Arbeitstagung 2007
First lecture

The first Arbeitstagungen with special emphasis on 1957, 1958 and 1962

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The Arbeitstagung 1957 took place from Saturday July 13 to Saturday July 20. Lecturers were Grothendieck, Atiyah, Kuiper, Hirzebruch, Tits, Grauert. These six people are the founding fathers of the AT. I welcomed Jacques Tits as the only founding father besides me who was present.

In 1957 Grothendieck lectured all of Saturday afternoon and Monday, Tuesday, Wednesday morning on

Kohärente Garben und verallgemeinerte
Riemann-Roch-Hirzebruch-Formel
auf algebraischen Mannigfaltigkeiten

My lecture had the title

$$c_n(\hat{\eta}) = (n-1)! \text{ generator } H^{2n}(S^{2n}, \mathbb{Z})$$

Speaking from a rather personal point of view, I reported only on these two talks. (Grothendieck: 12 hours, FH: 1 hour).

Let X be a projective algebraic manifold. The Grothendieck ring $K(X)$ is defined by the algebraic complex vector bundles and the coherent sheaves on X, where a vector bundle corresponds to the locally free sheaf of its sections. For an exact sequence $0 \to A \to B \to C \to 0$ of coherent sheaves we have in $K(X)$ the relation $B = A + C$. A coherent sheaf has a resolution in vector bundles and is in $K(X)$ an alternating sum of vector bundles. For a coherent sheaf S considered as element of $K(X)$ the RRH-expression is

$$(1) \qquad ch(S)\, td(TX) \in H^{ev}(X, \mathbb{Q})$$

where ch is the Chern character and $td(TX)$ the total Todd class of the tangent bundle of X. The evaluation of (1) on the fundamental cycle of X gives the holomorphic Euler number of X with coefficients in the sheaf S. This is the RRH-formula in Grothendieck's title (Of course, RRH was done originally only for a vector bundle S.)

For an algebraic map $f: X \to Y$ between projective algebraic manifolds Grothendieck considers the homomorphism

$$f_* : H^{ev}(X, \mathbb{Q}) \to H^{ev}(Y, \mathbb{Q})$$

and shows that f_* sends an RRH-expression for X into an RRH-expression for Y. More precisely

$$(2) \qquad f_*(ch(S)td(TX)) = ch(f_! S)td(TY))$$

where $f_! S$ is the alternating sum of the direct images of S given by the presheaves

$$Y \supset U \to H^q(f^{-1}U, S)$$

(2) is the Grothendieck Riemann-Roch theorem which gives the RRH-Formula if Y is a point.

A small indication of Grothendieck's proof in the case of an embedding $f: X \to Y$ (in the topological K-theory which came a little later, Atiyah-Hirzebruch) is:

Let N be the normal bundle of X in Y and $\bar{N}$ be a tubular neighborhood (remember; as an illustration only we switch to the topological category), then

$$(3) \qquad f_!(S) = S \cdot \sum (-1)^k \Lambda^k N^* \ .$$

The alternating sum can be lifted to $\bar{N}$ and vanishes on $\partial \bar{N}$ and can be extended to Y. (3) follows from (2) by using formulas for the Chern character relating the alternating sum to $td^{-1}(N)$.

In my lecture of July 16, 1957 (3.15 pm, in the morning there was only Grothendieck) I recalled that the RRH-expressions in (1) can be considered for topological complex vector bundles and a stable almost complex manifold X. Evaluating on the fundamental cycle of X we get the rational RR-numbers. It can be deduced from the signature theorem that these numbers are integers exc 2, i.e. their denominators are powers of 2 (Borel-Hirzebruch). For a complex vector bundle W over S^{2n} (with the trivial stable almost complex structure) the

RR-number is

$$ch(W)[S^{2n}] = \frac{\pm n c_n}{n!}[S^{2n}]$$

where c_n is the n-th Chern class of W.
Hence c_n is always divisible by $(n-1)!$ exc 2: $c_n/(n-1)!$ has a power of 2 as its denominator.
I reported in my lecture that Borel and I studied the positive spinor representation applied to the tangent bundle η of S^{2n}

$$SO(2n) \to U(2^{n-1})$$

to give a complex vector bundle $\hat{\eta}$ over S^{2n} whose Chern class c_n is $(n-1)!\cdot$ generator of
~~(4)~~ $H^{2n}(S^{2n}, \mathbb{Z})$. The exact homotopy sequence

$$(4) \quad \pi_{2n-1}(U(n)) \xrightarrow[c_n]{} \pi_{2n-1}(S^{2n-1}) \to \pi_{2n-2}(U(n-1)) \to \pi_{2n-2}(U(n))$$

shows that $\pi_{2n-2}(U(n-1))$ contains a cyclic subgroup whose order is $(n-1)!$ divided by a power of 2. Borel and I <u>conjectured</u>

$$(5) \qquad \pi_{2n-2}(U(n-1)) = \mathbb{Z}/_{(n-1)!}$$

Arbeitstagung 1958

Lectures by Abhyankar, Bott, Grauert, Grothendieck, Hirzebruch, Kervaire, Milnor, Puppe, Remmert, Serre, Stein, Thom

I reported mainly on Bott's lecture and mentioned the other underlined lectures, being rather short because of lack of time.

The homotopy group $\pi_i(U(n))$ is stable for $i < 2n$ and can be identified with $\pi_i(U)$, infinite unitary group. In a similar way $\pi_i(O)$ can be defined (infinite orthogonal group). Using Morse theory applied to the loop space of symmetric spaces Bott proved the periodicity theorems

$$\pi_i(U) = \mathbb{Z} \quad \text{for } i \text{ odd}$$
$$\pi_i(U) = 0 \quad \text{for } i \text{ even.}$$

For O the period equals 8. The groups beginning with $\pi_0(O)$ have the period

$$\mathbb{Z}_2,\ \mathbb{Z}_2,\ 0,\ \mathbb{Z},\ 0,\ 0,\ 0,\ \mathbb{Z}$$

In the unitary case

$\pi_{2n-1}(U(n))$ und $\pi_{2n-2}(U(n))$ are stable.

Bott shows that the generator of $\pi_{2n-1}(U(n))$ has Chern class $c_n = (n-1)!$ and because of (4) Bott obtains (5):

$$(5*) \qquad \pi_{2n-2}(U(n-1)) = \mathbb{Z}/(n-1)!$$

Bott announced his results in a PNAS note communicated by Steenrod on August 2, 1957. Bott periodicity was known ~~during~~ the first Arbeitstagung, but not to us! Bott told me about it in September 1957 in Princeton, I told it to Milnor who immediately saw that the divisibility by $(n-1)!$ implies that only the spheres S^1, S^3, S^7 are parallelisable. Kervaire had an independent proof about which he talked.

Milnor's lecture at AT 1958 concerned complex cobordism and I applied it in my lecture to prove that all RR-numbers are integers thus getting again the divisibility by $(n-1)!$ in a very round about way. Milnor and Bott had a correspondence about the parallelisability of spheres in December 1957/January 1958 (see Bott's Collected Papers).

Milnor uses the exact homotopy sequence

$$\pi_{2n-1}(SO(2n)) \xrightarrow{\alpha} \pi_{2n-1}(S^{2n-1}) \xrightarrow{\beta} \pi_{2n-2}(SO(2n-1))$$

and recalls that β maps a generator to the tangent bundle of S^{2n-1} and α maps a $SO(2n)$-bundle over S^{2n} to the Euler class $e_{2n}[S^{2n}]$ times a generator of $\pi_{2n-1}(S^{2n-1})$. The tangent bundle of S^{2n} is mapped by α to $2\times$ generator. Hence S^{2n-1} is parallelisable if and only if there exists an $SO(2n)$-bundle over S^{2n} with odd Euler class (equivalently: with Euler class one). After Milnor and Kervaire, Borel and I gave a different proof as follows. Let η be an $SO(2n)$-bundle over S^{2n}. For $n>1$, $w_2(\eta) = 0$ and the positive spinor representation can be applied to η to give an $U(2^{n-1})$-bundle $\hat{\eta}$ over S^{2n}. We calculated:

$$\frac{c_n(\hat{\eta})}{(n-1)!} = e_{2n}(\eta)/2 \qquad \text{for } n \text{ odd}$$

For $n = 2k$

$$\frac{c_{2k}(\hat{\eta})}{(2k-1)!} = -\frac{p_k(\eta)}{(2k-1)!} \cdot \frac{tg^{(2k-1)}(0)}{4} - \frac{e_{4k}(\eta)}{2}$$

$c_n(\hat{\eta})$ is divisible by $(n-1)!$ according to $(5*)$.
Also $p_k(\eta)/(2k-1)!$ is integral because $p_k(\eta)$ is up to sign the Chern class c_{2k} of the complex extension of η. We have $tg'(0)=1$, $tg^{(3)}(0)=2$ whereas all higher tangent derivatives are integers divisible by 4. (If η is the tangent bundle of S^{2n}, then $p_k(\eta)=0$ and this gives the result of my AT 1957 lecture.)

<u>Remark:</u> At the ICM 1958 Atiyah mentioned to me that Grothendieck's method (3) gives the integrality of the Todd genus of a stable almost complex manifold X by embedding X in an even-dimensional sphere. Discussion showed that one can ~~show~~ prove this way that all RR-numbers are integers. We were on our way to the topological K-theory and the "differentiable Riemann-Roch theorems".

Arbeitstagung 1962

Serge Lang gave the opening lecture "On the Nash embedding theorem à la Moser". Atiyah lectured on "Harmonic spinors". This was the first time the index theorem of Atiyah–Singer appeared in an Arbeitstagung based on work of Atiyah and Singer in Oxford not long before the Arbeitstagung. Much of it was still conjectural.

The $\hat{A}$-genus equals the RR-number

$$ch(L)\, td(X)\,[X]$$

if X is stable almost complex, with even first Chern class c_1 (equivalently $w_2 = 0$) and $c_1(L) = -c_1/2$. This RR-number depends only on ~~compact~~ the Pontrjagin classes of X and thus is defined for compact oriented differentiable manifolds with $w_2 = 0$. It is always an integer.

Now comes the break-through. Suppose $w_2 = 0$. Then we can apply the positive and the negative spinor representations to the tangent bundle T of X to obtain complex vector bundles

Δ^+ and Δ^-. The Dirac operator δ

$$\delta : \Gamma(\Delta^+) \to \Gamma(\Delta^-)$$

goes from the sections of Δ^+ to those of Δ^-. <u>The index equals $\hat{A}(X)$</u>.

S. Lang was so excited about Atiyah's talk that he wrote rather detailed notes which I still have.

Atiyah was a very active member of the Arbeitstagung. During the thirty ATs I organized from 1957 to 1991 he lectured 32 times including 16 opening lectures.

My talk became rather sketchy at the end. I mentioned a few titles of the talks of the ATs 1959, 1960, 1961. All titles since 1957 can be found in the web.

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