(\iota) = 1$.*

One can prove this statement by elementary computation using (6.1), (6.3) in Section 6. Another proof is given in Section 4 below. We will need the following:

$$\kappa(\delta^{-1}\gamma\delta) = \kappa(\gamma) \quad \text{for all} \quad \gamma \in \Gamma(9), \quad \delta \in \mathrm{SL}(2,\mathcal{O}). \tag{3.2}$$

For this we can refer to [1].

4 Cubic metaplectic forms. Let $\mathbf{H}$ be the 3-dimensional hyperbolic space realized as the set of pairs (z,v) with $z \in \mathbf{C}$, $v \in \mathbf{R}$, $v > 0$. The group $\mathrm{SL}(2,\mathbf{C})$ acts on the $\mathbf{H}$ in such a way that for the image γw of the point

$$w = (z,v) \in \mathbf{H} \quad \text{under} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbf{C}) \quad \text{one has}$$

$$\gamma w = \Big(\frac{(az+b)\overline{(cz+d)} + a\bar{c}v^2}{|cz+d|^2 + |c|^2v^2}, \frac{v}{|cz+d|^2 + |c|^2v^2}\Big).$$

The center of $\mathrm{SL}(2,\mathbf{C})$, consisting of $\pm I_2$ only acts trivially on $\mathbf{H}$, so there is the induced action of $\mathrm{PSL}(2,\mathbf{C})$. According to Poincaré, $\mathrm{PSL}(2,\mathbf{C})$ is just the group of all orientation preserving isometrices of $\mathbf{H}$. There is the second order differential operator whose action on the smooth functions $\mathbf{H} \to \mathbf{C}$ commute with the action of $\mathrm{SL}(2,\mathbf{C})$ on $\mathbf{H}$, that is the Laplace-Beltrami operator

$$\mathrm{D}_{\text{L-B}} = -v^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial v^2}\Big) + v\frac{\partial}{\partial v}, \quad x = \Re(z), \quad y = \Im(z).$$

Let Γ be a cofinite discrete subgroup of $\mathrm{SL}(2,\mathbf{C})$, such that the Kubota homomorphism is well defined on Γ. We say that $f\colon \mathbf{H} \to \mathbf{C}$ is a cubic metaplectic form on $\mathbf{H}$ under Γ if f is the $\mathrm{D}_{\text{L-B}}$-eigenfunction of moderate growth satisfying

$$f(\gamma w) = \kappa(\gamma) f(w) \quad \text{for all} \quad \gamma \in \Gamma, \quad w \in \mathbf{H}.$$

The moderate growth condition is not essential for our present goals. Anyway, the explicit statement is as follows: there is a polynomial P with real coefficients such that $|f(z,v)| < P(|z|, v, v^{-1})$ for all $(z,v) \in \mathbf{H}$.

We shall deal with the Kubota-Patterson cubic theta function $\Theta_{\text{K-P}}$, which is known to be a cubic metaplectic form under the group $\widetilde{\Gamma}(3)$. The $\Theta_{\text{K-P}}$ has been defined and studied originally by Kubota [13]. One has the Fourier expansion

$$\Theta_{\text{K-P}}(z,v) = v^{2/3} + (6\pi v)^{2/3} \sum_{\nu\in(\sqrt{-3})^{-3}\mathcal{O}\setminus\{0\}} \tilde{\tau}(\nu)\, Ai\big((6\pi|\nu|v)^{2/3}\big)\, e(\nu z)$$

with the Fourier coefficients $\tilde{\tau}(\nu)$ given by Patterson [15]; here Ai is the Airy function and $e(t) = \exp\big(2\pi i(t+\bar{t})\big)$. See also the authors exposition in [16]. We would like to show now that

the Kubota-Patterson cubic theta function $\Theta_{\text{K-P}}$ *is*
a cubic metaplectic form under the Bianchi group $\mathrm{B}(3)$.

One knows $\tilde{\tau}(-\nu) = \tilde{\tau}(\nu)$ for all ν, that yields $\Theta_{\text{K-P}}\big((-z,v)\big) = \Theta_{\text{K-P}}\big((z,v)\big)$ for all $(z,v) \in \mathbf{H}$. On the other hand, $(-z,v) = \iota(z,v)$. It follows that not only $\Theta_{\text{K-P}}(\gamma w) = \kappa(\gamma)\Theta_{\text{K-P}}(w)$ for all $\gamma \in \widetilde{\Gamma}(3)$, but also $\Theta_{\text{K-P}}(\gamma w) = \Theta_{\text{K-P}}(w)$ for $\gamma = \iota$. This observation gives the extension of the Kubota homomorphism κ described in Section 3, and proves the statement above.

5 Symplectic groups and $W(E_6)$. Recall, that by $W(E_6)$ we understand the Weyl group of the root system E_6. One knows, $W(E_6)$ is a group of order $51840 = 2^7\cdot 3^4\cdot 5$ isomorphic to the group of incidence preserving permutations of 27 lines on smooth cubic surface in the 3-dimensional projective space over algebraically closed field. We have the exact sequence

$$1 \to W(E_6)^+ \to W(E_6) \to \{1,R\} \to 1, \tag{5.1}$$

in which $W(E_6)^+$ is the commutator subgroup of $W(E_6)$. That is the simple group of order $25920 = 2^6\cdot 3^4\cdot 5$. By R we mean any element in $W(E_6)\setminus W(E_6)^+$ of order 2, so that R generates the cyclic subgroup $\{1,R\}$ of $W(E_6)$; for instance, one can take R to be one of the reflections which generate $W(E_6)$. In other words, $W(E_6)$ is a semidirect product of its subgroups $W(E_6)^+$ and $\{1,R\}$, so that (5.1) is a split extension. We should notice also that the mapping which takes $g \in W(E_6)^+$ to $R^{-1}gR \in W(E_6)^+$ is an outer involutory automorphism of $W(E_6)^+$. For the group $W(E_6)$ we can refer to Bourbaki [2], Manin [14], Coxeter [3].

We have two group isomorphisms

$$W(E_6)^+ \simeq \mathrm{PSp}(4,\mathbf{F}_3) = \left\{ \begin{array}{c} \text{projective symplectic} \\ \text{group of rank 2 over } \mathbf{F}_3 \end{array} \right\}, \tag{5.2}$$

$$W(E_6) \simeq \mathrm{PGSp}(4,\mathbf{F}_3) = \left\{ \begin{array}{c} \text{extended projective} \\ \text{symplectic group of rank 2 over } \mathbf{F}_3 \end{array} \right\}, \tag{5.3}$$

$$\text{the exact sequence} \quad 1 \to \mathrm{PSp}(4,\mathbf{F}_3) \to \mathrm{PGSp}(4,\mathbf{F}_3) \to \mathrm{C}_2 \to 1, \tag{5.4}$$

$$\text{and the cosets decomposition} \quad \mathrm{PGSp}(4,\mathbf{F}_3) = \mathrm{PSp}(4,\mathbf{F}_3) \cup \varrho\,\mathrm{PSp}(4,\mathbf{F}_3), \tag{5.5}$$

where $\mathbf{F}_3$ is the field consisting of 3 elements, and C_2 is the cyclic subgroup of order 2 in $\mathrm{PGSp}(4,\mathbf{F}_3)$ generated by $\varrho = \pm\,\mathrm{diag}(-1,1,1,-1)$.

For further convenience, let us describe the symplectic groups in more details. Let us agree to write 4×4 matrices like

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \text{where} \quad A,B,C,D \quad \text{are} \quad 2\times 2 \quad \text{matrices,} \tag{5.6}$$

and let $\Omega = \begin{pmatrix} 0_2 & -I_2 \\ I_2 & 0_2 \end{pmatrix}$, where 0_2 and I_2 are 2×2 zero and identity matrices. Then the symplectic group of rank 2 over $\mathbf{F}_3$ is

$$\mathrm{Sp}(4,\mathbf{F}_3) = \{\gamma \in \mathrm{GL}(4,\mathbf{F}_3) \mid \gamma\Omega\,{}^t\gamma = \Omega\},$$

where ${}^t\gamma$ is the matrix transpose to γ. For γ written as in (5.6), the condition $\gamma\Omega\,{}^t\gamma = \Omega$ is equivalent to

$$A\,{}^tB = B\,{}^tA, \quad C\,{}^tD = D\,{}^tC, \quad A\,{}^tD - B\,{}^tC = I_2. \tag{5.7}$$

By the extended symplectic group we mean

$$\mathrm{GSp}(4,\mathbf{F}_3) = \{\gamma \in \mathrm{GL}(4,\mathbf{F}_3) \mid \gamma\Omega\,{}^t\gamma = \pm\Omega\}.$$

The symplectic group $\mathrm{Sp}(4,\mathbf{F}_3)$ is a normal subgroup of index 2 in $\mathrm{GSp}(4,\mathbf{F}_3)$. We have the disjoint decomposition and the exact sequence

$$\begin{gathered} \mathrm{GSp}(4,\mathbf{F}_3) = \mathrm{Sp}(4,\mathbf{F}_3) \cup \varrho'\,\mathrm{Sp}(4,\mathbf{F}_3) \quad \text{with} \quad \varrho' = \mathrm{diag}(-1,1,1,-1), \\ 1 \to \mathrm{Sp}(4,\mathbf{F}_3) \to \mathrm{GSp}(4,\mathbf{F}_3) \to \mathrm{C}_2' \to 1, \end{gathered} \tag{5.8}$$

where C_2' is the cyclic subgroup of order 2 in $\mathrm{GSp}(4,\mathbf{F}_3)$ generated by ϱ'. So, $\mathrm{GSp}(4,\mathbf{F}_3)$ occurs to be semidirect product of its normal subgroup $\mathrm{Sp}(4,\mathbf{F}_3)$ and cyclic subgroup C_2'. The centers of both $\mathrm{Sp}(4,\mathbf{F}_3)$ and $\mathrm{GSp}(4,\mathbf{F}_3)$ consist of the $\pm I_4$ only. By definition,

$$\mathrm{PSp}(4,\mathbf{F}_3) = \mathrm{Sp}(4,\mathbf{F}_3)/\{\pm I_4\}, \quad \mathrm{PGSp}(4,\mathbf{F}_3) = \mathrm{GSp}(4,\mathbf{F}_3)/\{\pm I_4\},$$

and passing to quotient in (5.8) we obtain the desired cosets decomposition (5.5) and the exact sequence (5.4) for the projective groups.

The isomorphism (5.2) is given by Dieudonné [6], ch.4, with references to C. Jordan [11] and L. Dickson [4]. It seems very likely that the isomorphism (5.3) is known also, but we can not give reference. So let us give a short explanation on (5.3). By Dieudonné [5], every automorphism of the projective group $\mathrm{PSp}(4,\mathbf{F}_3)$ is induced (by passing to quotient) by an automorphism of $\mathrm{Sp}(4,\mathbf{F}_3)$. Loo-Keng Hua [10] and Dieudonné [5] showed also that every automorphism of $\mathrm{Sp}(4,\mathbf{F}_3)$ is conjugation by some element of $\mathrm{GSp}(4,\mathbf{F}_3)$. Hence, every automorphism of $\mathrm{PSp}(4,\mathbf{F}_3)$ is conjugation by some element of $\mathrm{PGSp}(4,\mathbf{F}_3)$. Then one can easily find that (up to isomorphism) $\mathrm{PGSp}(4,\mathbf{F}_3)$ is the only split extension of $\mathrm{PSp}(4,\mathbf{F}_3)$ by cyclic group of order 2 related to an outer involutory automorphism of $\mathrm{PSp}(4, \mathbf{F}_3)$. With this observation, we obtain isomorphism (5.3) just comparing (5.4) with (5.1).

6 Proof of Proposition 1. Let us begin with (a). The principal congruence subgroup $\Gamma(q)$, being the kernel of the homomorphism $\mathrm{SL}(2, \mathcal{O}) \to \mathrm{SL}(2, \mathcal{O}/q)$ induced by $\mathcal{O} \to \mathcal{O}/q$, is a normal subgroup of $\mathrm{SL}(2, \mathcal{O})$. One has the exact sequence

$$1 \to \Gamma(q) \to \mathrm{SL}(2, \mathcal{O}) \to \mathrm{SL}(2, \mathcal{O}/q) \to 1,$$

that yields isomorphism

$$\mathrm{SL}(2, \mathcal{O})/\Gamma(3) \simeq \mathrm{SL}(2, \mathcal{O}/3\mathcal{O}).$$

Hence, the index $\big[\mathrm{SL}(2, \mathcal{O}) : \Gamma(3)\big]$ equals to the order of $\mathrm{SL}(2, \mathcal{O}/3\mathcal{O})$. Next, one can easily find that the mapping $1 \mapsto 1$, $\sqrt{-3} \mapsto x$ induces a ring isomorphism $\mathcal{O}/3\mathcal{O} \to \mathbf{F}_3[x]/(x^2)$, so that we have a group isomorphism

$$\mathrm{SL}(2, \mathcal{O}/3\mathcal{O}) \simeq \mathrm{SL}(2, \mathbf{F}_3[x]/(x^2)).$$

Also we have a ring epimorphism $\mathbf{F}_3[x]/(x^2) \to \mathbf{F}_3$, which takes $a + bx \pmod{x^2}$ to $a \in \mathbf{F}_3$, and which henceforth induces an exact sequence

$$1 \to \Gamma(x) \to \mathrm{SL}(2, \mathbf{F}_3[x]/(x^2)) \to \mathrm{SL}(2, \mathbf{F}_3) \to 1,$$

where $\Gamma(x)$ is the principal congruence subgroup $\bmod\, x$ in $\mathrm{SL}(2, \mathbf{F}_3[x]/(x^2))$ consisting of the matrices

$$\begin{pmatrix} 1 + ax & bx \\ cx & 1 - ax \end{pmatrix}, \qquad a, b, c \in \mathbf{F}_3.$$

The $\mathrm{SL}(2, \mathbf{F}_3)$ is known to be a group of order 24. The order of $\Gamma(x)$ is 27. Hence, the order of $\mathrm{SL}(2, \mathbf{F}_3[x]/(x^2))$ occurs to be $27 \cdot 24 = 648$, and so are the order of $\mathrm{SL}(2, \mathcal{O}/3\mathcal{O})$ and the index $\big[\mathrm{SL}(2, \mathcal{O}) : \Gamma(3)\big]$, as required.

Now, given any $\gamma \in \mathrm{SL}(2, \mathcal{O})$, one can easily find that $\gamma \in \widetilde{\Gamma}(q)$ if and only if $\gamma \equiv \delta \pmod{q}$ for some $\delta \in \mathrm{SL}(2, \mathbf{Z})$, so if and only if $\gamma = \delta\gamma'$ for some $\delta \in \mathrm{SL}(2, \mathbf{Z})$, $\gamma' \in \Gamma(q)$. For $q = 3$, this remark yields

$$\widetilde{\Gamma}(3)/\Gamma(3) \simeq \mathrm{SL}(2, \mathbf{Z})/\big(\mathrm{SL}(2, \mathbf{Z}) \cap \Gamma(3)\big) \simeq \mathrm{SL}(2, \mathbf{Z}/3\mathbf{Z}),$$

that is a group of order 24, and so is the index $\big[\widetilde{\Gamma}(3) : \Gamma(3)\big]$. To conclude the proof of (a) notice that $\big[\mathrm{SL}(2, \mathcal{O}) : \widetilde{\Gamma}(3)\big] = 648/24 = 27$, just in view of the inclusions $\Gamma(3) \subset \widetilde{\Gamma}(3) \subset \mathrm{SL}(2, \mathcal{O})$.

Now, to prove (b), we notice that, according to the definition (2.2),

$$\begin{aligned} &\iota = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \text{so that} \quad \iota^2 = -I_2, \quad \text{and} \\ &\iota^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \iota = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{C}). \end{aligned} \tag{6.1}$$

It follows immediately that $\mathrm{SL}(2,\mathcal{O})$ is a normal subgroup of $\mathrm{B}(\mathcal{O})$, as well as $\widetilde{\Gamma}(3)$ is a normal subgroup of $\mathrm{B}(3)$, and that one has the cosets decompositions

$$\mathrm{B}(\mathcal{O}) = \mathrm{SL}(2,\mathcal{O}) \cup \iota\,\mathrm{SL}(2,\mathcal{O}), \tag{6.2}$$
$$\mathrm{B}(3) = \widetilde{\Gamma}(3) \cup \iota\,\widetilde{\Gamma}(3), \tag{6.3}$$

which imply (b). At last, (c) follows easily from (a) and (b), let us omit details.

7 Proof of Proposition 2. Assume that we have a disjoint decomposition (2.3). Substituting (2.3) into (6.2) and applying then (6.3) we obtain

$$\begin{aligned} \mathrm{B}(\mathcal{O}) = \{ \bigcup_{j=1,\ldots,27} \widetilde{\Gamma}(3)\,\sigma_j \} \cup \{ \bigcup_{j=1,\ldots,27} \iota\,\widetilde{\Gamma}(3)\,\sigma_j \} = \\ \bigcup_{j=1,\ldots,27} \{ \widetilde{\Gamma}(3) \cup \iota\,\widetilde{\Gamma}(3) \}\,\sigma_j = \bigcup_{j=1,\ldots,27} \mathrm{B}(3)\,\sigma_j, \end{aligned} \tag{7.1}$$

that is required decomposition (2.4), and it remains only to prove that it is disjoint. To this end, assume that $\mathrm{B}(3)\sigma_j = \mathrm{B}(3)\sigma_k$ for some $k \neq j$. Then $\sigma_j\sigma_k^{-1} \in \mathrm{B}(3)$. In view of (6.3) this means that either $\sigma_j\sigma_k^{-1} \in \widetilde{\Gamma}(3)$ or $\sigma_j\sigma_k^{-1} \in \iota\widetilde{\Gamma}(3)$. The first case is impossible because of (2.1) is disjoint. In the second case, we find $\sigma_j\sigma_k^{-1}\delta = \iota$ for some $\delta \in \widetilde{\Gamma}(3) \subset \mathrm{SL}(2,\mathcal{O})$. With this δ we have $\sigma_j\sigma_k^{-1}\delta \in \mathrm{SL}(2,\mathcal{O})$, while $\iota \notin \mathrm{SL}(2,\mathcal{O})$. Hence, we have that all the cosets in (7.1) are different, and so (7.1) is a disjoint decomposition.

8 Proof of Proposition 3. Point (a) is obvious, because of I_2 and $-I_2$ act trivially on $\mathbf{H}$. For (b) and (c) we should notice firstly, that both $\Gamma(3)$ and $\Gamma(9)$ are normal subgroups of $\mathrm{B}(\mathcal{O})$. Then, given $\gamma \in \Gamma(3)$ and any $j = 1, 2, \ldots, 27$ we have $\sigma_j\gamma\sigma_j^{-1} \in \Gamma(3) \subset \mathrm{B}(3)$, and so

$$f(\sigma_j\gamma w) = f(\sigma_j\gamma\sigma_j^{-1}\sigma_j w) = \kappa(\sigma_j\gamma\sigma_j^{-1})f(\sigma_j w).$$

This yields $\psi(\gamma) = \mathrm{diag}(\kappa(\sigma_1\gamma\sigma_1^{-1}), \kappa(\sigma_2\gamma\sigma_2^{-1}), \ldots, \kappa(\sigma_{27}\gamma\sigma_{27}^{-1}))$, that is (c). To prove (b) we only have to notice that $\kappa(\delta\gamma\delta^{-1}) = \kappa(\gamma)$ for all $\gamma \in \Gamma(9)$ and $\delta \in \mathrm{SL}(2,\mathcal{O})$ (see (3.2)), as well as for $\delta = \iota$, and hence for all $\delta \in \mathrm{B}(\mathcal{O})$. This yields $\kappa(\sigma_j\gamma\sigma_j^{-1}) = \kappa(\gamma)$, and hence $\psi(\gamma)$ is a scalar matrix for $\gamma \in \Gamma(9)$.

It is not so easy to prove (d). The statement to be proved do not depend on particular choice of σ_j in (1.1), (2.3), (2.4). Let us take the σ_j from the list given in [15]. This list includes

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}.$$

We do not have to specify all of σ_j in [15]. Let us notice only that $\sigma_j \notin \mathrm{B}(3)$ for $j = 2, 3, \ldots, 27$, and that $\sigma_j \in \mathrm{SL}(2,\mathcal{O})$ for $j = 1, 2, \ldots, 27$.

Let us define E to be the set of all matrices $\gamma \in \mathrm{B}(\mathcal{O})$ such that

$$w \mapsto \frac{\Theta_{\text{K-P}}(\gamma w)}{\Theta_{\text{K-P}}(w)} \quad \text{is a constant function} \quad \mathbf{H} \to \mathbf{C} \tag{8.1}$$

(depending on γ only). Clearly, E is a subgroup of $\mathrm{B}(\mathcal{O})$, and $\mathrm{B}(3) \subset E$. We would like to begin with proving that $E = \mathrm{B}(3)$. Assume that $\gamma \in E$, $\gamma \notin \mathrm{B}(3)$. We have the cosets decomposition (2.4) in Proposition 2, from which it follows that γ can be factored as $\gamma = \gamma'\sigma_k$ with some $k = 1, 2, \ldots, 27$ and $\gamma' \in \mathrm{B}(3) \subset E$. It follows that $\sigma_k \in E$, $\sigma_k \notin \mathrm{B}(3)$. So, if (8.1) has place for some $\gamma \notin \mathrm{B}(3)$ then (8.1) has place for at least one $\gamma = \sigma_k$, $k = 2, 3, \ldots, 27$. The author have used Maple V to write the procedure which returns numerical values of $\Theta_{\text{K-P}}(w)$. This procedure being applied to the functions (8.1) with $\gamma = \sigma_k$, $k = 2, 3, \ldots, 27$, shows that no one of them is a constant. We conclude that $E = \mathrm{B}(3)$.

Now, given $\gamma \in \mathrm{B}(\mathcal{O})$, the matrix $\psi(\gamma)$ is a diagonal one if and only if

$$w \mapsto \frac{\Theta_{\text{K-P}}(\sigma_k \gamma w)}{\Theta_{\text{K-P}}(\sigma_k w)} \quad \text{is a constant function} \quad \mathbf{H} \to \mathbf{C}$$

for every $k = 1, 2, \ldots, 27$ (depending on k and γ only). That is equivalent to $\sigma_k \gamma \sigma_k^{-1} \in E$ for all $k = 1, 2, \ldots, 27$. Since $E = \mathrm{B}(3)$, we conclude that $\psi(\gamma)$ is a diagonal matrix if and only if

$$\gamma \in P, \quad \text{where} \quad P = \bigcap_{k=1,\ldots,27} \sigma_k^{-1} \mathrm{B}(3) \sigma_k. \tag{8.2}$$

For (d) we only have to prove $P = \pm\Gamma(3)$. We have $\sigma_1 = I_2$, so that $P \subset \sigma_1^{-1}\mathrm{B}(3)\sigma_1 = \mathrm{B}(3)$. We know that $\pm\Gamma(3)$ is a normal subgroup of $\mathrm{B}(\mathcal{O})$, that yields $\pm\Gamma(3) = \pm\sigma_k^{-1}\Gamma(3)\sigma_k \subset \sigma_k^{-1}\mathrm{B}(3)\sigma_k$ for all k, so that $\pm\Gamma(3) \subset P$. On the other hand, we have

$$P \subset \mathrm{B}(3) \cap \sigma_2^{-1}\mathrm{B}(3)\sigma_2 \cap \sigma_3^{-1}\mathrm{B}(3)\sigma_3 \subset \pm\Gamma(3), \tag{8.3}$$

that implies $P = \pm\Gamma(3)$. The first inclusion in (8.3) is obvious. To prove the second one we need (6.3). Given $\gamma \in \mathrm{B}(3)$, it follows from (6.3) that either its entries $\in \mathcal{O}$ and then $\gamma \in \widetilde{\Gamma}(3)$, or its entries $\notin \mathcal{O}$ and then $\gamma \in \iota\,\widetilde{\Gamma}(3)$. A similar alternative takes place for $\gamma \in \sigma_k^{-1}\mathrm{B}(3)\sigma_k$ because of $\sigma_k \in \mathrm{SL}(2, \mathcal{O})$ for all k. Hence, given $\gamma \in P$, we have either

$$\gamma \in \widetilde{\Gamma}(3) \cap \sigma_2^{-1}\widetilde{\Gamma}(3)\sigma_2 \cap \sigma_3^{-1}\widetilde{\Gamma}(3)\sigma_3 \quad \text{or} \tag{8.4}$$

$$\begin{aligned} \gamma &\in \iota\widetilde{\Gamma}(3) \cap \sigma_2^{-1}\iota\widetilde{\Gamma}(3)\sigma_2 \cap \sigma_3^{-1}\iota\widetilde{\Gamma}(3)\sigma_3 \\ &= \iota\{\widetilde{\Gamma}(3) \cap \iota^{-1}\sigma_2^{-1}\iota\widetilde{\Gamma}(3)\sigma_2 \cap \iota^{-1}\sigma_3^{-1}\iota\widetilde{\Gamma}(3)\sigma_3\}. \end{aligned} \tag{8.5}$$

We should notice that $\iota^{-1}\sigma_k^{-1}\iota \in \mathrm{SL}(2, \mathcal{O})$ for both $k = 2$ and $k = 3$, and that $\widetilde{\Gamma}(3)$ consists of the matrices in $\mathrm{SL}(2, \mathcal{O})$ whose entries $\equiv 0, \pm 1 \pmod 3$. Then an elementary computation, which we omit for short, shows that the set in the right hand side in (8.4) equals to $\pm\Gamma(3)$, while the set in the right hand side in (8.5) is empty. This concludes the proof.

9 On monomorphism $\mathrm{SL}(2, \mathbf{F}_3[x]/(x^2)) \to \mathrm{Sp}(4, \mathbf{F}_3)$**.** The conditions (5.7) with $C = 0$ yield $A = {}^tD^{-1}$, $B = {}^tD^{-1}S$ with ${}^tS = S$. In particular, $\det D \neq 0$. Let

$$\mathbf{G} = \left\{ \begin{pmatrix} A & B \\ 0_2 & D \end{pmatrix} \in \mathrm{Sp}(4, \mathbf{F}_3) \;\middle|\; \det D = 1 \right\},$$

that is a subgroup of index 2 in the Siegel maximal parabolic subgroup of $\mathrm{Sp}(4,\mathbf{F}_3)$. Given $D \in \mathrm{SL}(2,\mathbf{F}_3)$ and a symmetric 2×2 matrix S with entries in $\mathbf{F}_3$, we have

$$\begin{pmatrix} {}^tD^{-1} & 0_2 \\ 0_2 & D \end{pmatrix} \in \mathbf{G} \quad \text{and} \quad \begin{pmatrix} I_2 & S \\ 0_2 & I_2 \end{pmatrix} \in \mathbf{G}. \tag{9.1}$$

Moreover, every $\gamma \in \mathbf{G}$ can be factored uniquely as

$$\gamma = \begin{pmatrix} {}^tD^{-1} & 0_2 \\ 0_2 & D \end{pmatrix} \begin{pmatrix} I_2 & S \\ 0_2 & I_2 \end{pmatrix} \tag{9.2}$$

with D and S as in (9.1). The number of a symmetric 2×2 matrices with entries in $\mathbf{F}_3$ is $3^3 = 27$, while the order of the group $\mathrm{SL}(2,\mathbf{F}_3)$ equals 24. Hence, $\mathbf{G}$ is the group of order $27\cdot 24 = 648$. We would like to prove that $\mathbf{G}$ is isomorphic to the group $\mathrm{SL}(2,\mathbf{F}_3[x]/(x^2))$ already considered in Section 6. It is clear from our considerations in Section 6 that every matrix $\delta \in \mathrm{SL}(2,\mathbf{F}_3[x]/(x^2))$ can be factored in $\mathrm{SL}(2,\mathbf{F}_3[x]/(x^2))$ uniquely as

$$\delta = D\begin{pmatrix} 1+ax & bx \\ cx & 1-ax \end{pmatrix}, \quad \text{where} \quad D \in \mathrm{SL}(2,\mathbf{F}_3), \quad a,b,c \in \mathbf{F}_3. \tag{9.3}$$

We can define $h\colon \mathrm{SL}(2,\mathbf{F}_3[x]/(x^2)) \to \mathbf{G}$ assining, for δ in (9.3), $h(\delta)$ to be the matrix γ in (9.2) with

$$S = \begin{pmatrix} -c & a \\ a & b \end{pmatrix}.$$

Clearly, h is a bijection. To prove that h is a group homomorphism one need the following: let $D \in \mathrm{SL}(2,\mathbf{F}_3)$, $a,b,c \in \mathbf{F}_3$, and $a',b',c' \in \mathbf{F}_3$; then

$${}^tD\begin{pmatrix} -c & a \\ a & b \end{pmatrix} D = \begin{pmatrix} -c' & a' \\ a' & b' \end{pmatrix} \quad \text{if and only if}$$

$$D^{-1}\begin{pmatrix} 1+ax & bx \\ cx & 1-ax \end{pmatrix} D = \begin{pmatrix} 1+a'x & b'x \\ c'x & 1-a'x \end{pmatrix}.$$

With this observation, a simple computation shows that the bijection h is a group homomorphism. Hence, h is an isomorphism. Let us omit the details.

10 Proof of Theorem. We have $n = [\mathrm{B}(\mathcal{O}) : \mathrm{B}(3)] = 27$, by Proposition 1 (c). Then, $\pm\Gamma(3)$ is a normal subgroup of $\mathrm{B}(\mathcal{O})$ and, according to Proposition 3 (c), (d), $\mathcal{Z}$ is the image of $\pm\Gamma(3)$ under the homomorphism ψ. In its turn, $\mathcal{E}$ is the image of $\mathrm{B}(\mathcal{O})$ under the ψ. Hence, $\mathcal{Z}$ is a normal subgroup of $\mathcal{E}$,

$$\mathcal{E}/\mathcal{Z} \simeq \mathrm{B}(\mathcal{O})/\pm\Gamma(3),$$

and that is a group of order 648, by Proposition 1 (c). On the other hand, we know that

$$W(E_6) \simeq \mathrm{PGSp}(4,\mathbf{F}_3).$$

So, the assertion of Theorem concerning the group $\mathcal{E}/\mathcal{Z}$ will be proved if we shall find some group monomorphism

$$\mathrm{B}(\mathcal{O})/\pm\Gamma(3) \xrightarrow{\widetilde{Q}} \mathrm{PGSp}(4,\mathbf{F}_3). \tag{10.1}$$

To this end, let us look at the diagramm

$$\begin{array}{ccc} \mathrm{PSp}(4,\mathbf{F}_3) \hookleftarrow \left\{\pm\begin{pmatrix} A & B \\ 0 & D\end{pmatrix} \in \mathrm{PSp}(4,\mathbf{F}_3) \,\middle|\, \det D = 1\right\} \\ \uparrow \\ \mathrm{SL}(2,\mathcal{O})/\pm\Gamma(3) \to \mathrm{PSL}(2,\mathcal{O}/3\mathcal{O}) \to \mathrm{PSL}(2,\mathbf{F}_3[x]/(x^2)) \end{array} \tag{10.2}$$

Here $\hookleftarrow$ is the inclusion of subgroup, $\uparrow$ is the isomorphism obtained by passing to quotient from the one constructed in Section 9, $\to$ are the isomorphisms obtained by passing to quotients from the ones constructed in Section 6. The composition

$$\mathrm{SL}(2,\mathcal{O})/\pm\Gamma(3) \xrightarrow{Q} \mathrm{PSp}(4,\mathbf{F}_3)$$

of all the homomorphisms in (10.2) is a group monomorphism, and we can use Q to define $\widetilde{Q}$ in (10.1).

We find in [7], 7.4, that three matrices

$$p_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1\end{pmatrix}, \quad p_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}, \quad p_3 = \begin{pmatrix} 1 & 0 \\ \omega & 1\end{pmatrix} \quad (\omega = (-1+\sqrt{-3})/2)$$

generate $\mathrm{SL}(2,\mathcal{O})$; these matrices and ι in (2.2) generate $\mathrm{B}(\mathcal{O})$. The canonical epimorphism $\mathrm{B}(\mathcal{O}) \to \mathrm{B}(\mathcal{O})/\pm\Gamma(3)$ takes p_1, p_2, p_3, ι to some elements, say p_1', p_2', p_3', ι', which generate the group $\mathrm{B}(\mathcal{O})/\pm\Gamma(3)$. Clearly, p_1', p_2', p_3' generate $\mathrm{SL}(2,\mathcal{O})/\pm\Gamma(3)$, and we have the disjoint decomposition

$$\mathrm{B}(\mathcal{O})/\pm\Gamma(3) = \mathrm{SL}(2,\mathcal{O})/\pm\Gamma(3) \cup \iota'\,\mathrm{SL}(2,\mathcal{O})/\pm\Gamma(3)$$

(compare with (6.2)). Let us notice that the above defined monomorphism Q satisfies

$$Q(\iota'^{-1}\gamma\iota') = \varrho^{-1}Q(\gamma)\varrho \quad \text{for all} \quad \gamma \in \mathrm{SL}(2,\mathcal{O})/\pm\Gamma(3). \tag{10.3}$$

Clearly, it is sufficient to verify (10.3) for generators of $\mathrm{SL}(2,\mathcal{O})/\pm\Gamma(3)$ only, say for $\gamma = p_1', p_2', p_3'$, and this can be done easily by inspection the homomorphisms involved into (10.2). Now, let us define $\widetilde{Q}$ by

$$\widetilde{Q}(\gamma) = Q(\gamma), \quad \widetilde{Q}(\iota'\gamma) = \varrho Q(\gamma) \quad \text{for all} \quad \gamma \in \mathrm{SL}(2,\mathcal{O})/\pm\Gamma(3).$$

Recall that $\iota'^2 = 1$ and $\varrho^2 = 1$. It follows easily from (10.3) that $\widetilde{Q}$ is a group monomorphism, that concludes the proof of Theorem.

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