Hopf algebra orbits on the prime spectrum of a module algebra

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Introduction

When G is an automorphism group of a ring R, the G-conjugates of a prime ideal P of R can be characterized as those primes that are minimal over an ideal of the form $\bigcap_{g\in X} g^{-1}P$ for a finite subset $X \subset G$. Although this is a trivial observation, the previous description has the advantage that it can be reformulated in purely Hopf algebraic terms. It is tempting to introduce the orbit equivalence relation on prime ideals of an H-module algebra for a Hopf algebra H. There appears to be no immediate answer as to when this can be accomplished. In this paper the existence of the orbit equivalence relation will be established for one class of module algebras. The H-stratification considered in [5] defines in general a coarser equivalence relation.

Throughout the paper the base ring k is an arbitrary commutative ring. We have to make the following assumption about the Hopf algebra which will not be repeated any more: H is the union of a directed family $\mathcal{F}$ of subcoalgebras such that each $C \in \mathcal{F}$ is a finitely generated projective k-module. When k is a field this assumption is clearly satisfied for any Hopf algebra. Let A be an H-module algebra. Given a subcoalgebra C of H and an ideal I of A, the ideal I_C of A is defined by the rule

$$I_C = \{a \in A \mid Ca \subset I\}.$$

For each ring R denote by $\operatorname{Spec}_f R$ the set of those prime ideals P of R for which there exists no infinite strictly ascending chain $P_0 \subset P_1 \subset \cdots$ in $\operatorname{Spec} R$ starting at $P_0 = P$. In other words, $P \in \operatorname{Spec} R$ is in $\operatorname{Spec}_f R$ if and only if the factor ring R/P satisfies ACC on prime ideals. For instance, $\operatorname{Spec}_f R = \operatorname{Spec} R$ when R is either left or right noetherian.

Theorem 0.1. *Let A be an H-module algebra module-finite over its center. Then there is an equivalence relation $\sim_H$ on $\operatorname{Spec}_f A$ such that for $P, P' \in \operatorname{Spec}_f A$ one has $P \sim_H P'$ if and only if P' is a prime minimal over P_C for some $C \in \mathcal{F}$.*

The proof of Theorem 0.1 is indirect. It will be shown that for $P \in \operatorname{Spec}_f A$ and $C \in \mathcal{F}$ the factor algebra A/P_C has an artinian classical quotient ring $Q(A/P_C)$. The rings $Q(A/P_C)$ with C running over $\mathcal{F}$ form an inverse system. Define

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$$L_P(A) = \varprojlim Q(A/P_C).$$

The inverse limit of discrete topologies makes $L_P(A)$ into a linearly compact algebra. There is also an H-module structure with respect to which $L_P(A)$ is an H-module algebra. It turns out that $P' \in \operatorname{Spec}_f A$ is a prime minimal over P_C for some $C \in \mathcal{F}$ if and only if there exists a continuous homomorphism of H-module algebras

$$\psi : L_P(A) \to L_{P'}(A)$$

compatible with the canonical maps $A \to L_P(A)$ and $A \to L_{P'}(A)$. The relation on $\operatorname{Spec}_f A$ determined by the existence of such a homomorphism is obviously reflexive and transitive. This relation is symmetric because ψ, if it exists, necessarily has a continuous inverse. The trickiest part is to prove that $\operatorname{Ker} \psi = 0$, which follows from

Theorem 0.2. *Let A be an H-module algebra module-finite over its center, and let $P \in \operatorname{Spec}_f A$. Then the linearly compact H-module algebra $L_P(A)$ is* topologically H-simple*, that is, $L_P(A)$ has no H-stable closed ideals other than* 0 *and $L_P(A)$.*

Restriction of the relation $\sim_H$ to only a part of $\operatorname{Spec} A$ is unavoidable in this approach. As an example, suppose that P is a prime ideal of A properly containing $g^{-1}P$ for some grouplike element $g \in H$. Here $C = k + kg$ is a subcoalgebra such that $P_C = g^{-1}P \subset P$. If A/P_C has an artinian classical quotient ring, the latter has to be simple since P_C is prime. In this case the canonical map $A/P_C \to A/P$ does not extend to a homomorphism of quotient rings, and $L_P(A)$ is not defined.

The existence of quotient rings $Q(A/P_C)$ has another application concerned with the H-semiprime version of Goldie's Theorem. In contrast to [16, Th. 0.1] the next result is valid without further restrictions on H. In fact even the bijectivity of the antipode $S : H \to H$ is not needed. Recall that A is *H-semiprime* if A has no nonzero nilpotent H-stable ideals.

Theorem 0.3. *Any noetherian H-semiprime H-module algebra A module-finite over its center Z has a quasi-Frobenius classical quotient ring $Q(A)$ isomorphic with $A \otimes_Z Q(Z)$ where $Q(Z)$ is the total ring of fractions of Z.*

By [16, Th. 2.2] $Q(A)$ is an H-module algebra with respect to a module structure extending that on A, and by [16, Lemma 4.2] $Q(A)$ is H-semisimple. One may wonder whether the module-finiteness of A over its center can be weakened to the assumption that A satisfies a polynomial identity. In fact quasi-Frobenius classical quotient rings exist for finitely generated noetherian PI Hopf algebras [18, Th. 0.2].

1. The quasi-Frobenius property in the module-finite case

The final result of this section, Theorem 1.8, will be used to show that the quotient ring $Q(A)$ in Theorem 0.3 is quasi-Frobenius. This result should be compared with [16, Th. 0.3]. For its proof we will need a strengthened version of [15, Th. 7.4] which will be offered in Proposition 1.5. Recall that an H-module algebra is *H-simple* (resp. *H-semisimple*) if it has no nonzero proper H-stable ideals (resp. if it is a finite direct product of H-simple H-module algebras).

For an algebra A and a coalgebra C (over the base ring k), we consider $\operatorname{Hom}(C, A)$ as an algebra with respect to the convolution multiplication (see [13] or [17]). Denote by $\mathcal{M}_A$ the category of right A-modules. If $M \in \mathcal{M}_A$, then $\operatorname{Hom}(C, M)$ is a right module over $\operatorname{Hom}(C, A)$ with respect to the convolution action

$$(\eta * \xi)(c) = \sum_{(c)} \eta(c_{(1)})\xi(c_{(2)})$$

where $\xi \in \operatorname{Hom}(C, A)$, $\eta \in \operatorname{Hom}(C, M)$ and $c \in C$. Similarly, if M is a left B-module where B is another algebra, then $\operatorname{Hom}(C, M)$ is a left module over $\operatorname{Hom}(C, B)$. When M is a B, A-bimodule, the two module structures on $\operatorname{Hom}(C, M)$ commute.

Further on we assume that A is an H-module algebra (see [13] or [17]). So A has a left H-module structure such that, when $\tilde{a} : H \to A$ is given by the rule $\tilde{a}(h) = ha$ for $a \in A$ and $h \in H$, the assignment $a \mapsto \tilde{a}$ defines an algebra homomorphism $\tau : A \to \operatorname{Hom}(H, A)$. Given a subcoalgebra C of H and an ideal I of A, the inclusion $C \to H$ and the projection $A \to A/I$ give rise to an algebra homomorphism

$$\operatorname{Hom}(H, A) \to \operatorname{Hom}(C, A/I).$$

Let $\tau_{C,I} : A \to \operatorname{Hom}(C, A/I)$ denote the composite of the latter with τ. It is immediate that $\operatorname{Ker} \tau_{C,I} = I_C$, the ideal defined in the introduction. When $I = 0$ we write τ_C instead of $\tau_{C,I}$, and $\tilde{a}$ will often denote $\tau_C(a)$, i.e. a linear map $C \to A$.

Note that I_H is the largest H-stable ideal of A contained in I. If C, D are two subcoalgebras with $C \subset D$ then $I_C \supset I_D$. Since H is the union of subcoalgebras in $\mathcal{F}$, we have $I_H = \bigcap_{C \in \mathcal{F}} I_C$.

If $M \in \mathcal{M}_A$, then A operates on $\operatorname{Hom}(C, M)$ via τ_C. We call this action of A *twisted* in order to distinguish it from the *untwisted* action which is implemented via the elements $\hat{a} \in \operatorname{Hom}(C, A)$ defined by the rule $\hat{a}(c) = \varepsilon(c)a$ for $a \in A$ and $c \in C$ where $\varepsilon : H \to k$ is the counit. Note that $(\eta * \hat{a})(c) = \eta(c)a$ for any $\eta \in \operatorname{Hom}(C, M)$.

The assumption about $\mathcal{F}$ implies that H has a flat underlying k-module. Therefore we obtain as a special case of [16, Lemma 1.1(iii)]:

Lemma 1.1. *If E is an injective in $\mathcal{M}_A$, then so too is* $\operatorname{Hom}(H, E)$ *with respect to the twisted action of A.*

Let $M \in \mathcal{M}_A$. A k-linear map $H \otimes M \to M$, denoted $h \otimes m \mapsto hm$, will be called a *quasi-measuring* of H on M if

$$h(ma) = \sum_{(h)} (h_{(1)}m)(h_{(2)}a)$$

for all $h \in H$, $m \in M$, $a \in A$. We say that a quasi-measuring satisfies the *Surjectivity Condition* if for each $C \in \mathcal{F}$ the linear transformation Φ of $\operatorname{Hom}(C, M)$ defined by the rule

$$\Phi(\eta)(c) = \sum_{(c)} c_{(1)}\eta(c_{(2)}), \qquad \eta \in \operatorname{Hom}(C, M) \text{ and } c \in C,$$

is surjective. Define $\hat{m}, \tilde{m} \in \operatorname{Hom}(C, M)$ for each $m \in M$ by the formulas

$$\hat{m}(c) = m\varepsilon(c), \qquad \tilde{m}(c) = cm.$$

The identity in the definition of a quasi-measuring can be rewritten as $\widetilde{ma} = \tilde{m} * \tilde{a}$. Thus the quasi-measurings of H on M are in a bijective correspondence with the $\mathcal{M}_A$-morphisms $M \to \operatorname{Hom}(H, M)$ where we consider the twisted action of A on $\operatorname{Hom}(H, M)$.

Let $C^* = \operatorname{Hom}(C, k)$. The canonical homomorphism $k \to A$ induces by functoriality a homomorphism of convolution algebras $C^* \to \operatorname{Hom}(C, A)$. The latter allows us to view $\operatorname{Hom}(C, M)$ as a right C^*-module for $M \in \mathcal{M}_A$.

Lemma 1.2. *For each $C \in \mathcal{F}$ the C^*-module* $\mathrm{Hom}(C, M)$ *is generated by the set $\{\hat{m} \mid m \in M\}$. When M is equipped with a quasi-measuring satisfying the surjectivity condition, the elements $\tilde{m}$ give another generating set.*

Proof. There is a k-linear bijection $\mathrm{Hom}(C, M) \cong M \otimes C^*$. Given $m \in M$ and $\xi \in C^*$, the map $\eta : C \to M$ corresponding to $m \otimes \xi$ is defined by the rule $\eta(c) = m\xi(c)$ for $c \in C$. Note that

$$(\hat{m} * \xi)(c) = \sum_{(c)} m\varepsilon(c_{(1)})\xi(c_{(2)}) = m\xi(c).$$

Thus $\eta = \hat{m} * \xi$. This proves the first part of the lemma. Since

$$\Phi(\eta)(c) = \sum_{(c)} c_1\bigl(m\xi(c_{(2)})\bigr) = \sum_{(c)} (c_{(1)}m)\xi(c_{(2)}) = (\tilde{m} * \xi)(c)$$

for all $c \in C$, we have $\Phi(\eta) = \tilde{m} * \xi$. Therefore the second part of the lemma follows from the surjectivity of Φ. $\square$

Recall that a ring R is said to be *weakly finite* if for each integer $n > 0$ every generating set for the free right R-module R^n containing exactly n elements is a basis for R^n. This is equivalent to the condition that all one-sided invertible $n \times n$-matrices with entries in R are invertible on both sides.

Lemma 1.3. *Let $C \in \mathcal{F}$ and $M \in \mathcal{M}_A$. Suppose that I is an ideal of A such that A/I is weakly finite, $M/MI \cong (A/I)^n$ in $\mathcal{M}_A$ and the A-module M/MI_C is n-generated. If M admits a quasi-measuring of H satisfying the surjectivity condition, then $M/MI_C \cong (A/I_C)^n$ in $\mathcal{M}_A$. More precisely, any set of n generators for M/MI_C is a basis over A/I_C.*

Proof. Let $v_1, \ldots, v_n$ generate M modulo MI_C. Each element of M can be written as $m = \sum v_i a_i + u$ for some $a_1, \ldots, a_n \in A$ and $u \in MI_C$. Then $\tilde{m} = \sum \tilde{v}_i * \tilde{a}_i + \tilde{u}$ in $\mathrm{Hom}(C, M)$. Note that

$$c(MI_C) \subset \sum_{(c)} (c_{(1)}M)(c_{(2)}I_C) \subset MI$$

for all $c \in C$. In particular, $\tilde{u}(c) = cu \in MI$ for all $c \in C$. So $\tilde{u}$ is contained in the kernel of the $\mathrm{Hom}(C, A)$-linear map

$$\pi : \mathrm{Hom}(C, M) \to \mathrm{Hom}(C, M/MI)$$

induced by the projection $M \to M/MI$, and $\pi(\tilde{m})$ is a $\mathrm{Hom}(C, A)$-linear combination of $\pi(\tilde{v}_1), \ldots, \pi(\tilde{v}_n)$. The projectivity of C as a k-module ensures that π is onto, whence $\pi(\tilde{v}_1), \ldots, \pi(\tilde{v}_n)$ generate the $\mathrm{Hom}(C, A)$-module $\mathrm{Hom}(C, M/MI)$ by Lemma 1.2. On the other hand, $\mathrm{Hom}(C, A)$ operates on the latter via the algebra homomorphism

$$\pi' : \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, A/I).$$

induced by the projection $A \to A/I$. Since M/MI is a direct sum of n copies of A/I, we deduce that $\mathrm{Hom}(C, M/MI)$ is a free $\mathrm{Hom}(C, A/I)$-module of rank n.

There is an algebra isomorphism $\mathrm{Hom}(C, A/I) \cong A/I \otimes C^*$. In general, if S is a weakly finite ring and $S \to R$ is a ring homomorphism such that the right S-module R_S is finitely generated projective, then R is weakly finite too. This follows, e.g., from [15, Lemma 2.1] since S is isomorphic with a subring of the endomorphism ring $\mathrm{End}\, S_R$. This observation applied with $S = A/I$ and $R = \mathrm{Hom}(C, A/I)$ shows that $\mathrm{Hom}(C, A/I)$ is weakly finite. It follows that $\pi(\tilde{v}_1), \ldots, \pi(\tilde{v}_n)$ are in fact a basis for $\mathrm{Hom}(C, M/MI)$ over $\mathrm{Hom}(C, A/I)$.

Note that $\tilde{a} \in \mathrm{Ker}\, \pi'$ for $a \in A$ if and only if $Ca \subset I$, i.e. $a \in I_C$. Suppose that $a_1, \ldots, a_n \in A$ are any elements such that $\sum v_i a_i \in MI_C$. Then $\sum \tilde{v}_i * \tilde{a}_i \in \mathrm{Ker}\, \pi$, and therefore $\sum \pi(\tilde{v}_i) * \pi'(\tilde{a}_i) = 0$. It follows that $\pi'(\tilde{a}_i) = 0$, i.e. $a_i \in I_C$ for each i. Hence the cosets of $v_1, \ldots, v_n$ modulo MI_C are linearly independent over A/I_C. $\square$

Corollary 1.4. *Let I be an ideal of A such that A/I is weakly finite and $I_H = 0$. Suppose that $M \in \mathcal{M}_A$ admits a quasi-measuring of H satisfying the surjectivity condition. If $M/MI \cong (A/I)^n$ and M is generated by n elements $v_1, \ldots, v_n$, then $v_1, \ldots, v_n$ are a basis for M over A.*

Proof. Suppose that $\sum v_i a_i = 0$ for some $a_1, \ldots, a_n \in A$. Since for each $C \in \mathcal{F}$ the images of $v_1, \ldots, v_n$ in M/MI_C give a basis for that A/I_C-module, we have $a_i \in \bigcap_{C \in \mathcal{F}} I_C = I_H = 0$ for all i. $\square$

If P is a maximal ideal of A such that A/P is artinian, then we put

$$r_P(M) = \frac{\mathrm{length}\, M/MP}{\mathrm{length}\, A/P}$$

where length stands for the composition series length in $\mathcal{M}_A$. If A is semilocal then A has finitely many maximal ideals and A/P is artinian for each of those. The artinian rings are known to be weakly finite.

Proposition 1.5. *Suppose that M is a finitely generated right A-module which admits a quasi-measuring of H satisfying the surjectivity condition. If A is semilocal and H-simple, then M^l is a free A-module for a suitable integer $l > 0$.*

Proof. Pick a maximal ideal P of A with the maximum value of $r_P(M)$. Let $r_P(M) = n/l$ for some integers $n \geq 0$ and $l > 0$. Then $(M/MP)^l \cong (A/P)^n$ since the two A/P-modules here have equal length. Since $r_{P'}(M) \leq n/l$ for any other maximal ideal P' of A, the A-module $(M/MP')^l$ is n-generated. By Nakayama's Lemma M^l is n-generated. Since P_H is a proper H-stable ideal of A, we must have $P_H = 0$. Corollary 1.4 applied with $I = P$ shows that $M^l \cong A^n$. $\square$

Denote by ${}_H\mathcal{M}_A$ the class of A-modules M equipped with a quasi-measuring $H \otimes M \to M$ defining a left H-module structure on M. The H-module structure on A is a quasi-measuring, if we regard A as a module over itself with respect to right multiplications. Therefore $A \in {}_H\mathcal{M}_A$.

Lemma 1.6. *Let $M \in {}_H\mathcal{M}_A$. Then the quasi-measuring of H on M satisfies the surjectivity condition. Moreover, it extends to a quasi-measuring on the injective hull $E = E(M)$ of M in $\mathcal{M}_A$, and the latter also satisfies the surjectivity condition.*

Proof. Define $\varphi, \varphi' : H \to \mathrm{End}_k\, M$ by the formulas

$$\varphi(h)v = hv, \qquad \varphi'(h)v = S(h)v \tag{$\dagger$}$$

for $h \in H$ and $v \in M$. Then φ' is a right inverse (actually two-sided inverse) of φ in the convolution algebra $\mathrm{Hom}(H, \mathrm{End}_k M)$ since

$$(\varphi * \varphi')(h)v = \sum_{(h)} h_{(1)}\big(S(h_{(2)})v\big) = \varepsilon(h)v$$

for all h and v. For each $C \in \mathcal{F}$ we may regard $\mathrm{Hom}(C, M)$ as a left module over $\mathrm{Hom}(C, \mathrm{End}_k M)$. In terms of this module structure the transformation Φ of $\mathrm{Hom}(C, M)$ is nothing else but $\eta \mapsto \varphi|_C * \eta$. Since $\varphi'|_C$ is a right inverse of $\varphi|_C$ in $\mathrm{Hom}(C, \mathrm{End}_k M)$, we have $\varphi|_C * \varphi'|_C * \eta = \eta$ for all $\eta \in \mathrm{Hom}(C, M)$. Hence Φ is surjective, as required.

Let $\theta_M : M \to \mathrm{Hom}(H, M)$ denote the $\mathcal{M}_A$-morphism such that $v \mapsto \tilde{v}$ for $v \in M$. Since $\mathrm{Hom}(H, E)$ with the twisted action of A is injective in $\mathcal{M}_A$ by Lemma 1.1, there exists an $\mathcal{M}_A$-morphism θ_E rendering commutative the diagram

$$\begin{array}{ccc} M & \xrightarrow{\theta_M} & \mathrm{Hom}(H, M) \\ {\scriptstyle \text{can.}}\downarrow & & \downarrow{\scriptstyle \text{can.}} \\ E & \xrightarrow{\theta_E} & \mathrm{Hom}(H, E) \end{array}$$

This θ_E corresponds to a quasi-measuring of H on E extending that on M. Define now $\varphi, \varphi' : H \to \mathrm{End}_k E$ by formulas (†), taking $v \in E$. We don't know whether φ' is a right inverse of φ any longer. However, $\varphi * \varphi' \in \mathrm{Hom}(H, \mathrm{End}_A E)$ since

$$\begin{aligned} (\varphi * \varphi')(h)(va) &= \sum_{(h)} h_{(1)}\big(S(h_{(2)})(va)\big) \\ &= \sum_{(h)} \big(h_{(1)}(S(h_{(4)})v)\big) \cdot \big(h_{(2)}S(h_{(3)})a\big) \\ &= \sum_{(h)} \big(h_{(1)}(S(h_{(2)})v)\big) \cdot a \end{aligned}$$

for all $h \in H$, $v \in E$ and $a \in A$. Let $C \in \mathcal{F}$, and let Φ, Φ' be the transformations of $\mathrm{Hom}(C, E)$ given by the assignments $\eta \mapsto \varphi|_C * \eta$ and $\eta \mapsto \varphi'|_C * \eta$, respectively. Then

$$(\Phi\Phi')(\eta) = \varphi|_C * \varphi'|_C * \eta$$

for all $\eta \in \mathrm{Hom}(C, E)$. Since $\varphi|_C * \varphi'|_C \in \mathrm{Hom}(C, \mathrm{End}_A E)$, the transformation $\Phi\Phi'$ is $\mathrm{Hom}(C, A)$-linear. In particular, $\Phi\Phi'$ commutes with the untwisted action of A. Since C is finitely generated projective as a k-module, so too is C^*, and there is a k-linear bijection $\mathrm{Hom}(C, E) \cong E \otimes C^*$ under which the untwisted action of A corresponds to the A-module structure on the first tensorand. The right A-module $E \otimes X$ is an injective hull of $M \otimes X$ for each finitely generated projective k-module X since both $E \otimes X$ and $M \otimes X$ are additive functors in X. In particular, $\mathrm{Hom}(C, E)$ with the untwisted action of A is an injective hull of $\mathrm{Hom}(C, M)$ in $\mathcal{M}_A$. But we have checked at the beginning of the proof that $\Phi\Phi'$ is identity on $\mathrm{Hom}(C, M)$. Then $\Phi\Phi'$ has to be a bijective transformation of $\mathrm{Hom}(C, E)$. It follows again that Φ is surjective. □

Proposition 1.7. *Let A be a semilocal left or right noetherian H-simple H-module algebra. If there exists $0 \neq M \in {}_H\mathcal{M}_A$ with a finitely generated injective hull in $\mathcal{M}_A$, then A is quasi-Frobenius.*

Proof. By Lemma 1.6 and Proposition 1.5 $E(M)^l \cong A^n$ in $\mathcal{M}_A$ for some integers $l, n > 0$. It follows that A is right selfinjective, and the conclusion is a classical fact [6, Th. 18] (also [9, Th. 13.2.1]). $\square$

In particular, A is quasi-Frobenius when the injective hull $E(A)$ of A in $\mathcal{M}_A$ is finitely generated. As is well known, not every artinian ring R has the property that the class of finitely generated right R-modules is closed under injective hulls. However, R does enjoy this property whenever R is a finitely generated module over its center [14] (also [1, Ch. II, Cor. 3.4]).

Theorem 1.8. *Let A be an artinian H-semiprime H-module algebra. If A is module-finite over its center, then A is quasi-Frobenius and H-semisimple.*

Proof. The H-semisimplicity of A is proved in [16, Lemma 4.2] (this easily follows also from Theorem 0.1 whose proof does not depend on Theorem 1.8). Thus A is isomorphic to $A_1 \times \cdots \times A_n$ where each A_i is an H-simple H-module algebra. The module-finiteness over center passes to all factors in this decomposition. Therefore $E(A_i)$ is a finitely generated A_i-module and A_i is quasi-Frobenius by Proposition 1.7, for each i. Hence A is quasi-Frobenius. $\square$

2. Linearly compact module algebras

Let R be a ring. A topology on an R-module M is *linear* if the open submodules of M form a neighbourhood base of 0. A linearly topologized R-module M is *linearly compact* if M is separated and for each set $\mathcal{B}$ of closed submodules directed by inverse inclusion the canonical map $M \to \varprojlim_{N\in\mathcal{B}} M/N$ is surjective. A topological ring R is *right linearly compact* if R is a linearly compact R-module via right multiplications. The general theory of linearly compact rings and modules was developed by Zelinsky [19] and Leptin [11]. The three properties below are standard (see also exercises in [3, Ch. III, §2]):

(LC1) *Any continuous homomorphism from a linearly compact module to a separated linearly topologized module has a linearly compact closed image.*

(LC2) *Inverse limits of linearly compact modules are linearly compact.*

(LC3) *Let M be the inverse limit of an inverse system $(M_\alpha, \psi_{\alpha\beta})$ of linearly compact modules indexed by a directed set. The canonical map $M \to M_\alpha$ is surjective for each α provided that all $\psi_{\alpha\beta}$ are surjective.*

Lemma 2.1. *Let M be a linearly compact R-module, (N_α) an indexed collection of closed submodules. Denote by I_α the annihilator of M/N_α in R. If $I_\alpha + I_\beta = R$ for each pair of indices $\alpha \neq \beta$, then the canonical map $M \to \prod M/N_\alpha$ is surjective.*

Proof. By the Chinese Remainder Theorem

$$M/(N_{\alpha_1} \cap \cdots \cap N_{\alpha_n}) \cong M/N_{\alpha_1} \times \cdots \times M/N_{\alpha_n}$$

for each finite subset of indices $\{\alpha_1, \cdots, \alpha_n\}$. It follows that $\prod M/N_\alpha$ is isomorphic with $\varprojlim_{N\in\mathcal{B}} M/N$ where $\mathcal{B}$ is the set of all finite intersections of submodules N_α. The definition of linear compactness yields the conclusion. $\square$

Lemma 2.2. *Let $(R_\alpha, \varphi_{\alpha\beta})$ be an inverse system of right linearly compact rings indexed by a directed set. If all $\varphi_{\alpha\beta}$ are surjective then the ring $R = \varprojlim R_\alpha$ is right linearly compact and the canonical map $R \to R_\alpha$ is surjective for each α.*

Proof. The surjectivity of $R \to R_\alpha$ follows from [4, Ch. I, Appendix, Th. 1] which can be applied by considering for each α the set $\mathfrak{S}_\alpha$ of cosets of closed right ideals of R_α. Now $(R_\alpha, \varphi_{\alpha\beta})$ may be regarded as an inverse system of linearly compact right R-modules. So R is right linear compact by (LC2). $\square$

Any artinian module is linearly compact with respect to the discrete topology. Therefore Lemma 2.2 can be applied in the case where all rings R_α are right artinian.

When we consider a linearly compact right module M over a right linearly compact ring R, we assume tacitly that M is a topological R-module, so that the module structure comes from a continuous map $M \times R \to M$.

Lemma 2.3. *Let R be a right linearly compact ring, M a linearly compact right R-module. Suppose that for each maximal open submodule W of M there exists a maximal ideal P of R such that $MP \subset W$ and the ring R/P is artinian. If $n \geq 0$ is an integer such that the R-module M/V is n-generated for each open submodule V of M, then M is n-generated.*

Proof. Denote by Ω the set of all maximal ideals of R which annihilate a factor module M/W for some maximal open submodule W of M. For $P \in \Omega$ the ring R/P is simple artinian. Hence $\operatorname{length} M/V \leq n \operatorname{length} R/P$ for any open submodule V of M such that P annihilates M/V. It follows that R satisfies DCC on submodules of this type. Any closed submodule N of M is an intersection of open submodules. If P annihilates M/N then N itself has to be open. In particular, this is valid when N is the closure of MP. The R-module $M/\overline{MP}$ is therefore n-generated by the hypothesis. By Lemma 2.1 the map $M \to \prod_{P\in\Omega} M/\overline{MP}$ is surjective. Hence there exist elements $v_1, \ldots, v_n \in M$ which generate M modulo $\overline{MP}$ for each P. The submodule M' of M generated by $v_1, \ldots, v_n$ is the image of a module homomorphism $\theta : R^n \to M$. Since R^n is linearly compact and θ is necessarily continuous, M' is closed in M by (LC1). Then M' is an intersection of open submodules of M. Suppose that $M' \subset V$ for some open submodule $V \neq M$. Since M/V is finitely generated, V is contained in a maximal submodule W of M. In this case M/W is annihilated by some $P \in \Omega$, and so $M' + \overline{MP} \subset W$. However, this contradicts the equality $M' + \overline{MP} = M$. Thus the only possibility is $M' = M$, so that $v_1, \ldots, v_n$ generate M. $\square$

Corollary 2.4. *Let $(R_\alpha, \varphi_{\alpha\beta})$ be an inverse system of semilocal right linearly compact rings indexed by a directed set and $(M_\alpha, \psi_{\alpha\beta})$ an inverse system with the same index set in which each M_α is an n-generated linearly compact right R_α-module and $\psi_{\alpha\beta} : M_\beta \to M_\alpha$ is an R_β-linear continuous map for each pair of indices $\alpha \leq \beta$. Put $R = \varprojlim R_\alpha$ and $M = \varprojlim M_\alpha$. Suppose that all $\varphi_{\alpha\beta}$ and all $\psi_{\alpha\beta}$ are surjective. Then the right R-module M is n-generated.*

Proof. We know already that R and M are linearly compact and the canonical maps $\varphi_\alpha : R \to R_\alpha$ and $\psi_\alpha : M \to M_\alpha$ are surjective. If V is an open submodule of M then $\psi_\alpha^{-1}(V_\alpha) \subset V$ for some α and an open submodule V_α of M_α. The R-module M/V is n-generated since it is an epimorphic image of $M/\psi_\alpha^{-1}(V_\alpha) \cong M_\alpha/V_\alpha$. Note that $\operatorname{Ker} \varphi_\alpha$ annihilates M/V. If V is a maximal submodule then M/V is a simple R_α-module; denoting $P = \varphi_\alpha^{-1}(P_\alpha)$ where P_α is the annihilator of M/V in R_α, we see that $MP \subset V$ and the ring $R/P \cong R_\alpha/P_\alpha$ is artinian. Thus the hypothesis of Lemma 2.3 is satisfied. $\square$

A *right linearly compact H-module algebra* is an H-module algebra A equipped with a topology with respect to which A is a right linearly compact ring and all elements of H operate on A as continuous transformations. If I is a closed ideal of such an A, then I_C is a closed ideal for any subcoalgebra C of H; in particular I_H is an H-stable closed ideal. We say that A is *topologically H-simple* if A has no H-stable closed ideals other than 0 and A. Denote by $\Omega(A)$ the set of all open maximal ideals of A.

Proposition 2.5. *Let A be a right linearly compact H-module algebra. Suppose that A has a neighbourhood base of 0 consisting of ideals K such that A/K is right artinian. Suppose also that $P_H = 0$ for at least one $P \in \Omega(A)$. If there exist integers $s, t \geq 0$ such that*

(a) length $A/P \leq s$ *for each $P \in \Omega(A)$,*

(b) *for each open ideal K of A any idempotent ideal of A/K is generated as a right ideal by at most t elements,*

then A is topologically H-simple.

Proof. Let $\mathcal{X}$ be the collection of all nonzero H-stable closed ideals of A. Denote $\Omega_1 = \{P \in \Omega(A) \mid P_H = 0\}$ and $\Omega_2 = \{P \in \Omega(A) \mid P_H \neq 0\}$. By the hypothesis $\Omega_1 \neq \varnothing$. If $P \in \Omega_1$ then $I + P = A$ for each $I \in \mathcal{X}$. It follows that $IJ \neq 0$, and in particular $I \cap J \in \mathcal{X}$, for any $I, J \in \mathcal{X}$. Let K be any open ideal of A. Since A/K is right artinian, the set of ideals of A of the form $I + K$ for some $I \in \mathcal{X}$ has a smallest element which we denote by $K^\dagger$. Denote by M the intersection of all $I \in \mathcal{X}$, and by N the intersection of all $K^\dagger$ for different K. Let $I \in \mathcal{X}$. Since I is closed, we have $I = \bigcap_K (I + K)$. The inclusions $K^\dagger \subset I + K$ show that $N \subset I$. Since this is valid for any I, we deduce $N \subset M$. On the other hand, each $K^\dagger$ contains some ideal from $\mathcal{X}$, and therefore $M \subset N$. Thus $M = N$.

The linear compactness of A ensures that $A \cong \varprojlim A/K$, and then $M \cong \varprojlim K^\dagger/K$. Suppose that K, L are any two open ideals of A. Then $K^\dagger = I + K$ and $L^\dagger = J + L$ for some $I, J \in \mathcal{X}$. The same equalities remain valid if we replace both I and J with $I \cap J$. Therefore $L^\dagger = K^\dagger + L$ whenever $K \subset L$. This shows that all maps in the inverse system of right A-modules $K^\dagger/K$ are surjective. Each of these modules is artinian, hence linearly compact. By (LC3) the canonical maps $M \to K^\dagger/K$ are surjective, i.e. $K^\dagger = M + K$ for each K. If $P \in \Omega_1$ then $P^\dagger = A$, so that $M + P = A$. In particular $M \neq 0$. Since M is a closed H-stable ideal, M is a smallest element of $\mathcal{X}$. We have then $M^2 = M$. It follows that each $K^\dagger/K$ is an idempotent ideal of A/K, and so $K^\dagger/K$ is generated as a right ideal by t elements. Corollary 2.4 ensures that M is a finitely generated right ideal of A.

Recall that $r_P(M)$ denotes (length M/MP)/(length A/P) where the lengths are computed in $\mathcal{M}_A$. We have $MI = M$ for each $I \in \mathcal{X}$. In particular $MP_H = M$, and therefore $MP = P$, for each $P \in \Omega_2$. The last equality shows that $r_P(M) = 0$ when $P \in \Omega_2$. For each $P \in \Omega(A)$ we have $0 \leq r_P(M) \leq t$; moreover, $r_P(M)$ is a fraction whose denominator is bounded by s in view of (a). There exist finitely many rational numbers with those properties. Therefore $r_P(M)$ attains a maximum value m at some P; clearly such a P can be taken in Ω_1.

Let $m = n/l$ for some integers $n \geq 0$ and $l > 0$. Then $r_P(M^l) \leq n$ for each $P \in \Omega(A)$. These inequalities together with Nakayama's Lemma ensure that the right A-module M^l/M^lK is n-generated for each open ideal K of A since the ring A/K is semilocal. Suppose that V is any open submodule of M^l. Since the action

$M^l \times A \to M^l$ is continuous, for each $u \in M^l$ there exists an open ideal K_u of A such that $uK_u \subset V$. As M^l is finitely generated, $M^l K \subset V$ for some open ideal K of A. It follows that the right A-module M^l/V is n-generated. If V is a maximal submodule, then M^l/V is a simple A/K-module; hence M^l/V is annihilated by some $P \in \Omega(A)$. Thus Lemma 2.3 can be applied. It shows that the right A-module M^l is n-generated.

There exists $P \in \Omega_1$ for which $r_P(M^l) = n$, and then $M^l/M^l P \cong (A/P)^n$ in $\mathcal{M}_A$. Since $M^l \in {}_H\mathcal{M}_A$, Lemma 1.6 and Corollary 1.4 yield $M^l \cong A^n$ in $\mathcal{M}_A$. But then $r_P(M) = m$ for all $P \in \Omega(A)$. Since $M \neq 0$, we have $n > 0$, i.e. $m > 0$, and it follows that $\Omega_2 = \varnothing$.

Let $I \in \mathcal{X}$. If $I \neq A$, then I is contained in an open ideal $K \neq A$; hence I is contained in some $P \in \Omega(A)$. However, the last inclusion implies that $I \subset P_H$, so that $P_H \neq 0$, and $P \in \Omega_2$. This contradiction entails $\mathcal{X} = \{A\}$. $\square$

Conditions (a) and (b) in Proposition 2.5 are easily verified in the case when for each open ideal K of A the ring A/K is generated as a module over its center $Z(A/K)$ by t elements. In fact (a) holds with $s = t$ since $Z(A/P)$ is a field for $P \in \Omega(A)$. The assumption that A/K is right artinian implies that $Z(A/K)$ is artinian by [2], and then (b) follows from the next lemma:

Lemma 2.6. *Let R be a Z-algebra where Z is a semiprimary commutative ring. Suppose that R is generated as a Z-module by t elements. Then any idempotent ideal I of R is a t-generated right ideal.*

Proof. Consider first a special case where $Z \cong K_1 \times \cdots \times K_n$ is a direct product of fields. Here $R \cong R_1 \times \cdots \times R_n$ where R_i is a K_i-algebra with $\dim_{K_i} R_i \le t$ for each $i = 1, \ldots, n$. Each right ideal of R is generated by t elements since this holds for right ideals of $R_1, \ldots, R_n$.

In general let $\mathfrak{n}$ denote the Jacobson radical of Z. Then $Z/\mathfrak{n}$ is a direct product of fields and $R/\mathfrak{n}R$ is a $Z/\mathfrak{n}$-algebra generated as a $Z/\mathfrak{n}$-module by t elements. By the previous step $(I + \mathfrak{n}R)/\mathfrak{n}R$ is a t-generated right ideal of $R/\mathfrak{n}R$. There exists a t-generated right ideal J of R such that $J \subset I \subset J + \mathfrak{n}R$. By induction

$$I^m \subset J + \mathfrak{n}^m R$$

for all integers $m > 0$. Indeed, if this inclusion holds for some m, then we obtain $I^{m+1} \subset (J + \mathfrak{n}^m R)I \subset J + I\,\mathfrak{n}^m \subset J + \mathfrak{n}^{m+1}R$. When m is sufficiently large, we have $\mathfrak{n}^m = 0$ since Z is semiprimary. However $I^m = I$ since I is idempotent. Hence $I = J$. $\square$

The next result presents a linearly compact version of Proposition 1.5. The assumption about the lengths of rings A/P is needed only to ensure that the function $P \mapsto r_P(M)$ attains the maximum value. The assumption that A is topologically H-simple can be weakened to the assumption that there exists $P \in \Omega(A)$ such that $P_H = 0$ and $r_{P'}(M) \le r_P(M)$ for all $P' \in \Omega(A)$.

Theorem 2.7. *Let A be a topologically H-simple right linearly compact H-module algebra, and let M be a finitely generated right A-module which admits a quasi-measuring of H satisfying the surjectivity condition (e.g. $M \in {}_H\mathcal{M}_A$). Suppose that A has a neighbourhood base of 0 consisting of ideals K such that A/K is semilocal. Suppose also that the lengths of the artinian rings A/P with $P \in \Omega(A)$ are bounded. Denote $M_0 = \bigcap MK$, the intersection over all open ideals K of A. Then:*

(i) *The rank function* $P \mapsto r_P(M)$, $P \in \Omega(A)$, *has a constant value, say* $r(M)$.

(ii) $(M/M_0)^l \cong A^n$ *in* $\mathcal{M}_A$ *for any integers* $n \geq 0$ *and* $l > 0$ *such that* $r(M) = n/l$.

(iii) *If* $\gcd\{\text{length}\, A/P \mid P \in \Omega(A)\} = 1$ *then* M/M_0 *is a free* A*-module.*

Proof. Consider the linear topology on M whose neighbourhood base of 0 is given by the submodules MK with K an open ideal of A. If $C \in \mathcal{F}$, then for each open ideal K there exists another open ideal L such that $CL \subset K$; this inclusion implies that $C(ML) \subset MK$. It follows that $CM_0 \subset M_0$. Furthermore, each k-linear map $C \to M/M_0$ can be lifted to a k-linear map $C \to M$. Hence $HM_0 \subset M_0$ and the induced quasi-measuring on M/M_0 satisfies the surjectivity condition. Now we can replace M with M/M_0, and so assume that the topology on M is separated. There exists an epimorphism $\theta : A^t \to M$ in $\mathcal{M}_A$ for some integer $t \geq 0$. Since θ is necessarily continuous and A^t is a linearly compact A-module, so too is M by (LC1). As in the proof of Proposition 2.6 we find $P \in \Omega(A)$ for which $r_P(M)$ attains a maximum value and show that the A-module M^l is n-generated when $n/l = r_P(M)$. Finally, $M^l \cong A^n$ by Corollary 1.4, whence $r_{P'}(M) = n/l$ for each $P' \in \Omega(A)$.

Note that $r(M) \cdot \text{length}\, A/P$ is an integer for each $P \in \Omega(A)$. Hence so too is $r(M)d$ where $d = \gcd\{\text{length}\, A/P \mid P \in \Omega(A)\}$. If $d = 1$ then $r(M)$ is an integer, and we may take $l = 1$. $\square$

Corollary 2.8. *Let* A *be as in Theorem* 2.7*. Then any* H*-stable closed right ideal* I *of* A *is generated by an idempotent. If* $\gcd\{\text{length}\, A/P \mid P \in \Omega(A)\} = 1$ *then necessarily either* $I = 0$ *or* $I = A$.

Proof. If M is a finitely generated linearly compact right A-module, then the submodules MK with K an open ideal of A give a neighbourhood base of 0 in M. Since the topology on M is separated, we have $M_0 = 0$. When $M \in {}_H\mathcal{M}_A$, Theorem 2.7 shows that M is projective in $\mathcal{M}_A$. In particular, this applies to $M = A/I$. We deduce that I is an $\mathcal{M}_A$-direct summand of A. Then also $M = I$ satisfies the hypotheses of Theorem 2.7. Since $A \cong I \oplus A/I$ in $\mathcal{M}_A$, we have $r(I) + r(A/I) = r(A) = 1$. If $\gcd\{\text{length}\, A/P \mid P \in \Omega(A)\} = 1$ then both $r(I)$ and $r(A/I)$ are integers, whence one of them have to equal 0. This means that either I or A/I is 0. $\square$

Lemma 2.9. *Let* $\varphi : R \to R'$ *be an injective continuous homomorphism where* R *is a right linearly compact ring and* R' *is a separated right linearly topologized ring. If* $\varphi(R)$ *is dense in* R' *and* R/K *is an artinian* R*-module for each open right ideal* K *of* R*, then* φ *is a bicontinuous isomorphism.*

Proof. We may regard R' as a separated linearly topologized right R-module. Hence $\varphi(R)$ is closed in R' by (LC1). The density of $\varphi(R)$ entails $\varphi(R) = R'$, which means that φ is bijective. The artinian hypothesis ensures that the topology on R is a coarsest separated right linear topology [11, Satz 4] (also [3, Ch. III, §2, Ex. 19]). Hence φ^{-1} is continuous. $\square$

3. Nice pairs and quotient rings

For each ring R let $\mathcal{C}(R)$ denote the set of regular elements, i.e. non-zero-divisors, of R. When $\mathcal{C}(R)$ is a left and right denominator set, $Q(R)$ will stand for the Ore localization of R with respect to $\mathcal{C}(R)$. An arbitrary overring Q of R is isomorphic to $Q(R)$ as an overring of R if and only if all elements of $\mathcal{C}(R)$ are invertible in Q

and for each $x \in Q$ there exist $s, t \in \mathcal{C}(R)$ such that $sx \in R$ and $xt \in R$; in this case Q is called a *classical quotient ring* of R.

Assuming that both $Q(R)$ and $Q(R')$ exist, a ring homomorphism $\varphi : R' \to R$ extends to a homomorphism $Q(R') \to Q(R)$ if and only if φ maps $\mathcal{C}(R')$ into $\mathcal{C}(R)$. When R' is a subring of R and $\mathcal{C}(R') \subset \mathcal{C}(R)$, the inclusion $R' \to R$ extends to an injective homomorphism $Q(R') \to Q(R)$; we say in this case that $Q(R')$ is a subring of $Q(R)$. Note that $Q(R)$, the total ring of fractions of R, always exists when R is commutative.

An arbitrary ring Q is a *quotient ring* if all elements of $\mathcal{C}(Q)$ are invertible in Q.

Lemma 3.1. *Let Q be a semilocal ring with a nil Jacobson radical J. Then:*

(i) *All factor rings of Q are quotient rings.*

(ii) *If $Q = Q(R)$ and K is an ideal of Q, then $Q/K \cong Q(R/I)$ where $I = R \cap K$.*

(iii) *If x is in the center of Q, then $Q = Qx^n \oplus \mathrm{Ann}_Q(x^n)$ for some integer $n > 0$.*

Proof. By [10, Ex. 3.10] each element of a semisimple artinian ring S is a product of an invertible element and an idempotent (one says that S is *unit regular*). In particular, this applies to $S = Q/J$. An element $u \in Q$ is invertible if and only if $u + J$ is an invertible element of Q/J. Thus for any $x \in Q$ there exists an invertible element $u \in Q$ such that ux is an idempotent modulo J, i.e. $ux - (ux)^2 \in J$. Since all elements of J are nilpotent, $(ux)^n(1 - ux)^n = 0$ for sufficiently large n.

(i) If $x \in \mathcal{C}(Q)$ and u is as in the previous paragraph, then $ux \in \mathcal{C}(Q)$. It follows that $y = 1 - ux$ is nilpotent, and $ux = 1 - y$ is invertible. Then x is invertible too. This shows that Q is a quotient ring. (This result, due to Asano, can be found in [10, Ex. 21.23].) If K is any ideal of Q then the Jacobson radical of Q/K coincides with $(J + K)/K$. Hence Q/K is semilocal with a nil Jacobson radical, and we may replace Q with Q/K.

(ii) We may identify R/I with a subring of Q/K. Let $\pi : R \to R/I$ denote the canonical map. If $s \in \mathcal{C}(R)$, then s is invertible in Q, whence $\pi(s)$ is invertible in Q/K. Each element of Q/K can be written as $\pi(a)\pi(s)^{-1}$ and as $\pi(s)^{-1}\pi(a)$ for some $a \in R$ and $s \in \mathcal{C}(R)$. It is clear from this that each regular element of R/I remains regular in Q/K; it is therefore invertible by (i).

(iii) If $x \in Q$ is a central element, then the equality $(ux)^n(1 - ux)^n = 0$ entails $(ux)^{2n} \in Qx^n$; hence $Qx^{2n} = Qx^n$ since u is invertible. Then $Q = Qx^n + \mathrm{Ann}_Q(x^n)$. The sum is direct since x^n and x^{2n} must have equal annihilators. $\square$

Lemma 3.2. *Let $\varphi : S \to R'$ and $\iota : R' \to R$ be ring homomorphisms. Suppose that ι is injective and S, R', R all have classical quotient rings. If ι and $\iota \circ \varphi$ extend to homomorphisms of quotient rings, then so too does φ. Moreover, the extension $Q(S) \to Q(R')$ is surjective whenever φ is surjective and $Q(S)$ is semilocal with a nil Jacobson radical.*

Proof. We may identify R' with a subring of R via ι and $Q(R')$ with a subring of $Q(R)$. If $s \in \mathcal{C}(S)$, then $\varphi(s)$ is invertible in $Q(R)$; hence $\varphi(s) \in \mathcal{C}(R')$, and $\varphi(s)^{-1} \in Q(R')$. It follows that the extension $\psi : Q(S) \to Q(R)$ has image in $Q(R')$. Thus $\psi\big(Q(S)\big)$ is a subring of $Q(R')$ which contains R' when $\varphi(S) = R'$. If $Q(S)$ is semilocal with a nil Jacobson radical, then $\psi\big(Q(S)\big)$ is a quotient ring by Lemma 3.1. In this case $t^{-1} \in \psi\big(Q(S)\big)$ for any $t \in \mathcal{C}(R')$, showing that $\psi\big(Q(S)\big) = Q(R')$. $\square$

Let R be a ring, Z a central subring of R. We say that R, Z are a *nice pair* if

(NP1) Z has finitely many minimal prime ideals, say $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$,

(NP2) $\mathcal{C}(Z) = Z \setminus \bigcup_{i=1}^n \mathfrak{q}_i$ and $\mathcal{C}(Z) \subset \mathcal{C}(R)$,

(NP3) there exists a faithful R-module V finitely generated over Z.

Recall that any minimal prime of a commutative ring consists of zero divisors, e.g. by [3, Ch. II, §2, Prop. 12]. Therefore the inclusion $\mathcal{C}(Z) \subset Z \setminus \bigcup_{i=1}^n \mathfrak{q}_i$ in (NP2) is automatic. Condition (NP3) holds in the case when R is module-finite over Z, which is of main interest to us. However, in the stated version (NP3) facilitates passing to any subring $S \subset R$ with $Z \subset S$: the pair S, Z is nice provided so is R, Z. We will assume in the proofs that V is a left R-module, just to fix the notation. Note also that (NP3) implies that all elements of R are integral over Z [3, Ch. V, §1, Th. 1].

Lemma 3.3. *If R, Z are a nice pair then R has a semilocal classical quotient ring $Q(R) \cong R \otimes_Z Q(Z)$. Moreover:*

(i) *All prime ideals of $Q(R)$ are maximal.*

(ii) *The minimal primes of R are the ideals $M \cap R$ with $M \in \operatorname{Spec} Q(R)$.*

(iii) *The minimal primes of Z are the ideals $P \cap Z$ with P a minimal prime of R.*

(iv) *$P \in \operatorname{Spec} R$ is a minimal prime if and only if the ring $Q(R/P)$ exists and the projection $R \to R/P$ extends to a homomorphism $Q(R) \to Q(R/P)$.*

Proof. The prime ideals of $Q(Z)$ are precisely the ideals $Q(Z)\mathfrak{q}_i$ for $i = 1, \ldots, n$. Thus $Q(Z)$ has finitely many prime ideals, and each of those is maximal. This means that $Q(Z)$ is semilocal and the Jacobson radical J of $Q(Z)$ is nil. Put $T = R \otimes_Z Q(Z)$ and $W = V \otimes_Z Q(Z)$.

Since $\mathcal{C}(Z) \subset \mathcal{C}(R)$, we may identify R with a subring of T. Each element of T can be written as as^{-1} with $a \in R$ and $s \in \mathcal{C}(Z)$. If as^{-1} annihilates W, then for each $v \in V$ there exists an element in $\mathcal{C}(Z)$ annihilating av. Since aV is a finitely generated Z-module, it is annihilated by some $t \in \mathcal{C}(Z)$. The faithfulness of V yields $ta = 0$, and so $a = 0$. This shows that W is a faithful T-module. Furthermore, W is a finitely generated module over the central subring $Q(Z)$ of T.

Denote by I the annihilator of the T-module W/JW. If $x \in I$, then xW is a finitely generated $Q(Z)$-submodule of W contained in JW. Hence $xW \subset J_x W$ where $J_x \subset J$ is a finite subset. We have $(Tx)^n W \subset J_x^n W$ for each integer $n > 0$ by induction. Since all elements of J are nilpotent, $J_x^n = 0$ for sufficiently large n. Then $(Tx)^n W = 0$, and so $(Tx)^n = 0$. Thus TxT is a nilpotent ideal of T, so that x is contained in the prime radical N of T. We conclude that $I \subset N$.

If $w_1, \ldots, w_m$ generate W over $Q(Z)$, then I coincides with the annihilator in T of the finite subset $\{w_1 + J, \ldots, w_m + J\} \subset W/JW$. Therefore T/I embeds as a $Q(Z)$-submodule into a direct sum of finitely many copies of W/JW. Since $Q(Z)/J$ is an artinian ring, the $Q(Z)$-module W/JW is artinian, whence so too are T/I and T/N. It follows that T/N is a semiprime artinian ring. Hence N coincides with the Jacobson radical of T, and T is semilocal with all prime ideals maximal.

The prime radical is always a nil ideal. In particular, N is nil. By Lemma 3.1 T is a quotient ring. Since $Q(Z)$ is a flat Z-algebra, we have $\mathcal{C}(R) \subset \mathcal{C}(T)$. In particular, all elements of $\mathcal{C}(R)$ are invertible in T. Since for each $x \in T$ there exists $s \in \mathcal{C}(Z)$ such that $sx = xs \in R$, we deduce that T is a classical quotient ring of R. Thus $Q(R) \cong T$, and (i) has been established.

(ii) We have $M \cap R \in \operatorname{Spec} R$ for each $M \in \operatorname{Spec} T$. This is straightforward since $IT = TI \cong I \otimes_Z Q(Z)$ for any ideal I of R, and therefore $I_1 I_2 \subset M \cap R$ for two ideals I_1, I_2 of R if and only if $I_1T \cdot I_2T \subset M$. Let $M_1, \dots, M_n$ be all prime ideals of T and $P_j = M_j \cap R$ for $j = 1, \dots, n$. Then $\bigcap P_j = N \cap R$. Since $N \cap R$ consists of strongly nilpotent elements of R, it is contained in the prime radical of R [10, Ex. 10.18A]. Hence every prime of R contains $\bigcap P_j$ and therefore one of the ideals $P_1, \dots, P_n$. On the other hand, each ideal of T is generated by elements in R, and in particular $M_j = P_jT$ for each j. Therefore there are no inclusions between $P_1, \dots, P_n$, which shows that these ideals are precisely the minimal primes of R.

(iii) Each $M_j \cap Q(Z)$ is a prime ideal of $Q(Z)$, and so $M_j \cap Q(Z) = \mathfrak{q}_i Q(Z)$ for some minimal prime $\mathfrak{q}_i$ of Z. Contracting further to Z, we get $P_j \cap Z = \mathfrak{q}_i$. Conversely, any given $\mathfrak{q}_i$ can be obtained in this way. To show this first observe that $\mathfrak{q}_i W \neq W$ since otherwise W would be annihilated by a nonzero element of $Q(Z)$ according to Nakayama's Lemma, but we have checked the faithfulness of W already. The finitely generated T-module $W/\mathfrak{q}_i W$ has a simple factor module. If M_j is the annihilator of the latter, then $\mathfrak{q}_i \subset M_j$, and therefore $M_j \cap Q(Z) = \mathfrak{q}_i Q(Z)$.

(iv) If $Q(R/P)$ exists, then P coincides with the kernel of the canonical map $R \to Q(R/P)$. Assuming that this map extends to a homomorphism $T \to Q(R/P)$, we get $P = K \cap R$, where K is the kernel of the latter. Now $K \neq T$ since $P \neq R$, and therefore $K \subset M_j$ for some maximal ideal of T. It follows that $P \subset P_j$, whence $P = P_j$ by the minimality of P_j. On the other hand, $Q(R/P_j) \cong T/M_j$ for each minimal prime of R by Lemma 3.1. $\square$

Lemma 3.4. *If R, Z are a nice pair then the following conditions are equivalent*:

(a) $Q(Z)$ *is artinian*, (b) $Q(R)$ *is artinian*, (c) $Q(R)$ *satisfies ACC on ideals.*

Proof. Condition (NP3) implies that R, regarded as a Z-module, embeds into a direct sum of finitely many copies of V. Hence $Q(R) \cong R \otimes_Z Q(Z)$, regarded as a $Q(Z)$-module, embeds into a direct sum of finitely many copies of $W = V \otimes_Z Q(Z)$. If $Q(Z)$ is artinian, then W is an artinian $Q(Z)$-module, whence $Q(R)$ is artinian.

Thus (a)$\Rightarrow$(b). The implication (b)$\Rightarrow$(c) is a consequence of the Hopkins-Levitzki Theorem. Finally, suppose that (c) holds. In this case W satisfies ACC on submodules of the form IW where I is an ideal of $Q(Z)$. In fact, if $W_0 \subset W_1 \subset \cdots$ is a chain of submodules of this form, then $W_i = I_i W$ for a chain of ideals $I_0 \subset I_1 \subset \cdots$ of $Q(Z)$ (we may take $I_i = \{a \in Q(Z) \mid aW \subset W_i\}$). There exists an integer $p > 0$ such that $I_i Q(R) = I_p Q(R)$ for all $i > p$. Since W is a $Q(R)$-module in a natural way, we have $W_i = I_i Q(R) W$ for each i, whence $W_i = W_p$ for $i > p$. Formanek's generalization of the Eakin-Nagata Theorem [7] (also [12, Th. 3.6]) now shows that $Q(Z)$ is noetherian. Any noetherian ring with a nil Jacobson radical is artinian. So (c)$\Rightarrow$(a). $\square$

The *height* and the *coheight* of a prime ideal P of a ring R are the supremums of the lengths, respectively, of strictly descending and strictly ascending chains in $\operatorname{Spec} R$ starting at P. The (classical) *Krull dimension* of R, denoted $\operatorname{Kdim} R$ in this paper, is the supremum of the lengths of arbitrary finite chains in $\operatorname{Spec} R$. Thus $\operatorname{coheight} P = \operatorname{Kdim} R/P$. We allow nonnegative integers and $+\infty$ as possible values for those quantities. However, the equalities in Lemma 3.5 and further results are valid also with the ordinal-valued refinements of height, coheight and Kdim.

Lemma 3.5. *Let R be a ring and Z a central subring of R. Suppose that R is either module-finite over Z or commutative and integral over Z. Let $P \in \operatorname{Spec} R$, and let*

$\mathfrak{p} = P \cap Z \in \operatorname{Spec} Z$. *Then* coheight $P =$ coheight $\mathfrak{p}$ *and* $P \in \operatorname{Spec}_f R$ *if and only if* $\mathfrak{p} \in \operatorname{Spec}_f Z$.

Proof. It is well-known that the going-up and the incomparability hold for the ring extension $Z \subset R$ under both assumptions. It follows that there exists a strictly ascending chain in $\operatorname{Spec} R$ starting at P of any given finite or infinite length if and only if there exists such a chain in $\operatorname{Spec} Z$ starting at $\mathfrak{p}$. $\square$

Lemma 3.6. *Let A be a commutative ring integral over subrings Z and Z'. If $\mathfrak{p}, \mathfrak{q}$ are prime ideals of A satisfying $\mathfrak{p} \cap Z = \mathfrak{q} \cap Z \in \operatorname{Spec}_f Z$ and $\mathfrak{p} \cap Z' \subset \mathfrak{q} \cap Z'$, then $\mathfrak{p} \cap Z' = \mathfrak{q} \cap Z' \in \operatorname{Spec}_f Z'$.*

Proof. Suppose $\mathfrak{p} \cap Z' \neq \mathfrak{q} \cap Z'$. By the going-up property there exists $\mathfrak{p}_1 \in \operatorname{Spec} A$ such that $\mathfrak{p} \subset \mathfrak{p}_1$ and $\mathfrak{p}_1 \cap Z' = \mathfrak{q} \cap Z'$. Then $\mathfrak{p}_1 \neq \mathfrak{p}$ and $\mathfrak{p}_1 \cap Z \supset \mathfrak{p} \cap Z = \mathfrak{q} \cap Z$. The inclusion here is proper by the incomparability. Another application of the going-up gives $\mathfrak{q}_1 \in \operatorname{Spec} A$ such that $\mathfrak{q} \subset \mathfrak{q}_1$ and $\mathfrak{q}_1 \cap Z = \mathfrak{p}_1 \cap Z$. Now $\mathfrak{q}_1 \neq \mathfrak{q}$ and $\mathfrak{q}_1 \cap Z' \supset \mathfrak{q} \cap Z' = \mathfrak{p}_1 \cap Z'$ with proper inclusion. Repeating this process, we obtain two infinite strictly ascending chains $\mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots$ and $\mathfrak{q} = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots$ in $\operatorname{Spec} A$. But this is impossible since $\mathfrak{p} \in \operatorname{Spec}_f A$ by Lemma 3.5 applied to the ring extension $Z \subset A$. The same lemma applied to $Z' \subset A$ yields $\mathfrak{p} \cap Z' \in \operatorname{Spec}_f Z'$. $\square$

Lemma 3.7. *Suppose that R, Z are a nice pair where Z is a domain satisfying ACC on prime ideals. Let R' be a subring of R and Z' a central subring of R'. If there exists a faithful R-module V' finitely generated over Z' then:*

(i) *Z' satisfies ACC on prime ideals.*

(ii) coheight $\mathfrak{q} = \operatorname{Kdim} Z$ *for any minimal prime $\mathfrak{q}$ of Z'.*

(iii) *R', Z' are a nice pair.*

(iv) *$Q(R)$ and $Q(R')$ are both artinian, and $Q(R')$ is a subring of $Q(R)$.*

Proof. Denote $A = ZZ'$, which is a commutative subring of R. Since R, Z are a nice pair, A is an integral extension of Z. The hypothesis about V' implies that A is integral over Z' as well.

Since $Q(Z)$ is a field, by Lemmas 3.3, 3.4 R and A have artinian classical quotient rings $Q(R) \cong R \otimes_Z Q(Z)$ and $Q(A) \cong A \otimes_Z Q(Z)$. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be all maximal ideals of $Q(A)$ and $\mathfrak{p}_i = \mathfrak{m}_i \cap A$ for $i = 1, \ldots, n$. An element $a \in A$ is regular if and only if a is invertible in $Q(A)$, i.e. $a \notin \mathfrak{m}_i$ for each i. Thus $\mathcal{C}(A) = A \setminus \bigcup_{i=1}^n \mathfrak{p}_i$. Since $\mathfrak{m}_i \cap Q(Z) = 0$, we have $\mathfrak{p}_i \cap Z = 0$ for each i.

Let $\mathfrak{q}_i = \mathfrak{p}_i \cap Z'$. Since the Jacobson radical $\bigcap \mathfrak{m}_i$ of $Q(A)$ is nilpotent and $\mathfrak{q}_i \subset \mathfrak{m}_i$ for each i, the ideal $\bigcap \mathfrak{q}_i$ of Z' is nilpotent too. It follows that any prime ideal of Z' contains one of $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$. By Lemma 3.6 there are no proper inclusions between $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$. Hence those ideals are precisely the minimal primes of Z'. Lemma 3.6 shows also that $\mathfrak{q}_1, \ldots, \mathfrak{q}_n \in \operatorname{Spec}_f Z'$. Then $\operatorname{Spec} Z' = \operatorname{Spec}_f Z'$, yielding (i). Lemma 3.5 applied to the ring extensions $Z \subset A$ and $Z' \subset A$ proves the equalities

$$\text{coheight}\, \mathfrak{q}_i = \text{coheight}\, \mathfrak{p}_i = \text{coheight}\, 0_Z = \operatorname{Kdim} Z$$

where 0_Z denotes the zero ideal of Z.

Clearly $Z' \setminus \bigcup_{i=1}^n \mathfrak{q}_i \subset \mathcal{C}(A)$ by the explicit description of $\mathcal{C}(A)$. Since each $\mathfrak{q}_i$ consists of zero divisors of Z', we must have $\mathcal{C}(Z') = Z' \setminus \bigcup_{i=1}^n \mathfrak{q}_i$ and $\mathcal{C}(Z') \subset \mathcal{C}(A)$. Since $Q(A)$ is a subring of $Q(R)$, we have also $\mathcal{C}(A) \subset \mathcal{C}(R)$, whence $\mathcal{C}(Z') \subset \mathcal{C}(R)$.

It is now clear that S, Z' are a nice pair for any subring S of R containing Z' in its center. In particular this is valid for $S = R'$ and for $S = A$.

Lemma 3.4 applied to the extension $Z' \subset A$ shows that $Q(Z')$ is artinian. We may also apply Lemmas 3.3, 3.4 to the extension $Z' \subset R'$ and conclude that $Q(R')$ is artinian and $Q(R') \cong R' \otimes_{Z'} Q(Z')$. Consider also the subring $S = R'Z$ of R. By Lemma 3.3 $Q(S) \cong S \otimes_Z Q(Z)$ and $Q(S) \cong S \otimes_{Z'} Q(Z')$. The first isomorphism shows that $Q(S)$ is a subring of $Q(R)$, the second shows that $Q(R')$ is a subring of $Q(S)$. Hence $Q(R')$ is a subring of $Q(R)$. $\square$

The assumption that Z coincides with the center of R is essential in Lemma 3.8.

Lemma 3.8. *Suppose that R is a ring module-finite over its center Z and Q is a classical quotient ring of R. If Q is semilocal with a nil Jacobson radical, then R, Z are a nice pair.*

Proof. Clearly Z is a central subring of Q. Hence $Z \cap M$ is a prime ideal of Z for each maximal ideal M of Q. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ be all distinct primes of Z having this form. Their intersection $\bigcap \mathfrak{q}_i$ is a nil ideal of Z since $\bigcap \mathfrak{q}_i$ is contained in the Jacobson radical of Q. Therefore the set $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ contains all minimal primes of Z. If $z \in Z \setminus \bigcup \mathfrak{q}_i$, then zQ is an ideal of Q contained in none of the maximal ideals. This implies that $zQ = Q$, so that z is regular in R. To complete the proof we have only to show that there are no proper inclusions between $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$, so that all these ideals are minimal primes of Z.

Let $\mathfrak{q}$ be any maximal element of the set $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ ordered by inclusion. We may assume $\mathfrak{q} = \mathfrak{q}_1$ after reindexing. Since $\mathfrak{q} \not\subset \mathfrak{q}_i$ for each $i \neq 1$, we have $\mathfrak{q} \not\subset \bigcup_{i \neq 1} \mathfrak{q}_i$. Pick any element $x \in \mathfrak{q} \setminus \bigcup_{i \neq 1} \mathfrak{q}_i$. Replacing x with x^n where n is as in Lemma 3.1(iii), we get $Q = xQ \oplus \operatorname{Ann}_Q(x)$. Let $I_1 = R \cap xQ$ and $I_2 = R \cap \operatorname{Ann}_Q(x)$. Also put $R_1 = R/I_1$ and $Z_1 = Z/(Z \cap I_1)$.

By Lemma 3.1(ii) $Q/xQ \cong Q(R_1)$. The Jacobson radical of Q/xQ coincides with J/xQ where J is the intersection of those maximal ideals M of Q for which $x \in M$. We have $Z \cap M = \mathfrak{q}$ for each M appearing here by the choice of x. Hence $Z \cap J = \mathfrak{q}$. The hypothesis about Q implies that J/xQ is nil. Hence $\mathfrak{q}Z_1$ is a single minimal prime ideal of Z_1. If $s \in Z \setminus \mathfrak{q}$, then $s \notin M$ for any maximal ideal M of Q with $x \in M$. The coset $s + xQ$ has to be invertible in the semilocal ring Q/xQ, which means that $s + I_1$ is a regular element of R_1. By the hypothesis R_1 is module-finite over Z_1. This shows that R_1, Z_1 are a nice pair. By Lemma 3.3 $Q(R_1) \cong R_1 \otimes_{Z_1} Q(Z_1)$.

We have $I_2 Q = \operatorname{Ann}_Q(x)$, whence $I_2 \cdot Q(R_1) = Q(R_1)$. Each element of $I_2 \cdot Q(R_1)$, in particular 1, can be written as $\varphi(u)\varphi(s)^{-1}$ for some $u \in I_2$ and $s \in Z \setminus \mathfrak{q}$ where $\varphi : R \to Q(R_1)$ is the canonical map. Therefore we can find u and s such that $\varphi(u) = \varphi(s)$, i.e. $u - s \in I_1$. Since s is central in R, we get

$$au - ua = a(u - s) - (u - s)a \in I_2 \cap I_1 = 0$$

for all $a \in R$. Thus $u \in Z$. It follows that $u \equiv s \pmod{\mathfrak{q}}$ since $Z \cap I_1 \subset Z \cap J = \mathfrak{q}$. In particular, $u \notin \mathfrak{q}$. Since $xu \in I_1 \cap I_2 = 0$, we have $u \in \mathfrak{p}$ for each prime ideal $\mathfrak{p}$ of Z with $x \notin \mathfrak{p}$. This shows that $u \in \mathfrak{q}_i$, and so $\mathfrak{q}_i \not\subset \mathfrak{q}$, whenever $i \neq 1$. In other words, each maximal element in $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ is also a minimal element of this set. $\square$

4. Final results

Throughout the whole section we assume that A is an H-module algebra module-finite over a central subring Z. If $P \in \operatorname{Spec} A$, then $\mathfrak{p} = P \cap Z$ is a prime ideal of Z and A/P is a module-finite ring over its central subring $Z/\mathfrak{p}$. Since $Z/\mathfrak{p}$ is a domain, the pair A/P, $Z/\mathfrak{p}$ is nice, whence $A/P \otimes_{Z/\mathfrak{p}} Q(Z/\mathfrak{p})$ is an artinian classical quotient ring of A/P by Lemmas 3.3, 3.4.

Let C be a coalgebra and B an algebra. Suppose that C is finitely generated projective as a k-module. Then $\operatorname{Hom}(C, B) \cong B \otimes C^*$ as algebras. If B has an artinian classical quotient ring $Q(B)$, then $\operatorname{Hom}\big(C, Q(B)\big)$ is an artinian classical quotient ring of $\operatorname{Hom}(C, B)$. This follows from the following observations. The homomorphism $B \otimes C^* \to Q(B) \otimes C^*$ induced by the canonical embedding $B \to Q(B)$ is injective since C^* is a projective k-module. Using the common denominator property of Ore localizations it is easy to see that for each $x \in Q(B) \otimes C^*$ there exist $s, t \in \mathcal{C}(B)$ such that $(s \otimes 1)x$ and $x(t \otimes 1)$ are in $B \otimes C^*$; clearly $s \otimes 1$ and $t \otimes 1$ are invertible in $Q(B) \otimes C^*$ and regular in $B \otimes C^*$. Since C^* is a finitely generated k-module, $Q(B) \otimes C^*$ is finitely generated as a right module and as a left module over the artinian subring $Q(B) \otimes 1$. Hence $Q(B) \otimes C^*$ is artinian, and in particular it is a quotient ring. Therefore all regular elements of $B \otimes C^*$ are invertible in $Q(B) \otimes C^*$ since they remain regular there. Thus $Q(B) \otimes C^*$ is indeed a classical quotient ring of $B \otimes C^*$.

If B' is another algebra with an artinian classical quotient ring and $B \to B'$ is a homomorphism which extends to a homomorphism of quotient rings, then the induced homomorphism $\operatorname{Hom}(C, B) \to \operatorname{Hom}(C, B')$ also extends to quotient rings. If C' is another coalgebra with a finitely generated projective underlying k-module and $C' \to C$ is a coalgebra homomorphism, then the induced algebra homomorphism $\operatorname{Hom}(C, B) \to \operatorname{Hom}(C', B)$ extends to a homomorphism of quotient rings. This follows from the functoriality of construction.

Let $C \in \mathcal{F}$. As we have just observed, the convolution algebra $R = \operatorname{Hom}(C, A/P)$ has an artinian classical quotient ring. In section 1 we defined a homomorphism $\tau_{C,P} : A \to R$ with kernel P_C. Let

$$\iota_{C,P} : A/P_C \to \operatorname{Hom}(C, A/P)$$

be the injection induced by $\tau_{C,P}$. Thus $\iota_{C,P}$ is an isomorphism of A/P_C onto the subalgebra $R' = \tau_{C,P}(A)$ of R.

Lemma 4.1. *Let $P \in \operatorname{Spec}_f A$ and $C \in \mathcal{F}$. Then:*

(i) *A/P_C, $Z/(Z \cap P_C)$ are a nice pair.*

(ii) *A/P_C has an artinian classical quotient ring.*

(iii) *$\iota_{C,P}$ extends to a homomorphism $Q(A/P_C) \to \operatorname{Hom}\big(C, Q(A/P)\big)$.*

(iv) *$P' \in \operatorname{Spec}_f A$ for any $P' \in \operatorname{Spec} A$ with $P_C \subset P'$.*

Proof. Let $\mathfrak{p}$, R, R' be as above. Note that $R \cong A/P \otimes C^*$ is module-finite over the subring $A/P \otimes 1$, and therefore over the central subring $K = Z/\mathfrak{p} \otimes 1$. If $s \in Z/\mathfrak{p}$ is nonzero, then $s \in \mathcal{C}(A/P)$ since A/P is a prime ring, and therefore $s \otimes 1 \in \mathcal{C}(R)$ since C^* is a flat k-module. If $s \otimes 1 = 0$ for such an element s, we must have $R = 0$, which entails $P_C = A$ (when k is a field, this happens only for $C = 0$). In this case all conclusions of Lemma 4.1 are trivial statements about zero rings. Otherwise $Z/\mathfrak{p}$ is mapped isomorphically onto K. Since $K \smallsetminus \{0\} \subset \mathcal{C}(R)$, the pair R, K is nice.

Also, R' is module-finite over its central subring $Z' = \tau_{C,P}(Z)$. Lemma 1.2 applied to $M = A$ shows that $\mathrm{Hom}(C, A) = \tau_C(A) \cdot C^*$, whence $\mathrm{Hom}(C, A)$ is left module-finite over $\tau_C(A)$. The canonical homomorphism $\mathrm{Hom}(C, A) \to R$ takes $\tau_C(A)$ to R'. Therefore R is left module-finite over R', and then also over Z'. By Lemma 3.5 $\mathfrak{p} \in \mathrm{Spec}_f Z$, i.e. the domain $K \cong Z/\mathfrak{p}$ satisfies ACC on prime ideals. Thus we meet all hypotheses of Lemma 3.7, and parts (i), (ii), (iii) are immediate. Lemma 3.7 shows also that $Z' \cong Z/(Z \cap P_C)$ satisfies ACC on prime ideals. Hence each prime $\mathfrak{p}'$ of Z containing $Z \cap P_C$ belongs to $\mathrm{Spec}_f Z$. In particular, this holds for $\mathfrak{p}' = Z \cap P'$ where P' is as in (iv), and Lemma 3.5 completes the proof of (iv). $\square$

Lemma 4.2. *Given $P \in \mathrm{Spec}_f A$ and $C, D \in \mathcal{F}$ with $C \subset D$, the canonical map $A/P_D \to A/P_C$ extends to a surjective homomorphism $Q(A/P_D) \to Q(A/P_C)$.*

Proof. Let $S = \mathrm{Hom}(D, A/P)$ and $S' = \tau_{D,P}(A) \cong A/P_D$. By Lemma 4.1 $Q(S')$ exists and is an artinian subring of $Q(S)$. There is a ring homomorphism $\varphi : S \to R$ obtained by restriction of linear maps $D \to A/P$ to C. By functoriality φ extends to a homomorphism $\psi : Q(S) \to Q(R)$. Since the diagram

$$\begin{array}{ccc} A & \xrightarrow{\tau_{D,P}} & S \\ {\scriptstyle\mathrm{id}}\downarrow & & \downarrow{\scriptstyle\varphi} \\ A & \xrightarrow{\tau_{C,P}} & R \end{array}$$

commutes, we have $\varphi(S') = R'$. Lemma 3.2 applied to the surjective homomorphism $\varphi|_{S'} : S' \to R'$ and the inclusion $R' \hookrightarrow R$ shows that ψ maps $Q(S')$ onto $Q(R')$. $\square$

Now we may define $L_P(A) = \varprojlim Q(A/P_C),\ C \in \mathcal{F}$. The artinian rings $Q(A/P_C)$ are left and right linearly compact in the discrete topology. By Lemma 2.2 $L_P(A)$ is left and right linearly compact in the inverse limit topology. Moreover, the canonical maps $L_P(A) \to Q(A/P_C)$ are surjective. The kernels of these maps give a neighbourhood base of 0 which consists of ideals of $L_P(A)$. We see also that $L_P(A)/K$ is an artinian module for each open one-sided ideal K of $L_P(A)$.

The collection of canonical maps $A \to Q(A/P_C)$ determines an algebra homomorphism $\alpha_P : A \to L_P(A)$. Clearly $\mathrm{Ker}\,\alpha_P = \bigcap_{C \in \mathcal{F}} P_C = P_H$.

For the purposes of the next lemma note that the earlier definition of the injection $\iota_{C,J}$ makes sense for an arbitrary ideal J of A, even if J is not prime.

Lemma 4.3. *Let $C, D, E \in \mathcal{F}$ be such that $CD \subset E$. Then the composite map*

$$\lambda : A/P_E \xrightarrow{\text{can.}} A/(P_C)_D \xrightarrow{\iota_{D,P_C}} \mathrm{Hom}(D, A/P_C)$$

extends to a ring homomorphism $Q(A/P_E) \to \mathrm{Hom}\big(D, Q(A/P_C)\big)$.

Proof. The ideal $(P_C)_D$ consists of all elements $x \in A$ such that $CDx \subset P$. Since $CD \subset E$, we have $P_E \subset (P_C)_D$, and so the canonical map $A/P_E \to A/(P_C)_D$ is defined. There is a commutative diagram

$$\begin{array}{ccc} A/P_E & \xrightarrow{\qquad\qquad\lambda\qquad\qquad} & \mathrm{Hom}(D, A/P_C) \\ {\scriptstyle\iota_{E,P}}\downarrow & & \downarrow{\scriptstyle\mathrm{Hom}(D,\,\iota_{C,P})} \\ \mathrm{Hom}(E, A/P) \xrightarrow{\nu} \mathrm{Hom}(C \otimes D, A/P) & \cong & \mathrm{Hom}\big(D, \mathrm{Hom}(C, A/P)\big) \end{array}$$

where ν is dual to the multiplication map $C \otimes D \to E$ which is a coalgebra homomorphism. In view of Lemma 4.1 all rings in the diagram have artinian classical quotient rings and both vertical arrows extend to homomorphisms of quotient rings. By functoriality ν also extends to quotient rings. Since the map on the right is injective, Lemma 3.2 yields the conclusion. $\square$

Lemma 4.4. *There is an H-module structure on $L_P(A)$ with respect to which $L_P(A)$ is a linearly compact H-module algebra and $\alpha_P : A \to L_P(A)$ is a homomorphism of H-module algebras.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} Q(A/P_E) & \longrightarrow & \operatorname{Hom}\big(D, Q(A/P_C)\big) \\ \downarrow & & \downarrow \\ \operatorname{Hom}\big(E, Q(A/P)\big) & \longrightarrow & \operatorname{Hom}\big(D, \operatorname{Hom}(C, Q(A/P))\big) \end{array}$$

obtained from the diagram in Lemma 4.3 by passing to quotient rings. For any fixed $D \in \mathcal{F}$ the set $\mathcal{I}_D = \{(C,E) \in \mathcal{F} \times \mathcal{F} \mid CD \subset E\}$ is directed by componentwise inclusion and both projections $\mathcal{I}_D \to \mathcal{F}$ are cofinal maps. The above diagrams with varying (C,E) form an inverse system. Passing to the inverse limit over $\mathcal{I}_D$, we obtain a commutative diagram

$$\begin{array}{ccc} L_P(A) & \longrightarrow & \operatorname{Hom}\big(D, L_P(A)\big) \\ \downarrow & & \downarrow \\ \operatorname{Hom}\big(H, Q(A/P)\big) & \longrightarrow & \operatorname{Hom}\big(D, \operatorname{Hom}(H, Q(A/P))\big) \end{array}$$

where all maps are algebra homomorphisms, and the vertical ones are injections since the inverse limit functors are left exact. By construction the map represented by the bottom arrow is dual to the multiplication map $H \otimes D \to H$. Thus $L_P(A)$ is isomorphic with an H-stable subalgebra of the H-module algebra $\operatorname{Hom}\big(H, Q(A/P)\big)$ on which the action of H is given by the rule $(g\xi)(h) = \xi(hg)$ for $g, h \in H$ and $\xi \in \operatorname{Hom}\big(H, Q(A/P)\big)$.

For any C, D, E as above there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \tau_D\ } & \operatorname{Hom}(D, A) \\ \downarrow & & \downarrow \\ Q(A/P_E) & \longrightarrow & \operatorname{Hom}\big(D,\ Q(A/P_C)\big). \end{array}$$

Passing to the inverse limit over $\mathcal{I}_D$, we deduce that α_P commutes with the action of D; since $D \in \mathcal{F}$ is arbitrary, α_P is H-linear. $\square$

The image of k in H is a subcoalgebra such that $P_k = P$. We may assume that $k \in \mathcal{F}$ enlarging $\mathcal{F}$ if necessary. Hence there is a canonical surjective homomorphism of algebras $L_P(A) \to Q(A/P)$. Its kernel, denoted V_P further on, is an open maximal ideal of $L_P(A)$ since the prime artinian ring $Q(A/P)$ has to be simple. For each subcoalgebra $C \subset H$ put

$$V_{P,C} = \{x \in L_P(A) \mid Cx \subset V_P\}.$$

Lemma 4.5. *We have $V_{P,D} = \mathrm{Ker}\big(L_P(A) \to Q(A/P_D)\big)$ for each $D \in \mathcal{F}$. In particular, $\{V_{P,D} \mid D \in \mathcal{F}\}$ is a neighbourhood base of 0 for the topology on $L_P(A)$ and $L_P(A)/V_{P,D} \cong Q(A/P_D)$ for each D. Furthermore, $V_{P,H} = 0$.*

Proof. For $C, D, E \in \mathcal{F}$ with $CD \subset E$ there is a commutative diagram

$$\begin{array}{ccc} L_P(A) & \xrightarrow{\tau_D} & \mathrm{Hom}\big(D, L_P(A)\big) \\ \downarrow & & \downarrow \\ Q(A/P_E) & \longrightarrow & \mathrm{Hom}\big(D, Q(A/P_C)\big). \end{array}$$

In fact the map at the top defining the H-module structure on $L_P(A)$ was constructed in Lemma 4.4 as the inverse limit of the bottom maps for varying C, E. Now take $C = k$ and $E = D$. Clearly $V_{P,D}$ coincides with the kernel of the composite map $L_P(A) \to \mathrm{Hom}\big(D, L_P(A)\big) \to \mathrm{Hom}\big(D, Q(A/P)\big)$. Since the map at the bottom of the diagram is injective, the first assertion is clear. Each nonzero element of $L_P(A)$ has a nonzero image in $Q(A/P_D)$ for some D. Hence $V_{P,H} = \bigcap_{D \in \mathcal{F}} V_{P,D} = 0$. $\square$

Let $t > 0$ be an integer such that A is a t-generated Z-module. Then A/P_C is a t-generated module over its center, and so is $Q(A/P_C)$ by Lemmas 4.1(i) and 3.3, for each $C \in \mathcal{F}$. If K is any open ideal of $L_P(A)$, then $V_{P,C} \subset K$ for some C, and therefore $L_P(A)/K$ is a factor ring of $Q(A/P_C)$. Hence $L_P(A)/K$ is a t-generated module over its center. We see that $L_P(A)$ satisfies all hypotheses of Proposition 2.5. Indeed, conditions (a) and (b) follow from Lemma 2.6. Thus $L_P(A)$ is topologically H-simple, as claimed in Theorem 0.2.

Suppose that B is another H-module algebra module-finite over its center and $P' \in \mathrm{Spec}_f\, B$. Let $\alpha_{P'} : B \to L_{P'}(B)$ be the canonical map.

Lemma 4.6. *Let $\varphi : A \to B$ be a homomorphism of H-module algebras. Suppose that $\varphi(P_C) \subset P'$ for some $C \in \mathcal{F}$ and the map $A/P_C \to B/P'$ induced by φ extends to a homomorphism of quotient rings. Then there exists a continuous homomorphism of H-module algebras $\psi : L_P(A) \to L_{P'}(B)$ rendering commutative the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\alpha_P} & L_P(A) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ B & \xrightarrow{\alpha_{P'}} & L_{P'}(B) \end{array}$$

Proof. Clearly φ maps $(P_C)_D$ into P'_D for each $D \in \mathcal{F}$. Hence $\varphi(P_E) \subset P'_D$ whenever $D, E \in \mathcal{F}$ satisfy $CD \subset E$, in which case we obtain a commutative diagram

$$\begin{array}{ccc} A/P_E & \xrightarrow{\lambda} & \mathrm{Hom}(D, A/P_C) \\ \downarrow & & \downarrow \\ B/P'_D & \xrightarrow{\iota_{D,P'}} & \mathrm{Hom}(D, B/P') \end{array}$$

where the vertical arrows are induced by φ, the top arrow represents the map constructed in Lemma 4.3 and the bottom arrow is the injection induced by $\tau_{D,P'}$. We know already that the latter two maps extend to homomorphisms of quotient rings. The hypothesis ensures that the same is valid for the map on the right of the diagram. By Lemma 3.2 this holds then for the map $A/P_E \to B/P'_D$ too. The set $\mathcal{J}_C = \{(D,E) \in \mathcal{F} \times \mathcal{F} \mid CD \subset E\}$ is directed by componentwise inclusion and both projections $\mathcal{J}_C \to \mathcal{F}$ are cofinal maps. The above diagrams with varying (D,E) form an inverse system. Passing first to quotient rings and then taking the inverse limit over $\mathcal{J}_C$, we arrive at the commutative diagram

$$\begin{array}{ccc} L_P(A) & \longrightarrow & \mathrm{Hom}\big(H, Q(A/P_C)\big) \\ \downarrow & & \downarrow \\ L_{P'}(B) & \longrightarrow & \mathrm{Hom}\big(H, Q(B/P')\big) \end{array}$$

We thus obtain a continuous algebra homomorphism $\psi : L_P(A) \to L_{P'}(B)$ satisfying $\psi \circ \alpha_P = \alpha_{P'} \circ \varphi$. It remains to verify that ψ commutes with the action of H. The proof of Lemma 4.4 shows that $L_{P'}(B)$ is mapped isomorphically onto an H-stable subalgebra of $\mathrm{Hom}\big(H, Q(B/P')\big)$; this is how the H-module structure on $L_{P'}(B)$ is obtained. The map on the right of the diagram is a homomorphism of H-module algebras by functoriality. The proof will be completed once we check that so too is the map on the top. Considering again the first diagram from the proof of Lemma 4.4, but now fixing C and letting D, E vary, we may pass to the inverse limit over $\mathcal{J}_C$. The resulting commutative diagram

$$\begin{array}{ccc} L_P(A) & \longrightarrow & \mathrm{Hom}\big(H, Q(A/P_C)\big) \\ \downarrow & & \downarrow \\ \mathrm{Hom}\big(H, Q(A/P)\big) & \longrightarrow & \mathrm{Hom}\big(H, \mathrm{Hom}(C, Q(A/P))\big) \end{array}$$

shows that the map on the top is H-linear since so are the remaining maps in the diagram, while the map on the right is injective. $\square$

Lemma 4.7. *Let $P, P' \in \mathrm{Spec}_f\, A$. Then the two conditions below are equivalent*:

(i) *P' is a prime minimal over P_C for some $C \in \mathcal{F}$,*

(ii) *there exists a continuous homomorphism of linearly compact H-module algebras $\psi : L_P(A) \to L_{P'}(A)$ such that $\psi \circ \alpha_P = \alpha_{P'}$.*

Moreover, the ψ in (ii) *is necessarily a bicontinuous isomorphism.*

Proof. (i)$\Rightarrow$(ii) This follows from Lemma 4.6 applied with $B = A$. Note that the canonical map $A/P_C \to A/P'$ extends to a homomorphism of quotient rings by Lemmas 4.1(i), 3.3(iv).

(ii)$\Rightarrow$(i) The ideals $V_{P',D} = \mathrm{Ker}\big(L'_P(A) \to Q(A/P'_D)\big)$ with $D \in \mathcal{F}$ give a neighbourhood base of 0 for the topology on $L_{P'}(A)$. Since $\psi^{-1}(V_{P',D})$ is open in $L_P(A)$, by Lemma 4.5 $V_{P,C} \subset \psi^{-1}(V_{P',D})$ for some $C \in \mathcal{F}$, and therefore ψ gives rise to a ring homomorphism $Q(A/P_C) \to Q(A/P'_D)$ which maps $a + P_C$ to $a + P'_D$ for each $a \in A$. The latter homomorphism is surjective since its image is an artinian subring of $Q(A/P'_D)$ containing A/P'_D (see Lemma 3.2). Hence $\mathrm{Im}\,\psi + V_{P',D} = L_{P'}(A)$ for each D, so that ψ has a dense image. Since $\mathrm{Ker}\,\psi$ is an H-stable closed ideal of A, we get $\mathrm{Ker}\,\psi = 0$ from Theorem 0.2. Now Lemma 2.9 shows that ψ has a continuous inverse.

If $D = k$, then $P'_D = P'$. The existence of a homomorphism $Q(A/P_C) \to Q(A/P')$ sending $a + P_C$ to $a + P'$ for $a \in A$ shows that $P_C \subset P'$, and by Lemma 3.3(iv) P'/P_C is a minimal prime of A/P_C. $\square$

Lemma 4.8. *Suppose that $P, P' \in \operatorname{Spec}_f A$ and $C, D \in \mathcal{F}$ satisfy $P_C \subset P'$ and $C \subset D$. Then P' is a prime minimal over P_C if and only if P' is a prime minimal over P_D.*

Proof. By Lemma 3.3(iv) P' is a prime minimal over P_C if and only if the canonical map $A/P_C \to A/P'$ extends to a ring homomorphism $Q(A/P_C) \to Q(A/P')$. Since $P_D \subset P_C \subset P'$, a similar characterization is valid with D replacing C. Any ideal of $Q(A/P_D)$ is generated by elements in A/P_D. In particular, the kernel K of the surjective homomorphism $Q(A/P_D) \to Q(A/P_C)$ from Lemma 4.2 is an ideal of $Q(A/P_D)$ generated by the kernel of the canonical map $A/P_D \to A/P_C$, that is by P_C/P_D. If $\varphi : Q(A/P_D) \to Q(A/P')$ is a ring homomorphism extending the canonical map $A/P_D \to A/P'$, then $K \subset \operatorname{Ker} \varphi$, and therefore φ factors through $Q(A/P_C)$. $\square$

As was pointed out in the introduction Theorem 0.1 is an immediate consequence of Lemma 4.7. Thus the equivalence relation $\sim_H$ is defined.

Lemma 4.9. *Suppose that $P, P', P_1 \in \operatorname{Spec}_f A$ satisfy $P \sim_H P'$ and $P_1 \subset P$. Then there exists $P_1' \in \operatorname{Spec}_f A$ such that $P_1 \sim_H P_1'$ and $P_1' \subset P'$.*

Proof. There exists $C \in \mathcal{F}$ such that P' is a prime minimal over P_C. We have then $P' \supset P_C \supset (P_1)_C$. Hence P' contains a prime P_1' minimal over $(P_1)_C$. $\square$

I don't know whether the going-up version of Lemma 4.9 is always true.

Proposition 4.10. *Given $P, P' \in \operatorname{Spec}_f A$ satisfying $P \sim_H P'$, we have*:

(i) coheight $P =$ coheight P',

(ii) *P is a maximal ideal of A if and only if so is P',*

(iii) *$P \not\subset P'$ unless $P = P'$.*

Assuming that $\operatorname{Spec}_f A = \operatorname{Spec} A$, *we also have*:

(iv) height $P =$ height P',

(v) *P is a minimal prime of A if and only if so is P',*

Proof. (i) In view of Lemma 3.5 the equality of coheights can be rewritten as coheight $\mathfrak{p} =$ coheight $\mathfrak{p}'$ where $\mathfrak{p} = Z \cap P$ and $\mathfrak{p}' = Z \cap P'$ are prime ideals of Z. Let $C \in \mathcal{F}$ be such that P' is a prime minimal over P_C. Since A/P_C, $Z/(Z \cap P_C)$ are a nice pair, it follows from Lemma 3.3(iii) that $\mathfrak{p}'$ is a prime minimal over $Z \cap P_C$. Lemma 3.7(ii) applied in the situation of Lemma 4.1 shows that $\operatorname{Kdim} Z/\mathfrak{p} =$ coheight $\mathfrak{q}$ for each minimal prime of $Z/(Z \cap P_C)$. Taking $\mathfrak{q} = \mathfrak{p}'/(Z \cap P_C)$, we obtain the desired equality.

(ii) A prime ideal is maximal if and only if its coheight equals 0.

(iii) There exist $C, D \in \mathcal{F}$ such that P' is a prime minimal over P_C and P is a prime minimal over P_D. Since $C + D$ is a finitely generated k-submodule of H, it is contained in some $E \in \mathcal{F}$. Then $P_E \subset P_C \cap P_D$. By Lemma 4.8 both P and P' are primes minimal over P_E.

(iv) Note that the equality $P' = P_1'$ in Lemma 4.9 entails $P \sim_H P_1$, and then $P = P_1$ by (iii). Hence for each chain in $\operatorname{Spec} A$ terminating at P there exists a chain of the same length terminating at P'.

(v) A prime ideal is minimal if and only if its height equals 0. $\square$

Lemma 4.11. *Suppose that $\operatorname{Spec} A = \operatorname{Spec}_f A$ and $\bigcap_{P \in \operatorname{Spec} A} P_H = 0$. Then any element $s \in A$ regular modulo each minimal prime of A is regular in A.*

Proof. Let P be a minimal prime of A. If $C \in \mathcal{F}$ and $P' \in \operatorname{Spec} A$ is minimal over P_C, then P' is minimal in $\operatorname{Spec} A$ by Proposition 4.10, whence s is regular modulo P'. By Lemma 4.1 A/P_C has an artinian classical quotient ring. The simple factor rings of $Q(A/P_C)$ are isomorphic with $Q(A/P')$ for primes P' minimal over P_C (cf. Lemma 3.3(ii)). Since the image of s in each $Q(A/P')$ is invertible, so is the image of s in $Q(A/P_C)$, whence s is regular modulo P_C.

Suppose that $a \in A$ satisfies $as = 0$ or $sa = 0$. Then $a \in \bigcap_{C\in\mathcal{F}} P_C = P_H$ for any minimal prime P. Hence s lies in the intersection of the ideals P_H for different minimal $P \in \operatorname{Spec} A$. This intersection is zero by the hypothesis. So $s = 0$. $\square$

The intersection $\bigcap_{P\in\operatorname{Spec} A} P_H$ is an H-stable ideal of A contained in the prime radical N of A. Suppose A is H-semiprime and noetherian. Then $\bigcap_{P\in\operatorname{Spec} A} P_H = 0$ since N is nilpotent. By [8, Prop. 7.5] $s \in A$ is regular modulo N if and only if s is regular modulo each minimal prime of A. Now Lemma 4.11 enables us to apply Small's criterion [8, Cor. 11.10], according to which A has an artinian classical quotient ring. By [16, Th. 2.2] $Q(A)$ is an H-module algebra with respect to a module structure extending that on A. If I is any H-stable nilpotent ideal of $Q(A)$, then $I \cap A$ is an H-stable nilpotent ideal of A. It follows that $I \cap A = 0$, and $I = 0$. Thus $Q(A)$ is H-semiprime.

Recall that A is *H-prime* if $IJ \neq 0$ for any pair of nonzero H-stable ideals of A. If $P_H = 0$ for some $P \in \operatorname{Spec} A$, then A is H-prime since none of the nonzero H-stable ideals of A is contained in P.

Lemma 4.12. *Suppose A is noetherian. Let P, P' be minimal primes of A. Then*:

(i) $P_H = P_C$ *for some* $C \in \mathcal{F}$.

(ii) $P \sim_H P'$ *if and only if* $P_H \subset P'$, *if and only if* $P_H = P'_H$.

Proof. Part (ii) follows from (i) and Theorem 0.1. To prove (i) we may replace A with A/P_H, and so assume that $P_H = 0$. In this case A is H-prime, hence H-semiprime. Therefore $Q(A)$ exists and is an artinian H-module algebra. We may identify A with an H-stable subalgebra of $Q(A)$. Now $M = P \cdot Q(A)$ is a maximal ideal of $Q(A)$ and $M \cap A = P$. Since $M_H \cap A \subset P_H = 0$, we have $M_H = 0$. The artinian property ensures that the family of ideals M_C with $C \in \mathcal{F}$ has a smallest member which must coincide with M_H. Hence $M_C = 0$ for some $C \in \mathcal{F}$. Since $P_C \subset M_C$, we get $P_C = 0$. $\square$

Corollary 4.13. *If A is noetherian and H-prime, then the minimal primes of A constitute a single $\sim_H$-equivalence class and $P_H = 0$ for each minimal prime P.*

Proof. Let $P_1, \ldots, P_n$ be all minimal primes of A. Since A is noetherian, a suitable product of these ideals is zero, and then a suitable product of the H-stable ideals $(P_1)_H, \ldots, (P_n)_H$ is also zero. The H-primeness of A ensures that $P_H = 0$ for at least one minimal prime P. But then $P_H \subset P_i$ for each $i = 1, \ldots, n$, whence $P_i \sim_H P$ and $(P_i)_H = 0$ by Lemma 4.12. $\square$

Lemma 4.14. *Suppose that A is noetherian. The pair A, Z is nice if either* (a) *A is H-prime or* (b) *A is H-semiprime and Z coincides with the center of A. Hence $Q(A) \cong A \otimes_Z Q(Z)$ in both cases.*

Proof. Under assumption (a) Lemma 4.12 and Corollary 4.13 show that $P_C = 0$ for any minimal prime P and a suitable $C \in \mathcal{F}$. The conclusion then repeats Lemma 4.1(i). Under assumption (b) the conclusion follows from Lemma 3.8. $\square$

If A is noetherian and H-semiprime, then $Q(A)$ is module-finite over its center by Lemma 4.14. Therefore $Q(A)$ is quasi-Frobenius by Theorem 1.8. The proof of Theorem 0.3 is now completed. I don't know whether A, Z are necessarily a nice pair when the noetherian hypothesis is dropped.

Both A and Z are equidimensional in the H-prime case:

Proposition 4.15. *If A is noetherian and H-prime, then for each minimal prime P of A and each minimal prime $\mathfrak{p}$ of Z we have*

$$\operatorname{coheight} P = \operatorname{Kdim} A = \operatorname{Kdim} Z = \operatorname{coheight} \mathfrak{p}.$$

Proof. The classical Krull dimension of a ring is the supremum of the coheights of its prime ideals. Therefore the first equality follows from Proposition 4.10(i) and Corollary 4.13. Since A, Z are a nice pair, by Lemma 3.3(iii) the minimal primes of Z are precisely the contractions of the minimal primes of A. By Lemma 3.5 we have $\operatorname{coheight} P = \operatorname{coheight} P \cap Z$, whence the remaining equalities. □

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