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by

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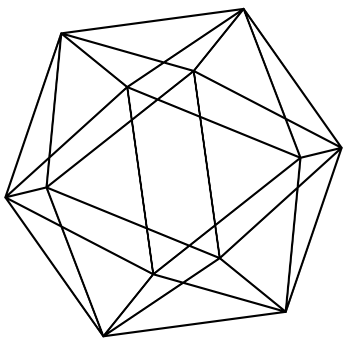

Max-Planck-Institut für Mathematik
Preprint Series 2011 (27)

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MPIM 11-27

A solution to the Gröbner Ring Conjecture

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May 7, 2011

Abstract

We prove that a valuation domain $\mathbf{V}$ has Krull dimension ≤ 1 if and only if for every finitely generated ideal I of $\mathbf{V}[X_1, \ldots, X_n]$, fixing the lexicographic order as monomial order, the ideal generated by the leading terms of the elements of I is also finitely generated. This proves the Gröbner ring conjecture. The proof we give is both simple and constructive. The same result is valid for Prüfer domains. As a "scoop", contrary to the common idea that Gröbner bases can be computed exclusively on Noetherian ground, we prove that computing Gröbner bases over $\mathbf{R}[X_1, \ldots, X_n]$, where $\mathbf{R}$ is a Prüfer domain, has nothing to do with Noetherianity, it is only related to the fact that the Krull dimension of $\mathbf{R}$ is ≤ 1 opening the doors to a wider class of rings over which Gröbner bases can be computed (the class of Prüfer domains of Krull dimension ≤ 1 instead of that of Dedekind domains).

MSC 2000 : 13C10, 19A13, 14Q20, 03F65.

Key words : Bezout domain, valuation domain, semihereditary ring, Gröbner ring conjecture, constructive mathematics.

Introduction

Recall that according to [12] a ring $\mathbf{R}$ is said to be *Gröbner* if for every $n \in \mathbb{N}$ and every finitely generated ideal I of $\mathbf{R}[X_1, \ldots, X_n]$, fixing a monomial order on $\mathbf{R}[X_1, \ldots, X_n]$, the ideal $\mathrm{LT}(I)$ generated by the leading terms of the elements of I is finitely generated. The *Gröbner ring conjecture* [6, 12] says that a valuation ring is Gröbner if and only if its Krull dimension is ≤ 1.

We prove (Theorem 2) that a valuation domain $\mathbf{V}$ satisfies the property "for any finitely generated ideal I of $\mathbf{V}[X_1, \ldots, X_n]$, fixing the lexicographic order as monomial order, the ideal $\mathrm{LT}(I)$ is finitely generated" if and only if its Krull dimension is ≤ 1. This proves the Gröbner ring conjecture (at least for the lexicographic monomial order), and also gives the first example of a class of non-Noetherian rings satisfying the property above. The proof we give is both simple and constructive. The same result is valid for Prüfer domains (Corollary 3).

As a "scoop", contrary to the common idea that Gröbner bases can be computed exclusively on Noetherian ground, we prove that computing Gröbner bases over $\mathbf{R}[X_1, \ldots, X_n]$, where $\mathbf{R}$ is a Prüfer domain, has nothing to do with Noetherianity, it is only related to the fact that the Krull dimension of $\mathbf{R}$ is ≤ 1 opening the doors to a wider class of rings over which Gröbner bases can be computed (the class of Prüfer domains of Krull dimension ≤ 1 instead of that of Dedekind domains).

It is worth pointing out that a solution to the Gröbner ring conjecture in case of one variable was obtained in [11].

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1 The Gröbner Ring Conjecture

Recall that a ring $\mathbf{R}$ has Krull dimension ≤ 1 if and only if

$$\forall a, b \in \mathbf{R},\ \ \exists n \in \mathbb{N},\ \ \exists\, x, y \in \mathbf{R} \ \ | \ \ a^n(b^n(1+xb)+ya)=0. \tag{1}$$

This is a constructive substitute for the classical abstract definition [2, 3, 7, 9]. For a valuation domain, it is easy to see that (1) amounts to the fact that the valuation group is archimedean.

Recall that a valuation domain $\mathbf{V}$ has Krull dimension ≤ 1 if and only if $\mathbf{V}\langle X\rangle$ (where for a ring $\mathbf{R}$, $\mathbf{R}\langle X\rangle$ denotes the localization of $\mathbf{R}[X]$ at monic polynomials) is a Bezout domain (see [9] for a constructive proof).

For any ring $\mathbf{R}$, one can define by induction the ring

$$\mathbf{R}\langle X_1,\ldots,X_n\rangle := (\mathbf{R}\langle X_1,\ldots,X_{n-1}\rangle)\langle X_n\rangle.$$

It is in fact the localization of the multivariate polynomial ring $\mathbf{R}[X_1,\ldots,X_n]$ at the monoid

$$S_n=\{p\in\mathbf{R}[X_1,\ldots,X_n]\ \mid\ \mathrm{LC}(p)=1\},$$

where $\mathrm{LC}(p)$ denotes the leading coefficient of p with respect to the lexicographic order on monomials with $X_1 < X_2 < \cdots < X_n$.

As mentioned above, if $\mathbf{V}$ is a valuation domain with dimension ≤ 1, then $\mathbf{V}\langle X_1,\ldots,X_n\rangle$ is a Bezout domain with Krull dimension ≤ 1.

The following lemma is immediate.

Lemma 1 *Let $\mathbf{A}$ be a ring. A term aX^k (where $a\in\mathbf{A}$ and $k\in\mathbb{N}$) belongs to an ideal of $\mathbf{A}[X]$ of the form $\langle b_\lambda X^{k_\lambda};\ \lambda\in\Lambda\rangle$, where $b_\lambda\in\mathbf{A}$ and $k_\lambda\in\mathbb{N}$, if and only if $a\in\langle b_\lambda;\ k_\lambda\leq k\rangle$.*

The following is the main result of this paper.

Theorem 2 *For a valuation domain $\mathbf{V}$, fixing the lexicographic order as monomial order, the following assertions are equivalent:*

1. *For any finitely generated ideal I of $\mathbf{V}[X_1,\ldots,X_n]$, the leading terms ideal $\mathrm{LT}(I)$ is also finitely generated.*
2. *If J is a finitely generated ideal of $\mathbf{V}[X_1,\ldots,X_n]$, then $J\cap\mathbf{V}$ is a principal ideal of $\mathbf{V}$.*
3. $\dim\mathbf{V}\leq 1$.

Proof. The implications "*1.* $\Rightarrow$ *2.* $\Rightarrow$ *3.*" are proved in [6] (see the proof of Theorem 11).

"*3.* $\Rightarrow$ *1.*" We proceed by induction on n. The result is obviously true for $n=0$. Let I be a finitely generated nonzero ideal of $\mathbf{V}[X_1,\ldots,X_n]$, say $I=\langle f_1,\ldots,f_s\rangle$. Denoting by $\mathbf{K}$ the quotient field of $\mathbf{V}$ and setting $\Delta := \gcd(f_1,\ldots,f_s)$ in $\mathbf{K}[X_1,\ldots,X_n]$, we have $I=\langle f_1,\ldots,f_s\rangle=\langle\Delta\, h_1,\ldots,\Delta\, h_s\rangle$ for some coprime polynomials $h_1,\ldots,h_s\in\mathbf{K}[X_1,\ldots,X_n]$. Replacing I by $\alpha\, I$ for an appropriate $\alpha\in\mathbf{V}\setminus\{0\}$, we may suppose that $\Delta, h_1,\ldots,h_s\in\mathbf{V}[X_1,\ldots,X_n]$. As $\mathbf{V}$ is a valuation domain, there is one coefficient a of one of the h_i's which divides all the others. Thus, one can write $I=a\,\Delta\,\langle g_1,\ldots,g_s\rangle$ where $\Delta, g_1,\ldots,g_s\in\mathbf{V}[X_1,\ldots,X_n]$, $\gcd(g_1,\ldots,g_s)=1$ in $\mathbf{K}[X_1,\ldots,X_n]$ and at least one of the g_i's is primitive. In particular, it follows that $\gcd(g_1,\ldots,g_s)=1$ in $\mathbf{V}[X_1,\ldots,X_n]$. As $\mathbf{V}\langle X_1,\ldots,X_n\rangle$ is a Bezout domain, denoting by $J=\langle g_1,\ldots,g_s\rangle$, we infer that

$$J\cap S_n\neq\emptyset.$$

Since proving that $\mathrm{LT}(I)$ is finitely generated amounts to proving that $\mathrm{LT}(J)$ is finitely generated, one may suppose that $I\cap S_n\neq\emptyset$. Moreover, by a change of variables "à la Nagata", we can suppose that I contains a monic polynomial at the variable X_n.

From now on, denoting by $\mathbf{A} = \mathbf{V}[X_1, \ldots, X_{n-1}]$, the leading terms of polynomials in $\mathbf{V}[X_1, \ldots, X_n]$ will be denoted using "LT" when considered as multivariate polynomials at the variables $X_1, \ldots, X_n$ and using "L" when considered as univariate polynomials at the variable X_n (i.e., in $\mathbf{A}[X_n]$). By virtue of Theorem 3 of [11] and its proof, with $\mathbf{A}$ as above and $X = X_n$, ($\mathbf{A}$ being coherent [5], see [1, 10] for a constructive proof), we have

$$\mathrm{L}(I) = \langle c_1(X_1, \ldots, X_{n-1})X_n^{\alpha_1}, \ldots, c_\ell(X_1, \ldots, X_{n-1})X_n^{\alpha_\ell}, X_n^m \rangle,$$

for some $\alpha_1 \leq \cdots \leq \alpha_\ell < m$ in $\mathbb{N}$ and $c_i \in \mathbf{A}$. One can rewrite $\{\alpha_1, \ldots \alpha_\ell\} = \{\beta_1, \ldots \beta_r\}$ with $\beta_1 < \cdots < \beta_r$. For $1 \leq j \leq r$, we set

$$\mathfrak{I}_j := \mathrm{LT}(\langle c_i \mid \alpha_i \leq \beta_j \rangle).$$

Now, for $f \in I$, let us denote by $\mathrm{LT}(f) = u\, X_1^{\gamma_1} \cdots X_{n-1}^{\gamma_{n-1}} X_n^{\gamma_n}$ and $\mathrm{L}(f) = (\cdots + u\, X_1^{\gamma_1} \cdots X_{n-1}^{\gamma_{n-1}})\, X_n^{\gamma_n}$ with $u \in \mathbf{V}$. If $\gamma_n \geq m$, then $\mathrm{LT}(f) \in \langle X_n^m \rangle$. Otherwise, as

$$\mathrm{LT}(f) = \mathrm{LT}(\mathrm{L}(f)),$$

then by writing $\mathrm{L}(f)$ as an element of $\langle c_1 X_n^{\alpha_1}, \ldots, c_\ell X_n^{\alpha_\ell}, X_n^m \rangle$ and using Lemma 1, one easily obtains that

$$\mathrm{L}(f) \in \mathfrak{I}_1.\, \langle X_n^{\beta_1} \rangle \ \vee\ \cdots\ \vee\ \mathfrak{I}_r.\, \langle X_n^{\beta_r} \rangle.$$

Thus,

$$\mathrm{LT}(I) = \mathfrak{I}_1.\, \langle X_n^{\beta_1} \rangle + \cdots + \mathfrak{I}_r.\, \langle X_n^{\beta_r} \rangle + \langle X_n^m \rangle.$$

By the induction hypothesis, all the $\mathfrak{I}_j$'s are finitely generated and thus so is $\mathrm{LT}(I)$.

□

Corollary 3 *For a Prüfer domain* $\mathbf{A}$*, fixing the lexicographic order as monomial order, the following assertions are equivalent:*

1. *For any finitely generated ideal* I *of* $\mathbf{A}[X_1, \ldots, X_n]$*, the leading terms ideal* $\mathrm{LT}(I)$ *is also finitely generated.*
2. *If* J *is a finitely generated ideal of* $\mathbf{A}[X_1, \ldots, X_n]$*, then* $J \cap \mathbf{A}$ *is finitely generated.*
3. $\dim \mathbf{A} \leq 1$.

Proof. The implications "*1.* ⇒ *2.* ⇒ *3.*" are as in [6].

"*3.* ⇒ *1.*" Follow the proof previously given for valuation domains applying the general dynamical technique as in [4, 6, 8, 9]. □

Remark 4 Over a valuation domain $\mathbf{V}$ (resp., a Prüfer domain $\mathbf{A}$) of Krull dimension ≤ 1, given an ideal I of $\mathbf{V}[X_1, \ldots, X_n]$ (resp., of $\mathbf{A}[X_1, \ldots, X_n]$), and fixing the lexicographic order as monomial order, $\mathrm{LT}(I)$ can be computed by computing a Gröbner basis (resp., a dynamical Gröbner basis) for I as explained in [6, 12] but of course there will be no need of noetherianity to ensure the termination of the generalized version (resp., the dynamical version of the generalized version) of Buchberger's algorithm, as it is ensured by the the fact that the Krull dimension is ≤ 1. Here, it is worth pointing out that the algorithm given in [12] which generalizes Buchberger's algorithm to Noetherian valuation rings contains a bug which is now corrected in a corrigendum [13] to this paper.

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