The Complete Classification Of Compactifications Of $\mathbb{C}^3$ Which Are Projective Manifolds With The Second Betti Number One

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Dedicated to Professor Dr. Friedrich Hirzebruch on his sixty-fifth birthday

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§0. Introduction.

Let (X, Y) be a smooth projective compactification of $\mathbb{C}^3$, namely, X is a smooth projective threefold and Y an analytic subvariety of X such that $X - Y$ is biholomorphic to $\mathbb{C}^3$. By the theorem of Hartogs, Y is of pure dimension two, namely, Y is a divisor on X.

Two smooth compactifications (X, Y) and (X', Y') are said to be isomorphic ,we write simply as $(X, Y) \cong (X', Y')$, if there exists a biholomorphic mapping $\varphi : X \longrightarrow X'$ such that $\varphi(Y) = Y'$.

We shall assume that the second Betti number $b_2(X) = 1$. Then Y is an irreducible ample divisor on X and $Pic\, X \cong \mathbf{Z} \cdot \mathcal{O}_X(Y)$, in particular, the canonical divisor K_X can be written as $K_X \sim -rY$ $(r \in \mathbf{Z}, 0 < r \leq 4)$ (cf.[**B-M**]). Thus X is a Fano threefold of the first kind (cf.[**Is**$_1$]). The integer r is called the "index" of X. Then we have the two cases:

(i) Y is normal, or
(ii) Y is non-normal irreducible.

In the case where Y is normal, we have proved the following

Theorem A ([Fu$_1$], [Fu$_2$], [F-N$_1$], [F-N$_2$], [P-S]). *Let (X, Y) be a smooth projective compactification of $\mathbb{C}^3$. Assume that Y is normal. Then we have the second Betti number $b_2(X) = 1$ and the index $r \geq 2$. Moreover,*

(1) $r = 4 \Longrightarrow (X, Y) \cong (\mathbb{P}^3, \mathbb{P}^2)$,

(2) $r = 3 \Longrightarrow (X, Y) \cong (\mathbb{Q}^3, \mathbb{Q}^2_0)$,

(3) $r = 2 \Longrightarrow (X, Y) \cong (V_5, H^0_5)$.

In particular, such a (X, Y) exists uniquely up to isomorphism, where

- $\mathbb{Q}^3$ *: a smooth hyperquardric in $\mathbb{P}^4$,*
- $\mathbb{Q}^2_0$ *: is a quardric cone in $\mathbb{P}^3$,*
- V_5 *: a linear section $Gr(2,5) \cap \mathbb{P}^6$ of the Grassmann variety $Gr(2,5) \hookrightarrow \mathbb{P}^9$ of lines in $\mathbb{P}^4$ by three hyperplanes in $\mathbb{P}^9$, which is the Fano threefold of the index two and degree 5 in $\mathbb{P}^6$,*
- H^0_5 *: a normal hyperplane section of V_5 with exactly one rational double point of A_4-type, which is a degenerated del Pezzo surface of degree 5 in $\mathbb{P}^5$.*

In the case where Y is non-normal irreducible, we have also proved the following

Theorem B ([P-S], [F-N$_1$]). *Let (X, Y) be a smooth projective compactification of $\mathbb{C}^3$. Assume that Y is non-normal irreducible. Then we have the index $r \leq 2$. Moreover, if the index $r = 2$, then $(X, Y) \cong (V_5, H^\infty_5)$, where H^∞_5 is a non-normal hyperplane section of V_5 whose singular locus is a line $\Sigma \cong \mathbb{P}^1$ in V_5 with the normal bundle $N_{\Sigma|X} \cong \mathcal{O}_\Sigma(-1) \oplus \mathcal{O}_\Sigma(1)$. In particular, H^∞_5 is a ruled surface swept out by lines on V_5 intersecting with the line Σ. Moreover such a (X, Y) exists uniquely up to isomorphism.*

By Theorem A and Theorem B, we have only to consider the case of $r = 1$. In this case, one sees X is a Fano threefold of the index $r = 1$ with $Pic X \cong \mathbf{Z} \cdot \mathcal{O}_X(-K_X)$. Here we call the number $g = \frac{1}{2}(-K_X)^3 + 1$ the "genus" of X (see [$\mathbf{Is}_1$]).

Recently, the author constructed two examples of the compactification (X, Y) of $\mathbb{C}^3$ with a non-normal irreducible divisor Y from the Mukai-Umemura's example [**M-U**] of the Fano threefold $U_{22} \hookrightarrow \mathbb{P}^{13}$,which is a special one among the Fano threefolds of the index $r = 1$ and the genus $g = 12$(see also [**M**], [**Pr**]), namely,

Theorem C ([$\mathbf{Fu}_2$], [$\mathbf{Fu}_3$], [$\mathbf{Fu}_4$], [**M**]). *Let U_{22} be the Mukai-Umemura's example of the Fano threefold. Then there exist non-normal hyperplane sections H_{22}^0 and H_{22}^∞ of U_{22} such that $U_{22} - H_{22}^0 \cong \mathbb{C}^3 \cong U_{22} - H_{22}^\infty$. The singular locus of H_{22}^0 (resp. H_{22}^∞) is the line ℓ in U_{22} with the normal bundle $N_{\ell|U_{22}} \cong \mathcal{O}_\ell(-2) \oplus \mathcal{O}_\ell(1)$, and $mult_\ell H_{22}^0 = 2$ (resp. $mult_\ell H_{22}^\infty = 3$). In particular, H_{22}^∞ is a ruled surface swept out by the conics which intersect the line ℓ.*

Remark 1. Mukai [**M**] and Prokhrov [**Pr**] proved that there is a 4-dimensional family (V_{22}^t, H_{22}^t) of compactifications of $\mathbb{C}^3$ containing (U_{22}, H_{22}^∞) such that $(V_{22}^t, H_{22}^t) \not\cong (V_{22}^s, H_{22}^s)$ if $t \neq s$, where V_{22}^t is a Fano threefold of the index $r = 1$ and the genus $g = 12$, which has the degree 22 in $\mathbb{P}^{13}$ by the anti-canonical embedding, and H_{22}^t is the non-normal hyperplane section of V_{22}^t whose singular locus is the line ℓ_t with the normal bundle $N_{\ell_t|V_{22}^t} \cong \mathcal{O}_{\ell_t}(-2) \oplus \mathcal{O}_{\ell_t}(1)$. In particular, H_{22}^t is a ruled surface swept out by conics intersecting the line ℓ_t. Therefore one can see that the compactification (X, Y) is not unique up to isomorphism in the case of $r = 1$.

On the other hand, Peternell asserts the following:

Theorem D ([**P**], [$\mathbf{P\text{-}S}_2$]). *Let (X, Y) be a smooth projective compactification of $\mathbb{C}^3$ with $b_2(X) = 1$. Assume that Y is non-normal and the index $r = 1$. Then,*

(I) X is a Fano threefold of the index $r = 1$ and the genus $g = 12$.

(II) Let E be the non-normal locus of Y equipped with the complex structure given by the conductor ideal sheaf. Let $\overline{Y}$ be the normalization of Y and let $\overline{E}$ be the preimage of E. Then

(1) E and $\overline{E}$ are reduced,

(2) Y is weakly normal, and

(3) E is a smooth rational curve and $\overline{E}$ consists of two smooth rational curves meeting at one point of order 2.

Unfortunately, Theorem D-(II) is not true. Indeed, the compactification (U_{22}, H_{22}^∞) in Theorem C does not satisfy the assertions (II)-(1) and (II)-(3) in Theorem D at all. In this example, E and $\overline{E}$ are both "non-reduced", and $\overline{E}$ consists of "three" smooth rational curves meeting at one point (see [$\mathbf{Fu}_3$]). Moreover, Theorem D-(II) plays a key role in the proof of Theorem D-(I) (for example, see the proof of Proposition (3.8) in [**P**]). Nevertheless, Theorem D-(I) is still true as we will prove in §2.

Our main result is the following:

Main Theorem. *Let (X, Y) be a smooth projective compactification of $\mathbb{C}^3$ with the second Betti number $b_2(X) = 1$. Assume that the index $r = 1$. Then*

(1) $(X, Y) \cong (V_{22}, H_{22}^{\infty})$ or (V_{22}, H_{22}^{0}), *where V_{22} is a Fano threefold of the index $r = 1$ with the genus $g = 12$, degree 22 in $\mathbb{P}^{13}$ by the anti-canonical embedding, and H_{22}^{∞} (resp. H_{22}^{0}) is a non-normal hyperplane section of V_{22},*

(2) *Let E be the non-normal locus of H_{22}^{∞} (or H_{22}^{0}) equipped with the complex structure given by the conductor ideal sheaf. Then $Z := E_{red}$ is a line on V_{22} with the normal bundle $N_{Z|V_{22}} \cong \mathcal{O}_Z(-2) \oplus \mathcal{O}_Z(1)$,*

(3) *$mult_E H_{22}^{\infty} = 3$ and $mult_Z H_{22}^{0} = 2$, in particular, H_{22}^{∞} is a ruled surface swept out by the conics intersecting with the line Z.*

Combining Theorem A and Theorem B with the main theorem above, we have finally

Theorem (cf. [Problem 27; Hi]). *Let (X, Y) be a smooth projective compactification of $\mathbb{C}^3$ with the second Betti number $b_2(X) = 1$. Then*

$(X, Y) \cong (\mathbb{P}^3, \mathbb{P}^2)$, $(\mathbb{Q}^3, \mathbb{Q}_0^2)$, (V_5, H_5^0), (V_5, H_5^{∞}), (V_{22}, H_{22}^0) *or* $(V_{22}, H_{22}^{\infty})$.

Remark 2. In [$\mathbf{Fu_4}$], it is shown how the compactifications $(V_{22}, H_{22}^{\infty})$ and (V_{22}, H_{22}^{0}) are constructed from the well-known compactification $(\mathbb{P}^3, \mathbb{P}^2)$ of $\mathbb{C}^3$.

This paper consists of three sections. First, in §1, we shall study the general properties of non-normal polarized surfaces of K3-type. Next, in §2, by applying the results obtained in §1, we shall give a new proof of Theorem D-(I). Finally, in §3, we shall give a proof of the Main Theorem.

Notation

- ω_V : dualizing sheaf of V
- $h^i(\mathcal{O}_V) = dim H^i(V, \mathcal{O}_V)$
- E_{red} : reduction of E
- $N_{Z|V}$: normal bundle of Z in V
- $mult_Z Y$: multiplicity of Y at a general point of Z
- $Bs|\mathcal{L}|$: base locus of the linear system $|\mathcal{L}|$ defined by the line bundle $\mathcal{L}$
- $b_i(V) := dim H^i(V; \mathbb{R})$: the i-th Betti number
- $\rho(V)$: Picard number of V
- $\chi(\mathcal{L}) := \sum_i (-1)^i h^i(\mathcal{L})$
- $\sim$: linear equivalence
- $\equiv$: numerical equivalence

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§1. Non-normal polarized surfaces of K3-type.

1. Let S be a non-normal irreducible reduced projective Gorenstein surface over $\mathbb{C}$. Let $\sigma : \overline{S} \longrightarrow S$ be the normalization, and $\mathcal{I} \subset \mathcal{O}_S$ be the conductor of σ defining closed subschemes $E := V_S(\mathcal{I})$ in S and $\overline{E} := V_{\overline{S}}(\mathcal{I})$ in $\overline{S}$. Let $\mu : \widehat{S} \longrightarrow \overline{S}$ be the minimal resolution and $B = \bigcup_{i=1} B_i$ be the exceptional set for μ. We put $\pi := \sigma \circ \mu : \widehat{S} \longrightarrow S$. Then we have the following:

(1.1) Lemma ([pp.165-pp.167 ; Mo]). $(i)\ \omega_{\overline{S}} \cong \sigma^*\omega_S \otimes \mathcal{I}$,

$(ii)\ \omega_{\overline{E}} \cong \sigma^*\omega_S \otimes \mathcal{O}_{\overline{E}}$,

$(iii)\ 0 \longrightarrow \mathcal{O}_S \longrightarrow \sigma_*\mathcal{O}_{\overline{S}} \longrightarrow \omega_S^{-1} \otimes \omega_E \longrightarrow 0$,

$(iv)\ 0 \longrightarrow \sigma_*\omega_{\overline{S}} \longrightarrow \omega_S \longrightarrow \omega_S \otimes \mathcal{O}_E \longrightarrow 0$,

$(v)\ 0 \longrightarrow \omega_{\overline{S}} \longrightarrow \sigma^*\omega_S \longrightarrow \sigma^*\omega_S \otimes \mathcal{O}_{\overline{E}} \longrightarrow 0,$

$(vi)\ 0 \longrightarrow \mathcal{O}_E \longrightarrow \sigma_*\mathcal{O}_{\overline{E}} \longrightarrow \omega_S^{-1} \otimes \omega_E \longrightarrow 0$.

(1.2) Definition. *Let $\mathcal{L}$ be a very ample line bundle on S. The pair $(S, \mathcal{L})$ is called a non-normal polarized surface of K3-type if*

(1) *S is a non-normal irreducible reduced projective Gorenstein surface* ,

(2) $\omega_S \cong \mathcal{O}_S$,

(3) $h^1(\mathcal{O}_S) = 0$, *and*

(4) *$\mathcal{L}$ is very ample on S.*

Applying (1.1), one can easily obtain the following:

(1.3) Lemma (cf. [Proposition 3.3, 3.5 ; P]). *Let $(S, \mathcal{L})$ be a non-normal polarized surface of K3-type. Then,*

$(i)\ \omega_{\overline{S}} \cong \mathcal{I} \iff K_{\overline{S}} \sim -\overline{E}$ *as a Weil divisor* ,

$(ii)\ \omega_{\overline{E}} \cong \mathcal{O}_{\overline{E}}$,

$(iii)\ h^1(\mathcal{O}_E) = 0$, *namely, each irreducible component E_i of E_{red} is a smooth rational curve* ,

$(iv)\ h^1(\mathcal{O}_{\overline{S}}) = h^0(\mathcal{O}_E) - 1$.

(1.4) Corollary. (a) $K_{\widehat{S}} \sim -\widehat{E} - \sum k_i B_i (k_i \in \mathbb{Z}\ , k_i \geq 0)$, *where $\widehat{E}$ is the proper transform of $\overline{E}$ in $\widehat{S}$.*

(b) *S is a rational or a ruled surface.*

Proof. Since $\omega_{\widehat{S}} = \mu^*\omega_{\overline{S}} \otimes \mathcal{O}(-\sum n_i B_i)$ for some $n_i \in \mathbb{Z}$ $(n_i \geq 0)$ and since $\omega_S \cong \mathcal{I}$, we have the assertion (a). By (a), we can easily see that $H^0(\widehat{S}; \mathcal{O}(mK_{\widehat{S}})) = 0$ for $m > 0, m \in \mathbb{Z}$. Thus, from the classification of surfaces, we conclude that $\widehat{S}$ is a rational or a ruled surface. This proves the assertion (b). □

(1.5) Proposition. *Let $(S,\mathcal{L})$ be as in (1.3). Then,*

(a) $H^i(S,\mathcal{L}) = 0$ *for* $i > 0$,

(b) $(\sigma^*\mathcal{L}\cdot\overline{E})_{\overline{S}} = 2(\mathcal{L}\cdot E)_S = 2\delta$, *where* $\delta := (\mathcal{L}\cdot E)_S > 0$, *in particular, if* E *is irreducible and reduced, then* $b_2(\overline{E}) \leq 2$,

(c) *There exists a smooth member* $\overline{C} \in |\sigma^*\mathcal{L}|$ *with the genus* $g(\overline{C}) = \frac{1}{2}d(\mathcal{L}) - \delta + 1$,

(d) $h^0(\sigma^*\mathcal{L}) = h^0(\mathcal{L}) + \delta - h^0(\mathcal{O}_E)$,

(e) $h^0(\mathcal{L}) = \frac{1}{2}d(\mathcal{L}) + 2$, *in particular,* $d(\mathcal{L}) := (\mathcal{L}^2)_S > 0$ *is even.*

(f) $\Delta(\overline{S},\sigma^*\mathcal{L}) = 2 + d(\mathcal{L}) + h^0(\mathcal{O}_E) - h^0(\mathcal{L}) - \delta$.

Proof. (a): Take a general (irreducible) member $C \in |\mathcal{L}|$. Since $H^1(S;\mathcal{O}_S) = 0$, we have $H^1(S;\mathcal{O}(-C)) = 0$, that is, $H^1(S;\mathcal{L}^{-1}) = 0$. Since $\omega_S \cong \mathcal{O}_S$, by the Serre duality theorem, we obtain $H^i(S;\mathcal{L}) \cong H^{2-i}(S;\mathcal{L}^{-1})$. This proves the assertion (a).

(b): In (1.1)-(iii),(v) and (vi), we put $\omega_S \cong \mathcal{O}_S$, then we obtain the following exact sequences:

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \sigma_*\mathcal{O}_{\overline{S}} \longrightarrow \omega_E \longrightarrow 0, \tag{1.5.1}$$

$$0 \longrightarrow \omega_{\overline{S}} \longrightarrow \mathcal{O}_{\overline{S}} \longrightarrow \mathcal{O}_{\overline{E}} \longrightarrow 0. \tag{1.5.2}$$

$$0 \longrightarrow \mathcal{O}_E \longrightarrow \sigma_*\mathcal{O}_{\overline{E}} \longrightarrow \omega_E \longrightarrow 0, \tag{1.5.3}$$

By (1.5.3), we have:

$$\begin{aligned}\chi(\sigma_*\mathcal{O}_{\overline{E}}\otimes\mathcal{L}) &= \chi(\mathcal{O}_E\otimes\mathcal{L}) + \chi(\omega_E\otimes\mathcal{L})\\ &= 2(\mathcal{L}\cdot E)_S + \chi(\mathcal{O}_E) + \chi(\omega_E)\\ &= 2(\mathcal{L}\cdot E)_S\\ &= 2\delta.\end{aligned} \tag{1.5.4}$$

On the other hand, since $\chi(\mathcal{O}_{\overline{E}}) = \chi(\mathcal{O}_{\overline{S}}) - \chi(\omega_{\overline{S}}) = 0$ by (1.5.2), we get

$$\begin{aligned}\chi(\sigma_*\mathcal{O}_{\overline{E}}\otimes\mathcal{L}) &= \chi(\mathcal{O}_{\overline{E}}\otimes\sigma^*\mathcal{L})\\ &= (\sigma^*\mathcal{L}\cdot\overline{E})_{\overline{S}} + \chi(\mathcal{O}_{\overline{E}})\\ &= (\sigma^*\mathcal{L}\cdot\overline{E})_{\overline{S}}.\end{aligned} \tag{1.5.5}$$

By (1.5.4) and (1.5.5), we conclude that $(\sigma^*\mathcal{L}\cdot\overline{E})_{\overline{S}} = 2(\mathcal{L}\cdot E)_S = 2\delta$. In particular, if E is irreducible and reduced, then we have $b_2(\overline{E}) \leq 2$.

(c): Since $Bs|\sigma^*\mathcal{L}| = \emptyset$, by the theorem of Bertini, there exists a smooth member $\overline{C} \in |\sigma^*\mathcal{L}|$. By the adjunction formula, $2g(\overline{C}) - 2 = \overline{C}(\overline{C} + \omega_{\overline{S}})$. Since $(\overline{C} \cdot \omega_{\overline{S}}) = (\sigma^*\mathcal{L} \cdot \omega_{\overline{S}}) = -2\delta$ and since $(\overline{C}^2)_{\overline{S}} = (\mathcal{L}^2)_S = d(\mathcal{L})$, we obtain $2g(\overline{C}) - 2 = d(\mathcal{L}) - 2\delta$. This proves the assertion (c).

(d): By operating $\otimes\mathcal{L}$ on (1.5.1), we obtain an exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \sigma_*\mathcal{O}_{\overline{S}} \otimes \mathcal{L} \longrightarrow \omega_E \otimes \mathcal{L} \longrightarrow 0.$$

Since $H^1(S; \mathcal{L}) = 0$ by (a), we obtain

(1.5.6) $$h^0(\sigma_*\mathcal{O}_{\overline{S}} \otimes \mathcal{L}) = h^0(\mathcal{L}) + h^0(\omega_E \otimes \mathcal{L}).$$

Since E is Cohen-Macaulay, $h^0(\omega_E \otimes \mathcal{L}) = h^1(\mathcal{O}_E \otimes \mathcal{L}^{-1})$. For a general member $C \in |\mathcal{L}|$, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_E(-C) \longrightarrow \mathcal{O}_E \longrightarrow \mathcal{O}_{E\cap C} \longrightarrow 0.$$

Since $h^1(\mathcal{O}_E) = 0$ and since $h^0(\mathcal{O}_{E\cap C}) = (\mathcal{L} \cdot E)_S = \delta$, we get

(1.5.7) $$\begin{aligned} h^0(\omega_E \otimes \mathcal{L}) &= h^1(\mathcal{O}_E \otimes \mathcal{L}^{-1}) \\ &= h^1(\mathcal{O}_E(-C)) \\ &= h^0(\mathcal{O}_{E\cap C}) - h^0(\mathcal{O}_E) \\ &= \delta - h^0(\mathcal{O}_E). \end{aligned}$$

On the other hand, since

$$h^0(\sigma_*\mathcal{O}_{\overline{S}} \otimes \mathcal{L}) = h^0(\sigma_*\mathcal{O}_{\overline{S}}(\sigma^*\mathcal{L})) = h^0(\sigma^*\mathcal{L}),$$

by (1.5.6) and (1.5.7), we have finally

$$h^0(\sigma^*\mathcal{L}) = h^0(\mathcal{L}) + \delta - h^0(\mathcal{O}_E).$$

(e): We can see that

$$\chi(\mathcal{L}^{\otimes m}) = \frac{1}{2}(\mathcal{L}^2)m^2 + am + \chi(\mathcal{O}_S)$$

for any m, where a is constant. Since $\omega_S \cong \mathcal{O}_S$, $\chi(\mathcal{L}^{\otimes m}) = \chi(\mathcal{L}^{-\otimes m})$. Hence $a = 0$, namely, $\chi(\mathcal{L}^{\otimes m}) = \frac{1}{2}(\mathcal{L}^2)m^2 + \chi(\mathcal{O}_S)$ for any m. Since $\chi(\mathcal{O}_S) = 2$ and $\chi(\mathcal{L}) = h^0(\mathcal{L})$, we have the assertion (d).

(f): By (c), one has easily

$$\begin{aligned} \Delta(\overline{S}, \sigma^*\mathcal{L}) &:= dim\overline{S} + deg\ \sigma^*\mathcal{L} - h^0(\sigma^*\mathcal{L}) \\ &= 2 + d(\mathcal{L}) - h^0(\mathcal{L}) - \delta + h^0(\mathcal{O}_E). \end{aligned}$$

The proof is completed. □

(1.6) Proposition. *Let $(S,\mathcal{L})$ be as in (1.3). Assume that $b_3(S)=0$. Then,*

(a) $\widehat{S}$ is a rational surface,

(b) $\overline{S}$ has at worst rational singularities,

(c) $h^1(\mathcal{O}_{\overline{S}})=h^2(\mathcal{O}_{\overline{S}})=0,\ b_1(\overline{S})=b_3(\overline{S})=0$,

(d) E_{red} is connected and has no cycle.

Proof. We have an exact sequence (cf. **[B-K]**):

$$
\begin{aligned}
(1.6.1)\qquad & H^1(S;\mathbb{Z})\longrightarrow H^1(\overline{S};\mathbf{Z})\oplus H^1(E;\mathbb{Z})\longrightarrow H^1(\overline{E};\mathbb{Z})\\
&\longrightarrow H^2(S;\mathbf{Z})\longrightarrow H^2(\overline{S};\mathbf{Z})\oplus H^2(E;\mathbf{Z})\longrightarrow H^2(\overline{E};\mathbf{Z})\\
&\longrightarrow H^3(S;\mathbf{Z})\longrightarrow H^3(\overline{S};\mathbf{Z})\longrightarrow 0
\end{aligned}
$$

Since $b_3(S)=0$, we have $b_3(\overline{S})=0$. It is known that $b_3(\widehat{S})=b_3(\overline{S})$ (cf. **[B]**). So we obtain $b_1(\widehat{S})=b_3(\widehat{S})=0$. Thus $\widehat{S}$ is a rational surface by (1.4) – (b). This proves (a). From the Leray spectral sequence we have:

$$
\begin{aligned}
(1.6.2)\qquad & 0\longrightarrow H^1(\overline{S};\mathcal{O}_{\overline{S}})\longrightarrow H^1(\widehat{S};\mathcal{O}_{\widehat{S}})\longrightarrow H^0(\overline{S};R^1\mu_*\mathcal{O}_{\widehat{S}})\\
&\longrightarrow H^2(\overline{S};\mathcal{O}_{\overline{S}})\longrightarrow .
\end{aligned}
$$

Since $\widehat{S}$ is rational and since

$$H^2(\overline{S};\mathcal{O}_{\overline{S}})\cong H^0(\overline{S};\omega_{\overline{S}})\cong H^0(\overline{S};\mathcal{I})=0,$$

we obtain $H^1(\overline{S};\mathcal{O}_{\overline{S}})=0=h^0(\overline{S};R^1\mu_*\mathcal{O}_{\widehat{S}})$. This proves (b) and (c). Finally, since $0=h^1(\mathcal{O}_{\overline{S}})=h^0(\mathcal{O}_E)-1$, we have $h^0(\mathcal{O}_E)=1$, thus E_{red} is connected. By (1.3) – (iii), $h^1(\mathcal{O}_E)=0$, so we have $h^1(\mathcal{O}_{E_{red}})=0$ (cf. [**(3.3)**; **P**]). Therefore E_{red} has no cycle. We complete the proof of the proposition. □

2. Next, we shall consider the adjoint line bundle $K_{\widehat{S}}+\pi^*\mathcal{L}$ on $\widehat{S}$, where $\pi:\widehat{S}\xrightarrow{\mu}\overline{S}\xrightarrow{\sigma}S$. Since $\mathcal{L}$ is very ample on S, $\pi^*\mathcal{L}$ is nef and big on $\widehat{S}$. By Kawamata vanishing theorem, we obtain

(1.7) Lemma. $H^i(\widehat{S};\mathcal{O}(K_{\widehat{S}}+\pi^*\mathcal{L}))=0$ *for* $i>0$.

(1.8) Corollary. $h^0(K_{\widehat{S}}+\pi^*\mathcal{L})=\frac{1}{2}d(\mathcal{L})-\delta+1-h^1(\mathcal{O}_{\widehat{S}})$.

Proof. We have easily

$$
\begin{aligned}
h^0(K_{\widehat{S}}+\pi^*\mathcal{L}) &= \chi(K_{\widehat{S}}+\pi^*\mathcal{L})\\
&= \frac{1}{2}\pi^*\mathcal{L}(\pi^*\mathcal{L}+K_{\widehat{S}})+\chi(\mathcal{O}_{\widehat{S}})\\
&= \frac{1}{2}(d(\mathcal{L})-2\delta)+1-h^1(\mathcal{O}_{\widehat{S}})\\
&= \frac{1}{2}d(\mathcal{L})-\delta+1-h^1(\mathcal{O}_{\widehat{S}}).
\end{aligned}
$$

□

Here we also make use of the same notations as in the paragraph **1**.

(1.9) Theorem. *Let $(S, \mathcal{L})$ be a non-normal polarized surface of K3-type. Then,*

(I). If $K_{\widehat{S}} + \pi^*\mathcal{L}$ is not nef, then we have either

(a) $(S, \mathcal{L}) \cong (Q_4, \mathcal{O}(1))$, *where $Q_4 \hookrightarrow \mathbb{P}^3$ is a non-normal irreducible quartic surface with $\delta := (\mathcal{L} \cdot E)_S = 3$, and $(\widehat{S}, \pi^*\mathcal{L}) \cong (\overline{S}, \sigma^*\mathcal{L}) \cong (\mathbb{P}^2, \mathcal{O}(2))$, or*

(b) *S is a (ruled) surface swept out by lines in $\mathbb{P}^{\frac{1}{2}d(\mathcal{L})+1}$. $\widehat{S}$ is a $\mathbb{P}^1$-bundle $\phi : \widehat{S} \longrightarrow \Gamma$ over a smooth curve Γ of the genus $g(\Gamma) = \frac{1}{2}d(\mathcal{L}) - \delta + 1$, and $(\pi^*\mathcal{L} \cdot f) = 1$ for a fiber f of ϕ . In particular, $\overline{S}$ is a cone over the curve Γ if $\overline{S} \not\cong \widehat{S}$.*

(II). If $K_{\widehat{S}} + \pi^*\mathcal{L}$ is nef, then we have either

(c) $(S, \mathcal{L}) \cong (S_4, \mathcal{O}(1)), (S_6, \mathcal{O}(1))$, *or* $(S_8, \mathcal{O}(1))$, *where $S_{d(\mathcal{L})} \hookrightarrow \mathbb{P}^{\frac{1}{2}d(\mathcal{L})+1}$ is a non-normal irreducible surface of degree $d(\mathcal{L})$, and $\delta := (\mathcal{L} \cdot E)_S = \frac{1}{2}d(\mathcal{L})$ with $d(\mathcal{L}) = 4, 6, 8$. In particular, $(\overline{S}, \sigma^*\mathcal{L}) \cong (\overline{S}, \omega_{\overline{S}}^{-1})$ and $\overline{S} \hookrightarrow \mathbb{P}^{d(\mathcal{L})}$ is a (normal) del Pezzo surface of degree $d(\mathcal{L}) = 4, 6, 8$,*

(d) *S is a (ruled) surface swept out by conics in $\mathbb{P}^{\frac{1}{2}d(\mathcal{L})+1}$. There is a $\mathbb{P}^1$-fibration $\phi : \widehat{S} \longrightarrow T$ over a smooth curve T , which has possibly singular fibers, such that $(\pi^*\mathcal{L} \cdot f) = 2$ and $K_{\widehat{S}} + \pi^*\mathcal{L} \equiv (\frac{1}{2}d(\mathcal{L}) - \delta)f$ for a general fiber f of ϕ , or*

(e) *$K_{\widehat{S}} + \pi^*\mathcal{L}$ is big.*

Proof. (I). Since $K_{\widehat{S}} + \pi^*\mathcal{L}$ is not nef, by Mori [**Mo**] (cf.[**KMM**]), there exist an extremal ray R and the contraction $\phi_R : \widehat{S} \longrightarrow W$ of the ray R such that

(i) W is smooth of *dim* $W \leq 2$,

(ii) $(K_{\widehat{S}} + \pi^*\mathcal{L}) \cdot R < 0$,

(iii) For any curve C, $\phi_R(C)$ is a point $\Longleftrightarrow C \in R$,

(iv) $\rho(\widehat{S}) = \rho(W) + 1$,

(v) ϕ_R has connected fibers .

(1.9.1) Claim. *dim* $W \leq 1$.

In fact, we assume that *dim* $W = 2$. Then ϕ_R is birational. Take a curve $C \in R$. Since $(K_{\widehat{S}} + \pi^*\mathcal{L}) \cdot C < 0$, one can easily see that C is the (-1)-curve on $\widehat{S}$ and $(\pi^*\mathcal{L} \cdot C) = 0$. Thus the curve C is contained in the exceptional set of $\mu : \widehat{S} \longrightarrow \overline{S}$. This is a contradiction, since $\mu : \widehat{S} \longrightarrow \overline{S}$ is the minimal resolution. Therefore *dim* $W \leq 1$. □

First, in the case of *dim* $W = 0$, since $\rho(\widehat{S}) = 1$, we have $\widehat{S} \cong \mathbb{P}^2$,hence, $\widehat{S} \cong \overline{S} \cong \mathbb{P}^2$. On the other hand, since $-(K_{\widehat{S}} + \sigma^*\mathcal{L})$ is ample and $d(\mathcal{L})$ is even, we obtain $d(\mathcal{L}) = 4$, that is, $\sigma^*\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^2}(2)$. By (1.3)-(iv) and (1.5), we have $h^0(\mathcal{L}) = 4$,$\delta = 3$. This proves (a).

Next, in the case of *dim* $W = 1$, since $\rho(\widehat{S}) = 2$, $\phi_R : \widehat{S} \longrightarrow \Gamma$ is a $\mathbb{P}^1$-bundle over a smooth curve $\Gamma := W$. For a fiber f of ϕ_R, we have $(K_{\widehat{S}} + \pi^*\mathcal{L}) \cdot f < 0$, hence $(\pi^*\mathcal{L} \cdot f) = 1$. Take a general smooth member $\widehat{C} \in |\pi^*\mathcal{L}|$. Since $(\pi^*\mathcal{L} \cdot f) = 1$,

$\widehat{C}$ is a section of ϕ_R. Thus we have $g(\Gamma) = g(\widehat{C}) = \frac{1}{2}d(\mathcal{L}) - \delta + 1$ by Proposition (1.5)-(c). If $\overline{S} \not\cong \widehat{S}$, then $\overline{S}$ is obtained from $\widehat{S}$ by blowing down the negative section of $\widehat{S}$. This proves (*b*).

(*II*): Since $K_{\widehat{S}} + \pi^*\mathcal{L}$ is nef, by the base point freeness theorem due to Kawamata (cf.[**KMM**]), we obtain $Bs|m(K_{\widehat{S}} + \pi^*\mathcal{L})| = \emptyset$ for $m >> 0$. By the contraction theorem (see [**KMM**]), there is a surjective morphism $\phi : \widehat{S} \longrightarrow T$ onto a normal variety T of $dim\ T \leq 2$ with connected fibers such that $K_{\widehat{S}} + \pi^*\mathcal{L} \sim \phi^*\mathcal{A}$ for an ample line bundle $\mathcal{A} \in Pic\,T$.

In the case of $dim\ T = 0$, we have $K_{\widehat{S}} = -\pi^*\mathcal{L}$. Suppose that $\widehat{S} \not\cong \overline{S}$, then, for each irreducible component B_i of the exceptional divisor B of $\mu : \widehat{S} \longrightarrow \overline{S}$, we have $(K_{\widehat{S}} \cdot B_i) = 0$, since $(\pi^*\mathcal{L} \cdot B_i) = 0$. This shows that B_i is the (-2)-curve on $\widehat{S}$. Thus $\overline{S}$ has at most rational double points, in particular, $\overline{S}$ is Gorenstein and $-K_{\overline{S}} = \sigma^*\mathcal{L}$ is ample on $\overline{S}$. Therefore $\overline{S}$ is a normal del Pezzo surface of degree $d(\mathcal{L})$ $(1 \leq d(\mathcal{L}) \leq 9)$ in $\mathbb{P}^{d(\mathcal{L})}$ (cf. [$\mathbf{B}_2$],[**H-W**]). Since $d(\mathcal{L})$ is even, we have $d(\mathcal{L}) = 2, 4, 6$, or 8.

(1.9.2) Claim. $d(\mathcal{L}) \neq 2$.

In fact, if $d(\mathcal{L}) = 2$, then the linear system $|\sigma^*\mathcal{L}|$ defines a two to one surjective morphism $\Phi_{|\sigma^*\mathcal{L}|} : \overline{S} \longrightarrow \mathbb{P}^2$. Thus $\mathcal{L}$ can not be very ample. This contradicts the assumption. Therefore $d(\mathcal{L}) \neq 2$. □

By (1.3)-(iv) and (1.5), one can easily get

$$(h^0(\mathcal{L}), d(\mathcal{L}), \delta) = (4,4,2),\ (5,6,3),\ (6,8,4)\ .$$

This proves (*c*).

In the case of $dim\ T = 1$, since $(K_{\widehat{S}} + \pi^*\mathcal{L}) \cdot f = 0$ for a general fiber f of ϕ, we have $f \cong \mathbb{P}^1$ and $(\pi^*\mathcal{L} \cdot f) = 2$. Since $(\pi(f) \cdot \mathcal{L}) = 2$, $\pi(f)$ is a conic in $\mathbb{P}^{\frac{1}{2}d(\mathcal{L})+1}$. This proves (*d*).

In the case of $dim\ T = 2$, since $(K_{\widehat{S}} + \pi^*\mathcal{L})^2 > 0$, we obtain (*e*).

Thus we complete the proof. □

(1.10) Proposition. *Let $(S, \mathcal{L})$ be as in (1.9)-(II), namely, $K_{\widehat{S}} + \pi^*\mathcal{L}$ is nef. Assume that* (1) $d(\mathcal{L}) > 4$ *and* (2) $h^1(\mathcal{O}_{\widehat{S}}) = 0$. *Then* $Bs|K_{\widehat{S}} + \pi^*\mathcal{L}| = \emptyset$.

Proposition (1.10) follows easily from the following:

(1.11) Proposition (cf. [S], [R]). *Let M be a non-singular projective surface and L a line bundle on M with $Bs|L| = \emptyset$ and $(L^2) > 4$. Assume that*

(1) $K_M + L$ *is nef ,*

(2) $h^1(\mathcal{O}_M) = 0$,

(3) *The singularities obtained by blowing down all the curves B with $(L \cdot B)_M = 0$ are at worst rational.*

Then $Bs|K_M + L| = \emptyset$.

Proof of Proposition (1.10).

By assumption (2) and the exact sequence (1.6.2), we obtain $H^0(\overline{S}; R^1\mu_*\mathcal{O}_{\widetilde{S}}) = 0$. Thus $\overline{S}$ has at worst rational singularities. Take any curve B with $(\pi^*\mathcal{L}\cdot B) = 0$. Then B must be contained in the exceptional set of μ, because $\sigma^*\mathcal{L}$ is ample on $\overline{S}$. Therefore, by (1.11), we complete the proof. □

Proof of Proposition (1.11).

Assume that there exists a base point $x \in M$ of the linear system $|K_M + L|$. Then, by Theorem 1-(i) and its proof in Reider **[R]**, there exist an effective divisor E on M passing through x, a vector bundle $\mathcal{E}$ of rank 2 on M, and exact sequences:

$$(1.11.a) \qquad 0 \longrightarrow \mathcal{O}_M(L-E) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_M(E) \longrightarrow 0,$$

$$(1.11.b) \qquad 0 \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{E} \longrightarrow \mathcal{J}_x \otimes \mathcal{O}_M(L) \longrightarrow 0$$

such that

(i) the composition map $\mathcal{O}_M(L-E) \longrightarrow \mathcal{E} \longrightarrow \mathcal{J}_x \otimes \mathcal{O}_M(L)$ is injective, where $\mathcal{J}_x$ is the ideal sheaf of x,

(ii) $L - 2E$ is big,

(iii) $(L\cdot E) = 1,\ (E^2) = 0$ or $(L \cdot E) = 0,\ (E^2) = -1$.

(1.11.1) Claim. $h^0(\mathcal{O}_M(E)) = 1$

In fact, suppose that $h^0(\mathcal{O}_M(E)) \geq 2$. We set $|E| = |C| + F$, where $|C|$ (resp.F) is the movable (resp. fixed) part of $|E|$. By (iii) above, we have $1 \geq (L\cdot E) = (L\cdot C) + (L\cdot F)$. Since $|C|$ is movable, we have $(L\cdot C) > 0$, hence, $(L\cdot C) = 1, (L\cdot F) = 0, (L\cdot E) = 1$, in particular, $(E^2) = 0$ by (iii). Taking into consideration that $Bs|L| = \emptyset$ and $(L\cdot C) = 1$, we can see that $\Phi_{|L|}(C)$ is a line in $\mathbb{P}^{dim|L|}$ for a general member C, where $\Phi_{|L|} : M \longrightarrow \mathbb{P}^{dim|L|}$ is a morphism defined by the linear system $|L|$. Thus we obtain $C \cong \mathbb{P}^1$ and $\mathcal{O}_C(L) \cong \mathcal{O}_{\mathbb{P}^1}(1)$. On the other hand, since $K_M + L$ is nef by assumption, we have

$$0 \leq (K_M + L)\cdot C = (K_M\cdot C) + 1 = -1 - (C^2),$$

that is, $(C^2) \leq -1$. This is a contradiction, since $|C|$ is movable. Therefore $h^0(\mathcal{O}_M(E)) = 1$. □

From $(1.11.a), (1.11.b), (1.11.1)$, we obtain

$$(1.11.2) \qquad 0 \longrightarrow H^0(M;\mathcal{O}_M(L-E)) \longrightarrow H^0(M;\mathcal{E}) \longrightarrow H^0(M;\mathcal{O}_M(E)) \longrightarrow 0.$$

$$(1.11.3)\quad 0 \longrightarrow H^0(M;\mathcal{O}_M) \longrightarrow H^0(M;\mathcal{E}) \longrightarrow H^0(M;\mathcal{J}_x \otimes \mathcal{O}_M(L)) \longrightarrow 0.$$

In fact, the composition map $\mathcal{O}_M \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_M(E)$ induces an isomorphism

$$H^0(M;\mathcal{O}_M) \cong H^0(M;\mathcal{O}_M(E)) \cong \mathbb{C}.$$

This yields a surjection

$$H^0(M;\mathcal{E}) \longrightarrow H^0(M;\mathcal{O}_M(E)) \cong \mathbb{C}$$

in (1.11.2) and an isomorphism

$$(1.11.4)\qquad H^0(M;\mathcal{O}_M(L-E)) \cong H^0(M;\mathcal{J}_x \otimes \mathcal{O}_M(L))$$

Now, from an exact sequence

$$0 \longrightarrow \mathcal{J}_x \otimes \mathcal{O}_M(L) \longrightarrow \mathcal{O}_M(L) \longrightarrow \mathbb{C}(x) \longrightarrow 0$$

we obtain

(1.11.5)
$$\begin{aligned} 0 \longrightarrow H^0(M;\mathcal{J}_x \otimes \mathcal{O}_M(L)) &\longrightarrow H^0(M;\mathcal{O}_M(L)) \longrightarrow \mathbb{C} \\ \longrightarrow H^1(M;\mathcal{J}_x \otimes \mathcal{O}_M(L)) &\longrightarrow H^1(M;\mathcal{O}_M(L)) \longrightarrow 0\,. \end{aligned}$$

Since $Bs|L| = \emptyset$, we have an isomorphism

$$(1.11.6)\qquad H^1(M;\mathcal{J}_x \otimes \mathcal{O}_M(L)) \cong H^1(M;\mathcal{O}_M(L))\,.$$

From (1.11.a), since $h^1(\mathcal{O}_M) = 0$, we obtain an injection

$$(1.11.7)\qquad H^1(M;\mathcal{E}) \hookrightarrow H^1(M;\mathcal{J}_x \otimes \mathcal{O}_M(L)).$$

From (1.11.a), (1.11.2), we also have an injection

$$(1.11.8)\qquad H^1(M;\mathcal{O}_M(L-E)) \hookrightarrow H^1(M;\mathcal{E}).$$

By (1.11.7), (1.11.8), we obtain an injection

$$(1.11.9)\qquad H^1(M;\mathcal{O}_M(L-E)) \hookrightarrow H^1(M;\mathcal{O}_M(L)).$$

Next, from an exact sequence

$$0 \longrightarrow \mathcal{O}_M(L-E) \longrightarrow \mathcal{O}_M(L) \longrightarrow \mathcal{O}_E(L) \longrightarrow 0\,,$$

we have

(1.11.10)

$$0 \longrightarrow H^0(M;\mathcal{O}_M(L-E)) \longrightarrow H^0(M;\mathcal{O}_M(L)) \longrightarrow H^0(E;\mathcal{O}_E(L)) \longrightarrow H^1(M;\mathcal{O}_M(L-E)) \hookrightarrow H^1(M;\mathcal{O}_M(L))$$

By (1.11.4), (1.11.5), (1.11.9), we conclude $H^0(E;\mathcal{O}_E(L)) \cong \mathbb{C}$. Since $Bs|L| = \emptyset$, we obtain $\mathcal{O}_E(L) \cong \mathcal{O}_E$, Thus $(L \cdot E) = 0$, in particular $(E^2) = -1$ by (*iii*).

Let $\varphi : M \longrightarrow S$ be the contraction of all curves B with $(L \cdot B) = 0$. By an exact sequence

$$0 \longrightarrow \mathcal{O}_M(-E) \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_E \longrightarrow 0\,,$$

we have

$$0 = R^1\varphi_*\mathcal{O}_M \longrightarrow H^1(E;\mathcal{O}_E) \longrightarrow R^2\varphi_*\mathcal{O}_M(-E) = 0\,,$$

that is, $H^1(E;\mathcal{O}_E) = 0$. Therefore

$$\begin{aligned} 1 \le h^0(\mathcal{O}_E) &= \chi(\mathcal{O}_E) \\ &= \chi(\mathcal{O}_M) - \{\frac{1}{2}(-E)(-E-K_M) + \chi(\mathcal{O}_M)\} \\ &= -\frac{1}{2}(K_M+E)\cdot E \end{aligned}$$

Thus we obtain $-(K_M+E)\cdot E \ge 2$, that is, $-(E^2) \ge (K_M \cdot E) + 2 \ge 2$, since $K_M + L$ is nef and $(L \cdot E) = 0$. This contradicts the fact that $(E^2) = -1$ above. The proof is completed $\square$

§2. A Fano threefold of index one as a compactification of $\mathbb{C}^3$.

1. Let us recall some facts on Fano threefolds of index $r = 1$ obtained by Iskovskih ([**Is**$_1$] , [**Is**$_2$]) and Takeuchi [**T**].

Let $V := V_{2g-2} \hookrightarrow \mathbb{P}^{g+1}$ be an anti-canonically embedded Fano threefold of index $r = 1$ with $Pic\, V \cong \mathbf{Z} \cdot \mathcal{O}_X(H)$, where $H \sim -K_V$ is a hyperplane section and $g = \frac{1}{2}(-K_V^3) + 1$ is the genus of V. Then,

(2.1) Lemma. *(1)([***Corollary 1 ; Is$_2$***]). V contains a one dimensional family of lines, and V does not contain cones if $g \geq 4$.*

*(2)([***Proposition 3-(iv) ; Is$_2$***]). The line Z on V intersects at most finite many other lines on V if $g \geq 7$.*

*(3)([***Proposition 2 ; Is$_2$***]). V contains a two dimensional family of conics such that a generic point $v \in V$ is contained in a finite number of conics from this family if $g \geq 5$.*

*(4)([***Theorem 4.4-(iii) ; Is$_1$***]). There is only a finite number of conics passing through each point $v \in V$ if $g \geq 10$.*

We assume below that the genus $g \geq 7$. Let $Z \subset V$ be a line on V. Then we have the normal bundle either

$$\begin{cases} (\alpha_1)\ N_{Z|V} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1} \ \ or \\ (\beta_1)\ N_{Z|V} \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(1). \end{cases}$$

Let $\tau : V' \longrightarrow V$ be the blowing-up of V along Z and let $Z' := \tau^{-1}(Z)$ be the exceptional ruled surface. Now, the line Z intersects at most finitely many lines $Z_1, Z_2, \cdots, Z_m$ $(m \geq 0)$ if $g \geq 5$ by (2.1)-(2), let $Z_1', Z_2', \cdots, Z_m'$ be the proper images of Z_i's on V' and Z_0' be the negative section of Z' if $N_{Z|V}$ has the type (β_1) above. We put $H' := \tau^* H - Z'$. Then,

(2.2) Lemma ([Lemma 2 ; Is$_2$]). *There is a birational map, called a flop $\chi : V' \cdots > V^+$ with the following properties:*

(2.2.1) *V^+ is a non-singular projective threefold.*

(2.2.2) *$\chi : V' - \bigcup_{i=0}^m Z_i' \cong V^+ - \bigcup_{i=0}^m Z_i^+$ (isomorphic), where Z_i^+ is the proper image of Z_i' with respect to χ for $0 \leq i \leq m$.*

(2.2.3) *If H^+ and Z^+ are proper images of H' and Z' with respect to χ, then we have $-K_{V^+} \sim H^+$,$(H^+ \cdot Z_i^+) = 0$ and $(H^+ - Z^+) \cdot Z_i^+ = 1$.*

Let D be a generic conic intersecting the line Z and let Q be the ruled surface swept out by conics intersecting the line Z. Let D^+ and Q^+ be the proper images of D and Q in V^+. Then,

(2.3) Lemma ([Proposition 1 ; Is$_2$]). *There exists a surjective morphism $\varphi : V^+ \longrightarrow W \hookrightarrow \mathbb{P}^{g-6}$ $(g \geq 7)$ onto a smooth projective variety W of $1 \leq dim\, W \leq 3$ such that*

(2.3.1) *φ has connected fibers,* (2.3.2) $\varphi(D^+)$ *is a point of* W *for a generic conic* D^+,*and dim* $\varphi(Q^+) \leq 1$

(2.3.3) $\mathcal{O}_{V^+}(H^+ - Z^+) \cong \varphi^*\mathcal{O}_W(1)$.

In particular, $R = \mathbb{R}_+[D^+]$ *is an extremal ray and* φ *is the contraction morphism of the ray* R. *Moreover,*

(2.3.4) *If* $g = 7$, *then* $W = \mathbb{P}^1$ *and* $\varphi : V^+ \longrightarrow \mathbb{P}^1$ *is a bundle whose fibers are irreducible del Pezzo surface of degree 5.*

(2.3.5) *If* $g = 8$, *then* $W = \mathbb{P}^2$ *and* $\varphi : V^+ \longrightarrow \mathbb{P}^2$ *is a standard conic bundle with discriminant curve* $\Delta \hookrightarrow \mathbb{P}^2$ *of degree 5.*

(2.3.6) *If* $g = 9$, *then* $W = \mathbb{P}^3$ *and* $\varphi : V^+ \longrightarrow \mathbb{P}^3$ *is the blowing-up of* $\mathbb{P}^3$ *along a smooth curve* Δ *of genus* $g(\Delta) = 3$, *deg* $\Delta = 7$ *lying on a unique cubic surface* $F_3 = \varphi(Z^+)$, *and* $Q^+ \sim 3H^+ - 4Z^+$.

(2.3.7) *If* $g = 10$, *then* $W = \mathbb{Q}^3 \hookrightarrow \mathbb{P}^4$ *is a non-singular hyper-quardric and* $\varphi : V^+ \longrightarrow \mathbb{Q}^3$ *is the blowing-up of* $\mathbb{Q}^3$ *along a smooth curve* Δ *of genus* $g(\Delta) = 2$, *deg* $\Delta = 7$ *lying on a unique surface* $F_4 = \varphi(Z^+) \hookrightarrow \mathbb{Q}^3$ *cut out by a quardric in* $\mathbb{P}^4$, *and* $Q^+ \sim 2H^+ - 3Z^+$.

(2.3.8) *If* $g = 12$, *then* $W = V_5 \hookrightarrow \mathbb{P}^6$ *is the Fano threefold* V_5 *of degree 5 in* $\mathbb{P}^6$*(the section of the Plücker embedding of the Grassmann variety* $Gr(2,5)$ *of lines in* $\mathbb{P}^4$ *by three hyperplanes) and* $\varphi : V^+ \longrightarrow V_5$ *is the blowing-up of a smooth rational curve* Δ *of degree 5 lying on a unique hyperplane section* $F_5 = \varphi(Z^+)$ *of* V_5, *and* $Q^+ \sim H^+ - 2Z^+$.

Remark 3. The composition $\pi_{2Z} := \varphi \circ \chi \circ \tau^{-1} : V \cdots > W \hookrightarrow \mathbb{P}^{g-6}$ is the double projection from the line Z.

2. Let D be a smooth conic on $V := V_{2g-2}$ $(g \geq 10)$. Then we have the normal bundle either

$$\begin{cases} (\alpha_2)\ N_{D|V} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \text{ or} \\ (\beta_2)\ N_{D|V} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1). \end{cases}$$

Let $\lambda : V'' \longrightarrow V$ be the blowing-up of V along the conic D and let $D'' := \lambda^{-1}(D)$ be the exceptional ruled surface. The conic D intersects at most finitely many lines $Z_1, \cdots, Z_n$ $(n \geq 1)$. Let $Z_1'', \cdots, Z_n''$ be the proper images of Z_i'' on V''. We put $H'' := \lambda^* H - D''$. Then,

(2.4) Lemma ([K]). *There exists a flop* $\chi' : V'' \cdots > V^\flat$ *with the following properties:*

(2.4.1) $V^\flat$ *is a non-singular projective threefold.*

(2.4.2) $\chi' : V'' - \bigcup_{i=1}^n Z_i'' \cong V^\flat - \bigcup_{i=1}^n Z_i^\flat$ *(isomorphic), where* $Z_i^\flat$ *is the proper image of* Z_i'' *with respect to* χ' *for* $1 \leq i \leq n$.

(2.4.3) *If* $H^\flat$ *and* $D^\flat$ *are proper images of* H'' *and* D'' *with respect to* χ', *then we have* $-K_{V^\flat} \sim H^\flat$,$(H^\flat \cdot Z_i^\flat) = 0$ *and* $(H^\flat - D^\flat) \cdot Z_i^\flat = 1$.

Let γ be a generic conic intersecting the conic D and let F be a ruled surface swept out by conics intersecting the conic D. Let $\gamma^\flat$ and $F^\flat$ be the proper images of γ and F in $V^\flat$ respectively. Then,

(2.5) Lemma ([(2.8.1)-(B); T]). *Assume that $g \geq 9$. Then there exists a surjective morphism $\psi : V^\flat \longrightarrow U \hookrightarrow \mathbb{P}^{g-8}$ onto a smooth projective variety U of $1 \leq \dim U \leq 3$ such that*

(2.5.1) *ψ has connected fibers,*

(2.5.2) *$\psi(\gamma^\flat)$ is a point of U for a generic conic $\gamma^\flat$, and $\dim \psi(F^\flat) \leq 1$*

(2.5.3) $\mathcal{O}_{V^\flat}(H^\flat - D^\flat) \cong \psi^*\mathcal{O}_U(1)$.

In particular, $R = \mathbb{R}_+[\gamma^\flat]$ is an extremal ray and ψ is the contraction morphism of the ray R. Moreover,

(2.5.4) *If $g = 9$, then $U \cong \mathbb{P}^1$ and $\psi : V^\flat \longrightarrow \mathbb{P}^1$ is a bundle whose fibers are irreducible del Pezzo surface of degree 6.*

(2.5.5) *If $g = 10$, then $U \cong \mathbb{P}^2$ and $\psi : V^\flat \longrightarrow \mathbb{P}^2$ is a conic bundle with discriminant curve Δ of degree 4.*

(2.5.6) *If $g = 12$, then $U \cong \mathbb{Q}^3 \hookrightarrow \mathbb{P}^4$ and $\psi : V^\flat \longrightarrow \mathbb{Q}^3$ is the blowing-up of $\mathbb{Q}^3$ along a smooth rational curve Δ of degree 6. In particular, $F^\flat \sim 2H^\flat - 3D^\flat$.*

Remark 4. In (2.5.5), let Θ be a generic quartic curve intersecting the conic D at two points and let $\Theta^\flat$ be a proper image of Θ in $V^\flat$. Then $\Theta^\flat$ is a generic fiber of the conic bundle $\psi : V^\flat \longrightarrow \mathbb{P}^2$. In particular, we have $(\Theta^\flat \cdot D^\flat) = (H^\flat \cdot \Theta^\flat) = 2$.

3. Let (X, Y) be a smooth projective compactification of $\mathbb{C}^3$ with the second Betti number $b_2(X) = 1$ and the index $r = 1$, namely, $-K_X \sim Y$. Then X is a Fano threefold of index one and Y is a non-normal irreducible ample divisor on X with $Pic\,X \cong \mathbb{Z} \cdot \mathcal{O}_X(Y)$ (cf.[**Fu**$_2$]). Moreover we have

(2.6) Lemma (cf.[B-M], [Is$_1$]). *(1) $H^i(X; \mathcal{O}_X) = 0$, $H^i(X; \mathcal{O}_X(Y)) = 0$ for $i > 0$,*

(2) $H^i(X; \mathbb{Z}) \cong H^i(Y; \mathbb{Z})$ for $i > 0$,

(3) $H^1(X; \mathbb{Z}) = 0$, $\quad H^2(X; \mathbb{Z}) \cong \mathbb{Z}$,

(4) $\omega_Y \cong \mathcal{O}_Y$,

(5) $H^1(Y; \mathcal{O}_Y) = 0$.

It is proved by Shokulov [**Sh**] that there exists a smooth member $H \in |-K_X|$, which is a K3-surface. We may assume that $C := H \cap Y$ is irreducible. By the adjunction formula, we have

$$\begin{aligned} p_a(C) &= \frac{1}{2}(C^2)_H + 1 \\ &= \frac{1}{2}(-K_X^3)_X + 1 \end{aligned}$$

The integer $g := \frac{1}{2}(-K_X^3)_X + 1$ is called the genus of X. Then we have

(2.7) Lemma ([Is₁]). *$X \cong V_{2g-2}$ $(2 \le g \le 10$ or $g = 12)$, and $(g, h^{1,2})$ is as follows:*

g	2	3	4	5	6	7	8	9	10	12
$h^{1,2}$	52	30	20	14	10	5	5	3	2	0

Table 1

,where $h^{1,2} = \frac{1}{2}b_3(X)$.

We put $\mathcal{L} := \mathcal{O}_Y(-K_X) \cong \mathcal{O}_Y(Y)$. Then $\mathcal{L}$ is very ample if $g \ge 3$ and $Bs|\mathcal{L}| = \emptyset$ if $g = 2$. Thus $(Y, \mathcal{L})$ is a non-normal polarized surface of K3-type if $g \ge 3$

(2.8) Lemma (cf. Proposition (1.5)). *(i) $H^i(Y; \mathcal{L}) = 0$ for $i > 0$,*

(ii) $d(\mathcal{L}) := (\mathcal{L}^2) = (-K_X^3)_X = 2g - 2$,

Let $\sigma : \overline{Y} \longrightarrow Y$ be the normalization and $\mathcal{I}$ the conductor of σ. Let $E := V_Y(\mathcal{I})$ (resp. $\overline{E} = V_{\overline{Y}}(\mathcal{I})$) be the closed subscheme defined by $\mathcal{I}$ in Y (resp. $\overline{Y}$). Let $\mu : \widehat{Y} \longrightarrow \overline{Y}$ be the minimal resolution and $B = \bigcup_{i=1} B_i$ the exceptional set of μ. Let $\widehat{E}$ be the proper transform of $\overline{E}$ in $\widehat{Y}$. We set $\pi : \widehat{Y} \xrightarrow{\mu} \overline{Y} \xrightarrow{\sigma} Y$.

By (1.4), (1.5), we obtain

(2.9) Lemma. *(i). $-K_{\overline{Y}} \sim \overline{E}$ as a Weil divisor, $-K_{\widehat{Y}} \sim \widehat{E} + \sum_i k_i B_i$ $(k_i \ge 0, k_i \in \mathbf{Z})$, in particular $\widehat{Y}$ is a rational or a ruled surface,*

(ii). $g(\overline{C}) = g - \delta$ for a general smooth member $\overline{C} \in |\sigma^*\mathcal{L}|$, where $\delta := (\mathcal{L} \cdot E)_Y$,

(iii). $(\sigma^*\mathcal{L} \cdot \overline{E})_{\overline{Y}} = 2\delta$,

(iv). If E is irreducible reduced, then $b_2(\overline{E}) \le 2$,

(v). Let E_0 be an irreducible component of E_{red}. Suppose that the number $^\#\{\sigma^{-1}(E_0)\}$ of irreducible components of $\sigma^{-1}(E_0)$ (analytic inverse image) is more than three. Then $mult_{E_0} Y \ge 3$

Proof. We have only to prove the assertion (v). Since E_0 is a non-normal locus of Y, we have $mult_{E_0} Y \ge 2$. Assume that $mult_{E_0} Y = 2$. Then a general hyperplane section $C \in |\mathcal{L}|$ has multiplicity two at a generic intersection point p. Thus the pull-back $\overline{C}$ of C in $\overline{Y}$ intersects $\sigma^{-1}(E_0)$ at two points (with multiplicity) over p. This is absurd since the number $^\#\{\sigma^{-1}(E_0)\} \ge 3$. □

Now, we shall consider an exact sequence (**[B-K]**):

(2.10)
$$0 \longrightarrow \mathbf{Z} \cong H^2(Y;\mathbf{Z}) \longrightarrow H^2(\overline{Y};\mathbf{Z}) \oplus H^2(E;\mathbf{Z})$$
$$\longrightarrow H^2(\overline{E};\mathbf{Z}) \longrightarrow H^3(Y;\mathbf{Z}) \longrightarrow H^3(\overline{Y};\mathbf{Z}) \longrightarrow 0.$$

Then,

(2.11) Lemma. *(a)* $b_3(X)+b_2(\overline{Y})+b_2(E)=2h^1(\mathcal{O}_{\hat{Y}})+b_2(\widehat{E})+1$,*in particular,* $b_2(\widehat{E})\geq b_3(X)+b_2(\overline{Y})-2h^1(\mathcal{O}_{\hat{Y}})$,

(b) $\frac{1}{2}b_3(X)+\frac{1}{2}\leq h^1(\mathcal{O}_{\hat{Y}})+\delta$.

Proof. Since $b_2(\widehat{E})=b_2(\overline{E})$, by (2.10), we obtain

$$b_3(Y)+b_2(\overline{Y})+b_2(E)=b_3(\overline{Y})+b_2(\widehat{E})+1.$$

Since $b_3(Y)=b_3(X)$ by (2.6)-(2) and since

$$b_3(\overline{Y})=b_3(\widehat{Y})=b_1(\widehat{Y})=2h^1(\mathcal{O}_{\hat{Y}})$$

(cf.$[B_1]$), we have the assertion (a). Next, by (2.9)-(iii), one obtain that $b_2(\overline{E})\leq 2\delta$. On the other hand, since

$$b_3(X)+2\leq b_3(X)+b_2(\overline{Y})+b_2(E),$$

we have $b_3(X)\leq 2h^1(\mathcal{O}_{\hat{Y}})+2\delta-1$. This proves (b). □

(2.12) Proposition. $K_{\hat{Y}}+\pi^*\mathcal{L}$ *is nef, in particular,* $(K_{\hat{Y}}+\pi^*\mathcal{L})^2\geq 0$.

Proof. Assume that $K_{\hat{Y}}+\pi^*\mathcal{L}$ is not nef. Then by (1.10)-(I) we have either

(1) $\widehat{Y}=\overline{Y}\cong\mathbf{P}^2$

or

(2) $\widehat{Y}$ is a $\mathbf{P}^1$-bundle $\phi:\widehat{Y}\longrightarrow\Gamma$ over a smooth curve Γ of $g(\Gamma)=g-\delta$.

(1.12.1). *The case (1) cannot occur.*

In fact, since $d(\mathcal{L})=2g-2$ $(2\leq g\leq 12,\ g\neq 11)$, one can easily see that $\sigma^*\mathcal{L}\cong\mathcal{O}_{\mathbf{P}^2}(2)$ and $g=3$. Let $\overline{C}\in|\sigma^*\mathcal{L}|$ be a smooth member. Then $\overline{C}$ is a smooth conic in $\mathbf{P}^2$, hence $0=g(\overline{C})=g-\delta=3-\delta$, that is, $\delta=3$. From the Table 1, we have $b_3(X)=60$ since $g=3$. Thus by (2.11)-(b) we obtain $30=\frac{1}{2}b_3(X)<\delta=3$. This is a contradiction. □

Thus we have the case (2). Then since $h^1(\mathcal{O}_{\hat{Y}})=g(\Gamma)=g-\delta$, by (2.11)-(b), we have $b_3(X)<2g$. From the Table 1, we obtain $g\geq 7$. We put $\ell_t:=\phi^{-1}(t)$ for $t\in\Gamma$. Since $(\pi^*\mathcal{L}\cdot\phi^{-1}(t))=1$ for all $t\in\Gamma$, ℓ_t is a line on X, and thus Y is a ruled surface swept out by the family $\{\ell_t\}$ of lines. If $\widehat{Y}\neq\overline{Y}$, then $\overline{Y}$ is obtained from $\widehat{Y}$ by blowing down the negative section of $\widehat{Y}$. Thus Y is a cone. But this cannot happen because of (2.1)-(2). Therefore we have $\widehat{Y}=\overline{Y}$.

(2.12.2) Claim. *Any line* ℓ_t *can not be a singular locus of* Y.

In fact, assume that some line $\ell_t=:Z$ is a singular locus of Y. Then we have $mult_ZY=2$. Otherwise, we have $mult_ZY\geq 3$. Hence any conic intersecting the line Z is always contained in Y. Thus Y is a ruled surface swept out by conics intersecting the line Z by (2.1)-(3). This shows that the $\mathbf{P}^1$-bundle $\overline{Y}$ contains infinitely many rational curves γ with $(\sigma^*\mathcal{L}\cdot\gamma)=2$. Since the rational curve γ can not be a fiber, we have $\overline{Y}\cong\mathbf{F}_d$ (the Hirzebruch surface of degree d) , in particular, $g(\Gamma)=g-\delta=0$. Let s_0 be the section of $\overline{Y}$ with $s_0^2=-d\leq 0$. Then the curve

γ can be written as $\gamma \sim as_0 + bf$, where f is a fiber and $a, b \in \mathbf{Z}$. Taking into consideration that $\gamma \cong \mathbf{P}^1$ and $(\sigma^*\mathcal{L}\cdot\gamma) = 2$, we obtain $a = b = 1$, $(\sigma^*\mathcal{L}\cdot s_0) = 1$ and $-s_0^2 = n \leq 1$. On the other hand, since $(\sigma^*\mathcal{L}\cdot f) = 1$, we can write as $\sigma^*\mathcal{L} \sim s_0 + kf$ for some $k \in \mathbf{Z}$. Since $1 = (\sigma^*\mathcal{L}\cdot s_0) = -n + k$ and $2g - 2 = -n + 2k$, we have $g = 2$. This contradicts the fact $g \geq 7$. Thus we must have $mult_Z Y = 2$ if the line Z is a singular locus of Y.

Now, we put $V := X (= V_{2g-2},\ g \geq 7)$. In order to avoid the confusion, we use the same notations as in (2.2) and (2.3). Since $mult_Z Y = 2$, the lines $Z_1, \cdots, Z_m$ intersecting the line Z is always contained in Y. By (2.1)-(1), we can see that $\ell_t \cap (Z_0 \cup Z_1 \cup \cdots \cup Z_m) = \emptyset$ for almost all $t \in \Gamma$. Let $H^+, Z^+, Z_0^+, \cdots, Z_m^+ \cdots$ be as in (2.2) and (2.3), and let Y^+, ℓ_t^+ be the proper images of Y, ℓ_t respectively. Then we have $(\ell_t \cap Z^+) = \emptyset$ for almost all $t \in \Gamma$ and $Y^+ \sim H^+ - Z^+$. Since $(H^+ - Z^+ \cdot \ell_t) = 1$ for almost all $t \in \Gamma$ and since $Y^+ \sim H^+ - Z^+ \sim \varphi^*G$ for $G \in |\mathcal{O}_W(1)|$, one can easily see that $g \geq 9$. Since $\varphi(\ell_t)$ is a line on W , we have $F_i \cap \varphi(\ell_t) \neq \emptyset$ for $i = 3, 4, 5$, where $F_i := \varphi(Z^+)$. This is impossible becuase the blowing-up center Δ is not a hyperplane section for $g \geq 9$. Therefore any line ℓ_t cannot be a singular locus of Y. The claim is proved. □

We shall continue the proof of the proposition. By (2.9), we have $-K_{\overline{Y}} \sim \overline{E}$. Since any ℓ_t cannot be a singular locus, $\overline{E}$ contain no fiber as its irreducible component. For a fiber f, we obtain $2 = (-K_{\overline{Y}}\cdot f) = (\overline{E}\cdot f)$. This shows that either

(α) $\overline{E} = 2\overline{E_0}$ with $(\overline{E}_0 \cdot f) = 1$,

(β) $\overline{E} = \overline{E}_1 + \overline{E}_2$ with $(\overline{E}_i \cdot f) = 1$ for $i = 1, 2$, or

(γ) $\overline{E}$ is irreducible reduced.

In the cases $(\alpha), (\gamma)$, we have $b_2(\overline{E}) = b_2(E) = 1$. Since $b_2(\overline{Y}) = 2$, by (2.11)-(a), we obtain $b_3(X) = 2h^1(\mathcal{O}_{\overline{Y}}) - 1$. This cannot happen, since $b_3(X)$ is even. In the case (β), since $b_2(\overline{E}) = 2 \geq b_2(E)$ and since $b_3(X) = 2h^1(\mathcal{O}_{\overline{Y}}) + b_2(E) - 1$ is even, we have $b_2(E) = 1$ and $b_3(X) = 2(g - \delta)$. Since $-K_{\overline{Y}} \sim \overline{E}_1 + \overline{E}_2$, by the adjunction formula, we obtain $g(\overline{E}_i) = 1 - \frac{1}{2}(\overline{E}_1 \cdot \overline{E}_2) \leq 1$, hence $b_3(X) \leq 2$. By the Table 1, we have $g = 12$ and $b_3(X) = 0$, hence we obtain $\overline{E}_i \cong \mathbf{P}^1, \overline{E}_1^2 + \overline{E}_2^2 = 4$, $(\overline{E}_1 \cdot \overline{E}_2) = 2$ and $(\sigma^*\mathcal{L}\cdot\overline{E}_i) = \delta = 12$ for $i = 1, 2$, in particular, $\overline{Y} \cong \mathbb{F}_d$ $(d \geq 0)$. Moreover one can easily show that $\overline{Y} \cong \mathbf{P}^1 \times \mathbf{P}^1$ or $\mathbb{F}_2$. In the case where $\overline{Y} \cong \mathbb{F}_2$, $\overline{E}_i$'s are sections with $\overline{E}_i^2 = 2$ for $i = 1, 2$. Thus $Y - E$ contains a smooth rational curve with the self-intersection number -2. This cannot occur since $Pic\, Y \cong \mathbf{Z}\cdot\mathcal{L}$. Therefore we obtain $\overline{Y} \cong \mathbf{P}^1 \times \mathbf{P}^1$. Moreover, since $H_1(\overline{E}; \mathbf{Z}) = 0$ and since $(\overline{E}_1 \cdot \overline{E}_2) = 2$, $\overline{E}_1$ is tangent to $\overline{E}_2$. On the other hand, we consider an exact sequence over $\mathbf{Z}$ or $\mathbf{R}$:

$$0 = H^1(E) \longrightarrow H_c^2(Y, E) \longrightarrow H^2(Y) \longrightarrow H^2(E) \longrightarrow$$
$$\longrightarrow H_c^3(Y, E) \longrightarrow H^3(Y) \longrightarrow 0.$$

Since $b_2(Y) = b_2(E) = 1$, we have

$$\begin{aligned} H^3(Y;\mathbb{R}) &\cong H^3_c(Y,E;\mathbb{R}) \cong H^3_c(\overline{Y},\overline{E};\mathbb{R}) \\ &\cong H_1(\overline{Y}-\overline{E};\mathbb{R}) \\ &\cong H_1(\mathbb{P}^1\times\mathbb{P}^1-(\overline{E}_1\cup\overline{E}_2);\mathbb{R}) \\ &\neq 0\,. \end{aligned}$$

This contradicts the fact $H^3(Y;\mathbb{R}) = H^3(X;\mathbb{R}) = 0$. Therefore $K_{\widehat{Y}} + \pi^*\mathcal{L}$ is nef. By (1.10)-(II), we have $(K_{\widehat{Y}} + \pi^*\mathcal{L})^2 \geq 0$. The proof is completed. □

Remark 4. Let $X := U_{22} \hookrightarrow \mathbb{P}^{13}$ be the Mukai-Umemura's example of the Fano threefold of the index $r = 1$ and the genus $g = 12$ (**[M-U]**). Then there exists a non-normal hyperplane section Y such that (i) $\overline{Y} \cong \mathbb{P}^1\times\mathbb{P}^1$, (ii) $K_{\widehat{Y}} + \pi^*\mathcal{L}$ is "not" nef, (iii) $\overline{E} = 2\overline{E_0}$ ($\overline{E_0}$ is a diagonal) is non-reduced, here we use the same notations as above. In our proof of (2.12), we use the conditions $b_2(Y) = 1$ and $H^3(Y;\mathbb{Z}) \cong H^3(X;\mathbb{Z})$ effectively.

(2.13) Lemma. *(1).* $\delta + 2h^1(\mathcal{O}_{\widehat{Y}}) \leq \frac{1}{2}(g+3)$ *if* $(K_{\widehat{Y}} + \pi^*\mathcal{L})^2 = 0$.

(2). $\delta + 3h^1(\mathcal{O}_{\widehat{Y}}) \leq \frac{1}{3}(g+8)$ *if* $(K_{\widehat{Y}} + \pi^*\mathcal{L})^2 > 0$.

Proof. (1). Since $8 - 8h^1(\mathcal{O}_{\widehat{Y}}) \geq K^2_{\widehat{Y}} = 4\delta - 2g + 2$, we have the claim (1).

(2). Since $(K_{\widehat{Y}} + \pi^*\mathcal{L})^2 > 0$, by the Kawamata vanisning theorem, we obtain $H^i(\widehat{Y};\mathcal{O}_{\widehat{Y}}(2K_{\widehat{Y}} + \pi^*\mathcal{L})) = 0$ for $i > 0$. Thus we have

$$\begin{aligned} h^0(2K_{\widehat{Y}} + \pi^*\mathcal{L}) &= \chi\,(2K_{\widehat{Y}} + \pi^*\mathcal{L}) \\ &= \frac{1}{2}(2K_{\widehat{Y}} + \pi^*\mathcal{L})(K_{\widehat{Y}} + \pi^*\mathcal{L}) + \chi(\mathcal{O}_{\widehat{Y}}) \\ &= K^2_{\widehat{Y}} - 3\delta + g - h^1(\mathcal{O}_{\widehat{Y}}) \\ &\geq 0. \end{aligned}$$

Since $8 - 8h^1(\mathcal{O}_{\widehat{Y}}) \geq K^2_{\widehat{Y}}$, one can get easily (2). □

(2.14) Corollary. $g \geq 9$.

Proof. We put $q := h^1(\mathcal{O}_{\widehat{Y}})$. Then, combining (2.11)-(b) with (2.13), we have

$$\text{(2.14.1)} \qquad \frac{1}{2}(b_3(X)+1) \leq \delta + q \leq \delta + 2q \leq \frac{g+3}{2} \quad if \quad (K_{\widehat{Y}} + \pi^*\mathcal{L})^2 = 0.$$

$$\text{(2.14.2)} \qquad \frac{1}{2}(b_3(X)+1) \leq \delta + q \leq \delta + 3q \leq \frac{g+8}{3} \quad if \quad (K_{\widehat{Y}} + \pi^*\mathcal{L})^2 > 0.$$

From the Table 1, one can easily see that $g \geq 9$. □

4. Next, we shall prove that $g = 12$. This can be done by proving that $g \neq 9, 10$. For the proof, we need the following:

(2.15) Lemma. *(a). Assume that $g \geq 9$ and that there is a line $Z \hookrightarrow Y$ with $mult_Z Y \geq 2$. If Y is a ruled surface swept out by conics intersecting the line Z, then $g = 12$. In particular, if $mult_Z Y \geq 3$, then $g = 12$.*

(b). Assume that $g \geq 10$. Then there exists no conic $D \hookrightarrow Y$ such that $mult_D Y \geq 3$.

Proof. Consider the bouble projection from the line Z. In order to avoid the confusion, we use the same notations as in (2.2) and (2.3).

(a): By (2.3.6),(2.3.7) and (2.3.8), we obtain $Q^+ := Y^+ \sim 3H^+ - 4Z^+, 2H^+ - 3Z^+$ and $H^+ - 2Z^+$ if $g = 9, 10$ and 12 respectively. Since Y is a hyperplane section, we have $Y^+ \sim H^+ - 2Z^+$, that is, $g = 12$. If $mult_Z Y \geq 3$, then any conic intersecting the line Z is always contained in Y. Thus by (2.1)-(3), one can see that Y is a ruled surface swept out by conics intersecting the line Z. The assertion (a) is proved.

(b). Similarly, since $mult_D Y \geq 3$, Y is a ruled surface swept out by conics intersecting the conic D. If $g = 12$, then by (2.5.6) we have $F^\flat := Y^\flat \sim 2H^\flat - 3D^\flat$.Thus Y cannot be a hyperplane section. If $g = 10$, then, by (2.5.5), $\psi(F^\flat) = \psi(Y^\flat)$ coincides with the discriminant locus Δ of the conic bundle $\psi : V^\flat \longrightarrow \mathbb{P}^2$. Since $deg\ \Delta = 4$ and since Y is a hyperplane section, this cannot occur. The proof is completed. □

Noe, since $g \geq 9$ by (2.14), we obtain $d(\mathcal{L}) := 2g - 2 \geq 16$. According to (1.10)-(II), we have the following two cases:

(2.16.A) There is a surjective morphism $\phi : \widehat{Y} \longrightarrow T$ over a smooth curve T whose generic fiber f is a smooth rational curve with $(\pi^*\mathcal{L} \cdot f) = 2$, in particular, there is a numerical equivalence $K_{\widehat{Y}} + \pi^*\mathcal{L} \equiv (g - \delta - 1)f$ (where, $g \geq 9$).

(2.16.B) $(K_{\widehat{Y}} + \pi^*\mathcal{L})^2 > 0$.

(2.17) Lemma. $g \neq 9$.

Proof. Assume that $g = 9$. Then we have $b_3(X) = 6$ by the Table 1. We shall derive a contradiction.

First, in the case (2,16.A), by (2.14.1), we obtain

$$4 \leq \delta + q \leq \delta + 2q \leq 6$$

Since $\delta \geq 1$, we have $q \leq 2$. Moreover, we obtain

(i) $q = 2$ and $\delta = 2$,

(ii) $q = 1$ and $3 \leq \delta \leq 4$,

(iii) $q = 0$ and $4 \leq \delta \leq 6$.

We put $\widehat{E} := \sum \widehat{E}_i$ ($\widehat{E}_i$: irreducible subscheme, not necessarily reduced).

The case (i) : Since $q = 2$, we have $K_{\widehat{Y}}^2 = -8$, that is, $\widehat{Y} \xrightarrow{\phi} T$ is a $\mathbb{P}^1$-bundle over T. Since $b_2(\widehat{E}) \geq 3$ by (2.11)-(a) and since $\delta = 2$, applying (2.9)-(iii), we obtain

$$4 = (\pi^*\mathcal{L} \cdot \widehat{E}) \geq \sum_{i=1}^{3} (\pi^*\mathcal{L} \cdot \widehat{E}_i).$$

Thus there exists a component $\widehat{E}_{i_0} \cong \mathbb{P}^1$ such that $(\pi^*\mathcal{L} \cdot \widehat{E}_{i_0}) = 1$. This $\widehat{E}_{i_0}$ must be a section. This is absurd since the genus of the base curve T is equal to two.

The case (ii) : Since $q = 1$, we have $b_2(\widehat{E}) \geq 5$ by (2.11)-(a). First, in the case of $\delta = 4$, we have $K_{\widehat{Y}}^2 = 0$, that is, $\widehat{Y} \xrightarrow{\phi} T$ is a $\mathbb{P}^1$-bundle over T. Since

$$8 = (\pi^*\mathcal{L} \cdot \widehat{E}) \geq \sum_{i=1}^{5} (\pi^*\mathcal{L} \cdot \widehat{E}_i),$$

there is a component $\widehat{E}_{i_0}$ such that $(\pi^*\mathcal{L} \cdot \widehat{E}_{i_0}) = 1$. By the same reason as in the case (i) above, we can derive a contradiction. Similarly, in the case of $\delta = 3$, then we obtain

$$6 = (\pi^*\mathcal{L} \cdot \widehat{E}) \geq \sum_{i=1}^{5} (\pi^*\mathcal{L} \cdot \widehat{E}_i).$$

Thus there is a component $\widehat{E}_{i_0}$ such that $(\pi^*\mathcal{L} \cdot \widehat{E}_{i_0}) = 1$. If $b_2(\widehat{E}) = 5$, then $b_2(E) = 1$ by (2.11)-(a). Thus $\pi(\widehat{E}_{i_0}) = E$ is a line, and the number $^\#\{\sigma^{-1}(E)\} = 5$. By (2.9)-(v), we have $mult_E Y \geq 3$. By (2.15)-(a), we obtain $g = 12$. This contradicts the assumption. If $b_2(\widehat{E}) = 6$, $b_2(E) \leq 2$. Moreover, we have $(\pi^*\mathcal{L} \cdot \widehat{E}_i) = 1$ for all i $(1 \leq i \leq 6)$. By the same reason as above, $b_2(E) \neq 1$. In case of $b_2(E) = 2$, E consists of two lines E_1 and E_2. Since $b_2(\widehat{E}) = 6$, we obtain $^\#\{\sigma^{-1}(E_i)\} \geq 3$ for $i = 1$ or 2. This implies $multi_{E_i} Y \geq 3$, hence $g = 12$. Therefore we have a contradiction.

The case (iii) : We have $b_2(\overline{E}) \geq 7$ by(2.11)-(a). In the case of $\delta = 6$, we have $K_{\widehat{Y}}^2 = 8$, that is, $\widehat{Y} \xrightarrow{\phi} T$ is a $\mathbb{P}^1$-bundle over $T \cong \mathbb{P}^1$. Moreover we obtain

$$12 = (\pi^*\mathcal{L} \cdot \widehat{E}) \geq \sum_{i=1}^{7} (\pi^*\mathcal{L} \cdot \widehat{E}_i).$$

Thus we have a component $\widehat{E}_{i_0} \cong \mathbb{P}^1$ such that $(\pi^*\mathcal{L} \cdot \widehat{E}_{i_0}) = 1$,which is also a section of ϕ. Then $\pi(\widehat{E}_{i_0}) =: E_{i_0}$ is a line. Since $\ell_t \cap E_{i_0} \neq \emptyset$ for any $t \in T$, where $\gamma_t := \pi(\phi^{-1}(t))$ is a conic. Thus Y is a ruled surface swept out by conics $\{\gamma_t\}$ intersecting the line E_{i_0}. By (2.15)-(a), we have $g = 12$. This is a contradiction. In the case of $\delta = 5$, we have

$$10 = (\pi^*\mathcal{L} \cdot \widehat{E}) \geq \sum_{i=1}^{7} (\pi^*\mathcal{L} \cdot \widehat{E}_i).$$

Since $b_2(E) = 1, \leq 2, \leq 3, \leq 4$ if $b_2(\widehat{E}) = 7, 8, 9, 10$ respectively, one can easily see that there is a line $E_0 \subset E$ such that the number $^{\#}\{\sigma^{-1}(E_0)\} \geq 3$. Thus we have $mult_{E_0} Y \geq 3$. By (2.15)-(a), we obtain $g = 12$, which is a contradiction. In case of $\delta = 4$, by a similar argument, we can also derive the same contradiction as above. Therefore $g \neq 9$ in the case (2.16.A).

Next, in the case (2.16.B), by (2.14.2), we obtain

$$\frac{7}{2} \leq \delta + q \leq \delta + 3q \leq \frac{17}{3}.$$

Since $\delta \geq 1$, we have $q \leq 1$. If $q = 1$, then by the inequality above we obtain $\frac{5}{2} \leq \delta \leq \frac{8}{3}$, hence $\delta \notin \mathbb{Z}$. Thus we have $q = 0$ and $4 \leq \delta \leq 5$. In particular, $b_2(\widehat{E}) \geq 7$ by (2.11)-(a). If $\delta = 5$, then we have

$$10 = (\pi^* \mathcal{L} \cdot \widehat{E}) \geq \sum_{i=1}^{7} (\pi^* \mathcal{L} \cdot \widehat{E}_i).$$

Since $b_2(E) = 1, \leq 2, \leq 3, \leq 4$ if $b_2(\widehat{E}) = 7, 8, 9, 10$ respectively. By an argument similar to the case (2.16.A) above, one can show that there is a line $E_0 \subset E$ such that $mult_{E_0} Y \geq 3$. Thus we have $g = 12$ by (2.15). This is a contradiction. Similarly, in the case of $\delta = 4$, one can derive a contradiction. Therefore $g \neq 9$. The proof of (2.17) is completed. □

(2.18) Lemma. $g \neq 10$.

Proof. Assuming $g = 10$, we shall derive a contradiction. From the Table 1, one sees $b_3(X) = 4$.

First, in the case (2.16.A), we have the following

(2.18.1). *(1) Let $B = \bigcup_i B_i$ be the exceptional set of the minimal resolution $\widehat{Y} \xrightarrow{\mu} \overline{Y}$. Then each irreducible component B_i is contained in a singular fiber of $\widehat{Y} \xrightarrow{\phi} T$, in particular, $\overline{Y}$ has at most rational double points.*

(2) There exists an irreducible component $\widehat{E}_0 \subset \widehat{E}$ such that the restriction $\phi|_{\widehat{E}_0} : \widehat{E}_0 \longrightarrow T$ is surjective.

In fact, assume that some B_i is not contained in any singular fiber. Then the restriction $\phi|_{B_i} : B_i \longrightarrow T$ is surjective. We put $y_i := \pi(B_i) \in Y$ (a point on Y). Then for generic $t \in T$, $\gamma_t = \pi(\phi^{-1}(t)) \subset Y \hookrightarrow X$ is a conic passing through the point y_i. This is a contradiction because of (2.1)-(vi). Thus the exceptional set B is contained in singular fibers. Let A_j be any irreducible component of a singular fiber. Then we have $(K_{\widehat{Y}} + \pi^* \mathcal{L}) \cdot A_j = (g - \delta - 1)(f \cdot A_j) = 0$. Thus we obtain either $(-K_{\widehat{Y}} \cdot A_j) = (\pi^* \mathcal{L} \cdot A_j) = 1$ or $(-K_{\widehat{Y}} \cdot A_j) = (\pi^* \mathcal{L} \cdot A_j) = 0$. This shows that A_j is a (-1)-curve or a (-2)-curve, and hence any irreducible component of B is a (-2)-curve. Therefore $\overline{Y}$ has at most rational double points. The assertion (1) is proved. Next, since $-K_{\widehat{Y}} \sim \widehat{E} + \sum_i B_i$ and since $(-K_{\widehat{Y}} \cdot f) = 2$ for a general fiber f, we obtain $(\widehat{E} \cdot f) = 2$. This proves the assertion (2). □

(2.18.2). *(1)* $b_2(\overline{Y}) \geq 2$. *(2)* $b_2(\widehat{E}) \geq 5 - 2q + b_2(E)$.

In fact, let $f_1, \cdots, f_N$ be singular fibers, $1 + \alpha_i$ the number of irreducible components of f_i and β_i the number of irreducible components of f_i other than the exceptional set B. By (2.18.1), we have $b_2(B) = \sum_{i=1}^{N}(1 + \alpha_i - \beta_i)$. Since $b_2(\widehat{Y}) = b_2(\overline{Y}) + b_2(B)$, we have

$$b_2(\widehat{Y}) = 2 + \sum_{i=1}^{N} \alpha_i = \sum_{i=1}^{N}(1 + \alpha_i - \beta_i) + b_2(\overline{Y}).$$

This yields $b_2(\overline{Y}) - 2 = \sum_{i=1}^{N}(\beta_i - 1) \geq 0$. In particular, $b_2(\overline{Y}) = 2$ iff there exists unique (-1)-curve in each singular fiber. This proves the assertion (1). By (2.11)-(a), we obtain the assertion (2). □

Now, by (2.14.1), we have

$$\frac{5}{2} \leq \delta + q \leq \delta + 2q \leq \frac{13}{2}.$$

This implies that

(i)' $q = 2$ and $1 \leq \delta \leq 2$,

(ii)' $q = 1$ and $2 \leq \delta \leq 4$ or

(iii)' $q = 0$ and $3 \leq \delta \leq 6$.

The case (i)' : Since $\delta \leq 2$, we have $b_2(E) \leq 2$ and $2 \leq b_2(\widehat{E}) \leq 4$ by (2.18.2)-(2). In the case of $\delta = 2$, we have

$$4 = (\pi^*\mathcal{L} \cdot \widehat{E}) \geq \sum_{i=1}^{2}(\pi^*\mathcal{L} \cdot \widehat{E}_i).$$

If $b_2(\widehat{E}) = 2$, then $b_2(E) = 1$. This shows that E is a line or a conic) and $(\pi^*\mathcal{L} \cdot \widehat{E}_i) \leq 2$ for $i = 1, 2$. Thus $\widehat{E}_i \cong \mathbb{P}^1$ for $i = 1, 2$. Similarly, one can also show that $(\pi^*\mathcal{L} \cdot \widehat{E}_i) \leq 2$ for all i for the case of $b_2(\widehat{E}) \geq 3$. Thus $\widehat{E} \cong \mathbb{P}^1$ for all i. By (2.18.1)-(2), we have a contradiction because the genus of the base curve T is equal to 2. In the case of $\delta = 1$, we have $(\pi^*\mathcal{L} \cdot \widehat{E}_i) = 1$ for $i = 1, 2$. By the same reason as above, we have a contradiction. Therefore $q \neq 2$.

The case (ii)' : By (2.18.2)-(2), we obtain $b_2(\widehat{E}) \geq 4$. In the case of $\delta = 4$, we have

$$4 = (\pi^*\mathcal{L} \cdot \widehat{E}) \geq \sum_{i=1}^{4}(\pi^*\mathcal{L} \cdot \widehat{E}_i).$$

If $b_2(\widehat{E}) \leq 5$, then $b_2(E) \leq 2$, and there is a line (or a conic) $E_0 \subset E$ such that the number $^{\#}\{\sigma^{-1}(E_0)\} \geq 3$. Hence $mult_{E_0}Y \geq 3$ by (2.9)-(v). By (2.15), this cannot happen in our case. If $b_2(\widehat{E}) = 6$, then $b_2(E) \leq 3$ and $(\pi^*\mathcal{L} \cdot \widehat{E}_i) \leq 2$ for all i. Thus $\widehat{E}_i \cong \mathbb{P}^1$ for all i. Since $q = 1$, this cannot happen. For the cases $b_2(\widehat{E}) \geq 6$, one can easily show that either $(\pi^*\mathcal{L} \cdot \widehat{E}_i) \leq 2$ for all i or there is a line (or an irreducible conic) $E_0 \subset E$ such that the number $\#\{\sigma^{-1}(E_0)\} \geq 3$. Thus we also have a contradiction. Similarly, in the case of $\delta \leq 3$, one can derive a contradiction. Therefore $q \neq 1$.

The case (iii)' : By (2.18.2)-(2), we have $b_2(\widehat{E}) \geq 6$, and

$$12 = (\pi^*\mathcal{L} \cdot \widehat{E}) \geq \sum_{i=1}^{6} (\pi^*\mathcal{L} \cdot \widehat{E}_i).$$

In the case of $\delta = 6$, if $b_2(\widehat{E}) \leq 9$, then, taking an account of $b_2(E) \leq 4$, one can easily show that there is a line (or a conic) $E_0 \subset E$ such that the number $\#\{\sigma^{-1}(E_0)\} \geq 3$. So we have $mult_{E_0} Y \geq 3$. This cannot occur in our case by (2.15).

If $b_2(\widehat{E}) \geq 10$, then one can see that the number $\#\{\widehat{E}_i; (\pi^*\mathcal{L} \cdot \widehat{E}_i) = 1\} \geq 8$. For each $\widehat{E}_i$ with $(\pi^*\mathcal{L} \cdot \widehat{E}_i) = 1$, since $(K_{\widehat{Y}} \cdot \widehat{E}_i) + 1 \geq 0$, we have the self-intersection number $\widehat{E}_i^2 \leq -1$. On the other hand, since $K_{\widehat{Y}}^2 = 4\delta - 18 = 6$, $\widehat{Y}$ can be obtained from the relatively minimal model $\mathbb{F}_n$ $(n \geq 0)$ (Hirzebruch surface) by bolwing up two times. Thus one can see that $\widehat{Y}$ cannot contain so much $\widehat{E}_i$'s with the negative intersection number. In the case of $\delta = 5$, we have

$$10 = (\pi^*\mathcal{L} \cdot \widehat{E}) \geq \sum_{i=1}^{6} (\pi^*\mathcal{L} \cdot \widehat{E}_i).$$

If $b_2(\widehat{E}) \leq 9$, then there is a line (or a conic) E_0 such that the number $\#\{\sigma^{-1}(E_0)\} \geq 3$. This cannot happen in our case as we have seen. If $b_2(\widehat{E}) = 10$, then we have $(\pi^*\mathcal{L} \cdot \widehat{E}_i) = 1$ for all i. Thus there is a line $\widehat{E}_{i_0} \subset E$ such that $\gamma_t \cap E_{i_0} \neq \emptyset$ for a generic $t \in T$. Thus Y is a ruled surface swept out by conics $\{\gamma_t\}$ intersecting the line E_{i_0}. This cannot happen in our case by (2.15). For the cases $\delta \leq 4$, by a similar argument, one can get easily a contradiction. Consequently, we have $g \neq 10$ in the case (2.16.A).

Next, in the case (2.16.B), since $b_3(X) = 4$, by (2.14.2), we obtain

$$\frac{5}{2} \leq \delta + q \leq \delta + 3q \leq 6.$$

Hence we have either

(i)" $q = 1$ and $2 \leq \delta \leq 3$ or

(ii)" $q = 0$ and $3 \leq \delta \leq 6$.

The case (i)" : First, in the case of $\delta = 3$, by (2.13)-(2), we obtain $0 \leq K_{\widehat{Y}}^2 \leq 3\delta - 9 = 0$, that is, $K_{\widehat{Y}}^2 = 0$. Thus $\widehat{Y}$ is a $\mathbb{P}^1$-bundle $\nu : \widehat{Y} \longrightarrow T$ over an elliptic curve $T \cong \mathbb{T}^1$. Moreover since $e := b_2(\widehat{E}) \geq 3$ by (2.11), we obtain

$$6 = \sum_{i=1}^{e} (\pi^*\mathcal{L} \cdot \widehat{E}_i) \geq \sum_{i=1}^{3} (\pi^*\mathcal{L} \cdot \widehat{E}_i).$$

If $b_2(\widehat{E}) = 3$, then $b_2(E) = 1$ and there exists a component $\widehat{E}_j \subset \widehat{E}$ such that $(\pi^*\mathcal{L} \cdot \widehat{E}) \leq 2$. Thus $E = \pi(\widehat{E}_j)$ is a line or a conic and we have the number $^\#\{\sigma^{-1}(E)\} = 3$. This cannot happen as we have seen before. If $b_2(\widehat{E}) \geq 4$, then there exists a component $\widehat{E}_i \subset \widehat{E}$ such that $(\pi^*\mathcal{L} \cdot \widehat{E}_i) = 1$. This $\widehat{E}_i \cong \mathbb{P}^1$ must be a fiber of $\nu : \widehat{Y} \longrightarrow T$, hence we have $(K_{\widehat{Y}} \cdot \widehat{E}_i) = -2$. Since $K_{\widehat{Y}} + \pi^*\mathcal{L}$ is nef, this cannot occur.

Next, in the case of $\delta = 2$, we have

$$4 = \sum_{i=1}^{e}(\pi^*\mathcal{L} \cdot \widehat{E}_i) \geq \sum_{i=1}^{3}(\pi^*\mathcal{L} \cdot \widehat{E}_i).$$

By the same reason as above, we may assume $b_2(\widehat{E}) \geq 4$. Then we obtain $(\pi^*\mathcal{L} \cdot \widehat{E}_i) = 1$ for all i $(1 \leq i \leq 4)$, hence $\widehat{E}_i \cong \mathbb{P}^1$ is irreducible and reduced for all i. Since $q = 1$ and since $K_{\widehat{Y}} + \pi^*\mathcal{L}$ is nef, we have $\widehat{E}_i^2 < 0$ for all i. Let $\nu : \widehat{Y} \longrightarrow T$ be the ruling over an elliptic curve T. Then $\widehat{E}_i$'s are all contained in singular fibers of ν, hence $(\widehat{E}_i \cdot \widehat{E}_j) \leq 1$ for $i \neq j$. We claim that $(\widehat{E}_i \cdot \widehat{E}_j) = 0$ for $i \neq j$. In fact, if $(\widehat{E}_i \cdot \widehat{E}_j) = 1$ for some $i \neq j$, then, since

$$-K_{\widehat{Y}} \sim \sum_{i=1}^{4}\widehat{E}_i + \sum_{i=1}^{N} k_i B_i \; (k_i \in \mathbb{Z}\,, k_i > 0),$$

by the adjunction formula, we have $B_i \cong \mathbb{P}^1$ and $k_i = 1$ for all i. Since $(-K_{\widehat{Y}} \cdot f) = 2$ and $(\widehat{E}_i \cdot f) = 0$ $(1 \leq i \leq 4)$ for a general fiber f of ν, there exists a component $B_i \not\cong \mathbb{P}^1$. This is a contradiction. Therefore we have $(\widehat{E}_i \cdot \widehat{E}_j) = 0$ for $i \neq j$. Let $\widehat{Y}_0 := \widehat{Y}/\widehat{E}$ be a normal projective surface obtained by contracting the disjoint rational curves $\widehat{E}_i$ $(1 \leq i \leq 4)$. Then $\widehat{Y}_0$ has at most rational singularities. Let $f_0 \subset \widehat{Y}_0$ be the image of a general fiber f of ν. Then f_0 does not pass through the singularities of $\widehat{Y}_0$ and the self-intersection number $f_0^2 = 0$. Thus we have $b_2(\widehat{Y}_0) \geq 2$. On the other hand, since $2 \leq b_2(\widehat{Y}_0) = b_2(\widehat{Y}) - b_2(\widehat{E}) = b_2(\widehat{Y}) - 4$, we obtain $b_2(\widehat{Y}) \geq 6$, hence $K_{\widehat{Y}}^2 \leq -4$. This is a contradiction since $K_{\widehat{Y}}^2 \geq 3\delta - 9 = -3$.

The case (ii)" : By (2.11) and (2.13)-(2), we have $b_2(\widehat{E}) \geq 5$. First, in the case of $\delta = 6$, since $K_{\widehat{Y}}^2 \geq 3\delta - 10 = 8$, one can see that $\widehat{Y} \cong \mathbf{F}_n$ (Hirzebruch surface of degree n). Let $\Phi := \Phi_{|K_{\widehat{Y}} + \pi^*\mathcal{L}|} : \widehat{Y} \longrightarrow \mathbb{P}^3$ be a morphism defined by the linear system $|K_{\widehat{Y}} + \pi^*\mathcal{L}|$, which is free from the base point by (1.10). Since $(K_{\widehat{Y}} + \pi^*\mathcal{L})^2 = 2$, we obtain $\widehat{Y} \cong \mathbb{P}^1 \times \mathbb{P}^1$, or $\mathbb{F}_2$. Let s_0 (resp. s_2) and f be the minimal section and a fiber of $\mathbb{P}^1 \times \mathbb{P}^1$ (resp. $\mathbb{F}_2$). Then one can easily show $\pi^*\mathcal{L} \sim 3s_0 + 3f$ (resp. $3s_2 + 6f$). Thus we have no irreducible curve ℓ with $1 \leq (\pi^*\mathcal{L} \cdot \ell) \leq 2$. On the other hand, since

$$12 = \sum_{i=1}^{e}(\pi^*\mathcal{L} \cdot \widehat{E}_i) \geq \sum_{i=1}^{5}(\pi^*\mathcal{L} \cdot \widehat{E}_i),$$

there exists a component $\widehat{E}_i$ such that $(\pi^*\mathcal{L} \cdot \widehat{E}_i) \leq 2$. This is a contradiction.

Next, in the case of $\delta = 5$, we have

$$10 = \sum_{i=1}^{e}(\pi^*\mathcal{L}\cdot\widehat{E}_i) \geq \sum_{i=1}^{5}(\pi^*\mathcal{L}\cdot\widehat{E}_i),$$

If $e = b_2(\widehat{E}) \leq 7$, then one can easily see that there exists a line or a conic $E_0 \subset E$ such that the number $^\#\{\sigma^{-1}(E_0)\} \geq 3$. This cannot happen as we have seen before. So we may assume that $e = b_2(\widehat{E}) \geq 8$. Then there exist irreducible components $\widehat{E}_1, \cdots, \widehat{E}_{e_0}$ ($e_0 \geq 6$) with $(\pi^*\mathcal{L}\cdot\widehat{E}_i) = 1$ for $1 \leq i \leq e_0$. Thus $\widehat{E}_i$'s $(1 \leq i \leq e_0)$ are reduced. Since $K_{\widehat{Y}} + \pi^*\mathcal{L}$ is nef, we have $(K_{\widehat{Y}}\cdot\widehat{E}_i) + 1 \geq 0$, that is, $\widehat{E}_i^2 < 0$ for all i $(1 \leq i \leq e_0)$. Since $q = 0$, Y is rational , hence E is connected and E_{red} has no cycle by an argument similar to (1.6).Thus, applying the adjunction formular to the curves $\widehat{E}_i$ $(1 \leq i \leq e_0)$, one can show $(\widehat{E}_i \cdot \widehat{E}_j) = 0$ for $i \neq j$,$(1 \leq i,j \leq e_0)$. Let $\widehat{Y}_0 := \widehat{Y}/\widehat{E}_0$, where $\widehat{E}_0 := \bigcup_{i=1}^{e_0}\widehat{E}_i$, be the contraction of the disjoint exceptional curves $\widehat{E}_0$. Then $\widehat{Y}_0$ has at most rational singularities, and we have $b_2(\widehat{Y}) = b_2(\widehat{Y}_0) + b_2(\widehat{E}_0) \geq 1 + e_0 \geq 7$. On the other hand, since $K_{\widehat{Y}}^2 \geq 3\delta - 10 = 5$, we have $b_2(\widehat{Y}) \leq 5$. This is a contradiction.

Similarly, in the case of $\delta = 4$, we may assume $e_0 = b_2(\widehat{E}) \geq 8$. Then one can find irreducible components $\widehat{E}_i \subset \widehat{E}$ with $(\pi^*\mathcal{L}\cdot\widehat{E}_i) = 1$ $(1 \leq i \leq e_0)$. In particular, we have $(\widehat{E}_i \cdot \widehat{E}_j) = 0$ for $i \neq j$ $(1 \leq i,j \leq e_0)$ and $b_2(\widehat{Y}) \geq e_0 + 1 \geq 9$ by the same arguments as above. On the other hand, since $K_{\widehat{Y}}^2 \geq 3\delta - 10 = 2$, we obtain $b_2(\widehat{Y}) \leq 8$. This is a contradiction.

Finally, in the case of $\delta = 3$, one can easily show that there exists a line $E_0 \subset E$ such that the number $^\#\{\sigma^{-1}(E_0)\} \geq 3$. This cannot happen in our case. Therefore we have $g \neq 10$ in the case (2.16.B). This completes the proof of (2.18). □

By (2.17) and (2.18), we conclude the following:

(2.19) Theorem (cf.[P],[P-S$_2$],[Fu$_2$]). *Let (X, Y) be a smooth projective compactification of $\mathbb{C}^3$ with the second Betti number $b_2(X) = 1$ and the index $r = 1$. Then X is a Fano threefold of index one and the genus $g = 12$, which is anti-canonically embedded into $\mathbb{P}^{13}$ with the degree 22, and Y is a non-normal hyperplane section of X, in particular, Y is rational.*

§3. The structure of V_{22} as a compactification of $\mathbf{C}^3$.

1. Let (X, Y) be a smooth projective compactification of $\mathbf{C}^3$ with $b_2(X) = 1$ and the index $r = 1$. Then by (2.19) $X \cong V_{22} \hookrightarrow \mathbf{P}^{13}$ and Y is a non-normal hyperplane section of X. We use the notations of §2.

By (1.6) and (2.11) , we have

(3.1) Lemma. *(1) $\widehat{Y}$ is a rational surface,*

(2) $\overline{Y}$ has at most rational singularities,

(3) $h^1(\mathcal{O}_{\overline{Y}}) = h^2(\mathcal{O}_{\overline{Y}}) = 0 = b_1(\overline{Y})$,

(4) E_{red} is connected and has no cycle,

(5) $b_2(\overline{Y}) + b_2(E) = b_2(\widehat{E}) + 1$.

According to (2.16.A) and (2.16.B), we have two cases :

(A) There is a surjective morphism $\phi : \widehat{Y} \longrightarrow T \cong \mathbf{P}^1$ such that $(\pi^*\mathcal{L} \cdot f) = 2$ for a generic fiber $f \cong \mathbf{P}^1$, in particular, $K_{\widehat{Y}} + \pi^*\mathcal{L} \sim (11 - \delta)f$.

(B) $(K_{\widehat{Y}} + \pi^*\mathcal{L})^2 > 0$.

⋆ The structure of (X,Y) in the case (A).

In (2.18.1),(2.18.2), we have proved

(3.2) Lemma. *(1) Let $B = \bigcup_i B_i$ be the exceptional set of the minimal resolution $\widehat{Y} \xrightarrow{\mu} \overline{Y}$. Then each irreducible component B_i is contained in a singular fiber of $\widehat{Y} \xrightarrow{\phi} T \cong \mathbf{P}^1$, in particular, $\overline{Y}$ has at most rational double points.*

(2) There exists an irreducible component $\widehat{E}_0 \subset \widehat{E}$ such that the restriction $\phi|_{\widehat{E}_0} : \widehat{E}_0 \longrightarrow T \cong \mathbf{P}^1$ is surjective.

(3) Let A_j be an irreducible component of a singular fiber of ϕ. Then A_j is either the (-1)-curve with $(\pi^*\mathcal{L} \cdot A_j) = 1$ or the (-2)-curve with $(\pi^*\mathcal{L} \cdot A_j) = 0$.

(4) $b_2(\overline{Y}) \geq 2$, in particular the equality holds if and only if there exists exactly one (-1)-curve A_j with $(\pi^*\mathcal{L} \cdot A_j) = 1$ in each singular fiber of ϕ.

(5) $b_2(\widehat{E}) = b_2(\overline{Y}) + b_2(E) - 1 \geq 2$.

(6) $\delta \leq 7$.

2. Let $\widehat{E}_{i_0} \subset \widehat{E}$ be an irreducible component with $(\widehat{E}_{i_0} \cdot f) \neq 0$ for a generic fiber f of ϕ. Since $(-K_{\widehat{Y}} \cdot f) = (\widehat{E} \cdot f) = 2$ by (3.2)-(1), the number of such a $\widehat{E}_{i_0}$ is at most two.

(3.3) Lemma. *$E_0 := \pi(\widehat{E}_{i_0}) \hookrightarrow Y \hookrightarrow X$ is a line on X.*

Proof. The proof will be divided into several steps.

(3.3.1). *Let $\widehat{A}$ be an irreducible curve with $(\pi^*\mathcal{L}\cdot\widehat{A}) \leq 2$ and $(\widehat{A}\cdot f) \neq 0$, where f is a generic fiber of ϕ. Then $A := \pi(\widehat{A})$ is a line on X with $A \subset E$. In particular, E_0 cannot be a conic.*

In fact, by assumption, A is a line or a conic on X. If A is a conic, then Y is a ruled surface swept out by conics $\{\gamma_t\}$, where $\gamma_t := \pi(\phi^{-1}(t))$ for a generic $t \in T$. According to (2.5.6), Y cannot be a hyperplane section. This is a contradiction. Thus A is a line on X. Since $K_{\widehat{Y}} + \pi^*\mathcal{L} \sim (11-\delta)f$, we obtain $(K_{\widehat{Y}}\cdot\widehat{A}) \geq (9-\delta) > 0$ by (3.2)-(7). On the other hand, since $-K_{\widehat{Y}}$ is effective, we obtain $(K_{\widehat{Y}}\cdot A) \geq 0$ unless $A \subset E$. This implies $A \subset E$. □

(3.3.2). *There exists an irreducible component $\widehat{E}_i \subset \widehat{E}$ such that $\phi(\widehat{E}_i)$ is a point of $T \cong \mathbf{P}^1$.*

In fact, assuming the contrary, then we have $(\widehat{E}_i\cdot f) \neq 0$ for each irreducible component $\widehat{E}_i \subset E$. Since $b_2(\widehat{E}) \geq 2$ by (3.2)-(5) and since $(\widehat{E}\cdot f) = 2$, we obtain $\widehat{E} = \widehat{E}_1 + \widehat{E}_2$, where $(\widehat{E}_1\cdot f) = (\widehat{E}_2\cdot f) = 1$. By (3.2)-(6), we have $K^2_{\widehat{Y}} = 4\delta - 22 \leq 6$, that is , $b_2(\widehat{Y}) \geq 4$. Thus $\phi : \widehat{Y} \longrightarrow T$ has at least a singular fiber $\phi^{-1}(0) =: f_0 \sim \sum_{i=0}^{m}\lambda_i B_i$ $(\lambda_i \in \mathbf{Z},\ \lambda_i > 0)$. By (3.2)-(3), we may assume that $B_0^2 = -1$, $(\pi^*\mathcal{L}\cdot B_0) = 1$ and $B_i^2 = -2$, $(\pi^*\mathcal{L}\cdot B_i) = 0$ $(1 \leq i \leq m)$. Since $H_1(\overline{E};\mathbf{Z}) = 0$, we have $H_1(\widehat{E}\cup B;\mathbf{Z}) = 0$, namely, $\widehat{E}\cup B$ has no cycle. Hence, applying the adjunction formula, we obtain $(\widehat{E}_1\cdot\widehat{E}_2) = 0$ or 2. In the case of $(\widehat{E}_1\cdot\widehat{E}_2) = 2$, by the adjunction formula, we have easily $\widehat{E}\cap B = \emptyset$. Hence we have

$$
\begin{aligned}
2 &= (-K_{\widehat{Y}}\cdot f) = (\widehat{E}\cdot f) \\
&= (\widehat{E}\cdot f_0) = (\widehat{E}_1\cdot f_0) + (\widehat{E}_2\cdot f_0) \\
&= (\widehat{E}_1\cdot B_0) + (\widehat{E}_2\cdot B_0).
\end{aligned}
$$

This implies $(-K_{\widehat{Y}}\cdot B_0) = 2$. This is a contradiction since B_0 is a (-1)-curve. In the case of $(\widehat{E}_1\cdot\widehat{E}_2) = 0$, applying the adjunction formula, one sees that the number of the singular fibers is equal to one. Moreover since the singular fiber contains exactly one (-1)-curve and since the other components are all (-2)-curves, we obtain a linear equivalence

$$-K_{\widehat{Y}} \sim \widehat{E}_1 + \widehat{E}_2 + B_1 + 2B_2 + 3B_3 + 2B_4 + 2B_5,$$

where

$$
\begin{aligned}
&(\widehat{E}_1\cdot B_4) = (\widehat{E}_2\cdot B_5) = 1, \\
&(B_4\cdot B_5) = 0, (B_3\cdot B_i) = 1\ (i = 2,4,5), \\
&(B_{i+1}\cdot B_i) = 1\ (i \leq 2).
\end{aligned}
$$

In particular, the number of irreducible components of the singular fiber f_0 is equal to 6. This yields $b_2(\widehat{Y}) = 7$, that is, $K^2_{\widehat{Y}} = 3$. Since $K^2_{\widehat{Y}} = 4\delta - 22$, we get $\delta = \frac{25}{4} \notin \mathbf{Z}$. This is a contradiction. This proves (3.3.2). □

We shall prove (3.3) below. Assume that $E_0 \subset E$ is not a line. Since the hyperplane section Y is a ruled surface swept out by the conics $\{\gamma_t\}$ intersecting E_0, E_0 cannot be a conic by (2.5.6), that is, $deg\, E_0 = (-K_X \cdot E_0)_X \geq 3$. According to (3.3.2), there is an irreducible component E_1 of $\widehat{E}$ such that $\phi(\widehat{E}_1)$ is a point. We put $E_1 := \pi(\widehat{E}_1) \cong \mathbb{P}^1$. Then since $(\pi^*\mathcal{L} \cdot \widehat{E}_1) \leq 2$ (the equality holds only if $\widehat{E}_1$ is a regular fiber of ϕ), $E_1 \subset E$ is a line or a conic. Since $deg\, E_0 \geq 3$, we have $E_1 \neq E_0$. Let A be a line or a conic intersecting the curve E_1 and let $\widehat{A}$ be it's proper transform in $\widehat{Y}$. In the case of $A \not\subset E$, taking into account that $(K_{\widehat{Y}} \cdot \widehat{A}) < 0$ and $K_{\widehat{Y}} + \pi^*\mathcal{L} = (11-\delta)f$, $\widehat{A}$ is contained in a fiber of ϕ, hence we have $\gamma_t \cap A = \emptyset$ for a generic $t \in T$. In the case of $A \subset E$. By (3.3.1), if $(\widehat{A} \cdot f) \neq 0$, then A is a line and Y is a ruled surface swept out by the conics γ_t intersecting the line A. Taking $\widehat{A}$ instead of E_0, the lemma is proved. So we have only to consider the case of $(\widehat{A} \cdot f) = 0$, that is, $\phi(\widehat{A})$ is a point. In this case, we also have $\gamma_t \cap A = \emptyset$ for a generic $t \in T$.

Now we put $E_1 =: Z$ (resp. $=: D$) if E_1 is a line (resp. a conic) and consider the double projection from the line Z (resp. conic D). In order to avoid the confusion, we use the same notations as in (2.2),(2.3),(2.4),(2.5), where A is considered as a flopping curve Z_i. By the observation above, we have $Z^+ \cap \gamma_t^+ = \emptyset$, $Q^+ \cap \gamma_t^+ = \emptyset$ (resp. $D^\flat \cap \gamma_t^\flat = \emptyset$, $F^\flat \cap \gamma_t^\flat = \emptyset$), where γ_t^+ (resp. $\gamma_t^\flat$) is the proper image of a generic conic γ_t in V^+ (resp. $V^\flat$). Thus we obtain $\varphi(Z^+) \cap \varphi(\gamma_t^+) = \emptyset$ (resp. $\psi(D^\flat) \cap \psi(\gamma_t^\flat) = \emptyset$). This is a contradiction because $\varphi(Z^+)$ and $\varphi(D^\flat)$ are ample (see (2.3.8),(2.5.6)). Therefore $E_0 \subset E$ is a line on X. This completes the proof of (3.3). □

3. Let $Z := E_0 \subset E$ be the line in (3.3), and we put $V := X$. Then $Q := Y$ is a ruled surface swept out by conics meeting Z. Let us consider the double projection π_{2Z} from the line Z. Then we have

$$\begin{array}{ccc} V' & \overset{\chi}{\dashrightarrow} & V^+ \\ \tau \downarrow & & \downarrow \varphi \\ V & \overset{\pi_{2Z}}{\dashrightarrow} & V_5 \cong W . \end{array}$$

Since

$$\begin{aligned} \mathbb{C}^3 \cong X - Y = V - Q &\cong V' - (Q' \cup Z') \\ &\cong V^+ - (Q^+ \cup Z^+) \\ &\cong W - F_5 \\ &\cong V_5 - F_5, \end{aligned}$$

one sees that (V_5, F_5) is a smooth compactification of $\mathbb{C}^3$, where we use the notations of (2.2),(2.3). By Theorem B (see Introduction), we obtain $F_5 \cong H_5^\infty$ or H_5^0. Moreover, $\Delta := \varphi(Q^+) \subset F_5$ is a smooth rational curve of degree 5 and $L_i := \varphi(Z_i^+) \subset F_5$ $(0 \leq i \leq m)$ is a line on V_5 which is a 2-chord for Δ

(3.4) Lemma. *The non-normal locus Σ of H_5^∞ is unique 2-chord for Δ, in particular, $\Delta \cap \Sigma = \{2p\}$ (double points).*

Proof. Let $\sigma : \overline{H_5^\infty} \longrightarrow H_5^\infty$ be the normalization and $\overline{\Sigma}$ be the analytic inverse image of Σ. Then it is known that $\overline{H_5^\infty} \cong \mathbf{F}_3$. Let s_3 be the negative section of $\mathbf{F}_3$. Then there is a fiber f_0 such that $\overline{\Sigma} = s_3 + f_0$ and $\sigma^*\Delta = s_3^\infty + f_0$, where $s_3^\infty \sim s_3 + 3f_3$ is an infinite section of $\mathbb{F}_3$ (cf.[**Fu**$_1$], [**F-N**$_2$], [**P-S**$_1$]). Let f_t $(t \neq 0)$ be a general fiber of $\mathbb{F}_3$. Since $(\sigma^*\Delta \cdot f_t) = 1$, the line $\sigma(f_t)$ cannot be a 2-chord for Δ. On the other hand, since $(\sigma^*\Delta \cdot \overline{\Sigma}) = 2$, the line Σ is a (unique) 2-chord for Δ. We put $p := \sigma(f_0)$. Then we have easily $\Delta \cap \Sigma = \{2p\}$ (double points). □

(3.5) Lemma([Fu$_1$]). *H_5^0 contains exactly one line Σ_0 passing through the rational double point p_0 of A_4-type.*

Under the notations above, we have the following:

(3.6) Proposition. *(1). The normal bundle $N_{Z|X}$ has the type $\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.*

(2). There exists no other line intersecting the line Z.

(3). $E_{red} = Z$, that is, the reduction E_{red} of the non-normal locus of Y is a line on X.

(4). $F_5 \cong H_5^\infty$.

Proof. (1): Assume that $N_{Z|X} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$, and let $Z_1^+, \cdots, Z_m^+$ be as in (2.2). Then we have $Z' \cong \mathbf{F}_1$, and L_i's are all 2-chords for Δ. Let f_t' be a general fiber, which are not intersecting the curves Z_i' $(1 \leq i \leq m)$. Let f_t^+ be it's proper image in V^+. In the case where $F_5 \cong H_5^\infty$, by (3.4), we have $m = 1$, in particular, $\varphi(f_t^+)$ is a conic with $\varphi(f_t^+) \cap L_1 = \emptyset$, where $L_1 = \Sigma$ is the non-normal locus of H_5^∞. This cannot occur since $H_5^\infty - \Sigma \cong \mathbb{C}^2$. In the case where $F_5 \cong H_5^0$, $\varphi(f_t^+)$ is a conic not passing through the singularity of H_5^0. Since $Pic\ H_5^0 \cong \mathbf{Z} \cdot (-K_{H_5^0})$, by an easy argument, one gets a contradiction.

(2): This follows directly from (3.4) and (3.5).

(3): Assume that E has an irreducible component other than $E_0 = Z$. By (2), we have the degree $deg\ E \geq 2$. Since $Y^+ := Q^+ \xrightarrow{\varphi} \Delta$ is a $\mathbb{P}^1$-bundle, it is smooth. Since $V' - Z_0' \cong V^+ - Z_0^+$, $Y' = Q'$ is smooth outside Z_0'. This contradicts the assumption.

(4): Assume that $F_5 \cong H_5^0$. Let $\mu : \widehat{H}_5^0 \longrightarrow H_5^0$ be the minimal resolution and let $B = \cup_{i=1}^4 B_i := \mu^{-1}(p_0)$ be the exceptional set of μ, where $p_0 = Sing\ H_5^0$. Then it is known that B is a linear tree of the (-2)-curves, and we have the following relation:

$$(B_i \cdot B_{i+1}) = 1\ (1 \leq i \leq 3), \quad (B_i \cdot B_j) = 0 \ \text{ if } \ |i-j| > 1,$$
$$(\widehat{\Sigma}_0 \cdot B_3) = 1, \quad (\widehat{\Sigma}_0 \cdot B_i) = 0 \text{ if } i \neq 3$$

, where $\widehat{\Sigma}_0$ is the proper transform of the line Σ_0 in $\widehat{H}_5^0$ (see [**Fu**$_1$]).

Since $H^2(\widehat{H}_5^0; \mathbf{Z}) \cong \bigoplus_{i=1}^4 \mathbf{Z}[B_i] \bigoplus \mathbf{Z}[\widehat{\Sigma}_0]$, the proper transform $\widehat{\Delta}$ of Δ in $\widehat{H}_5^0$ is written as follows:

$$\widehat{\Delta} \sim \sum_{i=1}^{4} k_i B_i + 5\widehat{\Sigma}_0,$$

for some $k_i \in \mathbf{Z}$.

If $p_0 \notin \Delta$, then since $(-K_{H_5^0} \cdot \Delta) = 5$, we have $\Delta^2 = 3$, hence we obtain $(\widehat{\Delta} \cdot \widehat{\Sigma}_0) = \frac{3}{5} \notin \mathbf{Z}$. Thus we have $p_0 \in \Delta$. Since Δ is a smooth curve passing through the rational double point p_0 of A_4-type, there exists exactly one component B_j such that $(\widehat{\Delta} \cdot B_j) = 1$, $(\widehat{\Delta} \cdot B_i) = 0$ $(i \neq j)$. Applying the adjunction formula, one gets $k_1 = \frac{j+5}{5} \notin \mathbf{Z}$ $(1 \leq j \leq 4)$. This is a contradiction. Therefore $F_5 \cong H_5^\infty$. The proof is completed. □

(3.7) Proposition (cf.[Is$_2$]). *Let Σ and Δ be as above. The inverse birational map $\pi_{2Z}^{-1} : V_5 - - - \succ V = V_{22}$ is given by the linear system $|\mathcal{O}_{V_5}(3) \otimes \mathcal{J}_\Sigma^2|$, where $\pi_{2Z}^{-1} = \tau \circ \chi^{-1} \circ \varphi^{-1}$ and $\mathcal{J}_\Sigma$ is the ideal sheaf of Σ.*

We put $H_{22}^\infty := \pi_{2Z}^{-1}(\Delta)$. Then we have just proved that $V_{22} - H_{22}^\infty \cong \mathbf{C}^3$ and H_{22}^∞ is a ruled surface swept out by conics intersecting the line $Z := E_{red} = \pi_{2Z}^{-1}(H_5^\infty)$. Consequently, under the notations above, we have :

(3.8) Proposition. *Let (X, Y) be a smooth projective compactification of $\mathbf{C}^3$ with $b_2(X) = 1$ and the index $r = 1$. Let $\pi : \widehat{Y} \xrightarrow{\mu} \overline{Y} \xrightarrow{\sigma} Y$ be the minimal resolution and put $\mathcal{L} := \mathcal{O}_Y(-K_X)$. Then*

(1). $K_{\widehat{Y}} + \pi^*\mathcal{L}$ is nef, and

(2). $(X, Y) \cong (V_{22}, H_{22}^\infty)$ if $(K_{\widehat{Y}} + \pi^*\mathcal{L})^2 = 0$.

Remark 5(Fu$_3$]). In the case of $\Delta \cap \Sigma = \{2p\}$ (double points), one has $\delta = 4$ and $\phi : \widehat{Y} \longrightarrow T \cong \mathbf{P}^1$ has exactly one singular fiber

$$f_0 := \bigcup_{i=1}^{13} B_i \cup \widehat{E}_1 \cup \widehat{E}_2.$$

Moreover, we obtain an linear equvalence

$$-K_{\widehat{Y}} \sim 2\widehat{E}_0 + 3\widehat{E}_1 + 3\widehat{E}_2 + \sum_{i=1}^{7}(3+i)B_i + \sum_{i=1}^{6}(3+i)B_{14-i} \ ,$$

where

$$(\widehat{E}_0 \cdot B_7) = (\widehat{E}_1 \cdot B_1) = (\widehat{E}_2 \cdot B_{13}) = 1, \ (\widehat{E}_i \cdot \widehat{E}_j) = 0 \ (i \neq j),$$
$$(B_i \cdot B_{i+1}) = 1, \ (B_i \cdot B_j) = 0 \ (|i-j| > 1),$$

and $(\widehat{E}_0 \cdot f) = 1$ for a general fiber f of ϕ.

The singularity of $\overline{Y}$ can be obtained from $\widehat{Y}$ by blowing down the linear tree of (-2)-curves $\bigcup_{i=1}^{13} B_i$, hence, $\overline{Y}$ has a rational double point of A_{13}-type as a singularity. Since $\widehat{E} = 2\widehat{E}_0 + 3\widehat{E}_1 + 3\widehat{E}_2$, $\overline{E} = V_{\overline{Y}}(\mathcal{I})$ is non-reduced (cf. Theorem D-(II)). Moreover, we have $H_{22}^\infty - E \cong \mathbf{C}^2$.

⋆ The structure of (X,Y) in the case (B).

4. Let $E_0 \subset E_{red}$ be any irreducible component of the non-normal locus E_{red} of Y. By assumption, $K_{\widehat{Y}} + \pi^*\mathcal{L}$ is nef and big. Then

(3.9) Proposition. $d := deg\, E_0 = (H \cdot E_0)_X = 1$, *where* H *is a hyperplane section of* $X = V_{22}$.

The proof is given in several steps.

(3.9.1). $mult_{E_0} Y = 2$.

Proof. Assume that $mult_{E_0} Y \geq 3$. Then any conic intersecting E_0 is always contained in Y. Hence Y is a ruled surface swept out by conics intersecting E_0 (see (2.1)-(iv)). Take a generic conic $\gamma \subset Y$ with $\gamma \cap E_0 \neq \emptyset$, and let $\widehat{\gamma}$ be the proper transform of γ in $\widehat{Y}$. Since $K_{\widehat{Y}} + \pi^*\mathcal{L}$ is nef and since $-K_{\widehat{Y}} = \widehat{E} + B$ is effective, we obtain $0 > (K_{\widehat{Y}} \cdot \widehat{\gamma}) \geq -(\pi^*\mathcal{L} \cdot \widehat{\gamma}) = -2$, that is, $(K_{\widehat{Y}} \cdot \widehat{\gamma}) = -1$ or -2 for a generic conic $\gamma \subset Y$. Since the (-1)-curves cannot make a continuous family, we conclude that $(K_{\widehat{Y}} \cdot \widehat{\gamma}) = -2$, that is, $(K_{\widehat{Y}} + \pi^*\mathcal{L} \cdot \widehat{\gamma}) = 0$ for a generic conic $\gamma \subset Y$. This shows that $(K_{\widehat{Y}} + \pi^*\mathcal{L})^2 = 0$, since $Bs|K_{\widehat{Y}} + \pi^*\mathcal{L}| = \emptyset$. This contradicts the assumption. Therefore we have $mult_{E_0} Y = 2$. □

(3.9.2). $d \leq 4$.

Proof. We shall first show that $\delta := (H \cdot E) \leq 6$. In fact, since $K_{\widehat{Y}} + \pi^*\mathcal{L}$ is nef and big, by the Kawamata vanishing theorem, we have $h^i(2K_{\widehat{Y}} + \pi^*\mathcal{L}) = 0$ for $i > 0$. By the Riemann-Roch theorem, we obtain $0 \leq h^0(2K_{\widehat{Y}} + \pi^*\mathcal{L}) = K_{\widehat{Y}}^2 - 3\delta + 12$, hence, we have $8 \geq K_{\widehat{Y}}^2 \geq 3\delta - 12$. This yields $\delta \leq 6$.

Let $\tau : X' \longrightarrow X$ be the blowing up of X along E_0 and let $E_0' := \tau^{-1}(E_0)$ be the exceptional ruled surface. Let Y' be the proper transform of Y in X'. Then we have $Y' \sim \tau^*H - 2E_0'$ by (3.9.1) and $(E_0')^3 = -c_1(N_{E_0|X}) = 2 - d$ (cf.[**Is**$_1$]). Let us consider an exact sequence

$$0 \longrightarrow \mathcal{O}_{X'}(E_0') \longrightarrow \mathcal{O}_{X'}(\tau^*H - E_0') \longrightarrow \mathcal{O}_{Y'}(\tau^*H - E_0') \longrightarrow 0.$$

Since $h^i(\mathcal{O}_{X'}(E_0')) = 0$ for $i > 0$ by the Kawamata vanishing theorem, we obtain the surjection

$$\mathbb{C}^{13-d} \cong H^0(\mathcal{O}_{X'}(\tau^*H - E_0') \longrightarrow H^0(\mathcal{O}_{Y'}(\tau^*H - E_0')) \cong \mathbb{C}^{12-d} \longrightarrow 0.$$

Since $Bs|\mathcal{O}_{X'}(\tau^*H - E_0')| = \emptyset$, we also have $Bs|\mathcal{O}_{Y'}(\tau^*H - E_0')| = \emptyset$. Let $\psi : X' \longrightarrow \mathbb{P}^{12-d}$ be a morphism defined by the complete linear system $|\mathcal{O}_{X'}(\tau^*H - E_0')|$ on X' and let $\psi' : Y' \longrightarrow \mathbb{P}^{11-d}$ be the restriction on Y'. Then we obtain $18 - 3d = (\tau^*H - E_0')^2(\tau^*H - 2E_0') \geq deg\, \psi'(Y') \geq codim\ \psi'(Y') + 1 = 10 - d$. This yields $d \leq 4$. □

(3.9.3). $d \leq 3$ *if* $E = E_0$ *is irreducible and reduced.*

Proof. By (3.9.2), we have $d \leq 4$. We assume that $d = 4$. Under the notations in (3.9.2), we have a (birational) morphism $\psi : Y' \longrightarrow M := \psi(Y') \hookrightarrow \mathbb{P}^7$, where $degM = codimM + 1 = 6$. Is is well-known that M is a rational scroll or a cone over a rational curve of degree 6 in $\mathbb{P}^6$. Take a smooth hyperplane section H containing E_0. Since $(H \cdot E_0) = 4$ and since $(E_0 \cdot E_0)_H = -2$, we obtain an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \longrightarrow N_{E_0|X} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(4) \longrightarrow 0.$$

This yields $N_{E_0|X} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$, where $(a,b) = (-2,4)$, $(-1,3)$, $(0,2)$, $(1,1)$, hence $E_0' \cong \mathbb{F}_t$ $(t = 0,2,4,6)$. We also have $\mathcal{O}_{E_0'}(Y') = \mathcal{O}_{E_0'}(-K_{E_0'}) = \mathcal{O}_{E_0}(2s_t + (t+2)f)$, where s_t (resp. f) is the negative section (resp. a fiber) of the Hirzebruch surface $\mathbb{F}_t$. We put $A := E_0' \cap Y'$.

(3.9.3.1). *Y' is normal.*

In fact, assume that Y' is non-normal. Then the non-normal locus is contained in $A = E_0' \cap Y'$ since E_0 is irreducible. Take a general hyperplane section H of X. Let A_0 be an irreducible component of A with $\tau^*H \cdot A_0 \neq 0$, here A_0 is not a fiber of $E_0' \cong \mathbb{F}_t$. Since $mult_{E_0} Y = 2$, Y' is smooth at a general point of A_0. Thus Y' is non-normal along a fiber $f_0 \subset E_0'$. On the other hand, since $(\tau^*H - E_0') \cdot f_0 = 1$, M has a singularity along the line $\psi(f_0)$ on M. This is absurd since M is normal. □

(3.9.3.2). *Y' has at most rational double points, in particular, the normalization $\overline{Y}$ is Gorenstein.*

In fact, let $g : \widehat{Y}' \longrightarrow Y'$ be the minimal resolution. Consider the following exact sequence of cohomology:

$$0 \longrightarrow H^1(\mathcal{O}_{Y'}) \longrightarrow H^1(\mathcal{O}_{\widehat{Y}'}) \longrightarrow H^0(R^1 g_* \mathcal{O}_{\widehat{Y}'}) \longrightarrow H^2(\mathcal{O}_{Y'}) \longrightarrow .$$

Since $\widehat{Y}'$ is rational and since $H^2(\mathcal{O}_{Y'}) = H^0(\mathcal{O}_{Y'}(-E_0')) = 0$, we get $H^0(R^1 g_* \mathcal{O}_{\widehat{Y}'}) = 0$, hence Y' has at most rational singularities. Since Y' is Gorenstein, we have the claim. □

(3.9.3.3). $\overline{Y} \cong \widehat{Y}'$.

We have only to prove that $A = E_0' \cap Y'$ contains no fiber of $E_0' \cong \mathbb{F}_t$. In fact, assume the contrary and let $f_0 \subset A$ be a fiber of E_0'. Then there is a birational morphism $h : \widehat{Y}' \longrightarrow \widehat{Y}$ such that $h(\widehat{f}_0)$ is a smooth point of M, where $\widehat{f}_0$ is the proper transform of f_0 in $\widehat{Y}'$. Hence $\widehat{f}_0$ is a (-1)-curve on $\widehat{Y}'$. We put $\mathcal{L}' := \tau^*H|_{Y'}$ and $\widehat{\mathcal{L}}' := g^*\mathcal{L}'$. Since $K_{Y'} + \mathcal{L}' = (\tau^*H - E_0')|_{Y'}$ is nef and big, so is $K_{\widehat{Y}'} + \widehat{\mathcal{L}}' = g^*(K_{Y'} + \mathcal{L}')$. Hence we have

$$0 \leq (K_{\widehat{Y}'} + \widehat{\mathcal{L}}') \cdot \widehat{f}_0 = -1 + (\widehat{\mathcal{L}}' \cdot \widehat{f}_0) = -1.$$

This is a contradiction. Therefore A contains no fiber of E_0'. This implies $Y' \cong \overline{Y}$. □

(3.9.3.4). *$b_2(M) = 1$, that is, M is a cone.*

In fact, since $mult_{E_0} Y = 2$, we obtain $b_2(A) \leq 2$. Taking into consideration that $X' - (Y' \cup E_0') \cong \mathbb{C}^3$, one sees $b_2(Y') = b_2(Y' \cap E_0') = b_2(A) \leq 2$. On the other hand, there is a line Z_1 on X meeting E_0 by (2.1). Then the proper transform Z_1' of Z_1 in Y' is blown down to a point of M since $(\tau^*H - E_0') \cdot Z_1' = 0$. This implies that $b_2(Y') = 2$ and $b_2(M) = 1$. □

(3.9.3.5). *Y is a ruled surface swept out by rational curves of degree three meeting E_0.*

According to (3.9.3.3), we have

$$K_{\overline{Y}} + \sigma^*\mathcal{L} = K_{Y'} + \mathcal{L}' = (\tau^*H - E_0')|_{Y'} \qquad \text{(3.9.3.5-a)}$$

and

$$(3.9.3.5\text{-b}) \qquad K_{\widehat{Y}} + \pi^*\mathcal{L} = \mu^*(K_{\overline{Y}} + \sigma^*\mathcal{L}).$$

Let L be a generic line on the cone $M \subset \mathbf{P}^7$ and let L' (resp. $\widehat{L}$) be the proper transform of L in $Y' = \overline{Y}$ (resp. $\widehat{Y}$). Since $(\tau^*H - E_0') \cdot L' = 1$, we get $(K_{\widehat{Y}} + \pi^*\mathcal{L}) \cdot \widehat{L} = 1$. One can easily see that the self-intersection number $(\widehat{L}^2)_{\widehat{Y}} = 0$, hence $(K_{\widehat{Y}} \cdot \widehat{L}) = -2$. This yields $(\pi^*\mathcal{L} \cdot \widehat{L}) = 3$, that is, $(H \cdot \pi(\widehat{L}))_X = 3$. This proves (3.9.3.5). □

(3.9.3.6). *$2K_{\widehat{Y}} + \pi^*\mathcal{L}$ is not nef.*

There is a line Z_1 meeting E_0 by (2.1). Let $\widehat{Z}_1$ be it's proper transform in $\widehat{Y}$. Since $Z_1 \neq E_0$, we obtain $(K_{\widehat{Y}} \cdot \widehat{Z}_1) < 0$. This implies $(2K_{\widehat{Y}} + \pi^*\mathcal{L} \cdot \widehat{Z}_1) = 2(K_{\widehat{Y}} \cdot \widehat{Z}_1) + 1 < 0$. Thus we have the claim. □

By (3.9.3.6) and the Cone theorem [**KMM**], one has three cases:

(i) $\widehat{Y} \cong \mathbb{P}^2$,

(ii) $\widehat{Y} \cong \mathbb{F}_n$ or

(iii) There is a (-1)-curve $\ell \subset \widehat{Y}$ such that $(\pi^*\mathcal{L} \cdot \ell) = 1$.

By an easy argument, one can exclude the first two cases, namely, $\widehat{Y} \not\cong \mathbb{P}^2$, $\mathbb{F}_n$. Thus we have the last case (iii).

Now, let $\phi' : \widehat{Y} \longrightarrow \widetilde{Y}_1$ be the blowing-bown of the (-1)-curve ℓ. If there is a (-1)-curve $\ell_1 \subset \widetilde{Y}_1$ with $(\widetilde{\mathcal{L}}_1 \cdot \ell_1) = 1$, then blow down it, where $\widetilde{\mathcal{L}}_1 := \phi'_*(\pi^*\mathcal{L})$. Repeating this process finitely many times, one has a birational morphism $\phi : \widehat{Y} \longrightarrow \widetilde{Y}$ onto a smooth projective surface $\widetilde{Y}$ satisfying

(a) $K_{\widehat{Y}} + \pi^*\mathcal{L} = \phi^*(K_{\widetilde{Y}} + \widetilde{\mathcal{L}})$, where $\widetilde{\mathcal{L}}) := \phi_*(\pi^*\mathcal{L})$.

(b) $2K_{\widetilde{Y}} + \widetilde{\mathcal{L}}$ is not nef.

(c) $(K_{\widehat{Y}})^2 = (K_{\widetilde{Y}})^2 + k$, $(-K_{\widetilde{Y}} \cdot \widetilde{\mathcal{L}}) = 8 + k$, $(\widetilde{\mathcal{L}})^2 = 22 + k$, for some positive integer k.

In fact, (a) and (c) are clear. To prove (b), take a general line L on M. Let $\widetilde{L}$ be the proper image of L in $\widetilde{Y}$. Since $(2K_{\widetilde{Y}} + \widetilde{\mathcal{L}}) \cdot \widetilde{L} = (K_{\widetilde{Y}} \cdot \widetilde{L}) + 1 < 0$, we have (b).

By construction, there is no (-1)-curve $\widetilde{\ell}$ with $(\widetilde{\mathcal{L}} \cdot \widetilde{\ell}) = 1$. Thus we have $\widetilde{Y} \cong \mathbf{P}^2$ or $\mathbb{F}_m$ by the Cone theorem. In the case of $\widetilde{Y} \cong \mathbb{P}^2$, $-(2K_{\widetilde{Y}} + \widetilde{\mathcal{L}})$ is ample on $\widetilde{Y} = \mathbf{P}^2$. This yields $deg\, \widetilde{\mathcal{L}} = 5$ and $k = 3$. By (c), we obtain $15 = (-K_{\widetilde{Y}} \cdot \widetilde{\mathcal{L}}) = 8 + 3 = 11$. This is a contradiction. Thus we have $\widetilde{Y} \cong \mathbf{F}_m$. Indeed, we have easily

(1) $\widetilde{Y} \cong \mathbb{F}_2$ and $\widetilde{\mathcal{L}} \sim 3s_2 + 8f$ or

(2) $\widetilde{Y} \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\widetilde{\mathcal{L}} \sim 3s_0 + 5f$.

From this, one sees $K_{\widetilde{Y}} + \widetilde{\mathcal{L}}$ is ample on $\widetilde{Y}$. This shows that $\phi : \widehat{Y} \longrightarrow \widetilde{Y}$ is given by the linear system $|K_{\widehat{Y}} + \pi^*\mathcal{L}|$, in particular, we have $\widetilde{Y} \cong M$ by (3.9.3.5-a and -b). This is absurd since $b_2(M) = 1$ by (3.9.3.4). The proof of (3.9.3) is completed. □

(3.9.4). *E_{red} contains no irreducible component E_0 of $d = degE_0 = 3$.*

Proof. In fact, assume that there is such an irreducible component E_0. Let us consider the double projection $\pi_{2E_0} : V \cdots \succ \mathbb{P}^2$ from the cubic curve E_0. By an argument similar to (2.3)-(2.7) in Takeuchi [T], we obtain a diagram:

$$\begin{array}{ccc} V' & \overset{\chi}{- - \succ} & V^+ \\ \sigma \downarrow & & \downarrow \varphi \\ V & \overset{\pi_{2E_0}}{- - \succ} & \mathbb{P}^2, \end{array}$$

where $\sigma : V' \longrightarrow V$ is the blowing up along E_0 with the exceptional ruled surface $E_0' := \sigma^{-1}(E_0)$, $\chi : V' - - \succ V^+$ is a flop, and $\varphi : V^+ \longrightarrow \mathbb{P}^2$ is a conic bundle over $\mathbb{P}^2$.

Let $Y' \sim \sigma^* H - 2E_0'$ be the proper transform of Y' in V', and let Y^+, E_0^+, H^+ be the proper transforms of Y', E_0', $H' := \sigma^* H - E_0'$ in V^+ respectively. Then E_0^+ is normal Gorenstein surface with at most rational double points. Moreover, we have $Y^+ = \varphi^* L$ for some line L on $\mathbb{P}^2$. For a generic fiber ℓ^+ of φ, we obtain $(H^+ \cdot \ell^+) = (E_0^+ \cdot \ell^+) = 2$. Since $-K_{E_0^+} = (H^+ - E_0^+)|_{E_0^+}$ and $(K_{E_0^+})^2 = (H^+ - E_0^+)^2 \cdot E_0^+ = 2$, $-K_{E_0^+}$ is nef big and $Bs| - K_{E_0^+}| = \emptyset$. This implies that the restriction $\varphi|_{E_0^+} : E_0^+ \longrightarrow \mathbb{P}^2$, which is defined by the linear system $| - K_{E_0^+}|$, is a double covering over $\mathbb{P}^2$. Thus the intersection $A^+ := Y^+ \cap E_0^+ = \varphi^{-1}(L) \cap E_0^+$ consists of at most two irreducible components, that is, $b_2(A^+) \leq 2$.

Now, since

$$V' - (Y' \cup E_0') \cong V^+ - (Y^+ \cup E_0^+) \cong \mathbb{C}^3,$$

we obtain

$$2 = b_2(V^+) = b_2(Y^+ \cup E_0^+) = b_2(Y^+) + b_2(E_0^+) - b_2(A^+),$$

hence,

$$b_2(Y^+) + b_2(E_0^+) = 2 + b_2(A^+) \leq 4. \tag{3.9.4.a}$$

Let $Z_0^+ \subset Y^+$ be the proper transform of the line $Z_1 \subset Y$ intersecting the cubic E_0. The flop $\chi : V' - - \succ V^+$ yields a new rational curve Z_0^+ which is contained in E_0^+. This shows that $b_2(E_0^+) \geq 3$, hence we have $b_2(Y^+) = 1$ by (3.9.4.a). This is impossible because the restriction $\varphi : Y^+ \longrightarrow L$ is a conical fibering. This proves (3.9.4). □

(3.9.5). *E_{red} contains no irreducible component D of $d = deg\, D = 2$.*

Proof. Assume the contrary and take a conic $D \subset E_{red}$. Then we consider the double projection $\pi_{2D} : X \longrightarrow \mathbb{Q}^3 \hookrightarrow \mathbb{P}^4$ from the conic D. In order to avoid the confusion, we use the same notations as in (2.5) and (2.6). We put $V := X$, and consider the following diagram:

$$\begin{array}{ccc} V'' & \overset{\chi}{- - \succ} & V^\flat \\ \lambda \downarrow & & \downarrow \psi \\ V & \overset{\pi_{2D}}{- - \succ} & U = \mathbb{Q}^3 \hookrightarrow \mathbb{P}^4 \end{array}$$

Then we have (cf.[T]):

(1) The number n of lines meeting the conic D is equal to four (counted with multiplicity) (see [**(2.8.2); T**]).
(2) $N_{Z_i|V''} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, or $\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}$ for $1 \leq i \leq n \leq 4$.
(3) $N_{D|V} \cong \mathcal{O}_D \oplus \mathcal{O}_D$, or $\mathcal{O}_D(-1) \oplus \mathcal{O}_D(1)$, that is, $D'' := \lambda^{-1}(D) \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_2$ (see [**(1.5)-(1.7); T**]).
(4) $Y^\flat := \chi'_*(Y'') \sim H^\flat - D^\flat$, where $Y'' \sim \lambda^* H - 2D''$ is the proper transform of Y in V''.
(5) $F^\flat := \chi'_*(F'') \sim 2H^\flat - 3D^\flat$, where $F'' \sim 2\lambda^* H - 5D''$ is the proper transform of the ruled surface F swept out by conics intersectiong the conic D.
(6) $F^\flat \cdot Z_i^\flat = 3$ for $1 \leq i \leq n \leq 4$.
(7) $\mathcal{O}_{V^\flat}(H^\flat - D^\flat) = \psi^* \mathcal{O}_U(1)$.
(8) $(H^\flat)^3 = 16,\ (H^\flat)^2 \cdot D^\flat = 4,\ H^\flat \cdot (D^\flat)^2 = -2,\ (D^\flat)^3 = -4$.

Moreover we put $S := \psi(D^\flat)$, $\Delta := \psi(F^\flat) \subset S$, $Q := \psi(Y^\flat)$, $\Sigma := \psi(Y^\flat \cap D^\flat) \subset Q \cap S$. Then,

(9) $Q \hookrightarrow U$ is a hyperplane section of $U = \mathbb{Q}^3$ and $S \sim 2Q$ is a normal del Pezzo surface of degree $(\omega_S^{-1})^2 = 4$. In particular, the minimal resolution $\widehat{D}^\flat$ of $D^\flat$ is obtained from $\mathbb{P}^2$ by the blowing-up of 5 points in (almost) general position, hence $b_2(D^\flat) \leq 6$. Δ is a smooth rational curve of degree $(\Delta \cdot Q) = 6$. Moreover, $deg \Sigma = (H^\flat - D^\flat) \cdot Y^\flat \cdot D^\flat = 4$.
(10) $(H^\flat \cdot \psi^{-1}(t)) = (D^\flat \cdot \psi^{-1}(t)) = 1$ for $t \in \Delta$.
(11) $b_2(Y^\flat \cap D^\flat) = b_2(Y^\flat) + b_2(D^\flat) - 2$ and $b_2(Y'') = b_2(Y'' \cap D'')$. This follows from the fact that $V'' - (Y'' \cup D'') \cong \mathbb{C}^3 \cong V^\flat - (Y^\flat \cup D^\flat)$, $b_2(Y'') = b_2(D'') = b_2(V^\flat) = 2$. In particular, since $Z_i^\flat \subset D^\flat$, we have $b_2(D^\flat) = 2+n$.

(a) The case of $D'' \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Let s_0 and f_0 be the section and a fiber of D'' . Let $s_0^\flat$ and $f_0^\flat$ be the proper transforms of s_0 and f_0 in $V^\flat$ respectively. Since $H^\flat \cdot s_0^\flat = 2$, the image $\psi(s_0^\flat)$ is not a point by (10). We put $\Delta'' := F'' \cap D'' \sim 5s_0 + 4f_0$ in D''. Then we obtain the virtual genus $p_a(\Delta'') = 12$. One can show that Δ'' is an irreducible curve with at most four singular points (infinitely near points allowed) (see [**Pagoda; Re**]).
This implies that

$$b_2(\Sigma) = b_2(Y^\flat \cap D^\flat) = b_2(Y^\flat) + b_2(D^\flat) - 2 = b_2(Y^\flat) + n \geq n + 2$$

by (11). On the other hand, since $deg\ \Sigma = 4$, we obtain $b_2(\Sigma) \leq 4$. Thus we have $n \leq 2$.

In case of $n = 2$, we have easily $b_2(\Sigma) = 4$, and $b_2(Y^\flat) = 2$. Thus Σ consists of four lines in $Q \cong \mathbb{Q}_0^2$. One can also show that the intersection $\Delta \cap Q$ consists of at least two points. Hence we have $b_2(Y^\flat) \geq 3$. This is a contradiction.

In case of $n = 1$, since $4 \geq b_2(\Sigma) = b_2(Y^\flat) + 1$, we have $b_2(Y^\flat) = 2$ or 3 , in particular, we have $Q \cong \mathbb{Q}_0^2$. On the other hand, it can be shown that the intersection $\Delta \cap Q$ consists of at least two points (resp. three points) if $b_2(Y^\flat) = 2$ (resp. $b_2(Y^\flat) = 3$). This is a contradiction because $b_2(Y^\flat) = b_2(Q) + {}^\#|Q \cap \Delta|$, where ${}^\#|Q \cap \Delta|$ is the number of points of the intersection $Q \cap \Delta$.

(b) The case of $D'' \cong \mathbb{F}_2$.

In this case, one can also show $F'' \cap D'' = \Delta'' \cup s_2$, where s_2 (resp. f_2) is the negative section (resp. a fiber) of $D'' \cong \mathbb{F}_2$ and $\Delta'' \sim 4s_2 + 9f_2$ is an irreducible curve with $p_a(\Delta'') = 12$. Then the proper transform $s_2^\flat \subset D^\flat$ of s_2 in $V^\flat$ is a fiber of the ruled surface $F^\flat = \psi^{-1}(\Delta)$. Since $-K_{D^\flat} = \psi^*Q|_{D^\flat}$ is nef and big, the minimal resolution $\widehat{D}^\flat$ of $D^\flat$ has no rational curve with the self-intersection number $-k$ ($k \geq 3$).This shows that $Z_i'' \cap s_2 = \emptyset$ (cf. **[Pagoda; Re]**).

By an arguement similar to the case (a), one obtains $b_2(Y^\flat) = 2$ and $\#|Q \cap \Delta| \geq 2$. This yields $2 = b_2(Y^\flat) \geq b_2(Q) + 2$, which is a contradiction. Therefore E_{red} contains no conic D in $V := X$. □

Proof of (3.9).

Since $\delta = (E \cdot H) \leq 6$ (see the proof of (3.9.2)), E_{red} consists of at most six irreducible components. If E_{red} contains a line E_0, then the other component of E_{red} is at most of degree three. In fact, taking the double projection π_{2E_0} : $V - -- \succ W = V_5 \hookrightarrow \mathbb{P}^6$, we can see that the image $\pi_{2E_0}(Y)$ is a non-normal hyperplane section of V_5, whose non-normal locus is a line on V_5 (cf. **[F-N$_2$]**, **[F-T]**, **[P-S$_1$]**). This implies that the degree of the other component of E_{red} is equal to three if it is neither a line nor a conic. The proof of (3.9) follows from this fact and (3.9.2)–(3.9.5). □

5. By (3.9), we know that the non-normal locus E_{red} of Y contains a line $Z := E_0$ in $V = X := V_{22} \hookrightarrow \mathbb{P}^{13}$. It is also known by **[Is$_1$]** that the normal bundle is either

(a) $N_{Z|V} \cong \mathcal{O}_Z(-1) \oplus \mathcal{O}_Z$

or

(b) $N_{Z|V} \cong \mathcal{O}_Z(-2) \oplus \mathcal{O}_Z(1)$.

Now, let us consider the double projection $\pi_{2Z} : V - -- \succ W = V_5 \hookrightarrow \mathbb{P}^6$. In order to avoid the confusion, we use the same notations as in (2.2), (2.3).

Then we have:

$$\begin{array}{ccc} V' & \overset{\chi}{- - \succ} & V^+ \\ \tau \downarrow & & \downarrow \varphi \\ V & \overset{\pi_{2Z}}{- - \succ} & W = V_5 \hookrightarrow \mathbb{P}^6 \end{array}$$

Let $Y' \sim \tau^*H - 2Z'$ be the proper transform of Y in V' and $Q' \sim \tau^*H - 3Z'$ the proper transform of the ruled surface Q swept out by conics meeting the line Z. We put $Y^+ := \chi_*(Y') \sim H^+ - Z^+$ and $Q^+ := \chi_*(F') \sim H^+ - 2Z^+$. Then $\varphi : V^+ \longrightarrow W = V_5$ is a blowing-up along the smooth rational curve Δ of degree 5 lying a unique hyperplane section $F_5 := \varphi(Z^+)$ of V_5. Hence $Q^+ = \varphi^{-1}(\Delta)$ is a $\mathbb{P}^1$-bundle over $\Delta \cong \mathbb{P}^1$. We put $F_5^0 := \varphi(Y^+)$, which is a hyperplane section of V_5 (see (2.3.8) and paragraph **3**).

(3.10) Proposition. *Each irreducible component Z of the non-normal locus E_{red} of Y has the normal bundle $N_{Z|V} \cong \mathcal{O}_Z(-2) \oplus \mathcal{O}_Z(1)$.*

Proof. Assume the contrary. Let $Z \subset E_{red}$ be a line with the normal bundle $N_{Z|V} \cong \mathcal{O}_Z(-1) \oplus \mathcal{O}_Z$. Then we obtain $Z' := \tau^{-1}(Z) \cong \mathbb{F}_1$. Let s_1 and f_1 be the negative section and a fiber of $Z' \cong \mathbb{F}_1$ respectively. Then we have:

(3.10.1). *Z^+ is normal.*

In fact, if Z^+ is non-normal, then so is $F_5 = \varphi(Z^+)$. Then the singular locus of F_5 is a line on V_5 and the normalization $\overline{F}_5$ of is isomorphic to $\mathbb{F}_1$ or $\mathbb{F}_3$ (cf. **[F-N$_2$]**, **[F-T]**). Since Z^+ has singularities at most along Z_i^+, there is exactly one line Z_1 meeting the line Z and hence $\varphi(Z_1^+)$ is the singular locus of F_5. In particular, F_5 is a ruled surface swept out by lines meeting the line $\varphi(Z_1^+)$. Let f_1^+ be the proper image of a general fiber f_1 in Z^+. Since $(H^+ - Z^+) \cdot f_1^+ = 2$, $\varphi(f_1^+) \subset F_5$ is a conic on V_5. Let $\overline{\varphi(f_1^+)}$ be the proper transform of $\varphi(f_1^+)$ in $\overline{F}_5$. One can easily show that there is no such family of conics $\{\overline{\varphi(f_1^+)}\}$ in $\overline{F}_5$. This proves (3.10.1). ☐

(3.10.2). *$Y' \cap Z' =: \Delta'$ is irreducible, in particular, there are three lines Z_i ($1 \leq i \leq 3$) meeting Z.*

In fact, $F_5 = \varphi(Z^+)$ is a normal del Pezzo surface of degree 5 with at most rational double points. Such a del Pezzo surface is completely classified in [**(8.4)**,**(8.5)**; **C-T**]. Then, using the relations

$$b_2(Y') = b_2(Y' \cap Z'),$$
$$b_2(Y^+ \cap Z^+) = b_2(Y^+) + b_2(Z^+) - 2,$$

one can show that $Y' \cap Z'$ contains neither the section s_1 nor a fiber f_1. Moreover, since $Y' \cdot Z' \sim 3s_1 + 4f_1$, one sees that $\Delta' \sim 3s_1 + 4f_1$ is irreducible. Since $\Delta = \varphi(Q^+)$ is a smooth rational curve and since $p_a(\Delta') = 3$, one can easily see that Δ' has exactly three double points. This implies that there are three flopping lines Z_i' ($1 \leq i \leq 3$) passing through these double points. This proves (3.10.2). ☐

Now, by (3.10.2), we have

$$b_2(Y^+ \cap Z^+) = b_2(Y^+) + b_2(Z^+) - 2 = b_2(Y^+) + 3 \geq 5.$$

On the other hand, since $Y' \cap Z' = \Delta'$ is irreducible, we obtain $b_2(Y^+ \cap Z^+) \leq 4$. This is a contradiction. This completes the proof of (3.10). ☐

6. Take an irreducible component $Z \subset E_{red}$. Then Z is a line on $V := X = V_{22}$ with the normal bundle $N_{Z|V} \cong \mathcal{O}_Z(-2) \oplus \mathcal{O}_Z(1)$ by (3.10), hence $Z' \cong \mathbb{F}_3$. Let s_3, f_3 be the negative section and a general fiber of $Z' \cong \mathbb{F}_3$. Let s_3^+, f_3^+ be their proper transforms in Z^+. Then we obtain $(Z' \cdot s_3) = 1 = -(Z^+ \cdot s_3^+)$, $(H' \cdot s_3) = (H^+ \cdot s_3^+) = 0$ and $(H^+ \cdot f_3^+) = 1$, in particular, $s_3^+ \subset Z^+$. Since $Q' \cdot Z' \sim 3s_3 + 7f_3$, the negative section s_3 must be an irreducible component of $Q' \cap Z'$.

(3.11) Lemma. *$Q' \cap Z'$ contains a fiber.*

Proof. Assume the contrary. Take an infinite section $s_\infty \sim s_3 + 3f_3$ of Z' and let s_∞^+ be its proper transform on Z^+. We may assume that s_∞^+ does not pass through the singular points of Z^+. Since $(H^+ \cdot s_\infty^+) = 4$ and $(s_\infty^+)^2 = 3$, we obtain $(Z^+ \cdot s_\infty^+) = -1$. This yields $(H^+ - Z^+) \cdot s_\infty^+ = 5$. Thus $\varphi(s_\infty^+) \subset F_5$ is a smooth rational curve of degree 5 with $Sing\, F_5 \cap \varphi(s_\infty^+) = \emptyset$. Since $(\omega_{F_5}^{-1} \cdot \varphi(s_\infty^+)) = 5$, we obtain $p_a(\varphi(s_\infty^+)) = 1$ by the adjunction formula. This is absurd because $\varphi(s_\infty^+)$ is a smooth rational curve. □

Let $\Delta^+ \subset Q^+ \cap Z^+$ be the irreducible component such that $\varphi(\Delta^+) = \Delta \subset F_5 = \varphi(Z^+)$ and $\Delta' \subset Q' \cap Z'$ the proper image of Δ^+ in Z'. Since $Q' \cap Z'$ contains the negative section s_3 and some fiber, we obtain either $\Delta' \sim 2s_3 + af_3$ or $s_3 + bf_3$ for some positive integers a, b. In the case of $\Delta' \sim 2s_3 + af_3$, since $(\Delta' \cdot f_3) = 2$ for a general fiber f_3, we obtain

$$2 = (Q^+ \cdot f_3^+) = (H^+ \cdot f_3^+) - 2(Z^+ \cdot f_3^+) = 1 - 2(Z^+ \cdot f_3^+),$$

which is absurd. Hence we obtain $\Delta' \sim s_3 + bf_3$ $(3 \le b \le 6)$ and $(Q^+ \cdot f_3^+) = 1$. Taking into consideration that $Q^+ \sim H^+ - 2Z^+$, one has $(Z^+ \cdot f_3^+) = 0$, and $(H^+ - Z^+) \cdot f_3^+ = 1$ for a general f_3^+. This shows that $\varphi(f_3^+) \subset F_5$ is a line on V_5 and thus F_5 is a ruled surface swept out by lines $\{\varphi(f_3^+)\}$ which intersect the line $\Sigma := \varphi(s_3^+) \subset F_5$. Hence F_5 is a non-normal hyperplane section of V_5. It is proved that the normalization $\overline{F}_5$ is isomorphic to $\mathbb{F}_3$ or $\mathbb{F}_1$ (cf.**[Fu$_1$]**, **[F-N$_2$]**, **[F-T]**). Moreover, we have the following:

Proposition (3.12). *(1). $Q' \cap Z' = \Delta' \cup A_1 \cup B_1$, where Δ', A_1, B_1 are smooth rational curves with $\Delta' \sim s_3 + 4f_3$, $A_1 \sim 2s_3$, $B_1 \sim 3f_3$ (as closed subschemes of $Z' \cong \mathbb{F}_3$).*

(2). $F_5 = \varphi(Z^+)$ is a non-normal del Pezzo surface of degree 5 whose non-normal locus is the line $\Sigma = \varphi(A_1^+)$ with the normal bundle $N_{\Sigma|V_5} \cong \mathcal{O}_\Sigma(-1) \oplus \mathcal{O}_\Sigma(1)$, where A_1^+ is the proper transform of A_1 in Z^+. In particular, F_5 is a ruled surface swept out by lines on $W = V_5$ meeting the line Σ.

(3). The image $\varphi(B_1^+) =: p$ is a point on $\Delta \subset F_5$ and $\Delta \cap \Sigma = \{p\}$, where B_1^+ is the proper transform of B_1 in Z^+.

(4). F_5 is obtained from the normalization $\overline{F}_5 \cong \mathbb{F}_3$ by identifying the negative section with a fiber of $\mathbb{F}_3$.

7. Next, we shall consider the surface $F_5^0 = \varphi(Y^+)$. Since $Y' \cdot Z' \sim 2s_3 + 5f_3$, the negative section s_3 must be contained in $Y' \cap Z'$. This implies $s_3^+ \subset Y^+$, namely, the line $\Sigma = \varphi(s_3^+) = \varphi(A_1^+)$ is contained in F_5^0. Since $p = \varphi(B_1^+) = \Delta \cap \Sigma \in F_5^0$, we obtain $B_1^+ \subset Y^+$. This shows that $Y' \cap Z'$ also contains a fiber f_3 of $Z' \cong \mathbb{F}_3$. Thus one sees that $Y' \cap Z' = A_2 \cup B_2$, where A_2, B_2 are smooth rational curves with $A_2 \sim 2s_3$, $B_2 \sim 5f_3$ (as closed subschemes of Z'). Let A_2^+ and B_2^+ be the proper transforms of A_2 and B_2 in Z^+ respectively. Then we have $\Sigma = \varphi(A_1^+) = \varphi(A_2^+)$ and $p = \varphi(B_1^+) = \varphi(B_2^+)$. Taking into consideration that $b_2(Y^+ \cap Z^+) = b_2(Y^+) + b_2(Z^+) - 2$, we obtain $b_2(Y^+) = 2$. This yields $b_2(F_5^0) = 1$

since $\Delta \cap F_5^0 \neq \emptyset$. On the other hand, the singular locus of F_5^0 is at most contained in the line Σ. Since F_5 is a unique hyperplane section of V_5 which has the line Σ as a non-normal locus, F_5^0 must be normal. In particular, since $b_2(F_5^0) = 1$, it has exactly one rational double point p of A_4-type (cf.[**Fu**$_1$], see also Case (A)).

It is known that $V_5 - F_5 \cong \mathbb{C}^3 \cong V_5 - F_5^0$ (cf. [**Fu**$_1$]). We put $\mathring{V}_5 := V_5 - F_5^0$, $\mathring{\Delta} := \mathring{V}_5 \cap \Delta$, $\mathring{F}_5 := \mathring{V}_5 \cap F_5$. Then we have easily $\mathring{V}_5 \supset \mathring{F}_5 \supset \mathring{\Delta}$.

From the defining equation of V_5 in $\mathbb{P}^6$ (cf. [**M-U**]), one can construct a polynomial automorphism $\alpha : \mathring{V}_5 \cong \mathbb{C}^3 \longrightarrow \mathbb{C}^3(x, y, z)$ such that

$$\alpha(\mathring{F}_5) = \{x = 0\}$$
$$\alpha(\mathring{\Delta}) = \{x = y = 0\},$$

where $x,\ y,\ z$ are coordinate functions of $\mathbb{C}^3$ (see [**Fu**$_5$]). This yields

$$\varphi^{-1}(\mathring{V}_5) - \mathring{F}_5{}^* \cong \mathbb{C}^3,$$

where $\mathring{F}_5{}^*$ is the proper transform of $\mathring{F}_5$ in $\varphi^{-1}(\mathring{V}_5)$.

On the other hand, since

$$\begin{aligned} X - Y &= V - Y \\ &\cong V' - (Y' \cup Z') \\ &\cong V^+ - (Y^+ \cup Z^+) \\ &\cong \varphi^{-1}(\mathring{V}_5) - \mathring{F}_5{}^* \\ &\cong \mathbb{C}^3, \end{aligned}$$

one sees that the compactification (X, Y) really exists in the case (B).

Conversely, take two compactifications (V_5, H_5^∞) and (V_5, H_5^0) of $\mathbb{C}^3$ with the index $r = 2$ satisfying:

(1) $H_5^\infty \cap H_5^0 = \Sigma := Sing\, H_5^\infty$, ($\Sigma$ is a line with the normal bundle $N_{\Sigma|V_5} \cong \mathcal{O}_\Sigma(-1) \oplus \mathcal{O}_\Sigma(1)$.

(2) $Sing\, H_5^0 =: p \in \Sigma$, (the point p is the rational double point of A_4-type) (cf.[**Fu**$_1$], [**F-N**$_2$], [**Fu**$_5$]).

One can easily see that there exists a smooth rational curve Δ of degree 5 contained in H_5^∞ such that $\Delta \cap \Sigma = \Delta \cap H_5^0 = \{p\}$.

Then the linear system $|\mathcal{O}_{V_5}(3) \otimes \mathcal{J}_\Delta^{\otimes 2}|$ on V_5 defines an inverse birational mapping $\pi_{2Z}^{-1} : V_5 - -- \succ V_{22} \hookrightarrow \mathbb{P}^{13}$ (see (3.7)).

Now, we put $H_{22}^0 := \pi_{2Z}^{-1}(F_5^0)$. Then (V_{22}, H_{22}^0) is a compactification of $\mathbb{C}^3$ and H_{22}^0 is a non-normal hyperplane section of V_{22} with the non-normal locus $E = \pi_{2Z}^{-1}(H_5^\infty)$. Moreover, $Z := E_{red}$ is a line with the normal bundle $N_{Z|V_{22}} \cong \mathcal{O}_Z(-2) \oplus \mathcal{O}_Z(1)$. By construction, we have $mult_Z H_{22}^0 = 2$.

Therefore we conclude:

(3.13) Proposition. $(X, Y) \cong (V_{22}, H_{22}^0)$ *if* $(K_{\hat{Y}} + \pi^*\mathcal{L})^2 > 0$.

By (3.8) and (3.13), the proof of main theorem is completed. □

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