A three-manifold invariant via the Kontsevich integral

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A three-manifold invariant via the Kontsevich integral*

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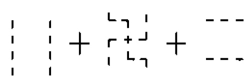
Key words: invariants of three-manifolds, invariants of knots and links, Casson-Walker invariant, Kontsevich integral, chord diagram, Kirby move.

We construct an invariant for closed, oriented three-manifolds from the Kontsevich integral $\hat{Z}$, which includes Lescop's generalization of the Casson-Walker invariant. Combining this result and a formula for computing the Kontsevich integral in [17], we can compute the Casson-Walker invariant combinatorially in terms of q-tangles (non-associative tangles in [3]). To get a three-manifold invariant, we impose the following three-term (3T) relation to the space of chord diagrams.



$= 0$ (3T relation).

In this relation, dotted lines present chords and the three chord diagrams are identical except within discs where they are as above.

The Kontsevich integral $\hat{Z}$ has values in the space $\mathcal{A}$ of chord diagrams subject to the four-term relation [13, 2, 17]. We know that a three-manifold is obtained by the surgery on a non-oriented framed link in S^3 [22]. Oriented three-manifolds obtained from two links are homeomorphic if one of the links are obtained from the other by a sequence of Kirby moves given in Figure 2 [12]. Framings of the links are given by the blackboard framings and the part of L'_1 in L' parallel to L_2 is actually parallel on the blackboard. We construct a three-manifold invariant by taking Kirby move invariant part

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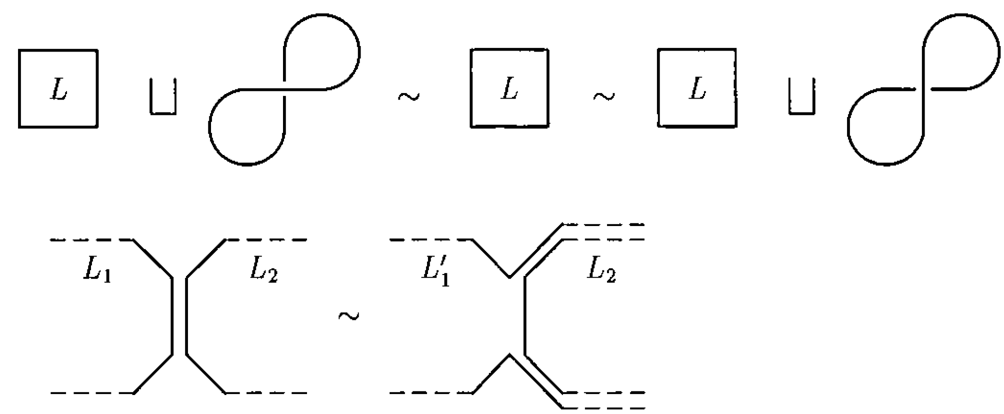
from the Kontsevich integral $\hat{Z}_f$ of oriented framed links constructed in [17]. We modify $\hat{Z}_f$ so that it has a good property with respect to the KII moves. Let $\nu = Z_f(U)^{-1}$, where U is the Morse knot given in Figure . This is the factor introduced in [2, 17] to normalize the effect of maximal and minimal points. For an ℓ-component link L, let $\check{Z}_f(L) = \hat{Z}_f(L)\#(\nu, \nu, \cdots, \nu)$. This means that we connect-sum ν to each string of $\hat{Z}_f(L)$. We take a certain quotient $\bar{\mathcal{A}}$ of $\mathcal{A}$ so that the image of $\check{Z}_f(L)$ is stable under the KII moves. Let $\Lambda'(L)$ denote this image of $\check{Z}_f(L)$. Our construction of $\Lambda'(L)$ is compatible with the structure of the category of the ribbon Hopf algebra in [26] and so $\Lambda'(L)$ factors the Jones-Witten invariant in [23, 26] without the normalization factor for the KI moves. We will write detail of the related facts elsewhere.

FIGURE 1. The diagram U.

(KI)

(KII)



$$L = L_1 \sqcup L_2 \sqcup \cdots \sqcup L_k \sim L' = L'_1 \sqcup L_2 \sqcup \cdots \sqcup L_k$$

FIGURE 2. Kirby Moves

In this paper we study $\Lambda'(L)$ modulo one more relation (stable equivalence, see Definition) concretely, and we show that it consists of the order of the first homology group and the Casson invariant. For a $\mathbb{Z}/r\mathbb{Z}$-homology 3-sphere (r : odd prime), we already know that the Jones-Witten invariant [11, 28] dominate the Casson-Walker invariant [24]. The element $\check{Z}_f(L)$ in $\bar{\mathcal{A}}$ dominates both the Jones-Witten invariant and the Casson invariant. However, we still don't know a relation between them for general three-manifolds.

The main results (contents of Section 1) are announced in [14].

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1. Modification of the universal Vassiliev-Kontsevich invariant for three-manifolds

We use notations in [17, 19]. Let C be a chord diagram with a distinguished string s, and let k be the number of end points of chords on s. Let $\Delta(C)$ denote the sum of 2^k diagrams obtained by adding a string parallel to s and changing each point on s as in Figure 3. Note that this is different to the coproduct introduced in [2].

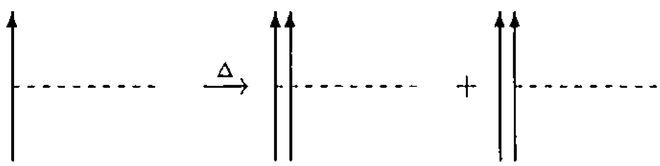
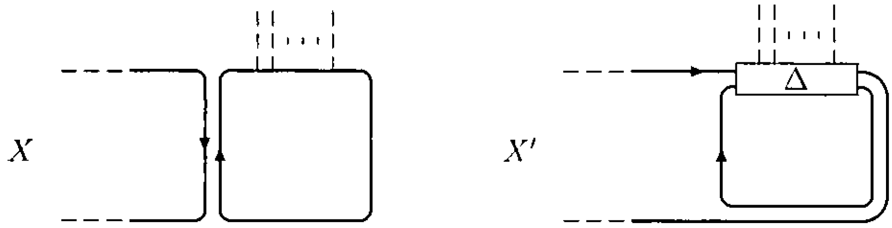


FIGURE 3. Parallel of a chord diagram

Proposition 1. *Let L and L' be two links as in the KII move in Figure 2. Let $\check{Z}_f(L) = \sum_{X:\text{chord diagram}} \check{c}_X\, X$. Then $\check{Z}_f(L') = \sum_{X:\text{chord diagram}} \check{c}_X\, X'$, where X' is obtained from X as in Figure 4.*

This is proved in Section 2.

Let $\mathcal{A}^{(\ell)}$ denote the $\mathbb{C}$-linear space spanned by the chord diagrams on a disjoint union of ℓ S^1's subject to the four-term relation. We add two types of relations to $\mathcal{A}^{(\ell)}$. The first one is for orientations of strings. Let D be a chord diagram and let D' the chord diagram obtained by changing the orientation of a string s of D. Then we impose $D' \sim (-1)^{e(s)} D$, where $e(s)$ denotes the number of end points on s. We call this the orientation independence relation. The second relation is for the KII move given in Figure 5. We call



$$\check{Z}_f(L) = \sum_{X:\text{chord diagram}} \check{c}_X\, X \qquad \check{Z}_f(L') = \sum_{X:\text{chord diagram}} \check{c}_X\, X'$$

FIGURE 4. Difference of $\check{Z}_f$ by the second Kirby move

it the KII relation of chord diagrams. Let

$$\bar{\mathcal{A}}^{(\ell)} = \mathcal{A}^{(\ell)} / (\text{Orientation independence relation,KII relation})\,,$$

and let $\Lambda'(L)$ be the image of $\check{Z}_f(L)$ in $\bar{\mathcal{A}}^{(\ell)}$ for an ℓ-component link L.

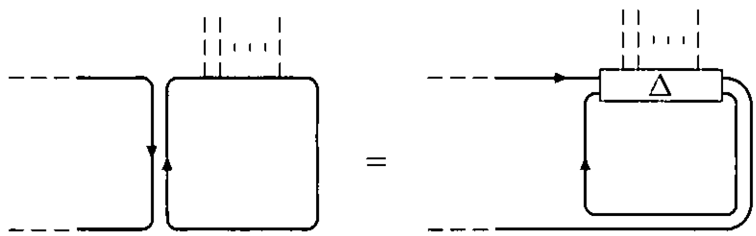


FIGURE 5. Relation for second Kirby moves

Proposition 1′. *$\Lambda'(L)$ is invariant under KII moves and orientation change of any component.*

We normalize a low degree part of $\Lambda'(L)$ for the KI moves.

Definition. Two elements D, D' in $\bar{\mathcal{A}}^{(\ell)}$ is called *stably equivalent* if $D \sqcup \Theta \sqcup \cdots \sqcup \Theta = D' \sqcup \Theta \sqcup \cdots \sqcup \Theta$ in $\bar{\mathcal{A}}^{(\ell+k)}$ for some $k \geq 0$, where Θ denotes the chord diagram on a circle with one chord. Let $\bar{\mathcal{A}}_1^{(\ell)}$ denote the set of stable equivalence classes of $\bar{\mathcal{A}}^{(\ell)}$.

Proposition 2. *$\bar{\mathcal{A}}_1^{(\ell)}$ is a two-dimensional vector space with basis $\{\Theta \sqcup \Theta \sqcup \cdots \sqcup \Theta, \Theta_2 \sqcup \Theta \sqcup \cdots \sqcup \Theta\}$, where Θ_2 denotes the chord diagram on a circle with two chords as in Figure 6.*

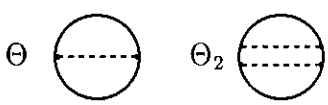


FIGURE 6. Θ and Θ_2

This is proved in Section 3.

For two elements $D_1 \in \bar{\mathcal{A}}_1^{(\ell_1)}$ and $D_2 \in \bar{\mathcal{A}}_1^{(\ell_2)}$, let $D_1\, D_2$ be the image of $D_1 \sqcup D_2 \in \bar{\mathcal{A}}_1^{(\ell_1+\ell_2)}$. For $\ell_1, \ell_2 \geq 1$, $\bar{\mathcal{A}}_1^{(\ell_1)}$ and $\bar{\mathcal{A}}_1^{(\ell_2)}$ are isomorphic by identifying the corresponding basis. Let $\bar{\mathcal{A}}_1^{(\infty)}$ be the space spanned by g_0 and g_1 which correspond to $\Theta \sqcup \Theta \sqcup \cdots \sqcup \Theta$ and $\Theta_2 \sqcup \Theta \sqcup \cdots \sqcup \Theta$ respectively and we identity $\bar{\mathcal{A}}_1^{(\ell)}$ with $\bar{\mathcal{A}}_1^{(\infty)}$. Then $\bar{\mathcal{A}}_1^{(\infty)}$ has two-dimensional algebra structure with multiplication given by the disjoint union, which is given by $g_0\, g_0 = g_0$, $g_0\, g_1 = g_1\, g_0 = g_1$, and $g_1\, g_1 = 0$. Note that $c_0\, g_0 + c_1\, g_1 \in \bar{\mathcal{A}}_1^{(\infty)}$ is invertible if and only if $c_0 \neq 0$. Let $\Lambda_1'(L)$ denote the image of $\Lambda'(L)$ in $\bar{\mathcal{A}}_1^{(\infty)}$. Then, for trivial knots $\infty_{\pm 1}$ with ± 1 framings, we have $\Lambda_1'(\infty_{+1}) = \frac{1}{2}\, g_0 + \frac{3}{8}\, g_1$, and $\Lambda_1'(\infty_{-1}) = -\frac{1}{2}\, g_0 + \frac{3}{8}\, g_1$. Hence $\Lambda_1'(\infty_{\pm 1})$ is invertible and we can modify $\Lambda_1'(L)$ for the KI moves as in the case of the Jones-Witten invariant. Let $\sigma_+(L)$ (resp. $\sigma_-(L)$) denote the number of positive (resp. negative) eigenvalues of the linking matrix B_L of L, and let

$$\Lambda_1(L) = 2^{\ell-\sigma_+(L)-\sigma_-(L)}\, \Lambda_1'(\infty_{+1})^{-\sigma_+(L)}\, \Lambda_1'(\infty_{-1})^{-\sigma_-(L)}\, \Lambda_1'(L).$$

Let $\Lambda_{1,0}(L)$ and $\Lambda_{1,1}(L)$ be the coefficients of $\Lambda_1(L)$ with respect to g_0 and g_1, i.e.

$$\Lambda_1(L) = \Lambda_{1,0}(L)\, g_0 + \Lambda_{1,1}(L)\, g_1.$$

For a framed link L and the corresponding three-manifold M_L, we have the following.

Theorem 1. *$\Lambda_1(L)$ is a topological invariant of the three-manifold M_L.*

Theorem 2. (1) *$\Lambda_{1,0}(L) = |H_1(M_L)|$, the order of the first homology group of M_L if $b_1(M_L) = 0$, and 0 if $b_1(M_L) > 0$, where $b_1(M_L)$ is the first Betti number of M_L.*
(2) *$\Lambda_{1,1}(L) = -3\, \tilde{\lambda}(M_L)$, where $\tilde{\lambda}(M_L)$ is twice Lescop's generalization* [21] *of the Casson-Walker invariant (Walker's normalizeation) $\lambda(M_L)$* [1, 27] *satisfying $\tilde{\lambda}(M_L) = |H_1(M_L)| \times \lambda(M_L)$ if $b_1(M_L) = 0$.*

Theorem 1 is a direct consequence of our construction of Λ_1. To prove Theorem 2, we use the fourth author's diagonalizing lemma given in [25, Corollary 2.5] and [24, Lemma 2.2]. According to the diagonalizing lemma, we can restrict our attention to algebraically split links, for which we can prove (1). See Section 4 for detail. To prove (2), adding to the diagonalizing lemma, we use Dehn surgery formula obtained in [10] and [21] which expresses the Casson-Walker invariant in terms of linking numbers and coefficients of the Conway polynomial [21]. For algebraically split links, this formula is rather simple and we can compare directly our invariant and their formula. Then we get (2). For detail, see Sections 5 and 6.

2. Proof of Proposition 1.

We prepare several lemmas. Suppose X is a one-dimensional oriented manifold whose components are numbered. A *chord diagram with support* X is a set consisting of a finite number of unordered pairs of distinct non-boundary points on X, regarded up to orientation and component preserving homeomorphisms. We view each pair of points as a chord on X and represent it as a dashed line connecting the two points. Let $\mathcal{A}(X)$ be the vector space over $\mathbb{C}$ spanned by all chord diagrams with support X, subject to the well-known 4-term relation (see, for example, [2, 17]). The vector space $\mathcal{A}(X)$ is graded by the number of chords, and, abusing notation, we use the same $\mathcal{A}(X)$ for the completion of this vector space with respect to this grading. When X is n numbered lines, $\mathcal{A}(X)$ is denoted by $\mathcal{P}_n$. All the $\mathcal{P}_n$ are algebras: the product of two chord diagrams D_1 and D_2 is obtained by placing D_1 on top of D_2. The algebra P_1 is commutative [2, 13].

We recall the associator $\Phi \in \mathcal{P}_3$ in [16, 17], which is equal to $Z_f(\,|\!/\!|\,)$, where $|\!/\!|$ presents the trivial q-tangle on three strings with brackets $(*(**))$ at the top and $((**)*)$ at the bottom. This associator corresponds to the associator of quasi-Hopf algebras in [5, 6] and is also studied in [3]. For $\mathbf{p} = (p_1, \cdots, p_g)$, $g(\mathbf{p}) = g$ is the length of $\mathbf{p}$, and $|\mathbf{P}| = p_1 + p_2 + \cdots + p_g$. For $\mathbf{p}$ and $\mathbf{r}$ with the same length g, $\mathbf{p} \geq \mathbf{r}$ means $p_i \geq r_i$, $\mathbf{p} > \mathbf{r}$ means $p_i > r_i$, $\mathbf{p} > 0$ means $p_i > 0$, and $\mathbf{p} \geq 0$ means $p_i \geq 0$ for $1 \leq i \leq g$. Let

$$\zeta(i_1, \ldots, i_k) = \sum_{m_1 < \cdots < m_k \in \mathbb{N}} \frac{1}{m_1^{i_1} \ldots m_k^{i_k}}, \tag{2.1}$$

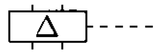
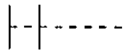
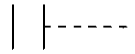
and

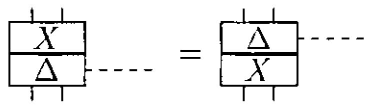
$$\tau(p_1, q_1, \ldots, p_n, q_n) = \zeta(\underbrace{1, \ldots, 1}_{p_1 - 1}, q_1 + 1, \underbrace{1, \ldots, 1}_{p_2 - 1}, q_2 + 1, \ldots, q_n + 1). \tag{2.2}$$

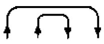
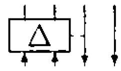
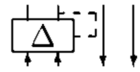
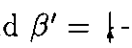
Then, as described at the end of [16],

$$\begin{aligned} \Phi = 1 + \sum_{k=2}^{\infty} \sum_{g \geq 1} \sum_{\substack{\mathbf{p}>0,\ \mathbf{q}>0, \\ |\mathbf{p}|+|\mathbf{q}|=k, \\ g(\mathbf{p})=g(\mathbf{q})=g}} & (-1)^{|\mathbf{q}|} \tau(p_1, q_1, \cdots, p_g, q_g) \times \\ & \sum_{\substack{g(\mathbf{r})=g(\mathbf{s})=g, \\ 0 \leq \mathbf{r} \leq \mathbf{p},\ 0 \leq \mathbf{s} \leq \mathbf{q}}} (-1)^{|\mathbf{r}|} \Big(\prod_{i=1}^{g} \binom{p_i}{r_i} \binom{q_i}{s_i}\Big) B^{|\mathbf{s}|} A^{p_1 - r_1} B^{q_1 - s_1} \ldots A^{p_g - r_g} B^{q_g - s_g} A^{|\mathbf{r}|}, \end{aligned} \tag{2.3}$$

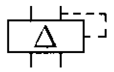
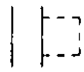
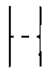
where A (resp. B) denotes the chord connecting the first and second (resp. the second and third) strings.

Proof of Proposition 1. Let  denote ε_1  $+\,\varepsilon_2$  , where $\varepsilon_i = 1$ (resp. -1) if the i–th string is oriented downward (resp. upward). Then, for any $X \in \mathcal{P}_2$, we have

(2.4) 

This is a special case of Lemma 2.1 in [19]. Let Δ_1 be a mapping from $\mathcal{P}_3$ to $\mathcal{P}_4$ applying Δ to the first string of $\mathcal{P}_3$. Let $E =$  , $\alpha =$  , $\alpha' =$  , $\beta =$  and $\beta' =$  . Note that $E\,\alpha = E\,\alpha'$ and $E\,\beta = E\,\beta'$. For $X \in \mathcal{P}_3$, we have

$$E\,\alpha\,\Delta_1(X) = E\,\Delta_1(X)\,\alpha', \qquad E\,\beta\,\Delta_1(X) = E\,\Delta_1(X)\,\beta'. \tag{2.5}$$

By applying (2.4) with $X = \alpha'$ and $X = \beta'$ to each end point on the first string of X, we get these formulas. Since  $= \varepsilon_1$  $+\,\varepsilon_2$  , we have

$$\alpha'\,\beta' = \beta'\,\alpha'. \tag{2.6}$$

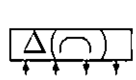
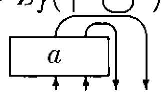
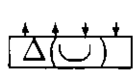
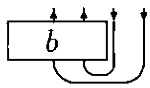
We use the above expression for Φ. Then, in $\Delta_1(\Phi)$, $\Delta_1(A) = -\alpha$ and $\Delta_1(B) = \beta$. By (2.5) and (2.6), we have $E\,\beta^{|\mathbf{s}|}\,\alpha^{p_1-r_1}\,\beta^{q_1-s_1}\ldots\alpha^{p_g-r_g}\,\beta^{q_g-s_g}\,\alpha^{|\mathbf{r}|} = E\,\alpha'^{|\mathbf{p}|}\,\beta'^{|\mathbf{q}|}$. Hence

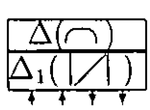
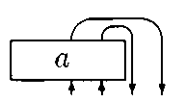
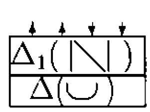
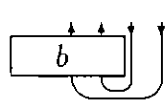
(2.7) $E\,\Delta_1(\Phi) = E\, +$

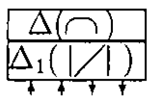
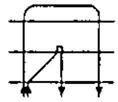
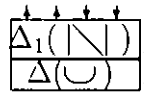
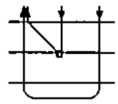
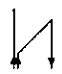
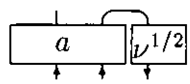
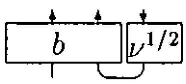
$$\sum_{k=2}^{\infty}\sum_{g\geq 1}\sum_{\substack{\mathbf{p}>0,\ \mathbf{q}>0,\\ |\mathbf{p}|+|\mathbf{q}|=k,\\ g(\mathbf{p})=g(\mathbf{q})=g}} \tau(p_1,q_1,\cdots,p_g,q_g) \sum_{\substack{g(\mathbf{r})=g(\mathbf{s})=g,\\ 0\leq\mathbf{r}\leq\mathbf{p},\ 0\leq\mathbf{s}\leq\mathbf{q}}} (-1)^{|\mathbf{p}|}(-1)^{|\mathbf{q}|}(-1)^{|\mathbf{r}|}\Big(\prod_{i=1}^{g}\binom{p_i}{r_i}\binom{q_i}{s_i}\Big)\,E\,\alpha'^{|\mathbf{p}|}\,\beta'^{|\mathbf{q}|}.$$

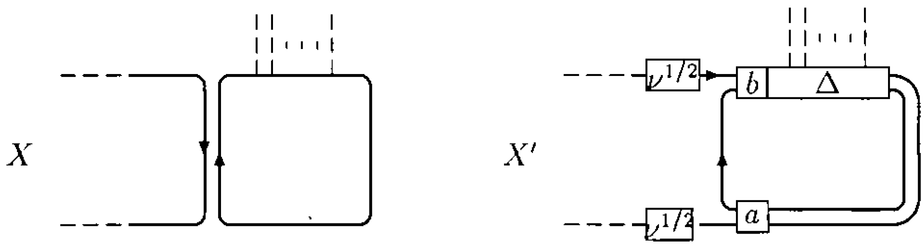
However, we know that $\sum_{r_i=0}^{p_i}(-1)^{r_i}\binom{p_i}{r_i} = 0$ for $p_i > 0$. Therefore, all the terms except the first one of the right hand side of the above expression for Φ vanish and so we get

$$E\,\Delta_1(\Phi) = E \tag{2.7}$$

Recall that ν is an element in $\mathcal{P}_2$ such that $\nu^{-1} = Z_f($  $)$. By the remark at the end of Section 5 in [19], we have $Z_f($  $) =$  and $Z_f($  $) =$  for some a and b such that $a\,b = \Delta(\nu)\,(\nu^{-1}\otimes\nu^{-1})$. Hence, by (2.7) and its opposite version, we get

(2.8) $Z_f($  $) =$  , $\quad Z_f($  $) =$  .

As q-tangles,  =  and ) =  where the brackets are as $(((**)*)*)$. On the other hand, $\hat{Z}_f$ is obtained from Z_f by adding $\nu^{1/2}$ at each maximal and minimal point. Hence, from (2,8), we get $\hat{Z}_f($  $)$ = , $\hat{Z}_f($  $)$ = . Therefore, the difference of $\hat{Z}_f(L)$ and $\hat{Z}_f(L')$ is given as in Figure 7. Hence, by adding extra ν's to each component to get $\check{Z}_f$ and using $a\,b = \Delta(\nu)\,(\nu^{-1}\otimes\nu^{-1})$, we get Proposition 1. □

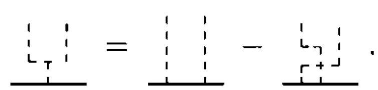


$\hat{Z}_f(L) = \sum_{X:\text{chord diagram}} \hat{c}_X\, X$ $\quad\quad$ $\hat{Z}_f(L') = \sum_{X:\text{chord diagram}} \hat{c}_X\, X'$

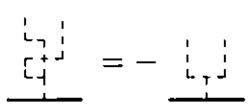
FIGURE 7. Difference of $\hat{Z}_f$ by the second Kirby move

3. PROOF OF PROPOSITION 2

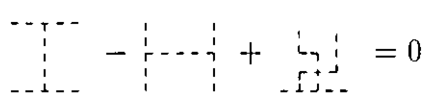
We extend the notion of chord diagrams as in [2]. Let



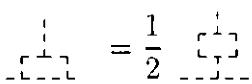
From this definition, the order of the edges at the branch point has meaning and we have

 (Flipping relation).

Then, according to the four-term relations, this element with a branch point on chords is well-defined in the space spanned by chord diagrams with the four-term relations. From the four-term relation, we have the following relation, which is called the *IHX relation* in [2].

 (IHX relation).

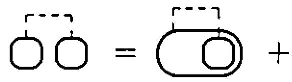
From the IHX and flip relations, we have

 (triangle reduction formula).

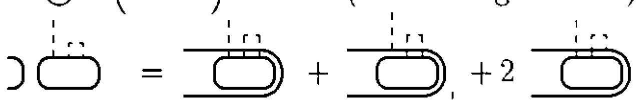
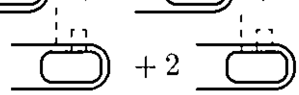
Let $\mathcal{A}^{(0)}$ denote the space spanned by chord diagrams with branch points subject to the IHX relation, and let $\bar{\mathcal{A}}_1^{(0)}$ denote the stably equivalent classes of chord diagrams without strings subject to the KII relation and the orientation independence relation.

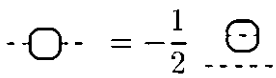
By using the KII relation and the orientation independence relation, we get several useful relations. From the orientation independence relation, we have

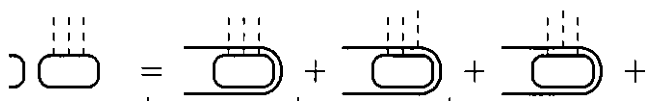
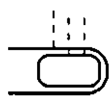
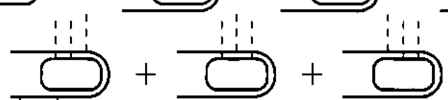
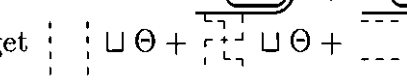
 (1 vanishing formula),

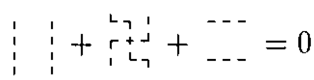
i.e. if a chord diagram has a string without any end point of a chord; then it is equal to 0 in $\bar{\mathcal{A}}^{(\ell)}$. Next, from the KII relation for chord diagrams, we have   and so  $= 0$. Hence, in $\bar{\mathcal{A}}_1^{(\ell)}$, we get

$$\bigcirc \sqcup \left(\sqcup^{\ell-1}\Theta\right) = 0 \qquad \text{(0 vanishing formula).}$$

Similarly, we have  + terms equal to 0 by the 0 and 1 vanishing formulas. Hence we have  $= 0$. This means that, in $\bar{\mathcal{A}}^{(\ell)}$,

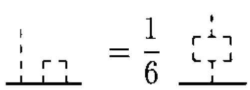
 (pushout formula).

We derive another important relation:   + terms equal to 0. Hence we have  $= 0$. By using the pushout formula, we get  $= 0$. Hence, in $\bar{\mathcal{A}}_1^{(\ell)}$,

 (3T relation).

This relation is called the *three-term (3T) relation.* From 3T relation, we get following two relations.

 (two legs reduction),

 (three legs reduction).

Moreover, for two chord diagrams D_1, $D_2 \in \bar{\mathcal{A}}_1^{(0)}$, we have

$$D_1 \bigsqcup D_2 = -D_1 \# D_2 - D_1 \# \overline{D}_2 \quad \text{in } \bar{\mathcal{A}}_1^{(0)} \qquad \text{(connected sum formula),}$$

where $D_1 \# D_2$ denote the connected sum of two diagrams at chords. $D_1 \# \overline{D}_2$ is also the connected sum at the same chords, but D_2 is attached to D_1 differently as in Figure 8.

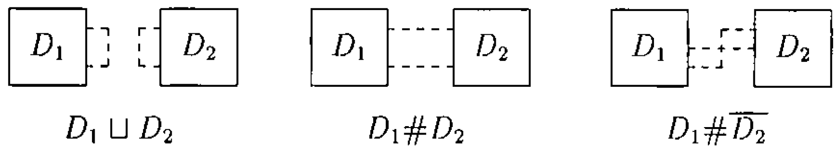
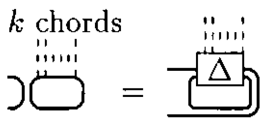


FIGURE 8. Connected sum of two chord diagrams without strings.

In the above, we get the 3T relation and the 0 vanishing formula from the KII relation for the chord diagrams. Conversely, we have the following.

Lemma 1. *The KII relation comes from the IHX, 3T, orientation independence relations, and the 0 vanishing formula, i.e.*

$$\bar{\mathcal{A}}_1^{(\ell)} = \mathcal{A}^{(\ell)} / (\textit{3T relation, orientation independence relation, 0 vanishing formula})\,.$$

Proof. We show 

by an induction on k. For $k = 0$, this formula is trivially satisfied. For $k = 1$, this formula is satisfied by the 0 vanishing formula. For $k = 2$, this formula comes from the 0 and 1 vanishing formulas, where the 1 vanishing formula is a consequence of the orientation independence relation. For $k > 2$, we reduce it to the case for fewer chords by using the two and three legs reductions. □

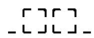
By using the above relations, we get the following lemma.

Lemma 2. *The space* $\bar{\mathcal{A}}_1^{(\ell)}$ *is spanned by*

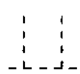
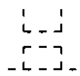
$$\{D \sqcup (\sqcup^\ell \Theta) \mid D : \textit{connected chord diagram in } \mathcal{A}^{(0)} \textit{ with less than three branch points}\}.$$

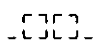
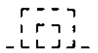
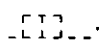
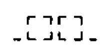
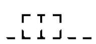
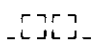
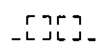
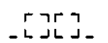
Proof. By using the two and three legs reduction (and 0, 1 vanishing formula), we can reduce any chord diagram to a linear combination of chord diagrams such that any of its strings has just two end points of chords. Then, by using the pushout formula for these diagrams, we get a linear combination of chord diagrams of the form $D \sqcup (\sqcup^\ell \Theta)$, where D is a chord diagram without a string. Here, by using the connected sum formula, we reduce D to a scalar multiple of a connected diagram.

Let θ denote the diagram  . By using the 3T relation, we can reduce a connected chord diagram with $2d$ branch points to a scalar multiple of a connected sum of d θ's.

Now we assume that D has $2d$ branch points with $d > 1$ and show that D vanishes in $\bar{\mathcal{A}}_1^{(\ell)}$. By the above argument, we may assume that D is a connected sum of d θ's, and, moreover, we may assume that D has a part given by  . From the 3T relation, we have

$$2 \text{ } \text{\textcircled{}} \qquad \tag{3.1}$$

(3.1) 2  +  −  $= 0.$

By using this, we have  $= -1/2$  $+1/2$ . Since  $=$  , and  $= 1/2$  by the triangle reduction formula, we have $5/4$  $= 0$, and so  $= 0$. $\square$

Proof of Proposition 2. For a chord diagram $D \in \bar{\mathcal{A}}_1^{(\ell)}$, let $\deg(D)$ be

$$\deg(D) = (\text{number of branch points and end points of } D)/2 - \ell,$$

and we call it the *degree of* D. All relations to define $\bar{\mathcal{A}}_1^{(\ell)}$ are homogeneous with respect to this degree. Lemma 2 shows that the degree 0 and 1 parts of $\bar{\mathcal{A}}_1^{(\ell)}$ are both at most 1-dimensional and other parts are all 0-dimensional. Therefore, it is enough to show that the elements $\sqcup^\ell\Theta$ and $\Theta_2 \sqcup \left(\sqcup^{\ell-1}\Theta\right)$ do not vanish. To do this, we list up all the non-zero chord diagrams of degree 0 and 1 in $\bar{\mathcal{A}}_1^{(\ell)}$ and show that they are reduced uniquely to scalar multiples of $\sqcup^\ell\Theta$ and $\Theta_2 \sqcup \left(\sqcup^{\ell-1}\Theta\right)$ by the IHX, 3T, orientation independence relations and 0 vanishing formula. We write down exactly all such non-zero diagrams as scalar multiples of $\sqcup^\ell\Theta$ and $\Theta_2 \sqcup \left(\sqcup^{\ell-1}\Theta\right)$, and check that all the relations are compatible with these elements.

Non-zero diagrams of degree 0 are disjoint union of chord diagrams with several components connected by chords as a chain in Figure 9. Then we have

$$D^{(k)} = 1/(-2)^{k-1} \left(\sqcup^k\Theta\right). \tag{3.2}$$

Hence, for a diagram $D \in \bar{\mathcal{A}}_1^{(\ell)}$ of degree 0 with j disjoint chains, $D = (-2)^{-(\ell-j)} \left(\sqcup^\ell\Theta\right)$.

Non-zero diagrams of degree 1 are disjoint union of one component of degree 1 and (zero or some) $D^{(k)}$'s. Non-zero connected diagrams of degree 1 are $D_{=}^{(k_1,k_2)}$, $D_x^{(k_1,k_2)}$ and $D_{\pm 1}^{(k_1,k_2,k_3)}$ in Figure 9. Then we have

$$\begin{aligned} D_{=}^{(k_1,k_2)} &= 1/(-2)^{k_1+k_2}\, \Theta_2 \sqcup \left(\sqcup^{k_1+k_2}\Theta\right), \\ D_x^{(k_1,k_2)} &= 1/(-2)^{k_1+k_2-1}\, \Theta_2 \sqcup \left(\sqcup^{k_1+k_2}\Theta\right), \\ D_{\pm 1}^{(k_1,k_2,k_3)} &= \pm 3\,(-2)^{-(k_1+k_2+k_3+2)}\, \Theta_2 \sqcup \left(\sqcup^{k_1+k_2+k_3+1}\Theta\right). \end{aligned} \tag{3.3}$$

So, combining (3.2) and (3.3), we get expressions for all the non-vanishing diagrams.

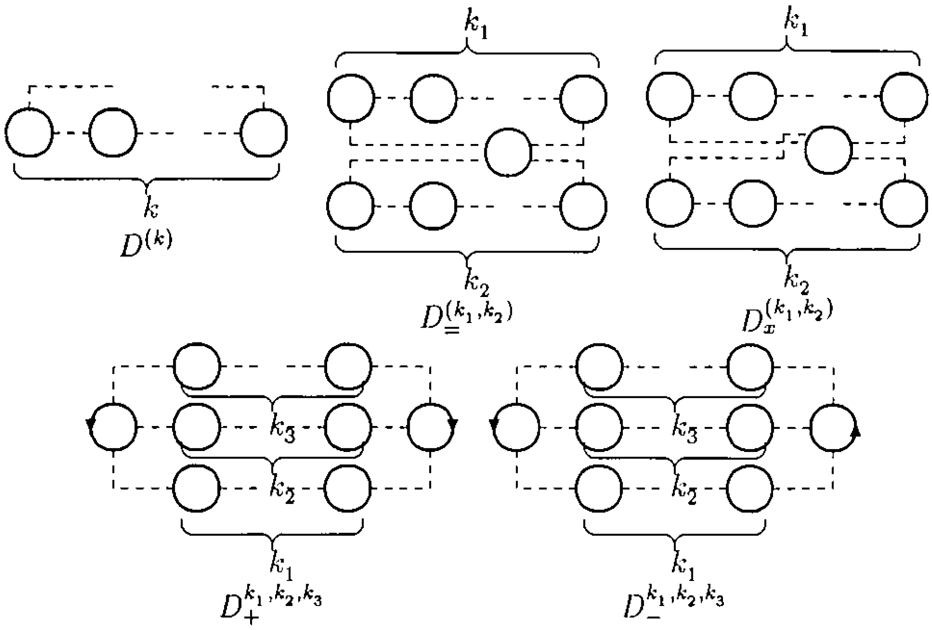


FIGURE 9. Non-trivial diagrams

Now we can check all the relation for the above elements. Computation is rather elementary and we omit the detail. □

4. ORDER OF THE FIRST HOMOLOGY GROUP

We first recall the fourth author's diagonalizing Lemma, by which we can restrict our work to simple cases. For a framed link L, let B_L denote the linking matrix of L. A framed link L is called an *algebraically split link* if B_L is a diagonal matrix.

Lemma 3 (Diagonalizing Lemma, [25, Corollary 2.5] and [24, Lemma 2.2]) *Let L be a framed link. There is an algebraically split link L' with a non-degenerate linking matrix such that $L \sqcup L'$ is equivalent to an algebraically split link by the Kirby moves.*

Proof of Theorem 2 (1). To prove (1), we compute the integral for a framed link L corresponding to the configuration of disjoint union of $D^{(k)}$. Let $w(K)$ be the writhe of a knot K and $\mathrm{lk}(K_1, K_2)$ be the algebraic linking number of knots K_1 and K_2. Here we give a coordinate in $\mathbb{R}^3$ by $(z,t) \in \mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$. We first assume that L is a knot. In this case, we need the coefficient of $D^{(1)}$, which is equals to Θ. The coefficient of Θ in $Z_f(L)$ is known to be a half of the writhe of L [17]. Next, we assume that L is a two component link with components L_1 and L_2. In this case, we need the coefficients of $D^{(2)}$

and $D^{(1)} \sqcup D^{(1)}$, which are given by the following integrals: The coefficient of $D^{(2)}$ is

$$\frac{1}{(2\pi\sqrt{-1})^2}\int_{t_1<t_2}\sum_{\substack{z_1\in L_1\cap\{t=t_1\},z_1'\in L_2\cap\{t=t_1\}\\ z_2\in L_1\cap\{t=t_2\},z_2'\in L_2\cap\{t=t_2\}}} \varepsilon_1\,\varepsilon_1'\,\varepsilon_2\,\varepsilon_2'\,d\log(z_1-z_1')\wedge d\log(z_2-z_2'),$$

where ε_i, (resp. ε_i') is 1 if the link L is oriented upward at the point (t_i, z_i) (resp. (t_i, z_i')), and is equal to -1 if L is oriented downward at this point. This integral is equal to

$$\frac{1}{2}\left(\frac{1}{2\pi\sqrt{-1}}\int_{t_1}\sum_{\substack{z_1\in L_1\cap\{t=t_1\}\\ z_1'\in L_2\cap\{t=t_1\}}}\varepsilon_1\,\varepsilon_1'\,d\log(z_1-z_1')\right)\left(\frac{1}{2\pi\sqrt{-1}}\int_{t_2}\sum_{\substack{z_2\in L_1\cap\{t=t_2\}\\ z_2'\in L_2\cap\{t=t_2\}}}\varepsilon_2\,\varepsilon_2'\,d\log(z_2-z_2')\right).$$

As in [13], we know that

$$\frac{1}{2\pi\sqrt{-1}}\int_{t_1}\sum_{z_1\in L_1\cap\{t=t_1\},z_1'\in L_2\cap\{t=t_1\}}\varepsilon_1\,\varepsilon_1'\,d\log(z-z') = \mathrm{lk}(L_1, L_2),$$

and so the coefficient is equal to

$$\frac{1}{2}\,\mathrm{lk}(L_1, L_2)^2.$$

As in the case of knot, the coefficient of $D^{(1)} \sqcup D^{(1)}$ is equal to

$$w(L_1)\,w(L_2).$$

Now consider the case for ℓ-component link L with components L_1, L_2, $\cdots$, L_ℓ. We first compute the coefficient of $D^{(\ell)}$ whose components corresponding to L_1, $\cdots$, L_ℓ are connected by chords in that order. It is given by the following integral:

$$\frac{1}{(2\pi\sqrt{-1})^\ell}\int_{t_1,t_2,\cdots,t_\ell}\sum_{\substack{z_1,\in L_1\cap\{t=t_1\},z_1',\in L_2\cap\{t=t_1\}\\ z_2,\in L_2\cap\{t=t_2\},z_2'\in L_3\cap\{t=t_2\}\\ \cdots\\ z_\ell,\in L_\ell\cap\{t=t_\ell\},z_\ell',\in L_1\cap\{t=t_\ell\}}}\bigwedge_{k=1}^{\ell}\varepsilon_k\,\varepsilon_k'\,d\log(z_k-z_k').$$

Let $L_{\ell+1}$ be L_1, then the above integral is equal to

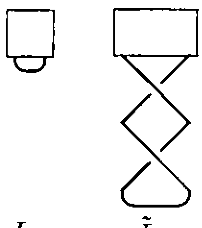
$$\prod_{k=1}^{\ell}\frac{1}{2\pi\sqrt{-1}}\int_{t_k}\sum_{\substack{z_k\in L_k\cap\{t=t_k\}\\ z_k'\in L_{k+1}\cap\{t=t_k\}}}\varepsilon_k\,\varepsilon_k'\,d\log(z_k-z_k').$$

Hence, the coefficient is equal to

$$\prod_{k=1}^{\ell}\mathrm{lk}(L_k, L_{k+1}).$$

Next we compute the coefficient of a disjoint union of several $D^{(k)}$'s. In this case, the result is a product of coefficients corresponding to every $D^{(k)}$ given above.

We show that $\Lambda_{1,0}(L)$ is equal to the absolute value of $\det B_L$. According to diagonalizing Lemma, we have to prove only for.algebraically split links. that L is an algebraically



L_- $\tilde{L}_-$

FIGURE 10. Add full twist to L_-.

split link since $\det B_{L\sqcup L'} = \det B_L \det B_{L'}$. Let $L = L_1 \cup L_2 \cup \cdots \cup L_\ell$ be an ℓ-component algebraically split framed link. In this case, $\mathrm{lk}(L_i, L_j) = 0$ if $i \neq j$ and so there is only one chord diagram of degree 0 whose coefficient does not vanish. That is $\bigsqcup_{i=1}^{\ell} \Theta$ and the coefficient is

$$\prod_{i=1}^{\ell} \frac{w(L_i)}{2}.$$

Therefore,

$$\Lambda_{1,0}(L) = 2^{\ell-\sigma_+(L)-\sigma_-(L)} \left(\frac{1}{2}\right)^{-\sigma_+(L)} \left(-\frac{1}{2}\right)^{-\sigma_-(L)} \prod_{i=1}^{\ell} \frac{w(L_i)}{2} = |\prod_{i=1}^{\ell} w(L_i)| = |\det B_L|,$$

since $\sigma_+(L)$ and $\sigma_-(L)$ are equal to the number of components of L with positive writhes and negative writhes respectively. This implies Theorem 2 (1) since $|\det B_L|$ is equal to the order of the first homology group of M_L if $b_1(M_L) = 0$, and is equal to 0 if $b_1(M_L) > 0$. $\square$

5. SKEIN RELATION FOR $\Lambda_{1,1}$

We make a skein relation for $\Lambda_{1,1}(L)$ for an algebraically split link L and then compare it with Lescop's formula. Let L_+ and L_- be two links which are identical except in a small ball B where $L_+ \cap B$ is a positive crossing and $L_- \cap B$ is a negative crossing.

We first consider the case that L_+ and L_- are knots. Let $\tilde{L}_-$ be the knot obtained from L_- by adding a positive full twist as in figure 10. Note that the writhes of L_+ and $\tilde{L}_-$ are equal. We separate $\Lambda_{1,1}(L_+)$ and $\Lambda_{1,1}(\tilde{L}_-)$ into the q-tangles corresponding to the crossing points in B and the added full twist, and the contribution from the other parts. Let $P_\pm$ be the integral from the crossing, T be that from the full twist, and Q be that from the other part. Then, P_+, P_- are given in [17] and

$$P_+ = \exp\left(\frac{\vdash\!\!\dashv}{2}\right) = |\ \ | + \frac{\vdash\!\!\dashv}{2} + \frac{\models\!\!\dashv}{8} + \cdots, \qquad P_- = \exp\left(-\frac{\vdash\!\!\dashv}{2}\right) = |\ \ | - \frac{\vdash\!\!\dashv}{2} + \frac{\models\!\!\dashv}{8} + \cdots.$$

The contribution of T is given by the connected sum of $\bigcirc + \Theta + \Theta_2/2 + \cdots$ to the integral of L_-. Let $P_\pm^{(k)}$, $T^{(k)}$ and $Q^{(k)}$ denote parts consisting of terms with chord diagrams with

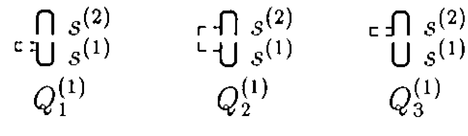


FIGURE 11. Chord diagrams in $Q^{(1)}$, where $s^{(i)}$ denote the sring corresponding to $K^{(i)}$.

k chords. Let $\Lambda'_{1,0}$ and $\Lambda'_{1,1}$ be coefficients of Λ'_1 with respect to g_0 and g_1 in Section 1, i.e., $\Lambda'_1(L) = \Lambda'_{1,0}(L)\, g_0 + \Lambda'_{1,1}(L)$. Then

$$\begin{aligned}\Lambda'_{1,1}(L_+) &= P_+^{(0)} Q^{(2)} + P_+^{(1)} Q^{(1)} + P_+^{(2)} Q^{(0)},\\ \Lambda'_{1,1}(L_-) &= P_-^{(0)} Q^{(2)} + P_-^{(1)} Q^{(1)} + P_-^{(2)} Q^{(0)}.\end{aligned}$$

Therefore, by the above formulas for $P_\pm$, we have

$$\begin{aligned}&\Lambda'_{1,1}(L_+) - \Lambda'_{1,1}(\tilde{L}_-)\\ &\quad= (P_+^{(0)} - P_-^{(0)})\, Q^{(2)} + ((P_+^{(1)} - P_-^{(1)})\, Q^{(1)} - (P_-^{(0)}\, Q^{(1)})\#\Theta +\\ &\qquad\qquad ((P_+^{(2)} - P_-^{(2)})\, Q^{(0)} - (P_-^{(1)}\, Q^{(0)})\#\Theta - (P_-^{(0)}\, Q^{(0)})\#\Theta_2/2.\end{aligned}$$

Since $(P_-^{(1)}\, Q^{(0)})\#\Theta = -\Theta_2/2$ and $(P_-^{(0)}\, Q^{(0)})\#\Theta_2/2 = \Theta_2/2$, we have

$$\begin{aligned}&\Lambda'_{1,1}(L_+) - \Lambda'_{1,1}(\tilde{L}_-)\\ &\quad= (P_+^{(1)} - P_-^{(1)})\, Q^{(1)} - (P_-^{(0)}\, Q^{(1)})\#\Theta\\ &\quad= \vdash\!\!-\!\!\dashv\ \cup Q^{(1)} - (\ |\ \ |\ \cup Q^{(1)})\#\Theta.\end{aligned}$$

We compute $Q^{(1)}$ exactly. Let $K^{(1)}$, $K^{(2)}$ be the two components obtained from $L_\pm$ by smoothing in B. There are three kinds of chord diagrams in $Q^{(1)}$ given in Figure 11. Let α_1, α_2, α_3 be the integrals for configurations corresponding to $Q_1^{(1)}$, $Q_2^{(1)}$, $Q_3^{(1)}$. Then $\alpha_1 = w(K^{(1)})/2$, $\alpha_2 = \mathrm{lk}(k^{(1)}, K^{(2)})$, and $\alpha_3 = w(K^{(2)})/2$. Hence

$$\begin{aligned}\Lambda'_{1,1}(L_+) - \Lambda'_{1,1}(\tilde{L}_-) =& \frac{w(K^{(1)}) + w(K^{(2)})}{2}\,\Theta_2 + \mathrm{lk}(K^{(1)}, K^{(2)})\ \bigoplus\\ &- \frac{w(K^{(1)}) + 2\,\mathrm{lk}(K^{(1)}, K^{(2)}) + w(K^{(2)})}{2}\,\Theta_2\\ =& -3\,\mathrm{lk}(K^{(1)}, K^{(2)})\,\Theta_2.\end{aligned}$$

Here we use the relation $\bigoplus = -2\,\Theta_2$. So we get

$$\Lambda'_{1,1}(L_+) - \Lambda'_{1,1}(\tilde{L}_-) = -3\,\mathrm{lk}(K^{(1)}, K^{(2)}). \qquad \text{(skein I')}$$

For a knot L, we have

$$\Lambda_{1,1}(L) = 2\,\mathrm{sign}(w(L)) \left(\Lambda'_{1,1}(L) - \frac{3}{4}\,\mathrm{sign}(w(L))\,\Lambda'_{1,0}(L)\right),$$

where $\operatorname{sign}(x) = 1$ if $x \geq 0$ and $\operatorname{sign}(x) = -1$ if $x < 0$. Since $w(L_+) = w(\tilde{L}_-)$ and $\Lambda_{1,0}(L) = |w(L)|$ for any knot L, we have

$$\Lambda_{1,1}(L_+) - \Lambda_{1,1}(L_-) = -6\operatorname{sign}(w(L_+))\operatorname{lk}(K^{(1)}, K^{(2)}), \qquad \text{(skein I)}$$

Now we consider a skein relation at a crossing point with strings of different components. Let $L_{+-} = L_{+-}^{(1)} \cup L_{+-}^{(2)} \cup L^{(3)} \cup \cdots \cup L^{(\ell)}$ be an ℓ-component algebraically split link with a positive crossing and a negative crossing of $L_{+-}^{(1)}$ and $L_{+-}^{(2)}$ in small balls B_1, B_2 respectively, $L_{--} = L_{--}^{(1)} \cup L_{--}^{(2)} \cup L^{(3)} \cup \cdots \cup L^{(\ell)}$ be a link obtained from L_{+-} by a crossing change in B_1, and $L_{-+} = L_{-+}^{(1)} \cup L_{-+}^{(2)} \cup L^{(3)} \cup \cdots \cup L^{(\ell)}$ be a link obtained from L_{--} by a crossing change in B_2. We compute the difference $\Lambda'_{1,1}(L_{+-}) - \Lambda'_{1,1}(L_{-+})$. Note that L_{-+} is an algebraically split link since so is L_{+-}.

We first compute $\Lambda'_{1,1}(L_{+-}) - \Lambda'_{1,1}(L_{--})$. Let K_1 be the knot obtained from $L_{+-}^{(1)}$ and $L_{+-}^{(2)}$ by smoothing in B_1, and $L_1 = K_1 \cup L^{(3)} \cup \cdots \cup L^{(\ell)}$. Let P_+, P_-, P_1 denote the contribution to the integral from the q-tangles corresponding to parts of L_{+-}, L_{--}, L_1 in B_1, and Q be that form other part. Note that Q is the same one for L_{+-}, L_{--}, L_1. We represents the chord diagrams in Q as in the figure 12. We know in [17] that

$$P_+ = \exp(\frac{\vdash\!\!-\!\!\dashv}{2}), \qquad P_- = \exp(-\frac{\vdash\!\!-\!\!\dashv}{2}), \qquad P_1 = \times.$$

Let $P_\pm^{(k)}$, $P_1^{(k)}$ and $Q^{(k)}$ denote parts consisting of terms with chord diagrams with k chords. Then, as before, we have

$$\Lambda'_{1,1}(L_{+-}) - \Lambda'_{1,1}(L_{--}) = \vdash\!\!-\!\!\dashv \cup Q^{(\ell)}, \qquad \Lambda'_{1,1}(L_1) = \times \cup Q^{(\ell)}.$$

The chord diagrams nontrivially contributing to Q^ℓ are listed in Figure 12. The integral corresponding to a middle chord of $-\!\circ\!-\!\circ\!-$ is given by the linking number of the two components containing the end points of the chord. We assumed that L_{+-} is an algebraically split link, and hence the integral corresponding to a chord diagram containing a part $-\!\circ\!-\!\circ\!-$ vanishes. By using this, we list up all chord diagrams of $Q^{(\ell)}$ which do not vanish in Figure 13. Let $Q_1^{(i,j)}, \cdots, Q_{33}$ be the above diagrams. Let $E_1^{(i,j)}, \cdots,$ E_{33} be diagrams obtained by inserting $\vdash\!\!-\!\!\dashv$ to $Q_1, \cdots, Q_{33}$, and let $F_1^{(i,j)}, \cdots, F_{33}$ be diagrams obtained by inserting $\times$ to $Q_1, \cdots, Q_{33}$. Then E_k (or $E_k^{(i)}$, $E_k^{(i,j)}$) $= e_k$ (or $e_k^{(i)}, e_k^{(i,j)}$) $\left(\sqcup^{\ell-1}\Theta\right) \sqcup \Theta_2$ and F_k (or $F_k^{(i)}$, $F_k^{(i,j)}$) $= f_k$ (or $f_k^{(i)}$, $f_k^{(i,j)}$) $\left(\sqcup^{\ell-1}\Theta\right) \sqcup \Theta_2$ in $\bar{\mathcal{A}}_1^{(\ell-1)}$ for $k = 1, 2, \cdots, 29$. By using relations in $\bar{\mathcal{A}}_1^{(\ell)}$, they are given as follows. $e_1^{(i,j)} = 3/16$, $e_2^{(i,j)} = -3/16$, $e_3^{(i)} = 1/4$, $e_4^{(i)} = -1/2$, $e_5^{(i)} = 1/4$, $e_6^{(i)} = -3/8$, $e_7^{(i)} = 3/8$, $e_8^{(i)} = -3/8$, $e_9^{(i)} = 3/8$, $e_{10}^{(i)} = 1/4$, $e_{11}^{(i)} = -1/2$, $e_{12}^{(i)} = 1/4$, $e_{13} = -1/2$, $e_{14} = 1$, $e_{15} = -1/2$, $e_{16} = 3/4$,

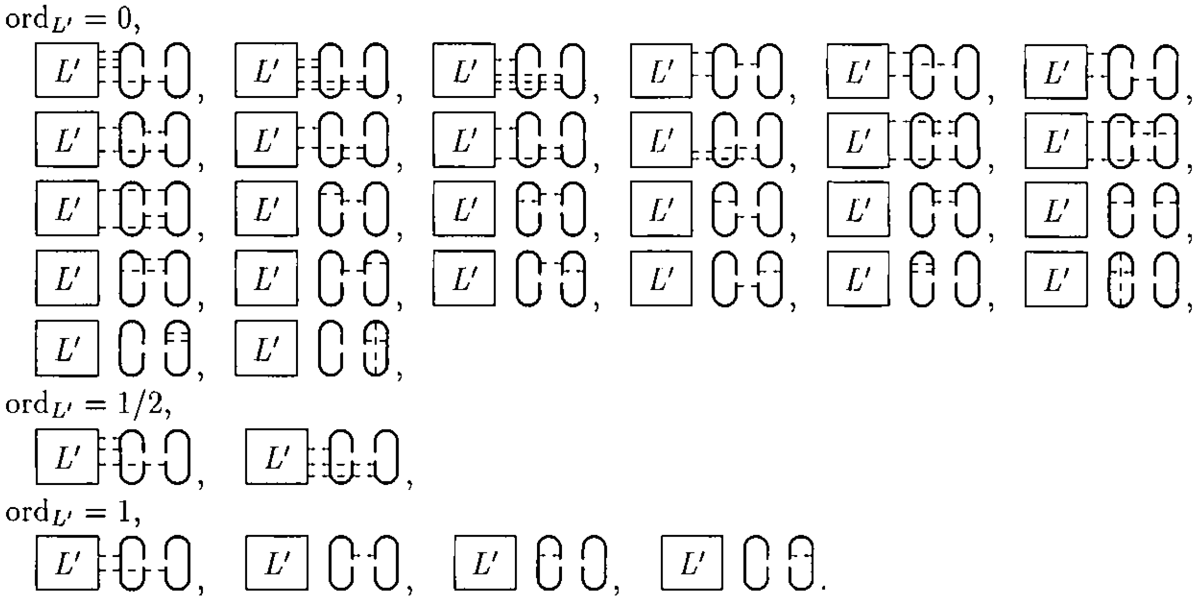


FIGURE 12. Non-trivial diagrams

$e_{17} = 0$, $e_{18} = -3/4$, $e_{19} = -1/2$, $e_{20} = 1$, $e_{21} = -1/2$, $e_{22} = 0$, $e_{23} = 0$, $e_{24} = 0$, $e_{25} = 0$, $e_{26}^{(i)} = -3/8$, $e_{27}^{(i)} = 3/8$, $e_{28}^{(i)} = 3/8$, $e_{29}^{(i)} = -3/8$, $f_1^{(i,j)} = -1/2$, $f_2^{(i,j)} = 1/4$, $f_3^{(i)} = -1/2$, $f_4^{(i)} = 1$, $f_5^{(i)} = -1/2$, $f_6^{(i)} = 1$, $f_7^{(i)} = -1/2$, $f_8^{(i)} = 1$, $f_9^{(i)} = -1/2$, $f_{10}^{(i)} = -1/2$, $f_{11}^{(i)} = 1$, $f_{12}^{(i)} = -1/2$, $f_{13} = 1$, $f_{14} = -2$, $f_{15} = 1$, $f_{16} = -2$, $f_{17} = 1$, $f_{18} = 1$, $f_{19} = 1$, $f_{20} = -2$, $f_{21} = 1$, $f_{22} = 1$, $f_{23} = -2$, $f_{24} = 1$, $f_{25} = -2$, $f_{26}^{(i)} = 3/4$, $f_{27}^{(i)} = -3/4$, $f_{28}^{(i)} = -3/4$, $f_{29}^{(i)} = 3/4$, Similarly, we have $E_{30}^{(i)} = 1/4\, D_1^{(\ell-3)} \sqcup (\sqcup^3\Theta)$, $E_{31} = -1/2\, D_1^{(\ell-2)} \sqcup (\sqcup^2\Theta)$, $E_{32} = 0$, $E_{33} = 0$, $F_{30}^{(i)} = -1/2\, D_1^{(\ell-3)} \sqcup (\sqcup^3\Theta)$, $F_{31} = D_1^{(\ell-2)} \sqcup (\sqcup^2\Theta)$, $F_{32} = D_1^{(\ell-2)} \sqcup (\sqcup^2\Theta)$, $F_{33} = D_1^{(\ell-2)} \sqcup (\sqcup^2\Theta)$. Let α_k (or $\alpha_k^{(i)}$, $\alpha_k^{(i,j)}$) denote the integral corresponding to Q_k (or $Q_k^{(i)}$, $Q_k^{(i,j)}$). Then we have

$$\begin{aligned}
&\Lambda'_{1,1}(L_{+-}) - \Lambda'_{1,1}(L_{--}) + \frac{1}{2}\Lambda'_{1,1}(L_1) = \\
&\quad -\sum_{3\leq i<j\leq \ell} (\alpha_1^{(i,j)} + \alpha_2^{(i,j)})/16 + \sum_{i=3}^{\ell} (\alpha_6^{(i)} + \alpha_7^{(i)} + \alpha_8^{(i)} + \alpha_9^{(i)})/8 - \alpha_{16}/4 + \alpha_{17}/2 - \alpha_{18}/4 \\
&\qquad + \alpha_{22}/2 - \alpha_{23} + \alpha_{24}/2 - \alpha_{25} + w(L_{+-}^{(1)})\, \Lambda'_{1,1}(L')/4 + w(L_{+-}^{(2)})\, \Lambda'_{1,1}(L')/4,
\end{aligned}$$

where $L' = L^{(3)} \cup L^{(4)} \cup \cdots \cup L^{(\ell)}$. Here, the last two terms correspond to integrals for F_{31} and F_{32}. We get $\Lambda'_{1,1}(L')$ by summing up integrals of all the possible configurations for D_1. We know that

$$\alpha_1^{(i,j)} + \alpha_2^{(i,j)} = \mathrm{lk}(L_{+-}^{(1)}, L^{(i)})\, \mathrm{lk}(L_{+-}^{(1)}, L^{(j)})\, \mathrm{lk}(L_{+-}^{(2)}, L^{(i)})\, \mathrm{lk}(L_{+-}^{(2)}, L^{(j)}) \prod_{\substack{k=3 \\ k\neq i,j}}^{\ell} \frac{w(L^{(k)})}{2}.$$

It is a product of linking numbers and so it is equal to 0 because L_{+-} is an algebraically split link. By the same reason, we have

$$\alpha_6^{(i)} + \alpha_7^{(i)} + \alpha_8^{(i)} + \alpha_9^{(i)} = \mathrm{lk}(L_{+-}^{(1)}, L^{(i)})\,\mathrm{lk}(L_{+-}^{(2)}, L^{(i)})\,\mathrm{lk}(L_{+-}^{(1)}, L_{+-}^{(2)}) \prod_{\substack{k=3 \\ k\neq i}}^{\ell} \frac{w(L^{(k)})}{2} = 0,$$

and

$$\alpha_{16} + \alpha_{18} = \frac{\mathrm{lk}(L_{+-}^{(1)}, L_{+-}^{(2)})^2}{2} \prod_{k=3}^{\ell} \frac{w(L^{(k)})}{2} = 0.$$

We also know that $\alpha_{17} = \left(\prod_{j=3}^{\ell} \frac{w(L^{(j)})}{2}\right) w(L_{+-}^{(1)})\, w(L_{+-}^{(2)})/4$. Moreover, $\alpha_{22} - 2\,\alpha_{23} = 5\,\Lambda'_{1,1}(L_{+-}^{(1)}) \left(\prod_{j=3}^{\ell} \frac{w(L^{(j)})}{2}\right)$ and $\alpha_{24} - 2\,\alpha_{25} = 5\,\Lambda'_{1,1}(L_{+-}^{(2)}) \left(\prod_{j=3}^{\ell} \frac{w(L^{(j)})}{2}\right)$. Hence, we have

$$\begin{aligned}\Lambda'_{1,1}(L_{+-}) - \Lambda'_{1,1}(L_{--}) + \frac{1}{2}\,\Lambda'_{1,1}(L_1) &= \frac{\Lambda'_{1,1}(L')}{4}\left(w(L_{+-}^{(1)}) + w(L_{+-}^{(2)})\right) \\ &+ \left(\prod_{j=3}^{\ell} \frac{w(L^{(j)})}{2}\right) \frac{w(L_{+-}^{(1)})w(L_{+-}^{(2)}) + 10\,\Lambda'_{1,1}(L_{+-}^{(1)}) + 10\,\Lambda'_{1,1}(L_{+-}^{(2)})}{4}.\end{aligned}$$

Similarly, let L_2 be a link obtained from L_{-+} by smoothing at the crossing in the ball B_2. Then we have

$$\begin{aligned}\Lambda'_{1,1}(L_{-+}) - \Lambda'_{1,1}(L_{--}) + \frac{1}{2}\,\Lambda'_{1,1}(L_2) &= \frac{\Lambda'_{1,1}(L')}{4}\left(w(L_{-+}^{(1)}) + w(L_{-+}^{(2)})\right) \\ &+ \left(\prod_{j=3}^{\ell} \frac{w(L^{(j)})}{2}\right) \frac{w(L_{-+}^{(1)})\,w(L_{-+}^{(2)}) + 10\,\Lambda'_{1,1}(L_{-+}^{(1)}) + 10\,\Lambda'_{1,1}(L_{-+}^{(2)})}{4}.\end{aligned}$$

Since $L_{+-}^{(i)}$ and $L_{-+}^{(i)}$ are of the same knot type for $i = 1$ and 2, we have

$$\Lambda'_{1,1}(L_{+-}) - \Lambda'_{1,1}(L_{-+}) = -\frac{1}{2}\left(\Lambda'_{1,1}(L_1) - \Lambda'_{1,1}(L_2)\right). \qquad \text{(skein II}'\text{)}$$

By using the relation between $\Lambda_{1,1}$ and $\Lambda'_{1,1}$, we get

$$\Lambda_{1,1}(L_{+-}) - \Lambda_{1,1}(L_{-+}) = -\mathrm{sign}(w(L_{+-}^{(1)})\,w(L_{+-}^{(2)})\,w(L_1))\,\left(\Lambda_{1,1}(L_1) - \Lambda_{1,1}(L_2)\right). \text{ (skein II)}$$

6. Coincidence with the Casson-Walker invariant

We show the equivalence of $\Lambda_{1,1}$ and $\tilde{\lambda}$ by using the diagonalizing Lemma, the Dehn surgery formula for $\tilde{\lambda}$, and the formulas (skein I) and (skein II) in the previous section. Let $\Lambda'_{1,0}$ and $\Lambda'_{1,1}$ be the degree 0 and 1 part of Λ'_1 respectively. For two links L_1 and L_2, let $\mathrm{L}_1 \sqcup L_2$ denote the split union of these two links. Then we have

$$\Lambda_{1,1}(L_1 \sqcup L_2) = \Lambda_{1,1}(L_1)\,\Lambda_{1,0}(L_2) + \Lambda_{1,0}(L_1)\,\Lambda_{1,1}(L_2). \tag{6.1}$$

Since $\tilde{\lambda}$ satisfies similar formula with respect to a split union of framed links, we get the formula in Theorem 2 (2) for L_1 if it is true for L_2 and $L_1 \sqcup L_2$. Hence, it is good enough to prove Theorem 2 (2) only for algebraically split links according to the diagonalizing Lemma.

For an ℓ-component algebraically split link $L = L_1 \cup L_2 \cup \cdots \cup L_\ell$, a Dehn surgery formula for the Casson invariant is obtained in [10, 20, 21], which is given by

$$\tilde{\lambda}(M_L) = -\sum_{i=1}^{\ell} \mathrm{sign}(w(L_i)) \left(\prod_{j\neq i} |w(L_j)|\right) \frac{(|w(L_i)|-1)(|w(L_i)|-2)}{12} +$$
$$\sum_{I\subset\{1,2,\cdots,\ell\}} 2\left(\prod_{i\in I}\mathrm{sign}(w(L_i))\right)\left(\prod_{i\notin I}|w(L_i)|\right) a_{|I|+1}(\bigcup_{i\in I} L_i). \qquad \text{(Dehn surgery formula)}$$

Here $|I|$ denotes the number of elements in I, and $a_k(L)$ is the coefficient of t^k of the Conway polynomial $\nabla_L(t)$ [4], which is defined by the following skein relation:

$$\nabla_{L_+}(t) - \nabla_{L_-}(t) = -t\,\nabla_{L_0}(t),$$

where L_+, L_-, L_0 are links identical except within a ball at which they are a positive crossing, negative crossing and their smoothing as usual. Note that there is a minus at the right hand side of the relation. Recall that $\tilde{\lambda}(M_L) = |H_1(M_L)|\,\lambda(M_L) = \prod_{i=1}^{\ell} |w(L_i)|\,\lambda(M_L)$ if M_L is a homology sphere.

Proof of Theorem 2 (2). For an ℓ-component algebraically split link L with non-degenerate linking matrix, we will prove that $-3\,(\prod_{i=1}^{\ell}|w(L_i)|)\,\lambda(M_L) = \Lambda_{1,1}(L)$. The computation of $\Lambda_{1,1}$ is reduced to those for split links by the relations (skein II). By using (6.1), $\Lambda_{1,1}$ for a split link is determined by $\Lambda_{1,1}$ and $\Lambda_{1,0}$ for each component. Moreover, $\Lambda_{1,1}$ of a knot is reduced to $\Lambda_{1,1}$ of trivial knots with framings by (skein I). Therefore, to prove Theorem 2 (2), it is enough to show the following three things:

(1) For any trivial knot L with a framing, $-3\,\tilde{\lambda}(M_K) = \Lambda_{1,1}(K)$.
(2) For any knot L, $-3\,\tilde{\lambda}(M_L)$ satisfies (skein I).
(3) For any link L, $-3\,\tilde{\lambda}(M_L)$ satisfies (skein II).

We first show (1). Since $Z_f(K) = \bigcirc + (w(K)/2)\,\Theta + (w(K)^2/8)\,\Theta_2 +$ (terms with more than two chords), and $\nu = \bigcirc + (1/24)\,(\Theta_2 - \oplus) +$ (terms with more than two chords), we have $\Lambda_{1,1}(K) = \mathrm{sign}(w(K))\,(w(K)^2 - 3\,|w(K)| + 2)/4$. On the other hand, $\lambda(M_K) = -\mathrm{sign}(w(K))\,(w(K)^2 - 3\,|w(K)| + 2)/12$ by the Dehn surgery formula. Hence (1) is true.

Now we show (2). Let L_+, L_-, $\tilde{L}_-$, $K^{(1)}$ and $K^{(2)}$ be knots as in the proof of (skein II). Since $w(L_+) = w(\tilde{L}_-)$, (2) comes from the relation $a_2(L_+) - a_2(\tilde{L}_-) = a_2(L_+) - a_2(L_-) =$

$-a_1(K^{(1)} \cup K^{(2)}) = \mathrm{lk}(K^{(1)}, K^{(2)})$ ((4.7) in [7] and Theorem 2 in [8]).

It remains to prove (3). We use the notations in (skein II). We also know that $w(L_1^{(1)}) = w(L_2^{(1)})$, $w(L_{+-}^{(1)}) = w(L_{-+}^{(1)})$ and $w(L_{+-}^{(2)}) = w(L_{-+}^{(2)})$. Then, we have

$$\tilde{\lambda}(M_{L_{+-}}) - \tilde{\lambda}(M_{L_{-+}}) =$$
$$2 \sum_{J \subset \{3,4,\cdots,\ell\}} \mathrm{sign}(w(L_\pm^{(1)})\, w(L_\pm^{(2)})) \left(\prod_{j\in J} |w(L^{(j)})|\right) \left(\prod_{i\in\{3,4,\cdots,\ell\}\setminus J} |w(L^{(i)})|\right) \times$$
$$\left(a_{|J|+3}(L_{+-}^{(1)} \cup L_{+-}^{(2)} \cup (\cup_{j\in J} L^{(j)})) - a_{|J|+3}(L_{-+}^{(1)} \cup L_{--}^{(2)} \cup (\cup_{j\in J} L^{(j)}))\right),$$

since the other terms of $\tilde{\lambda}(M_{L_{+-}})$ are identical to the correspoinding terms of $\tilde{\lambda}(M_{L_{-+}})$. Similarly, we have

$$\tilde{\lambda}(M_{L_1}) - \tilde{\lambda}(M_{L_2}) = 2 \sum_{J \subset \{3,4,\cdots,\ell\}} \mathrm{sign}(w(L_1^{(1)}))) \left(\prod_{j\in J} |w(L^{(j)})|\right) \left(\prod_{i\in\{3,4,\cdots,\ell\}\setminus J} |w(L^{(i)})|\right) \times$$
$$\left(a_{|J|+2}(L_1^{(1)} \cup (\cup_{j\in J} L^{(j)})) - a_{|J|+2}(L_2^{(1)} \cup (\cup_{j\in J} L^{(j)}))\right),$$

where $L_1 = L_1^{(1)} \cup L^{(3)} \cup \cdots \cup L^{(\ell)}$ and $L_2 = L_2^{(1)} \cup L^{(3)} \cup \cdots \cup L^{(\ell)}$. From the definition of ∇, we have

$$a_{|J|+3}(L_{+-}^{(1)} \cup L_{+-}^{(2)} \cup (\cup_{j\in J} L^{(j)})) - a_{|J|+3}(L_{--}^{(1)} \cup L_{--}^{(2)} \cup (\cup_{j\in J} L^{(j)})) = -a_{|J|+2}(L_1^{(1)} \cup (\cup_{j\in J} L^{(j)})),$$
$$a_{|J|+3}(L_{-+}^{(1)} \cup L_{-+}^{(2)} \cup (\cup_{j\in J} L^{(j)})) - a_{|J|+3}(L_{--}^{(1)} \cup L_{--}^{(2)} \cup (\cup_{j\in J} L^{(j)})) = -a_{|J|+2}(L_2^{(1)} \cup (\cup_{j\in J} L^{(j)})).$$

Hence, we have (3), completing the proof. □

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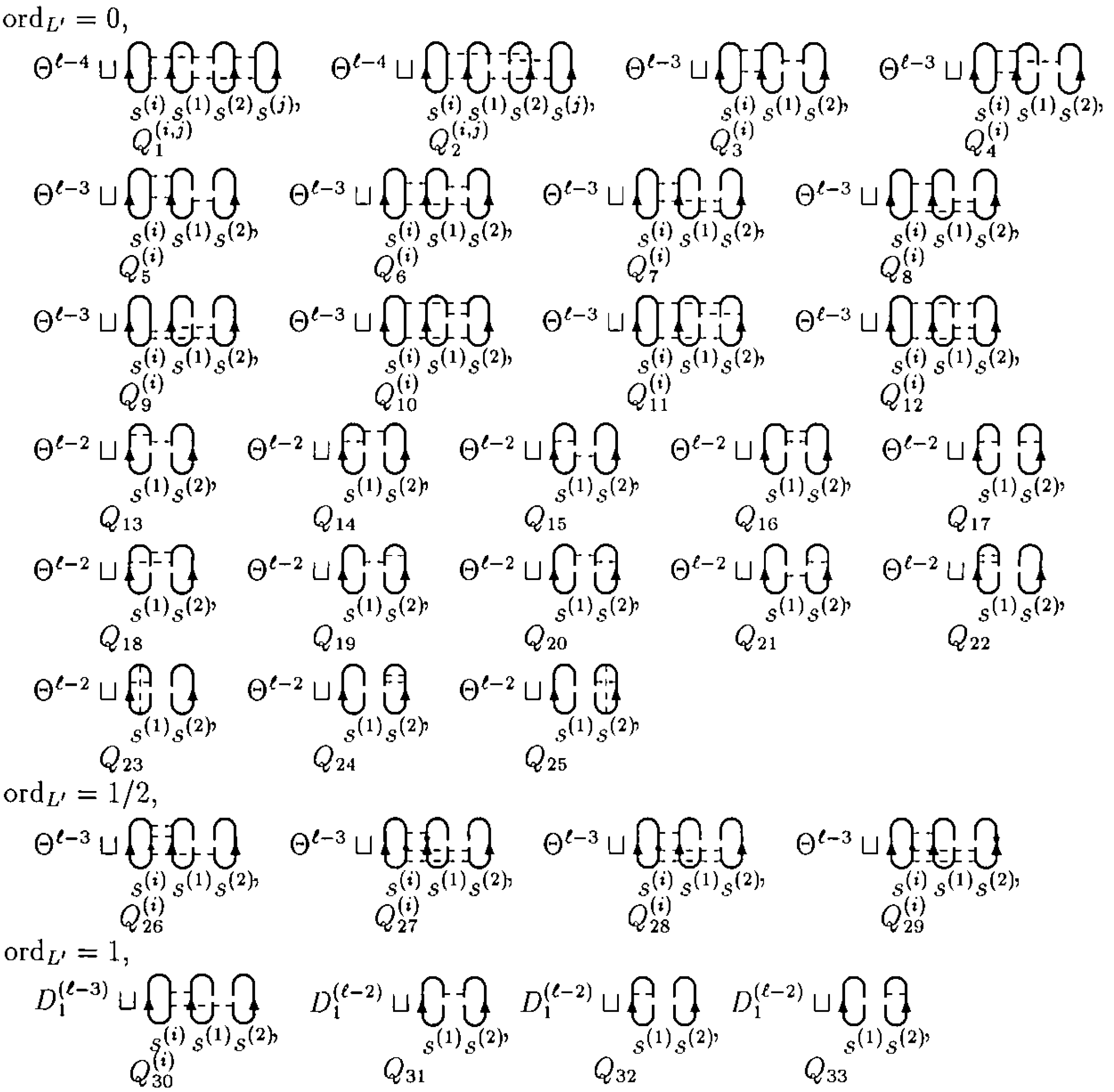

where $s^{(i)}$ denote the string corresponding to the i-th component of L_{+-}, Θ^j is a split union of j Θ's and $D_1^{(k)}$ is a non-vanishing chord diagram of order 1 with k components.

FIGURE 13. Non-vanishing chord diagrams in Q^ℓ for an algebraically split link