The analogs of Riemann and Penrose tensors on supermanifolds

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Abstract. The Spencer cohomology of certain $\mathbb{Z}$graded Lie superalgebras are completely computed. This cohomology is interpreted as analogs of Riemann and Penrose tensors on supermanifolds. The results make it manifest that there is no simple generalization of Borel-Weil-Bott's theorem for Lie superalgebras.

Contents

This paper expands the results that appeared mainly in not very accessible [P1-P8] listed below and complements [LPS] which should have been published in 1991. Meanwhile there appeared Grozman's package SuperLie that confirmed our — not very easy — calculations; for some of Grozman's independently obtained results (the cases of exceptional algebras) see [LPS].

[P1] Poletaeva, E. I. Spencer cohomology connected with some Lie superalgebras. In: Onishchik A. (ed.) *Problems in group theory and homological algebra* (Russian), 162–167, Matematika, Yaroslav. Gos. Univ., Yaroslavl, 1988. MR 1 175 010

[P2] Poletaeva, E. Structure functions on the usual and exotic symplectic and periplectic supermanifolds. In: Bartocci C. et al. (eds.) *Differential geometric methods in theoretical physics* (Rapallo, 1990), 390–395, Lecture Notes in Phys., 375, Springer, Berlin, 1991. MR 92h:58207

[P3] Poletaeva, E. I. Spencer cohomology of Lie superalgebras of vector fields.In: Onishchik A. (ed.) *Problems in group theory and homological algebra* (Russian), 168–169, Matematika, Yaroslav. Gos. Univ., Yaroslavl, 1990. MR 93c:17036

[P4] Leites, D.; Poletaeva, E. Analogues of the Riemannian structure for classical superspaces. Proceedings of the International Conference on Algebra, Part 1 (Novosibirsk, 1989), 603–612, Contemp. Math., 131, Part 1, Amer. Math. Soc., Providence, RI, 1992. MR 93k:58008

[P5] Poletaeva, E. Analogues of Riemann tensors for the odd metric on supermanifolds. Acta Appl. Math. 31 (1993), no. 2, 137–169. MR 94d:58166

[P6]Poletaeva, E. Penrose's tensors on super-Grassmannians. Math. Scand. 72 (1993), no. 2, 161–190. MR 94m:58008a

[P7] Poletaeva, E. Penrose's tensors. II. Math. Scand. 72 (1993), no. 2, 191–211.MR 94m:58008b

[P8] MR96d:58005 Leites, D.; Poletaeva, E. Supergravities and contact type structures on supermanifolds. Second International Conference on Algebra (Barnaul, 1991), 267–274, Contemp. Math., 184, Amer. Math. Soc., Providence, RI, 1995.

Our results also make it clear why one can not just "superize" metric in order to get Einstein-Hilbert's equations. Their true superizations — various SUGRAs — correspond to $\mathbb{Z}$-graded Lie superalgebras of depth $d > 1$. Such Lie superalgebras were discussed by Yu. Manin [M] but the corresponding structure functions were not calculated yet except in the cases considered by Grozman and Leites in

[GL1] Grozman, P.; Leites, D. From supergravity to ballbearings. *Supersymmetries and quantum symmetries* (Dubna, 1997), 58–67, Lecture Notes in Phys., 524, Springer, Berlin, 1999 MR 2000j:83090

There is, however, a paper where approach similar to the one described in what follows is applied to $\mathbb{Z}$-graded Lie superalgebras of depth $d = 1$ and the results are interpreted as supergravity since the tensor obtained after deleting all that depends on odd parameters are exactly the standard Riemannian tensor:

[GL2] Grozman, P.; Leites, D., An unconventional supergravity. *Noncommutative structures in mathematics and physics* (Kiev, 2000), 41–47, NATO Sci. Ser. II Math. Phys. Chem., 22, Kluwer Acad. Publ., Dordrecht, 2001 MR 1 893 452

Introduction

Structure functions

The main object of the study of Riemannian geometry is the properties of the Riemann tensor, which in turn splits into the Weyl tensor, the traceless Ricci tensor, and the scalar curvature. All these tensors are obstructions to the possibility of "flattening" the manifold on which they are considered. The word "splits" above means that at every point of the Riemannian manifold M^n for $n \neq 4$ the space of values of the Riemann tensor constitutes an $O(n)$-module which splits into the sum of three irreducible components (for $n = 4$ there are four of them, because the Weyl tensor splits additionally in this case) [ALV, Kob].

More generally, let $G \subset GL(n)$ be any Lie group, not necessarily $O(n)$. A reduction of the principal $GL(n)$-bundle on M to the principal G-bundle is called a *G-structure* on M.

Recall that on a manifold with a G-structure there is a canonical connection. For a Riemannian manifold this is the Levi-Civita connection. The so-called *structure functions* (SFs) constitute the complete set of obstructions to integrability of the canonical connection or, in other words, to the possibility of local flattening of a manifold with G-structure. The Riemann tensor is an example of a SF. Among the most known other examples of SFs are the following ones:

- a *conformal structure*, $G = O(n) \times \mathbb{R}^*$, SFs are called the *Weyl tensor*;
- Penrose's *twistor* theory, $G = S(U(2) \times U(2)) \times \mathbb{C}^*$, SFs-Penrose's tensors — split into two components called the "*α-forms*" and "*β-forms*";
- an almost *complex structure*, $G = GL(n; \mathbb{C}) \subset GL(2n; \mathbb{R})$, SFs are called the *Nijenhuis tensor*;
- an almost *symplectic structure*, $G = Sp(2n)$, no accepted name for SFs.

Spencer cohomology groups

Recall necessary definitions [St, Gu].

The simplest G-structure is the *flat* G-structure defined as follows. Let V be $\mathbb{K}^n$ with a fixed frame. Consider the bundle over V whose fiber over $v \in V$ consists of all frames obtained from the fixed one under the G-action, V being identified with T_vV.

Obstructions to identification of the $(k+1)$-st infinitesimal neighborhood of a point $m \in M$ on a manifold M with G-structure and that of a point of the flat manifold V with the above G-structure are called *structure functions of order k*. The identification is performed inductively and is possible provided the obstructions of lesser orders vanish. At each point of a manifold M SFs take values in certain cohomology groups, called *Spencer cohomology groups*. The corresponding complex is defined as follows. Let S^iV denote the i-th symmetric power of a vector space V and $Lie(G)$ denote the Lie algebra of the Lie group G. Set $\mathfrak{g}_{-1} = T_mM, \mathfrak{g}_0 = \mathfrak{g} = Lie(G)$ and for $i > 0$ put:

$$\begin{aligned} \mathfrak{g}_i &= \{X \in \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{i-1}) : X(v)(w, \dots) = X(w)(v, \dots) \text{ for any } v, w \in \mathfrak{g}_{-1}\} \\ &= (\mathfrak{g}_0 \otimes S^i(\mathfrak{g}_{-1})^*) \cap (\mathfrak{g}_{-1} \otimes S^{i+1}(\mathfrak{g}_{-1})^*). \end{aligned}$$

Now set $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0) = \oplus_{i\geq -1}\mathfrak{g}_i$. Suppose that the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is faithful .

Then $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0) \subset \mathfrak{vect}(n) = der\mathbb{K}[[x_1, \dots, x_n]]$, where $n = \dim \mathfrak{g}_{-1}$. It can be verified that the Lie algebra structure on $\mathfrak{vect}(n)$ induces such a structure on $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0)$. The Lie algebra $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0)$, usually abbreviated $\mathfrak{g}_*$, will be called the *Cartan prolongation* of the pair $(\mathfrak{g}_{-1},\ \mathfrak{g}_0)$.

Let E^iV be the i-th exterior power of a vector space V. Set

$$C^{k,s}_{\mathfrak{g}_0} = \mathfrak{g}_{k-s} \otimes E^s(\mathfrak{g}^*_{-1}).$$

Define the differentials $\partial^{k,s}_{\mathfrak{g}_0} : C^{k,s}_{\mathfrak{g}_0} \longrightarrow C^{k,s+1}_{\mathfrak{g}_0}$ as follows: for any $g_1, \dots, g_{s+1} \in \mathfrak{g}_{-1}$

$$(\partial^{k,s}_{\mathfrak{g}_0} f)(g_1, \dots, g_{s+1}) = \sum_i (-1)^i [f(g_1, \dots, \widehat{g_{s+1-i}}, \dots, g_{s+1}),\ g_{s+1-i}] \tag{1}$$

As expected, $\partial^{k,s}_{\mathfrak{g}_0}\partial^{k,s+1}_{\mathfrak{g}_0} = 0$. The cohomology of bidegree $(k,\ s)$ of this complex is called the $(k,\ s)$-th Spencer cohomology group $H^{k,s}_{\mathfrak{g}_0}$. It turns out that structure functions of order k on a manifold M with G-structure are sections of certain vector bundles over M with fiber over a point $m \in M$ isomorphic to $H^{k,2}_{\mathfrak{g}}(T_mM)$, where $\mathfrak{g} = Lie(G)$.

Generalized conformal structures

A generalization of the notion of conformal structure is a G-structure of type X, where X is a classical space, i.e., an irreducible compact Hermitian symmetric space (CHSS). These G-structures were introduced and intensively studied by A. Goncharov, who calculated the corresponding structure functions [G1, G2]. In his examples G is the reductive part of the stabilizer of a point of X. The usual conformal structure is the one that corresponds to $X = Q_n$, a quadric in the projective space. The complex grassmannian $X = Gr^4_2$ corresponds to Penrose's twistors.

Recall that Penrose's idea is to embed the Minkowski space M^4 into the complex Grassmann manifold Gr^4_2 of planes in $\mathbb{C}^4$ (or straight lines in $\mathbb{CP}^3$) and to express the conformal structure on M^4 in terms of the incidence relation of the straight lines in $\mathbb{CP}^3$ [Pe].

The conformal structure on M^4 is given by a field of quadratic cones in the tangent spaces to the points of M^4. In Penrose's case these cones possess two families of two-dimensional flat generators, the so-called "α-planes" and "β-planes." The geometry of these families is vital for Penrose's considerations. In particular, the Weyl tensor gets a lucid description in terms of these families.

It is interesting to include 4-dimensional Penrose theory into a more general theory of geometric structures. A. Goncharov has shown that there is an analogous field of quadratic cones for any irreducible compact Hermitian symmetric space X of rank greater then one [G2].

Let S be a simple complex Lie group, P its parabolic subgroup with the Levi decomposition $P = GN$, i.e., G is reductive and N is the radical of P. As one knows [He], N is Abelian if and only if $X = S/P$ is a CHSS, and in this case $G = G_0 \times \mathbb{C}^*$, where G_0 is semisimple.

Let $P_x = G_xN_x$ be the Levi decomposition of the stabilizer of $x \in X$ in S. Denote by C_x the cone of highest weight vectors in the G_x-module T_xX, i.e., each element in C_x is highest with respect to some Borel subgroup in G_x. Since $s \in S$ transforms C_x to C_{sx}, then with X there is associated the cone $C(X) \subset T_{\bar{e}}X$, where $\bar{e}$ is the image of the unit $e \in S$ in X.

Let $\mathrm{rk}(X) > 1$, i.e., $X \neq \mathbb{CP}^n$. Then on a manifold M a *generalized conformal structure* of type X is given if M is endowed with a family of cones C_m and $\mathbb{C}$-linear isomorphisms $A_m : T_{\bar{e}}X \longrightarrow T_mM$ such that $A_m(C(x)) = C_m$.

Goncharov has shown that a manifold M with generalized conformal structure of type X is a manifold with a $\tilde{G}$-structure, where $\tilde{G}$ is a group of linear automorphisms of the cone $C(X)$ and the connected component of the identity of this group is precisely G [G2].

The case of a simple Lie algebra $\mathfrak{g}_*$ over $\mathbb{C}$

The following remarkable fact, though known to experts, is seldom formulated explicitly [LRC, KN].

PROPOSITION. *Let $\mathbb{K}=\mathbb{C}$, $\mathfrak{g}_*=\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0)$ be simple. Then only the following cases are possible:*

1) $\mathfrak{g}_2\neq 0$*, then* $\mathfrak{g}_*$ *is either* $\mathfrak{vect}(n)$ *or its special subalgebra* $\mathfrak{svect}(n)$ *of divergence-free vector fields, or its subalgebra* $\mathfrak{h}(2n)$ *of Hamiltonian vector fields.*

2) $\mathfrak{g}_2=0,\ \mathfrak{g}_1\neq 0$*, then* $\mathfrak{g}_*$ *is the Lie algebra of the complex Lie group of automorphisms of a CHSS (see §3).*

Let $R(\sum_i a_i\pi_i)$ be the irreducible $\mathfrak{g}_0$-module with the highest weight $\sum_i a_i\pi_i$, where π_i is the i-th fundamental weight.

THEOREM (Serre [St]). *In case 1) of Proposition SFs can only be of order 1. More precisely: for* $\mathfrak{g}_*=\mathfrak{vect}(n)$ *and* $\mathfrak{svect}(n)$ *SFs vanish, for* $\mathfrak{g}_*=\mathfrak{h}(2n)$ *nonzero SFs are* $R(\pi_1)$ *for* $n=2$*, and* $R(\pi_1)\oplus R(\pi_3)$ *for* $n>2$.

When $\mathfrak{g}_*$ is a simple finite dimensional Lie algebra over $\mathbb{C}$ computation of SFs becomes an easy corollary of the Borel-Weil-Bott (BWB) theorem in a form due to W. Shmid [Sh], cf. work of A. Goncharov [G2]. Indeed, by definition,

$$\oplus_k H^{k,2}_{\mathfrak{g}_0}=H^2(\mathfrak{g}_{-1},\ \mathfrak{g}_*).$$

The BWB theorem implies that as a $\mathfrak{g}_0$-module, $H^2(\mathfrak{g}_{-1},\ \mathfrak{g}_*)$ has as many components as $H^2(\mathfrak{g}_{-1})$. Thanks to commutativity of $\mathfrak{g}_{-1}$ one has $H^2(\mathfrak{g}_{-1})=E^2\mathfrak{g}^*_{-1}$, which facilitates the count of components. The BWB theorem also gives the formula for the highest weights of these components.

Reduced structures

Let $X=S/P$, where $Lie(S)=\mathfrak{g}_*=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$ and $Lie(P)=\mathfrak{g}_0\oplus\mathfrak{g}_1$, be a CHSS. Let $\hat{\mathfrak{g}}_0$ be the semisimple part of $\mathfrak{g}_0=Lie(G)$. A $\hat{G}$-structure, where $Lie(\hat{G})=\hat{\mathfrak{g}}_0$, will be referred to as a *Riemannian structure* of type X. To reduce the structure group G to its semisimple part $\hat{G}$ is an action similar to distinguishing a metric from a conformal class on a conformal manifold.

The structure functions of the $\hat{G}$-structures form an analogue of the Riemann tensor for the metric. They include the structure functions of the G-structure and several other irreducible components, some of which are analogues of the traceless Ricci tensor or the scalar curvature.

More precisely, the structure functions of the G-structure are defined as the part of the structure functions of the $\hat{G}$-structure obtained by a reduction of the G-structure that does not depend on the choice of reduction. In other words, this is a generalized conformally invariant part of the structure functions of the $\hat{G}$-structure.

Since in the case of the Riemannian structure $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}}_0)=\mathfrak{g}_{-1}\oplus\hat{\mathfrak{g}}_0$,then there only exist SFs of orders 1 and 2. Though the BWB theorem doesn't work in this case, SFs are describable thanks to the following proposition:

Proposition. *[G2].* 1) $H^{1,2}_{\hat{\mathfrak{g}}_0} = H^{1,2}_{\mathfrak{g}_0}$;
2) $H^{2,2}_{\hat{\mathfrak{g}}_0} = H^{2,2}_{\mathfrak{g}_0} \oplus S^2(\mathfrak{g}^*_{-1})$.

The Riemannian structure in the classical case of Riemannian geometry will be considered next.

Einstein equations

Let $G = O(n)$. In this case $\mathfrak{g}_1 = \mathfrak{g}_{-1}$ and a 1-dimensional subspace is distinguished in $S^2(\mathfrak{g}_{-1})^*$. The sections through this subspace constitute a Riemannian metric g on M. The usual way to determine a metric on M is to define a matrix-valued function, but actually this function with values in symmetric matrices depends only on one functional parameter. The values of the Riemann tensor at a point of M constitute an $O(n)$-module $H^2(\mathfrak{g}_{-1},\ \mathfrak{g}_*)$, which contains a trivial component. Let a section through it be denoted by R. This trivial component is naturally realized as a submodule in a module isomorphic to $S^2(\mathfrak{g}_{-1})^*$.

Thus, there exist two matrix-valued functions: g and R, both preserved by $O(n)$. Now let R correspond to the Levi-Civita connection. The process of restoring R from g involves differentiations and in this way one gets a nonlinear pde, which constitutes one of the two conditions called *Einstein equations* [L4, LSV, LPS]:

$$R = \lambda g, \text{ where } \lambda \in R. \tag{EE_0}$$

The other condition is that the other component belonging to $S^2(\mathfrak{g}_{-1})^*$, the traceless Ricci tensor Ric, vanishes:

$$\text{Ric } = 0. \tag{EE_{ric}}$$

There is a close relation between G-structures and so-called F-structures, which are also of interest, in particular, because of their application to Penrose's geometry. This relation will now be explained.

F-structures and their structure functions

Recall that the notion of F-structure is a generalization of the notion of distribution, i.e., a subbundle in TM and the SFs of an F-structure generalize the notion of the Frobenius form [G2].

Let $V = T_mM$, $F \subset Gr_k(V)$ be a manifold with a transitive action of a subgroup $G_F \subset GL(V)$, $\mathbb{F}(M)$ be a subbundle of $Gr_k(TM)$, where the fiber of $Gr_k(TM)$ is $Gr_k(T_mM)$. The bundle $\mathbb{F}(M) \longrightarrow M$ is called an *F-structure* on M, if for any point m of M there is a linear isomorphism $I_m : V \longrightarrow T_mM$, which induces a diffeomorphism $I_m(F) = \mathbb{F}(m)$. A submanifold $Z \subset M$ of dimension k such that $T_zZ \subset \mathbb{F}(z)$ for any $z \in Z$ is called an integral submanifold. An F-structure is integrable if for any $z \in Z$ and for any subspace $V(z) \subset \mathbb{F}(z)$ there is an integral manifold Z with $T_zZ = V(z)$.

SFs of an F-structure are defined as follows. For $f \in F$ let $V_f \subset V$ be the subspace corresponding to f. Set

$$(T_fF)_{-1} = V/V_f, \quad (T_fF)_0 = T_fF.$$

Define

$$(T_fF)_s = ((T_fF)_{s-1} \otimes V_f^*) \cap ((T_fF)_{s-2} \otimes S^2V_f^*)$$

for $s > 0$, and

$$C^{k,s}_{T_fF} = (T_fF)_{k-s} \otimes E^sV_f^*$$

Define the differentials as in (1). Then the cohomology groups $H^{k,s}_{T_fF}$ are naturally defined. It turns out that the obstruction to integrability of order k of an F-structure on a manifold M is a section of a certain vector bundle over $\mathbb{F}(M)$ with fiber over a point $\psi \in \mathbb{F}(m)$

isomorphic to $H^{k,2}_{T_{I_m^{-1}(\psi)}F}$. Moreover, there exists a map $H^{k,s}_{\mathfrak{g}_F} \longrightarrow H^{k,s}_{T_fF}$, where $\mathfrak{g}_F = Lie(G_F)$ [G2].

The relation between SFs of a G_F-structure and the obstructions to integrability of an F-structure generalizes a theorem of Penrose, which states that the anti-selfdual part of the Weyl tensor on a 4-dimensional manifold with a conformal structure vanishes if and only if α-surfaces exist, in other words, the metric is α-integrable [AHS, Gi].

More precisely, for a generalized conformal structure of type X, where $X = Gr_m^{m+n}(\mathbb{C})$, there exist two families of m and n-dimensional flat generators–analogues of Penrose's α-planes and β-planes. When neither m nor n is equal to 1, i.e., the grassmannian is not a projective space, SFs decompose into the direct sum of two components, which are analogues of the self-dual and anti-self-dual parts of the Weyl tensor on a 4-dimensional manifold with a conformal structure. The integrability of each of two families of generators is equivalent to the vanishing of the corresponding component of the SFs.

Structure functions on supermanifolds

The necessary background on Lie superalgebras and supermanifolds is gathered in [L1, L2, L3, K1, M].

The classical superspaces (homogeneous compact Hermitian symmetric superspaces), which are the super analogues of CHSS, considered by Goncharov, are listed in [S1].

The above definitions of SFs are generalized to Lie superalgebras via the sign rule. However, in the super case new phenomena appear, which have no analogues in the classical case:

- Cartan prolongations of $(\mathfrak{g}_{-1},\ \mathfrak{g}_0)$ and of $(\Pi\mathfrak{g}_{-1},\ \mathfrak{g}_0)$ are essentially different;
- faithfulness of the $\mathfrak{g}_0$-action on $\mathfrak{g}_{-1}$ is violated in natural examples of supergrassmannians of subsuperspaces in an $(n,\ n)$-dimensional superspace when the center z of $\mathfrak{g}_0$ acts trivially. This will be explained in §9.
- the formulation of Serre's theorem and of the Proposition of §4 fails to be literally true for Lie superalgebras.

Description of results

In Chapter 1 I compute the SFs for the odd analogue of the metric on the supermanifolds and for several related G-structures (see §2 of Chapter 1). In this case $\mathfrak{g}_0 = Lie(G)$ is the periplectic Lie superalgebra, the special periplectic Lie superalgebra, or their central extensions. It turns out that unlike the classical case of Riemannian geometry, the $\mathfrak{g}_0$-module $H^{k,2}_{\mathfrak{g}_0}$ is not completely reducible, and I describe the Jordan-Hölder series for this module. Thus, my computations show that there is no analogue of (EE_0) for the odd metric.

In Chapter 2 and Chapter 3 I obtain an explicit description of the Spencer cohomology groups $H^{k,2}_{\mathfrak{g}_0}$ for simple finite-dimensional complex classical Lie superalgebras endowed with $\mathbb{Z}$-grading of depth 1: $\mathfrak{g} = \oplus_{i\geq -1}\mathfrak{g}_i$, where $\mathfrak{g}_0$ is the zero-th part of the grading.

It is known [K2, S2] that all such $\mathbb{Z}$-gradings are of the form $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, except for the case when $\mathfrak{g}$ is the special periplectic superalgebra considered in Chapter 1. Thus, the cohomology groups $H^{k,2}_{\mathfrak{g}_0}$ constitute the space of values of SFs of G-structures corresponding to homogeneous compact Hermitian symmetric superspaces, where G is a reductive complex Lie supergroup of classical type and $\mathfrak{g}_0 = Lie(G)$. The groups $H^{k,2}_{\hat{\mathfrak{g}}_0}$ correspond to structures of Riemannian type.

An important particular case is $\mathfrak{g} = \mathfrak{sl}(m|n)$, where $m \neq n$, corresponding to general supergrassmannians.

In Chapter 2 I consider a $\mathbb{Z}$-grading of $\mathfrak{g}$ for which $\mathfrak{g}_0$ is a reductive Lie algebra. Thus, the $\mathfrak{g}_0$-module $H^{k,2}_{\mathfrak{g}_0}$ is completely reducible, and for $m, n > 2$ decomposes into the direct sum of two irreducible components — super analogues of Penrose's tensors for the usual complex grassmannians (see Theorem 3.1 of Chapter 2).

The case $\mathfrak{g} = \mathfrak{sl}(n|n)$ is also interesting, because I discovered a phenomenon which has no an analogue in the classical case. Indeed, the center z of $\mathfrak{g}_0$ acts trivially on $\mathfrak{g}_{-1}$. If one retains the same definition of the Cartan prolongation, then it has the form of the semidirect sum $S^*(\mathfrak{g}_{-1}^*) \Subset \mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0/z)$ (the ideal is $S^*(\mathfrak{g}_{-1}^*)$) with the natural $\mathbb{Z}$-grading and Lie superalgebra structure, but this Lie superalgebra is not a subsuperalgebra of $\mathfrak{vect}(\dim \mathfrak{g}_{-1})$ anymore (see Theorem 1.1 of Chapter 2).

In Chapter 3 I describe the Spencer cohomology groups for the other $\mathbb{Z}$-gradings of depth 1 of $\mathfrak{sl}(m|n)$ and $\mathfrak{psl}(n|n)$ (see Theorem 1.4 and Theorem 1.5 of Chapter 3). These theorems show that the superspace of SFs can be not completely reducible, and I get the answer in terms of nonsplit exact sequences of $\mathfrak{g}_0$-modules.

Finally, in the cases when m or n are equal to 1, I get SFs of the Lie superalgebra of vector fields $\mathfrak{vect}(m|n)$ or of divergence-free vector fields $\mathfrak{svect}(m|n)$ (see Theorem 2.1 of Chapter 2 and Theorem 1.3 of Chapter 3).

Theorem 2.3 of Chapter 3 shows that the SFs for queer grassmannians constitute a module looking exactly the same as that for grassmannians of generic dimensions.

The case $\mathfrak{g} = \mathfrak{osp}(m|2n)$ is similar to Riemannian geometry. The SFs constitute an irreducible $\mathfrak{g}_0$-module, which is an analogue of the Weyl tensor, and the superspace of the SFs for the reduced structure decomposes into the direct sum of three components–the super analogues of the Weyl tensor, the traceless Ricci tensor, and the scalar curvature. I find the highest weights of these components (see Theorem 3.3 of Chapter 3).

Finally, I describe the Spencer cohomology groups for exceptional Lie superalgebras $D(\alpha)$ and AB_3 (see Theorem 4.3 and Theorem 5.3 of Chapter 3, respectively).

Algebraic methods

As in the classical case (Lie theory), computation of Spencer cohomology groups reduces to certain problems of representation theory. However, in the super case computations become much more complicated, because of the absence of complete reducibility. I could not directly apply the usual tools for computing (co)homology (spectral sequences and the Borel-Weil-Bott theorem) to superalgebras and had to retreat a step and apply these tools to the even parts of the considered Lie superalgebras. Then, using certain necessary conditions, I verified whether two modules over a Lie superalgebra that could be glued into an indecomposable module were glued or not.

My method of computing the structure functions is based on the Hochcshild-Serre spectral sequence [Fu]. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra and M be a $\mathfrak{g}$-module. On the superspace of k-dimensional cochains $C^k = C^k(\mathfrak{g}, M)$ define a filtration:

$$F^0C^k = C^k \supset F^1C^k \supset F^2C^k \supset \dots F^jC^k \supset \dots \supset F^{k+1}C^k = 0,$$

where $F^jC^k = \{c \in C^k | c(g_1, \dots, g_i, \dots, g_k) = 0 \text{ if } k-j+1 \text{ arguments belong to } \mathfrak{g}_0\}$, $0 \leq j \leq k+1$. Using this filtration define the usual corresponding spectral sequence $E_r^{p,q}$ [GM]. Thus, $H^2(\mathfrak{g}, M) = \oplus_{p+q=2} E_\infty^{p,q} = E_3^{2,0} \oplus E_3^{1,1} \oplus E_4^{0,2}$.

In particular, $E_1^{p,q} = H^q(\mathfrak{g}_0,\ M \otimes S^p\mathfrak{g}_1^*)$ [Fu]. Since, in the case of Spencer cohomology, $\mathfrak{g} = \mathfrak{g}_{-1} = (\mathfrak{g}_{-1})_0 \oplus (\mathfrak{g}_{-1})_1$ is a commutative Lie superalgebra, then

$$E_1^{p,q} = H^q((\mathfrak{g}_{-1})_0,\ \mathfrak{g}_* \otimes S^p(\mathfrak{g}_{-1})_1^*) = H^q((\mathfrak{g}_{-1})_0,\ \mathfrak{g}_*) \otimes S^p(\mathfrak{g}_{-1})_1^*.$$

Then in special cases I use the BWB theorem to compute $H^q((\mathfrak{g}_{-1})_0,\ \mathfrak{g}_*)$ as a module over $(\mathfrak{g}_0)_0$.

§1. Plans for the future; structure functions of Lie superalgebras of Cartan type

It is interesting to compute the Spencer cohomology groups $H^{k,s}_{\mathfrak{g}_0}$ for simple finite-dimensional Lie superalgebras of Cartan type with nonstandard $\mathbb{Z}$-gradings of depth 1. For example, let $\mathfrak{g} = \mathfrak{vect}(0|n)$. Define the $\mathbb{Z}$–grading as follows: $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0 = \Lambda(n-1) \Subset \mathfrak{vect}(0|n-1)$, $\Lambda(n-1)$ is a commutative ideal in $\mathfrak{g}_0$ isomorphic to the Grassmann algebra as the $\mathfrak{vect}(0|n-1)$-module, $\mathfrak{g}_1 \cong \Pi(\mathfrak{vect}(0|n-1))$ as the $\mathfrak{g}_0$-module, $\mathfrak{g}_{-1} \cong \Pi(\Lambda(n-1))$ as the $\mathfrak{g}_0$-module, and the action of $\Lambda(n-1)$ on $\mathfrak{g}_{-1}$ is the usual multiplication in $\Lambda(n-1)$.

It is much more difficult to compute the Spencer cohomology groups for Cartan Lie superalgebras than for classical ones, because in this case the number of the irreducible quotient modules in the Jordan-Hölder series does depend on n. Moreover, even for small n computations seem to be very complicated, because of the absence of complete reducibility [LPS].

Terminological conventions

$\mathfrak{cg}$ is the trivial central extension of the Lie superalgebra $\mathfrak{g}$;
$\Subset$ stands for the semidirect sum of the algebras with the ideal to the left;
Π is the functor of the change of parity;
$p(X)$ is the parity of homogeneous element X of a superspace;
S^iV is the i-th symmetric power of a vector (super)space V;
$S^*V = \oplus_{k\geq 0} S^kV$;
E^iV is the i-th exterior power of a vector (super)space V;
$\langle 1\rangle$ is a trivial module over a Lie superalgebra;
I will consider the following Lie superalgebras [L2, L3]:
$\mathfrak{gl}(m|n) = Mat(m|n;\ \mathbb{C})$,
$\mathfrak{sl}(m|n) = \{X \in \mathfrak{gl}(m|n) |\ strX = 0\}$,
$\mathfrak{psl}(n|n) = \mathfrak{sl}(n|n)/\langle 1_{2n}\rangle$, where 1_{2n} is the unit matrix;
$\mathfrak{vect}(m|n) = \mathfrak{der}\mathbb{C}[[x]]$ is the Lie superalgebra of vector fields;
$\mathfrak{svect}(m|n) = \{D \in \mathfrak{vect}(m|n) |\ \div (D) = 0\}$ is the Lie superalgebra of divergence-free vector fields;
$\mathfrak{h}(0|n) = \{D \in \mathfrak{vect}(0|n) |\ Dw = 0$ for the Hamiltonian form $w = \sum_{i=1}^n (d\xi_i)^2\}$ is the Lie superalgebra of Hamiltonian vector fields;
Other Lie superalgebras will be defined in Chapter 1 and Chapter 3.

CHAPTER 1

The analogues of the Riemannian tensors for the odd metric on supermanifolds

Periplectic superalgebras and their Cartan prolongations

Let $z = 1_{2n}$ be the unit matrix and $\tau = \mathrm{diag}(1_n,\ -1_n)$.

Let P be a nondegenerate supersymmetric odd bilinear form on a superspace V. Clearly, $\dim V = (n,\ n)$. Define the odd analogue of the symplectic Lie algebra, the periplectic Lie superalgebra $\mathfrak{pe}(n)$, and its special subsuperalgebra $\mathfrak{spe}(n)$, setting

$$\begin{aligned}&\mathfrak{pe}(n) = \{X \in \mathfrak{gl}(n|n)|\ X^{st}P + (-1)^{p(X)}PX = 0\},\\ &\mathfrak{spe}(n) = \mathfrak{pe}(n) \cap \mathfrak{sl}(n|n).\end{aligned}$$

Thus,

$$\mathfrak{pe}(n) = \mathfrak{spe}(n) \Subset \langle\tau\rangle.$$

Denote by $\varepsilon_1, \dots, \varepsilon_n$ the standard basis of the space dual to the space of diagonal matrices in $\mathfrak{gl}(n) \subset \mathfrak{pe}(n)$. Denote by V_λ the irreducible $\mathfrak{gl}(n)$-module with highest weight λ and highest vector v_λ and by X_λ the irreducible $\mathfrak{pe}(n)$-module with highest weight λ and an even highest vector.

Let $V = V_0 \oplus V_1$ be the standard (identity) $\mathfrak{pe}(n)$-module, $e_1, \dots, e_n$ be a basis of V_0, and $f_1, \dots, f_n$ be a basis of V_1 with respect to which the form P on V takes the form $P = \mathrm{antidiag}(1_n, 1_n)$. With respect to this basis the elements $X \in \mathfrak{pe}(n)$ are represented by matrices of the standard format $(n,\ n)$ [L2]:

$$X = \mathrm{diag}(A, -A^t) + \mathrm{antidiag}(B, C),\ \text{where } A \in \mathfrak{gl}(n),\ B^t = B,\ C^t = -C.$$

In what follows we will often use a natural abbreviation: e.g., $B_{1,n}$ stands for the matrix X whose components A and C are zero and all the entries of B are also zero except for $(1,\ n)$-th and $(n,\ 1)$-st.

Denote by $\tilde{e}_1, \dots, \tilde{e}_n$ and $\tilde{f}_1, \dots, \tilde{f}_n$ the basis of V^* dual to the above basis of V, i.e., $\tilde{f}_i(e_j) = \tilde{e}_i(f_j) = \delta_{ij}$. Since the form P preserved by $\mathfrak{pe}(n)$ is odd, then V^* and $\Pi(V)$ are isomorphic as $\mathfrak{pe}(n)$-modules. Notice that as $\mathfrak{pe}(n)$-modules,

$$\mathfrak{pe}(n) \cong \Pi(E^2V^*).$$

1.1. Lemma. *a) There exists a $\mathbb{Z}$-grading of the Lie superalgebra $\mathfrak{pe}(n+1)$ of the form*

$$\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

where

$$\begin{aligned}&\mathfrak{g}_{-1} = V,\\ &\mathfrak{g}_0 = \mathfrak{cpe}(n),\\ &\mathfrak{g}_1 = V^* = \Pi(V),\\ &\mathfrak{g}_2 = \Pi(\langle 1\rangle).\end{aligned}$$

b) There exists a $\mathbb{Z}$-grading of the Lie superalgebra $\mathfrak{spe}(n+1)$ of the form

$$\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

where

$$\begin{aligned}
\mathfrak{g}_{-1} &= V,\\
\mathfrak{g}_0 &= \mathfrak{spe}(n) \Subset \langle \tau + nz\rangle,\\
\mathfrak{g}_1 &= V^* = \Pi(V),\\
\mathfrak{g}_2 &= \Pi(\langle 1\rangle).
\end{aligned}$$

PROOF. Let $W = W_0 \oplus W_1$ be the standard (identity) $\mathfrak{pe}(n+1)$-module, $e_1, \dots, e_{n+1}$ be a basis of W_0, and $f_1, \dots, f_{n+1}$ be a basis of W_1 with respect to which the form P on W takes the form $P = \operatorname{antidiag}(1_{n+1},\ 1_{n+1})$. Denote by $\tilde e_1, \dots, \tilde e_{n+1}$ and $\tilde f_1 \dots, \tilde f_{n+1}$ the basis of W^* dual to the above basis of W, e.g., $\tilde f_i(e_j) = \tilde e_i(f_j) = \delta_{ij}$.

Note that

$$\mathfrak{pe}(n+1) = \Pi(E^2W^*) = \Pi(E^2W_1 \oplus W_0 \wedge W_1 \oplus S^2W_0).$$

Thus,

$$\mathfrak{pe}(n+1) = \langle e_i\tilde e_j, e_i \wedge \tilde f_j, f_i \wedge \tilde f_j\rangle (1 \le i,\ j \le n+1), \text{ where}$$

$$\begin{aligned}
e_i\tilde e_j &= (1/2)(e_i \otimes \tilde e_j + e_j \otimes \tilde e_i),\\
e_i \wedge \tilde f_j &= (1/2)(e_i \otimes \tilde f_j - f_j \otimes \tilde e_i),\\
f_i \wedge \tilde f_j &= (1/2)(f_i \otimes \tilde f_j - f_j \otimes \tilde f_i).
\end{aligned} \tag{1.1.1}$$

Note that the commutator in $\mathfrak{pe}(n+1)$ is defined as follows:

$$[w_i \wedge \tilde w_j,\ w_s \wedge \tilde w_t] = (1/2)(\tilde w_j(w_s)(w_i \wedge \tilde w_t) - (-1)^{p(w_s)}\tilde w_t(w_i)(w_s \wedge \tilde w_j)), \tag{1.1.2}$$

where $w_l \in \{e_1, \dots, e_{n+1}; f_1, \dots, f_{n+1}\}$ for $1 \le l \le n+1$.

Let $V = V_0 \oplus V_1 = \langle e_1, \dots, e_n; f_1, \dots, f_n\rangle$ be $(n,\ n)$-dimensional subsuperspace in W. Then

$$\begin{aligned}
&\Pi(E^2W_1) = \Pi(E^2V_1) \oplus V_1 \wedge \langle \tilde f_{n+1}\rangle,\\
&\Pi(W_0 \wedge W_1) = \Pi(V_0 \wedge V_1) \oplus V_0 \wedge \langle \tilde f_{n+1}\rangle \oplus V_1 \wedge \langle \tilde e_{n+1}\rangle \oplus \langle e_{n+1} \wedge \tilde f_{n+1}\rangle,\\
&\Pi(S^2W_0) = \Pi(S^2V_0) \oplus V_0 \wedge \langle \tilde e_{n+1}\rangle \oplus \langle e_{n+1}\tilde e_{n+1}\rangle.
\end{aligned}$$

Set

$$\begin{aligned}
\mathfrak{g}_{-1} &= V_0 \wedge \langle \tilde f_{n+1}\rangle \oplus V_1 \wedge \langle \tilde f_{n+1}\rangle,\\
\mathfrak{g}_0 &= \Pi(E^2V_1 \oplus V_0 \wedge V_1 \oplus S^2V_0) \oplus \langle e_{n+1} \wedge \tilde f_{n+1}\rangle,\\
\mathfrak{g}_1 &= V_0 \wedge \langle \tilde e_{n+1}\rangle \oplus V_1 \wedge \langle \tilde e_{n+1}\rangle,\\
\mathfrak{g}_2 &= \langle e_{n+1}\tilde e_{n+1}\rangle.
\end{aligned} \tag{1.1.3}$$

According to (1.1.2), formulas (1.1.3) indeed define a $\mathbb{Z}$-grading of $\mathfrak{pe}(n+1)$, described in Lemma 1.1.

In order to define a $\mathbb{Z}$-grading of $\mathfrak{spe}(n+1)$ we set

$$\begin{aligned}
\mathfrak{g}_0 = {} & \Pi(E^2V_1 \oplus S^2V_0) \oplus \langle \textstyle\sum_{i,j=1,n} a_{ij} e_i \wedge \tilde f_j | \sum_{i=1,n} a_{ii} = 0\rangle \oplus\\
& \langle (\textstyle\sum_{i=1,n} e_i \wedge \tilde f_i) - ne_{n+1} \wedge \tilde f_{n+1}\rangle.
\end{aligned}$$

Note that by (1.1.2) we have $2[e_{n+1} \wedge \tilde f_{n+1},\ \mathfrak{g}_i] = i\mathfrak{g}_i$ for $(-1 \le i \le 2)$. Hence, $-2e_{n+1} \wedge \tilde f_{n+1} = z$. Since by (1.1.1) $2\sum_{i=1,n} e_i \wedge \tilde f_i = \tau$, then

$$2(\sum_{i=1,n} e_i \wedge \tilde f_i - ne_{n+1} \wedge \tilde f_{n+1}) = \tau + nz.$$

Thus, $\mathfrak{g}_0 = \mathfrak{spe}(n) \Subset \langle \tau + nz\rangle$. This proves Lemma 1.1. □

1.2. Theorem. *Let $\mathfrak{g}_{-1} = V$. Then*

a) If $\mathfrak{g}_0 = \mathfrak{spe}(n)$, $\mathfrak{pe}(n)$, $\mathfrak{cspe}(n)$ or $\mathfrak{spe}(n) \Subset \langle a\tau + bz\rangle$, where $a, b \in \mathbb{C}$ are such that $a, b \neq 0$ and $b/a \neq n$, then $\mathfrak{g}_*(\mathfrak{g}_{-1}, \mathfrak{g}_0) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$.

b) If $\mathfrak{g}_0 = \mathfrak{cpe}(n)$ or $\mathfrak{spe}(n) \Subset \langle \tau + nz\rangle$, then $\mathfrak{g}_*(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ is either $\mathfrak{pe}(n+1)$ or $\mathfrak{spe}(n+1)$, respectively, in the $\mathbb{Z}$-grading described in Lemma 1.1.

Proof. Let us consider the case where $\mathfrak{g}_0 = \mathfrak{cpe}(n)$. By Lemma 1.1 we have

$$\mathfrak{pe}(n+1) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

where

$$\begin{aligned} \mathfrak{g}_{-1} &= V, \\ \mathfrak{g}_0 &= \mathfrak{cpe}(n), \\ \mathfrak{g}_1 &= V^* = \Pi(V), \\ \mathfrak{g}_2 &= \Pi(\langle 1\rangle). \end{aligned}$$

Therefore,

$$\mathfrak{pe}(n+1) \subset \mathfrak{g}_*(\mathfrak{g}_{-1}, \mathfrak{g}_0). \tag{1.2.1}$$

In fact, since $\mathfrak{spe}(n+1)$ is a simple Lie superalgebra, then it is transitive, (i.e., if there exists $g \in \mathfrak{g}_i (i \geq 0)$ such that $[\mathfrak{g}_{-1}, g] = 0$, then $g = 0$). It follows that $\mathfrak{g}_i \subset \mathfrak{g}_{i-1} \otimes \mathfrak{g}_{-1}^*$. The Jacobi identity implies $\mathfrak{g}_i \subset \mathfrak{g}_{i-2} \otimes S^2\mathfrak{g}_{-1}^*$. □

Let us find $\mathfrak{g}_1$.

1.3. Lemma. *As a $\mathfrak{gl}(n)$-module, $\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^*$ is the direct sum of irreducible $\mathfrak{gl}(n)$-submodules whose highest weights and highest vectors are listed in Table 1.*

Convention. Let v, w be elements of a vector space. Set

$$vw = (v \otimes w + (-1)^{p(v)p(w)} w \otimes v)/2, \;\; v \wedge w = (v \otimes w - (-1)^{p(v)p(w)} w \otimes v)/2.$$

Proof of Lemma 1.3. consists of:

a) a verification of the fact that vectors v from Table 1 are indeed highest with respect to $\mathfrak{gl}(n)$, i.e., $A_{i,j}v = 0$ for $i < j$,

b) a calculation of dimensions of the corresponding irreducible submodules by the formula from the Appendix. □

Let us show with the help of Table 1 that if $\lambda \neq \varepsilon_1, -\varepsilon_n$, then $v_\lambda \notin \mathfrak{g}_{-1} \otimes S^2\mathfrak{g}_{-1}^*$. Indeed, if $\lambda = -\varepsilon_{n-1} - 2\varepsilon_n$, then

$$v_\lambda(e_n)(e_{n-1}) = -f_n/2, v_\lambda(e_{n-1})(e_n) = 0;$$

if $\lambda = -\varepsilon_{n-2} - \varepsilon_{n-1} - \varepsilon_n$, then

$$v_\lambda(e_n)(e_{n-1}) = f_{n-2}/2, v_\lambda(e_{n-1})(e_n) = -f_{n-2}/2;$$

if $\lambda = \varepsilon_1 - 2\varepsilon_n$, then

$$v_\lambda(e_n)(f_1) = f_n/2, v_\lambda(f_1)(e_n) = 0;$$

if $\lambda = \varepsilon_1 + \varepsilon_2 - \varepsilon_n$, then

$$v_\lambda(f_2)(e_n) = -e_1/2, v_\lambda(e_n)(f_2) = 0;$$

if $\lambda = 3\varepsilon_1$, then

$$v_\lambda(f_1)(f_1) = e_1 \neq 0;$$

if $\lambda = 2\varepsilon_1 + \varepsilon_2$, then

$$v_\lambda(f_1)(f_1) = e_2/2 \neq 0.$$

Let $\lambda=\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n$. According to Table 1, $\mathfrak{g}_0\otimes\mathfrak{g}_{-1}^*$ contains two highest vectors of weight λ. Let

$$v_\lambda=k_1f_{n-1}\wedge\tilde f_n\otimes\tilde e_1+k_2(f_{n-1}\wedge\tilde e_1\otimes\tilde f_n-f_n\wedge\tilde e_1\otimes\tilde f_{n-1}),\ \text{where } k_1,\ k_2\in\mathbb{C},$$

be a linear combination of these vectors. The condition

$$v_\lambda(e_n)(e_{n-1})=v_\lambda(e_{n-1})(e_n)$$

implies $k_2=0$. Then the condition

$$v_\lambda(e_n)(f_1)=v_\lambda(f_1)(e_n)$$

implies $k_1=0$.

Let $\lambda=2\varepsilon_1-\varepsilon_n$. Let

$$v_\lambda=k_1f_n\wedge\tilde e_1\otimes\tilde e_1+k_2e_1\tilde e_1\otimes\tilde f_n,\ \text{where } k_1,\ k_2\in\mathbb{C},$$

be a linear combination of the highest vectors of weight λ which belong to $\mathfrak{g}_0\otimes\mathfrak{g}_{-1}^*$. The condition $v_\lambda(f_1)(f_1)=0$ implies $k_1=0$. Then the condition $v_\lambda(e_n)(f_1)=v_\lambda(f_1)(e_n)$ implies $k_2=0$. Therefore, if $\lambda\neq\varepsilon_1,\ -\varepsilon_n$, then $v_\lambda\notin\mathfrak{g}_{-1}\otimes S^2\mathfrak{g}_{-1}^*$, hence $v_\lambda\notin\mathfrak{g}_1$.

Let $\lambda=\varepsilon_1$. According to Table 1, $\mathfrak{g}_0\otimes\mathfrak{g}_{-1}^*$ has four highest vectors of weight λ. Let

$$v_\lambda=k_1\sum_{i=1}^n f_i\wedge\tilde e_i\otimes\tilde e_1+k_2\sum_{i=1}^n f_i\wedge\tilde e_1\otimes\tilde e_i+k_3\sum_{i=1}^n e_1\tilde e_i\otimes\tilde f_i+k_4\sum_{i=1}^n e_i\tilde f_i\otimes\tilde e_1,$$

where $k_1,\ k_2,\ k_3,\ k_4\in\mathbb{C}$, be their linear combination. Note that $v_\lambda\in\mathfrak{g}_{-1}\otimes S^2\mathfrak{g}_{-1}^*$ if and only if the following conditions are satisfied:

$$\begin{aligned}
&v_\lambda(f_1)(f_1)=0,\\
&v_\lambda(f_1)(f_i)=-v_\lambda(f_i)(f_1)\ \text{for } i\neq 1,\\
&v_\lambda(f_1)(e_1)=v_\lambda(e_1)(f_1),\\
&v_\lambda(f_1)(e_i)=v_\lambda(e_i)(f_1)\ \text{for } i\neq 1,\\
&v_\lambda(f_i)(e_i)=v_\lambda(e_i)(f_i)\ \text{for } i\neq 1,
\end{aligned}$$

which determine, respectively, the following system of linear equations:

$$\begin{aligned}
&k_1+k_2+k_4=0,\\
&k_1+k_4=-k_2,\\
&(-k_1-k_2+k_4)/2=k_3,\\
&-k_1+k_4=k_3,\\
&-k_2=k_3.
\end{aligned}$$

The solution of this system is

$$k_1=0,\ k_3=-k_2=k_4.$$

Therefore,

$$v_{\varepsilon_1}=-\sum_{i=1}^n f_i\wedge\tilde e_1\otimes\tilde e_i+\sum_{i=1}^n e_1\tilde e_i\otimes\tilde f_i+\sum_{i=1}^n e_i\tilde f_i\otimes\tilde e_1\in\mathfrak{g}_1. \tag{1.3.1}$$

Since $\mathfrak{g}_1$ is a $\mathfrak{pe}(n)$-module and v_{ε_1} is an odd vector, $\mathfrak{g}_1=V^*$.

Let us find $\mathfrak{g}_2$.

1.4. Lemma. *There exist the following nonsplit sequences of $\mathfrak{pe}(n)$-modules:*

$$0\longrightarrow X_{2\varepsilon_1}\longrightarrow E^2V^*\longrightarrow\Pi(\langle 1\rangle)\longrightarrow 0, \tag{1.4.1}$$

$$0\longrightarrow\Pi(\langle 1\rangle)\longrightarrow S^2V^*\longrightarrow X_{\varepsilon_1+\varepsilon_2}\longrightarrow 0. \tag{1.4.2}$$

Proof. First of all recall that $\Pi(E^2V^*)$ and $\mathfrak{pe}(n)$ itself are isomorphic $\mathfrak{pe}(n)$-modules and there exists the following nonsplit sequence of $\mathfrak{pe}(n)$-modules

$$0 \longrightarrow \mathfrak{spe}(n) \longrightarrow \mathfrak{pe}(n) \longrightarrow \langle\tau\rangle \longrightarrow 0. \tag{1.4.3}$$

Note that as a $\mathfrak{gl}(n)$-module, E^2V^* is isomorphic to

$$E^2V_0^* \oplus V_0^* \wedge V_0 \oplus S^2V_0, \text{ where } S^2V_0 = V_{2\varepsilon_1}, v_{2\varepsilon_1} = \tilde{e}_1^2. \tag{1.4.4}$$

Since $B_{i,j}v_{2\varepsilon_1} = 0$, then $v_{2\varepsilon_1}$ is a $\mathfrak{pe}(n)$-highest vector. From the simplicity of $\mathfrak{spe}(n)$ we get (1.4.1) after the change of parity.

Let us prove (1.4.2). Notice that as $\mathfrak{gl}(n)$-modules,

$$S^2V^* = S^2V_0^* \oplus V_0^* \cdot V_0 \oplus E^2V_0, \text{ where}$$

$$\begin{array}{l} S^2V_0^* = V_{-2\varepsilon_n}, v_{-2\varepsilon_n} = \tilde{f}_n^2, \\ V_0^* \cdot V_0 = V_{\varepsilon_1-\varepsilon_n} \oplus \langle v_0\rangle,\ v_{\varepsilon_1-\varepsilon_n} = \tilde{f}_n\tilde{e}_1,\ v_0 = \sum_{i=1}^n \tilde{f}_i\tilde{e}_i, \\ E^2V_0 = V_{\varepsilon_1+\varepsilon_2},\ v_{\varepsilon_1+\varepsilon_2} = \tilde{e}_1 \wedge \tilde{e}_2. \end{array} \tag{1.4.5}$$

Note that $\langle v_0\rangle$ is the trivial 1-dimensional $\mathfrak{pe}(n)$-module. Indeed,

$$B_{i,j}(v_0) = -\tilde{e}_j \wedge \tilde{e}_i + -\tilde{e}_i \wedge \tilde{e}_j = 0,\ C_{i,j}(v_0) = \tilde{f}_i\tilde{f}_j + \tilde{f}_j(-\tilde{f}_i) = 0.$$

Since $B_{i,j}v_{\varepsilon_1+\varepsilon_2} = 0$, then $v_{\varepsilon_1+\varepsilon_2}$ is a $\mathfrak{pe}(n)$-highest vector.

Let us prove that as $\mathfrak{gl}(n)$-modules,

$$X_{\varepsilon_1+\varepsilon_2} \cong V_{\varepsilon_1+\varepsilon_2} \oplus V_{\varepsilon_1-\varepsilon_n} \oplus V_{-2\varepsilon_n}.$$

Indeed,

$$\begin{array}{l} C_{2,n}(v_{\varepsilon_1+\varepsilon_2}) = v_{\varepsilon_1-\varepsilon_n},\ \ -B_{2,n}(v_{\varepsilon_1-\varepsilon_n}) = v_{\varepsilon_1+\varepsilon_2}, \\ C_{1,n}(v_{\varepsilon_1-\varepsilon_n}) = v_{-2\varepsilon_n},\ B_{1,n}(v_{-2\varepsilon_n}) = v_{\varepsilon_1-\varepsilon_n}. \end{array}$$

Finally, we get

$$v_0 = \frac{1}{2}\sum_{i=1}^{n-1} B_{i,n}A_{n,i} + nB_{n,n}v_{-2\varepsilon_n}.$$

This proves (1.4.2) and Lemma 1.4. □

Let us prove that if v_λ from $\mathfrak{g}_1 \otimes \mathfrak{g}_{-1}^*$ is a $\mathfrak{pe}(n)$-highest vector of weight either $\lambda = 2\varepsilon_1$ or $\varepsilon_1 + \varepsilon_2$, then $v_\lambda \notin \mathfrak{g}_0 \otimes S^2\mathfrak{g}_{-1}^*$. In fact, if $\lambda = 2\varepsilon_1$, then by (1.3.1) and (1.4.4) we have

$$v_\lambda = (-\sum_{i=1}^n f_i \wedge \tilde{e}_1 \otimes \tilde{e}_i + \sum_{i=1}^n e_1\tilde{e}_i \otimes \tilde{f}_i + \sum_{i=1}^n e_i\tilde{f}_i \otimes \tilde{e}_1) \otimes \tilde{e}_1.$$

Then

$$v_\lambda(f_1)(f_1) = -f_1 \wedge \tilde{e}_1 + \sum_{i=1}^n e_i\tilde{f}_i \neq 0.$$

Therefore, $v_\lambda \notin \mathfrak{g}_0 \otimes S^2\mathfrak{g}_{-1}^*$.

If $\lambda = \varepsilon_1 + \varepsilon_2$, then by (1.3.1) and (1.4.5)

$$\begin{aligned} v_\lambda = {} & (-\sum_{i=1}^n f_i \wedge \tilde{e}_1 \otimes \tilde{e}_i + \sum_{i=1}^n e_1\tilde{e}_i \otimes \tilde{f}_i + \sum_{i=1}^n e_i\tilde{f}_i \otimes \tilde{e}_1) \otimes \tilde{e}_2 \\ & - (-\sum_{i=1}^n f_i \wedge \tilde{e}_2 \otimes \tilde{e}_i + \sum_{i=1}^n e_2\tilde{e}_i \otimes \tilde{f}_i + \sum_{i=1}^n e_i\tilde{f}_i \otimes \tilde{e}_2) \otimes \tilde{e}_1. \end{aligned}$$

Thus,

$$v_\lambda(f_2)(e_i) = e_1\tilde{e}_i \neq 0 \text{ and } v_\lambda(e_i)(f_2) = 0.$$

Hence, $v_\lambda \notin \mathfrak{g}_0 \otimes S^2\mathfrak{g}_{-1}^*$.

Let $\lambda = 0$. According to Lemma 1.4, the Jordan-Hölder series of the $\mathfrak{pe}(n)$-module $\mathfrak{g}_1 \otimes \mathfrak{g}_{-1}^*$ contains two $\mathfrak{pe}(n)$-modules with highest weight 0. By Lemma 1.4 the sequence (1.4.1) is nonsplit and we have already proved that $\mathfrak{g}_2$ has no irreducible $\mathfrak{pe}(n)$-module with highest weight $2\varepsilon_1$. Therefore, either $\mathfrak{g}_2$ consists of one trivial $\mathfrak{pe}(n)$-module or $\mathfrak{g}_2 = 0$. But by (1.2.1)

$$\mathfrak{pe}(n+1) \subset \mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0).$$

Hence, $\mathfrak{g}_2 = \Pi(\langle 1\rangle)$.

Finally, let us show that $\mathfrak{g}_3 = 0$. By definition

$$\mathfrak{g}_3 = (\mathfrak{g}_2 \otimes \mathfrak{g}_{-1}^*) \cap (\mathfrak{g}_1 \otimes S^2\mathfrak{g}_{-1}^*).$$

Note that

$$\mathfrak{g}_2 \otimes \mathfrak{g}_{-1}^* = \Pi(\langle 1\rangle) \otimes V^* \cong V,$$

as $\mathfrak{pe}(n)$-modules. By (1.1.3) the $\mathfrak{pe}(n)$-highest vector in $\mathfrak{g}_2 \otimes \mathfrak{g}_{-1}^*$ is $v = e_{n+1}\tilde{e}_{n+1} \otimes \tilde{e}_1$.

By the explicit formula (1.1.2) of multiplication in $\mathfrak{pe}(n+1)$ we have

$$v(f_1)(e_1) = [e_{n+1}\tilde{e}_{n+1}, e_1 \wedge \tilde{f}_{n+1}] = e_1\tilde{e}_{n+1} \in \mathfrak{g}_1.$$

On the other hand, $v(e_1)(f_1) = 0$. Therefore, $v \notin \mathfrak{g}_1 \otimes S^2\mathfrak{g}_{-1}^*$. Hence, $\mathfrak{g}_3 = 0$.

Thus, Theorem 1.2 is proved for $\mathfrak{g}_0 = \mathfrak{cpe}(n)$. This result and part b) of Lemma 1.1 imply the statement of Theorem 1.2 for $\mathfrak{g}_0 = \mathfrak{spe}(n) \Subset \langle \tau + nz\rangle$.

Let us prove that $\mathfrak{g}_1 = 0$ for $\mathfrak{g}_0 = \mathfrak{spe}(n)$, $\mathfrak{pe}(n)$, $\mathfrak{cspe}(n)$, or $\mathfrak{spe}(n) \Subset \langle a\tau + bz\rangle$, where $a,\ b \in \mathbb{C}$, $a,\ b \neq 0$, and $b/a \neq n$.

Indeed, as has been shown, $\mathfrak{g}_1 = \Pi(V)$ for $\mathfrak{g}_0 = \mathfrak{cpe}(n)$ or $\mathfrak{spe}(n) \Subset \langle \tau + nz\rangle$, and by (1.3.1) the corresponding $\mathfrak{spe}(n)$-highest vector is

$$v_{\varepsilon_1} = -\sum_{i=1}^{n} f_i \wedge \tilde{e}_1 \otimes \tilde{e}_i + \sum_{i=1}^{n} e_1\tilde{e}_i \otimes \tilde{f}_i + \sum_{i=1}^{n} e_i\tilde{f}_i \otimes \tilde{e}_1.$$

Then

$$v_{\varepsilon_1}(f_1) = -f_1 \wedge \tilde{e}_1 + \sum_{i=1}^{n} e_i\tilde{f}_i \in \mathfrak{spe}(n) \Subset \langle \tau + nz\rangle.$$

Note that

$$-f_1 \wedge \tilde{e}_1 + \sum_{i=1}^{n} e_i\tilde{f}_i \notin \mathfrak{spe}(n).$$

Hence $v_{\varepsilon_1}(f_1) \notin \mathfrak{g}_0$ for $\mathfrak{g}_0 = \mathfrak{spe}(n) \Subset \langle a\tau + bz\rangle$, where $a,\ b \in \mathbb{C},\ a \neq 0$ and $b/a \neq n,$ or $a = 0$. Therefore,

$$\mathfrak{g}_1 = 0 \text{ for } \mathfrak{g}_0 = \mathfrak{spe}(n),\ \mathfrak{pe}(n),\ \mathfrak{cspe}(n), \text{ or } \mathfrak{spe}(n) \Subset \langle a\tau + bz\rangle,$$

where $a,\ b \in \mathbb{C},\ a,\ b \neq 0$ and $b/a \neq n$. Thus, in these cases

$$\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0.$$

The main theorem

The following theorem describes SFs for the odd analogues of Riemannian metric and various conformal versions.

THEOREM. *For the G-structures with the following Lie $(G) = \mathfrak{g}_0$ the nonzero SFs are of orders not exceeding 2 and as follows:*

order 1: *if $\mathfrak{g}_0 = \mathfrak{spe}(n)$ or $\mathfrak{spe}(n) \Subset \langle \tau + nz \rangle$ we have V^*;* order 2:*if $\mathfrak{g}_0 = \mathfrak{spe}(n)$, where $n > 3$, we have the following nonsplit exact sequence of $\mathfrak{spe}(n)$-modules:*

$$0 \longrightarrow X_{\varepsilon_1+\varepsilon_2} \longrightarrow H^{2,2}_{\mathfrak{g}_0} \longrightarrow \Pi(X_{2\varepsilon_1+2\varepsilon_2}) \longrightarrow 0;$$

if $\mathfrak{g}_0 = \mathfrak{spe}(3)$ another space is added to the SFs: we have the following nonsplit exact sequence of $\mathfrak{spe}(3)$-modules:

$$0 \longrightarrow X \longrightarrow H^{2,2}_{\mathfrak{g}_0} \longrightarrow \Pi(X_{3\varepsilon_1}) \longrightarrow 0,$$

where X is determined from the following nonsplit exact sequence of $\mathfrak{spe}(3)$-modules:

$$0 \longrightarrow X_{\varepsilon_1+\varepsilon_2} \longrightarrow X \longrightarrow \Pi(X_{2\varepsilon_1+2\varepsilon_2}) \longrightarrow 0;$$

if $\mathfrak{g}_0 = \mathfrak{spe}(n) \Subset \langle a\tau + bz \rangle$, where $a,\ b \in \mathbb{C}$ are such that $a = 0,\ b \neq 0$ or $a \neq 0,\ b/a \neq n$, then for $n > 2$ we have the following nonsplit exact sequence of $\mathfrak{spe}(n)$-modules:

$$0 \longrightarrow H^{2,2}_{\mathfrak{spe}(n)} \longrightarrow H^{2,2}_{\mathfrak{g}_0} \longrightarrow X_{2\varepsilon_1} \longrightarrow 0;$$

if $\mathfrak{g}_0 = \mathfrak{spe}(n) \Subset \langle \tau + nz \rangle$, then for $n > 3$ $H^{2,2}_{\mathfrak{g}_0} = \Pi(X_{2\varepsilon_1+2\varepsilon_2})$ is an irreducible $\mathfrak{spe}(n)$-module, for $n = 3$ we have the following nonsplit exact sequence of $\mathfrak{spe}(3)$-modules:

$$0 \longrightarrow \Pi(X_{2\varepsilon_1+2\varepsilon_2}) \longrightarrow H^{2,2}_{\mathfrak{g}_0} \longrightarrow \Pi(X_{3\varepsilon_1}) \longrightarrow 0;$$

if $\mathfrak{g}_0 = \mathfrak{cpe}(n),\ n > 2$, then

$$H^{2,2}_{\mathfrak{cpe}(n)} = \Pi(S^2(E^2V/\Pi(\langle 1 \rangle))/E^4V),$$

more precisely, we have the following nonsplit exact sequence of $\mathfrak{spe}(n)$-modules:

$$0 \longrightarrow H^{2,2}_{\mathfrak{spe}(n) \Subset \langle \tau + nz \rangle} \longrightarrow H^{2,2}_{\mathfrak{g}_0} \longrightarrow X_{2\varepsilon_1} \longrightarrow 0.$$

Proof of the main theorem

3.1. Calculation of SFs of order 1. Recall that the bidegree of the differentials in the Spencer complex is $(-1,\ 1)$. We will often refer to the following

LEMMA. *Let $(\mathfrak{g}_{-1},\ \mathfrak{g}_0)$ be an arbitrary pair, where $\mathfrak{g}_{-1}$ is a faithful module over a Lie superalgebra $\mathfrak{g}_0$, and let*

$$\mathfrak{g}_{k-1} \otimes \mathfrak{g}^*_{-1} \xrightarrow{\partial^{k+1,1}_{\mathfrak{g}_0}} \mathfrak{g}_{k-2} \otimes E^2\mathfrak{g}^*_{-1} \xrightarrow{\partial^{k,2}_{\mathfrak{g}_0}} \mathfrak{g}_{k-3} \otimes E^3\mathfrak{g}^*_{-1} \qquad (k \geq 1)$$

be the corresponding Spencer cochain sequence. Then

$$\operatorname{Im}\partial^{k+1,1}_{\mathfrak{g}_0} \cong (\mathfrak{g}_{k-1} \otimes \mathfrak{g}^*_{-1})/\mathfrak{g}_k. \qquad (3.1.1)$$

PROOF. By the definition of the Cartan prolongation

$$\mathfrak{g}_k = (\mathfrak{g}_{k-1} \otimes \mathfrak{g}^*_{-1}) \cap (\mathfrak{g}_{k-2} \otimes S^2\mathfrak{g}^*_{-1}).$$

Let $c \in \mathfrak{g}_{k-1} \otimes \mathfrak{g}^*_{-1}$. Then $c \in \mathfrak{g}_k$ if and only if $c(g_1)g_2 = (-1)^{p(g_1)p(g_2)}c(g_2)g_1$ for any (homogeneous) $g_1, g_2 \in \mathfrak{g}_{-1}$. On the other hand

$$\partial^{k+1,1}_{\mathfrak{g}_0}c(g_1, g_2) = -(-1)^{p(g_1)p(g_2)}c(g_2)g_1 + c(g_1)g_2.$$

Hence, $\operatorname{Ker}\partial^{k+1,1}_{\mathfrak{g}_0} = \mathfrak{g}_k$. This proves the Lemma. $\square$

In particular, to define $H^{1,2}_{\mathfrak{g}_0}$ we have the following Spencer cochain sequence:

$$\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^* \xrightarrow{\partial^{2,1}_{\mathfrak{g}_0}} \mathfrak{g}_{-1} \otimes E^2\mathfrak{g}_{-1}^* \xrightarrow{\partial^{1,2}_{\mathfrak{g}_0}} 0, \text{ where } \mathrm{Ker}\partial^{2,1}_{\mathfrak{g}_0} = \mathfrak{g}_1.$$

Let us prove that

$$H^{1,2}_{\mathfrak{g}_0} = 0 \text{ if either } \mathfrak{g}_0 = \mathfrak{cpe}(n) \text{ or } \mathfrak{g}_0 = \mathfrak{spe}(n) \Subset \langle a\tau + bz\rangle,$$

where $a = 0,\ b \neq 0$ or $a \neq 0,\ b/a \neq n$.

Let $\mathfrak{g}_0 = \mathfrak{spe}(n) \Subset \langle \tau\rangle = \mathfrak{pe}(n)$. Since $\mathfrak{pe}(n)$ and $\Pi(E^2V^*)$ are isomorphic $\mathfrak{g}_0$-modules, the $\mathfrak{pe}(n)$-module $\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^*$ is isomorphic to

$$\Pi(E^2V^*) \otimes V^* \cong E^2V^* \otimes V \cong \mathfrak{g}_{-1} \otimes E^2\mathfrak{g}_{-1}^* = \mathrm{Ker}\partial^{1,2}_{\mathfrak{g}_0}.$$

By part a) of Theorem 1.2 we have $\mathfrak{g}_1 = 0$. Therefore, by (3.1.1)

$$\mathrm{Im}\partial^{2,1}_{\mathfrak{g}_0} \cong \mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^* \cong \mathrm{Ker}\partial^{1,2}_{\mathfrak{g}_0},$$

i.e., $H^{1,2}_{\mathfrak{pe}(n)} = 0$.

Let $\mathfrak{g}_0 = \mathfrak{spe}(n) \Subset \langle a\tau + bz\rangle$, where $a = 0,\ b \neq 0$ or $a \neq 0,\ b/a \neq n$. By Theorem 1.2, $\mathfrak{g}_1 = 0$ for such $\mathfrak{g}_0$. Note that $\dim \mathfrak{g}_0 = \dim \mathfrak{pe}(n)$. Therefore,

$$\dim \mathrm{Im}\partial^{2,1}_{\mathfrak{g}_0} = \dim \mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^* = \dim \mathfrak{pe}(n) \otimes V^* = \dim E^2V^* \otimes V = \dim \mathrm{Ker}\partial^{1,2}_{\mathfrak{g}_0}.$$

Hence, $H^{1,2}_{\mathfrak{g}_0} = 0$.

Let $\mathfrak{g}_0 = \mathfrak{cpe}(n)$. Then by part b) of Theorem 1.2 $\mathfrak{g}_1 = V^*$. Note that

$$\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^* = (\mathfrak{pe}(n) \oplus \langle z\rangle) \otimes V^* = \mathfrak{pe}(n) \otimes V^* \oplus \langle z\rangle \otimes V^* \cong V \otimes E^2V^* \oplus V^*.$$

Therefore,

$$\mathrm{Im}\partial^{2,1}_{\mathfrak{g}_0} = V \otimes E^2V^* = \mathrm{Ker}\partial^{1,2}_{\mathfrak{g}_0}.$$

Hence, $H^{1,2}_{\mathfrak{cpe}(n)} = 0$.

Let us prove that

$$H^{1,2}_{\mathfrak{g}_0} = V^* \text{ if } \mathfrak{g}_0 = \mathfrak{spe}(n) \text{ or } \mathfrak{spe}(n) \Subset \langle \tau + nz\rangle.$$

Let $\mathfrak{g}_0 = \mathfrak{spe}(n)$. By part a) of Theorem 1.2 $\mathfrak{g}_1 = 0$. Therefore,

$$\mathrm{Im}\partial^{2,1}_{\mathfrak{g}_0} \cong \mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^* = \mathfrak{spe}(n) \otimes V^*.$$

As has been shown for the case $\mathfrak{g}_0 = \mathfrak{cspe}(n) = \mathfrak{spe}(n) \oplus \langle z\rangle$, we have

$$\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^* = (\mathfrak{spe}(n) \oplus \langle z\rangle) \otimes V^* \cong V \otimes E^2V^*.$$

Therefore,

$$\mathrm{Ker}\partial^{1,2}_{\mathfrak{spe}(n)}/\mathrm{Im}\partial^{2,1}_{\mathfrak{spe}(n)} \cong V^*.$$

Hence $H^{1,2}_{\mathfrak{spe}(n)} = V^*$.

Finally, let $\mathfrak{g}_0 = \mathfrak{spe}(n) \Subset \langle \tau + nz\rangle$. By part b) of Theorem 1.2 $\mathfrak{g}_1 = V^*$. Since

$$\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^* = (\mathfrak{spe}(n) \Subset \langle \tau + nz\rangle) \otimes V^*,$$

the Jordan-Hölder series for the $\mathfrak{spe}(n)$-module $\mathrm{Im}\partial^{2,1}_{\mathfrak{g}_0}$ contains the same irreducible quotient modules as that for the $\mathfrak{spe}(n)$-module $\mathfrak{spe}(n) \otimes V^*$. Since

$$\mathrm{Ker}\partial^{1,2}_{\mathfrak{g}_0} = V \otimes E^2V^* \cong (\mathfrak{spe}(n) \oplus \langle z\rangle) \otimes V^*,$$

then $H^{1,2}_{\mathfrak{g}_0} = V^*$. This proves Theorem 2.1 in the case of the SFs of the first order.

3.2. Hochschild-Serre spectral sequence. The continuation of the proof of Theorem 2.1 is based on the Hochschild-Serre spectral sequence. Let us recall the corresponding formulations in a form convenient for us, since the case of Lie superalgebras is hardly reflected in the literature (one might think that the union of [Fu] and [GM] should suffice, but the sign rule applied to the Lie algebra case does not completely solve the problem).

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra and M be a $\mathfrak{g}$-module. On the superspace of k-dimensional cochains $C^k = C^k(\mathfrak{g}, M)$ define a filtration:

$$F^0C^k = C^k \supset F^1C^k \supset F^2C^k \supset \ldots \supset F^jC^k \supset \ldots \supset F^{k+1}C^k = 0,$$

where

$$F^jC^k = \{c \in C^k |\ c(g_1, \ldots, g_i, \ldots, g_k) = 0 \text{ if } k-j+1 \text{ arguments belong to } \mathfrak{g}_0\},$$

$0 \leq j \leq k+1$. Set

$$Z_r^{p,q} = \{c \in F^pC^{p+q} |\ dc \in F^{p+r}C^{p+q+1}\}. \tag{3.2.1}$$

Finally, set

$$E_r^{p,q} = Z_r^{p,q}/(Z_{r-1}^{p+1,q-1} + dZ_{r-1}^{p-r+1,q+r-2}). \tag{3.2.2}$$

Notice that the differential d induces the differentials

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1} \tag{3.2.3}$$

and $E_{r+1}^{p,q} = H^{p,q}(E_r)$ [GM].

Since $d(F^jC^k) \subset F^jC^{k+1}$, on $H^k = H^k(\mathfrak{g}, M)$ we get the induced filtration such that $F^pH^k/F^{p+1}H^k = E_\infty^{p,q}$, where $p+q=k$.

We want to compute the group $H^2(V,\ \mathfrak{g}_*)$, where $V = V_0 \oplus V_1$ is the standard $\mathfrak{pe}(n)$-module, $\mathfrak{g}_* = \mathfrak{g}_*(V,\ \mathfrak{cpe}(n))$. The Hochschild-Serre spectral sequence corresponding to the subalgebra V_0 converges to $H^2(V,\ \mathfrak{g}_*)$. Thus, $H^2(V,\ \mathfrak{g}_*) = \oplus_{p+q=2} E_\infty^{p,q}$ and in order to compute the limit terms of the spectral sequence $E_\infty^{p,q}$ we have to consider three cases:

1) $p=2$, $q=0$. Then by formula (3.2.3) we have

$$\begin{gathered} E_1^{1,0} \xrightarrow{d_1^{1,0}} E_1^{2,0} \xrightarrow{d_1^{2,0}} E_1^{3,0}, \\ E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \xrightarrow{d_2^{2,0}} 0, \\ 0 \xrightarrow{d_3^{-1,2}} E_3^{2,0} \xrightarrow{d_3^{2,0}} 0. \end{gathered} \tag{3.2.4}$$

Therefore, $E_\infty^{2,0} = E_3^{2,0}$.

2) $p=1$, $q=1$. Then by formula (3.2.3) we have

$$\begin{gathered} E_1^{0,1} \xrightarrow{d_1^{0,1}} E_1^{1,1} \xrightarrow{d_1^{1,1}} E_1^{2,1}, \\ 0 \xrightarrow{d_2^{-1,2}} E_2^{1,1} \xrightarrow{d_2^{1,1}} E_2^{3,0}, \\ 0 \xrightarrow{d_3^{-2,3}} E_3^{1,1} \xrightarrow{d_3^{1,1}} 0. \end{gathered} \tag{3.2.5}$$

Therefore, $E_\infty^{1,1} = E_3^{1,1}$.

3) $p=0, q=2$. Then by formula (3.2.3) we have

$$\begin{gathered} 0 \xrightarrow{d_1^{-1,2}} E_1^{0,2} \xrightarrow{d_1^{0,2}} E_1^{1,2}, \\ 0 \xrightarrow{d_2^{-2,3}} E_2^{0,2} \xrightarrow{d_2^{0,2}} E_2^{2,1}, \\ 0 \xrightarrow{d_3^{-3,4}} E_3^{0,2} \xrightarrow{d_3^{0,2}} E_3^{3,0}, \\ 0 \xrightarrow{d_4^{-4,5}} E_4^{0,2} \xrightarrow{d_4^{0,2}} 0. \end{gathered} \tag{3.2.6}$$

Therefore, $E_\infty^{0,2} = E_4^{0,2}$.

3.3. Continuation of the proof. Notice that by [Fu] $E_1^{p,q} = H^q(\mathfrak{g}_0,\ M\otimes S^p\mathfrak{g}_1^*)$. Since in our case $V = V_0\oplus V_1$ is a commutative Lie superalgebra, then

$$E_1^{p,q} = H^q(V_0,\ \mathfrak{g}_*\otimes S^pV_1^*) = H^q(V_0,\ \mathfrak{g}_*)\otimes S^pV_1^*. \tag{3.3.1}$$

Let us calculate $H^q(V_0,\ \mathfrak{g}_*)$ for $q = 0, 1, 2$. By Lemma 1.1

$$\begin{array}{l}\mathfrak{g}_* = \mathfrak{pe}(n+1) = \mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1\oplus\mathfrak{g}_2,\ \text{where}\\ \mathfrak{g}_{-1} = V_0\oplus V_1, \mathfrak{g}_0 = \mathfrak{cpe}(n) = \mathfrak{sl}(n)\oplus\Pi(S^2V_0)\oplus\Pi(E^2V_1)\oplus\langle\tau\rangle\oplus\langle z\rangle,\\ \mathfrak{g}_1 = \Pi(V_1)\oplus\Pi(V_0),\\ \mathfrak{g}_2 = \Pi(\langle 1\rangle).\end{array} \qquad (3.3.2$$

Recall that as an $\mathfrak{sl}(n+1)$-module,

$$\mathfrak{pe}(n+1)\cong\mathfrak{sl}(n+1)\oplus E^2W_0^*\oplus S^2W_0\oplus\langle\ \mathrm{d}\ \rangle,$$

where d is $\mathrm{diag}(1_{n+1},\ -1_{n+1})$ and W_0 is the standard $\mathfrak{sl}(n+1)$-module. Clearly,

$$\begin{aligned} E^2W_0^* &= \Pi(E^2V_1)\oplus V_1,\\ S^2W_0 &= \Pi(S^2V_0)\oplus\Pi(V_0)\oplus\mathfrak{g}_2,\\ \mathfrak{sl}(n+1) &= \mathfrak{sl}(n)\oplus\langle\tau+nz\rangle\oplus V_0\oplus\Pi(V_1),\\ \mathrm{d}\ &= \tau - z.\end{aligned}$$

Let $\varepsilon_1,\dots,\varepsilon_{n+1}$ be the standard basis of the dual space to the space of the diagonal matrices in $\mathfrak{gl}(n+1)$ and the ordering is performed so that

$$\Delta_+ = \{\varepsilon_i-\varepsilon_j, i<j\},\ \Delta_- = \{\varepsilon_i-\varepsilon_j, i>j\}.$$

Let $E_{\varepsilon_i-\varepsilon_j}(i\neq j)$ be the corresponding root vectors. Then V_0 is the subspace of $\mathfrak{sl}(n+1)$ generated by

$$E_{\varepsilon_1-\varepsilon_{n+1}},\ E_{\varepsilon_2-\varepsilon_{n+1}},\dots,\ E_{\varepsilon_n-\varepsilon_{n+1}}.$$

Let V_λ be the irreducible $\mathfrak{sl}(n+1)$-module with highest weight λ. The BWB theorem says [Kos] that there exists a 1-1 correspondence between the irreducible components of $H^q(V_0,\ V_\lambda)$, considered as $\mathfrak{gl}(n)$-module, and elements $w\in W(\mathfrak{sl}(n+1))$ of length q from the Weyl group of the Lie algebra $\mathfrak{sl}(n+1)$ such that

$$w(\Delta_-)\cap\Delta_+\subset\{\varepsilon_1-\varepsilon_{n+1},\ \varepsilon_2-\varepsilon_{n+1},\dots,\varepsilon_n-\varepsilon_{n+1}\}. \tag{3.3.3}$$

Moreover, the highest weight of the $\mathfrak{gl}(n)$-module corresponding to w is equal to $w(\lambda+\rho)-\rho$, where $\rho = (\sum_{\alpha\in\Delta_+}\alpha)/2$.

Notice that $E^2W_0^*,\ S^2W_0,\ \mathfrak{sl}(n+1),\ \langle\tau-z\rangle$ are all irreducible $\mathfrak{sl}(n+1)$-modules with highest weights, respectively,

$$\varepsilon_1+\ldots+\varepsilon_{n-1},\ 2\varepsilon_1,\ 2\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_n,\ 0.$$

Let us find the highest weights of irreducible $\mathfrak{gl}(n)$-submodules of $H^q(V_0,\ V_\lambda)$ for each of the indicated λ.

1) $q=0$. The only element of the Weyl group of length 0 is the unit. Hence $w(\lambda+\rho)-\rho = \lambda$.

2) $q=1$. Let $\langle\alpha_1,\dots,\alpha_n\rangle$, where

$$\alpha_1 = \varepsilon_1-\varepsilon_2,\dots,\ \alpha_i = \varepsilon_i-\varepsilon_{i+1},\dots,\ \alpha_n = \varepsilon_n-\varepsilon_{n+1}$$

be the system of simple roots. The elements of the Weyl group of length 1 are reflections corresponding to the simple roots:

$$r_{\alpha_i} : \alpha \longrightarrow \alpha - \frac{2(\alpha_i, \alpha)}{(\alpha_i, \alpha_i)}\alpha_i.$$

Since the only element r_{α_i} satisfying (3.3.3) is r_{α_n}, then $w(\lambda+\rho)-\rho = r_{\alpha_n}(\lambda) - \alpha_n$. For

$$\lambda = \varepsilon_1 + \ldots + \varepsilon_{n-1},\ 2\varepsilon_1,\ 2\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n,\ 0$$

this expression is equal to, respectively:

$$-2\varepsilon_n,\ \varepsilon_1 - \varepsilon_2 - \ldots - \varepsilon_{n-1} - 2\varepsilon_n,\ \ -\varepsilon_2 - \varepsilon_3 - \ldots - \varepsilon_{n-1} - 3\varepsilon_n,\ \ -\varepsilon_1 - \varepsilon_2 - \ldots - \varepsilon_{n-1} - 2\varepsilon_n.$$

3) $q = 2$. The elements of length 2 are of the form $r_{\alpha_i}r_{\alpha_j}$. The only such element satisfying (3.3.3) is $r_{\alpha_n}r_{\alpha_{n-1}}$. Then

$$w(\lambda+\rho)-\rho = r_{\alpha_n}r_{\alpha_{n-1}}(\lambda) - \alpha_{n-1} - 2\alpha_n.$$

For $\lambda = \varepsilon_1 + \ldots + \varepsilon_{n-1}$, $2\varepsilon_1$, $2\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n$, 0 this expression is equal to, respectively:

$$\begin{aligned}
&-2\varepsilon_1 - 2\varepsilon_2 - \ldots - 2\varepsilon_{n-2} - 4\varepsilon_{n-1} - 4\varepsilon_n,\\
&-3\varepsilon_2 - 3\varepsilon_3 \text{ (if } n=3) \text{ or } -2\varepsilon_2 - \ldots - 2\varepsilon_{n-2} - 3\varepsilon_{n-1} - 3\varepsilon_n \text{ (if } n>3),\\
&-\varepsilon_1 - 3\varepsilon_2 - 4\varepsilon_3 \text{ (if } n=3) \text{ or } -\varepsilon_1 - 2\varepsilon_2 - \ldots - 2\varepsilon_{n-2} - 3\varepsilon_{n-1} - 4\varepsilon_n \text{ (if } n>3),\\
&-2\varepsilon_1 - \ldots - 2\varepsilon_{n-2} - 3\varepsilon_{n-1} - 3\varepsilon_n.
\end{aligned}$$

REMARK. We have obtained the weights with respect to $\mathfrak{gl}(n) = \mathfrak{sl}(n) \oplus \langle \tau + nz \rangle$ embedded into $\mathfrak{sl}(n+1)$. Now it is not difficult to rewrite these weights as the highest ones with respect to $\mathfrak{pe}(n)_0 = \mathfrak{sl}(n) \oplus \langle \tau \rangle$. We collect all our results in the following

3.4.

LEMMA. $\mathfrak{gl}(n) = \mathfrak{pe}(n)_0$ *module* $H^q(V_0,\ \mathfrak{g}_*)$ $(q = 0,\ 1,\ 2)$ *is the direct sum of irreducible submodules with the highest weights and highest vectors listed in Table 2.*

3.5. LEMMA. *Let* V_λ *be an irreducible* $\mathfrak{gl}(n)$*-module with highest weight* λ*. Then* $E_1^{p,0}(p = 1,\ 2,\ 3)$*,* $E_1^{p,1}(p = 0,\ 1,\ 2)$*,* $E_1^{p,2}$ $(p = 0,\ 1)$ *are the direct sums of irreducible* $\mathfrak{gl}(n)$*-submodules with highest weights given in the corresponding columns of Tables 3, 4, and 5, respectively.*

PROOF. By formula (3.3.1) the following $\mathfrak{gl}(n)$-modules are isomorphic: $E_1^{p,q} \cong H^q(V_0,\ \mathfrak{g}_*) \otimes S^pV_0$. Making use of the description of $H^q(V_0,\ \mathfrak{g}_*)$ as a $\mathfrak{gl}(n)$-module given in Lemma 3.4 we find the decomposition of the indicated tensor product into irreducible components described in Tables 3, 4, and 5. □

3.6. LEMMA. $E_0^{p,0}(p = 1,\ 2,\ 3)$ *are the direct sums of the irreducible* $\mathfrak{gl}(n)$*-modules with highest weights described in the corresponding columns of Table 6.*

PROOF. By formula (3.2.2) we have

$$E_0^{p,0} = Z_0^{p,0} = \mathfrak{g}_* \otimes S^pV_1^* = \mathfrak{g}_* \otimes S^pV_0.$$

Making use of the description of $\mathfrak{g}_*$ as $\mathfrak{gl}(n)$-module given in (3.3.2) we find the decomposition of the indicated tensor products into direct sum of the irreducible components described in Table 6. □

3.7. LEMMA. $E_\infty^{2,0}$ *is an irreducible* $\mathfrak{gl}(n)$*-module with highest weight* $2\varepsilon_1 + 2\varepsilon_2$.

PROOF. First, recall that $H^{1,2}_{\mathfrak{cpe}(n)} = 0$ by the already proved part of Theorem 2.1 for the case of SFs of order 1. Making use of Lemma 3.4, we note that if $E^{2,0}_\infty$ had contained a $\mathfrak{gl}(n)$-submodule belonging to either $V_{-\varepsilon_n} \otimes S^2V_0$ or $V_{\varepsilon_1} \otimes S^2V_0$, then this submodule would have belonged to $H^{1,2}_{\mathfrak{cpe}(n)}$. □

Therefore, with the help of Table 3 we deduce that $E^{2,0}_\infty$ has no $\mathfrak{gl}(n)$-submodules with highest weights $2\varepsilon_1 - \varepsilon_n$, ε_1, $3\varepsilon_1$, $2\varepsilon_1 + \varepsilon_2$.

Let us show that $E^{2,0}_\infty$ has no irreducible $\mathfrak{gl}(n)$-submodules with highest weights $4\varepsilon_1$, $3\varepsilon_1 + \varepsilon_2$, and $2\varepsilon_1$ either. More precisely, let us show that even $E^{2,0}_2$ does not have them.

Recall that the corresponding differentials act as follows:

$$E^{1,0}_1 \xrightarrow{d^{1,0}_1} E^{2,0}_1 \xrightarrow{d^{2,0}_1} E^{3,0}_1.$$

Note that according to (3.2.2), $E^{p,0}_1 = Z^{p,0}_1$ for $p = 1, 2, 3$. Let us show that $\mathrm{Ker} d^{2,0}_1$ has no components with weights $4\varepsilon_1$, $3\varepsilon_1 + \varepsilon_2$, and $2\varepsilon_1$. It follows from Tables 2 and 3 that the corresponding highest vectors in $E^{2,0}_1$ are

$$v_{4\varepsilon_1} = e_1\tilde{e}_1 \otimes \tilde{e}_1^2, \ v_{3\varepsilon_1+\varepsilon_2} = e_1\tilde{e}_2 \otimes \tilde{e}_1^2 - e_1\tilde{e}_1 \otimes \tilde{e}_1\tilde{e}_2, \ v_{2\varepsilon_1} = (\tau - z) \otimes \tilde{e}_1^2.$$

We remind the reader that the differentials d in our case are the same as the differentials $\partial^{k,s}_{\mathfrak{g}_0}$. Notice that if $c \in E^{2,0}_1$ then

$$d^{2,0}_1 c(f_1, f_1, f_1) = dc(f_1, f_1, f_1) = -3c(f_1, f_1)(f_1).$$

Therefore,

$$d^{2,0}_1 v_{4\varepsilon_1}(f_1, f_1) = 3e_1, \ d^{2,0}_1 v_{3\varepsilon_1+\varepsilon_2}(f_1, f_1) = (3/2)e_2, \ d^{2,0}_1 v_{2\varepsilon_1}(f_1, f_1) = -6f_1. \tag{3.7.1}$$

Hence,

$$v_{4\varepsilon_1}, \ v_{3\varepsilon_1+\varepsilon_2}, \ v_{2\varepsilon_1} \notin \mathrm{Ker} d^{2,0}_1.$$

Finally, let us prove that in $E^{2,0}_\infty$ there is an irreducible $\mathfrak{gl}(n)$-submodule of highest weight $2\varepsilon_1 + 2\varepsilon_2$. Notice that $E^{2,0}_2$ has submodule of highest weight $2\varepsilon_1 + 2\varepsilon_2$, since according to Table 3, this module is contained in $E^{2,0}_1$ and is not contained in either $E^{1,0}_1$ or $E^{3,0}_1$. Recall that the corresponding differentials act as follows:

$$E^{0,1}_2 \xrightarrow{d^{0,1}_2} E^{2,0}_2 \xrightarrow{d^{2,0}_2} 0.$$

Therefore, $\mathrm{Ker} d^{2,0}_2 = E^{2,0}_2$ has a component of weight $2\varepsilon_1 + 2\varepsilon_2$. By Table 4 $E^{0,1}_1$ has no components of weight $2\varepsilon_1 + 2\varepsilon_2$. Hence, neither $E^{0,1}_2$ nor $\mathrm{Im} d^{0,1}_2$ have such a component . Therefore, it must be in $E^{2,0}_3 = E^{2,0}_\infty$.

3.8. LEMMA. a) *As a $\mathfrak{gl}(n)$-module, $E^{1,1}_\infty$ can only have the irreducible submodules with the following highest weights, each of multiplicity not greater then 1:*

$$2\varepsilon_1, \ 2\varepsilon_1 + \varepsilon_2 - \varepsilon_n, \ \varepsilon_1 + \varepsilon_2 - 2\varepsilon_n, \ \textit{and} \ \varepsilon_1 - \varepsilon_n;$$

b) *$E^{1,1}_\infty$ has an irreducible $\mathfrak{gl}(n)$-submodule with highest weight $2\varepsilon_1$.*

PROOF. By Theorem 2.1 for the case of SFs of order 1 and Tables 2 and 4 we see that $E^{1,1}_\infty$ has no irreducible $\mathfrak{gl}(n)$-submodules of highest weight $\varepsilon_1 - 2\varepsilon_n$ and $-\varepsilon_n$, since they would have corresponded to SFs of order 1.

Let us show that there are no components of weight $2\varepsilon_1 - 2\varepsilon_n$ or 0 in $E^{1,1}_\infty$, more precisely, that even $E^{1,1}_2$ does not have them. Recall that the corresponding differentials act as follows:

$$E^{0,1}_1 \xrightarrow{d^{0,1}_1} E^{1,1}_1 \xrightarrow{d^{1,1}_1} E^{2,1}_1.$$

By (3.2.2) we have

$$\begin{aligned} E_1^{0,1} &= Z_1^{0,1}/(Z_0^{1,0}+dZ_0^{0,0}),\\ E_1^{1,1} &= Z_1^{1,1}/(Z_0^{2,0}+dZ_0^{1,0}),\\ E_1^{2,1} &= Z_1^{2,1}/(Z_0^{3,0}+dZ_0^{2,0}). \end{aligned} \tag{3.8.1}$$

By Tables 2 and 4 the highest vectors of weights $2\varepsilon_1-2\varepsilon_n$ and 0 in $E_1^{1,1}$ are, respectively,

$$v_{2\varepsilon_1-2\varepsilon_n}=(e_1\wedge\tilde f_n)\otimes\tilde e_1\wedge\tilde f_n \text{ and } v_0=\sum_{i=1}^n(\tau-z)\otimes\tilde f_i\wedge\tilde e_i.$$

We see that

$$dv_{2\varepsilon_1-2\varepsilon_n}(e_n,f_1,f_1)=-v_{2\varepsilon_1-2\varepsilon_n}(f_1,f_1)(e_n)-2v_{2\varepsilon_1-2\varepsilon_n}(e_n,f_1)(f_1)=-f_n/2\neq 0,$$
$$dv_0(e_1,f_1,f_1)=-v_0(f_1,f_1)(e_1)-2v_0(e_1,f_1)(f_1)=2f_1\neq 0.$$

Suppose that

$$dv_{2\varepsilon_1-2\varepsilon_n}\in Z_0^{3,0}+dZ_0^{2,0}.$$

Then there exist highest $\mathfrak{gl}(n)$-vectors $v'_{2\varepsilon_1-2\varepsilon_n}\in Z_0^{3,0}$ and $v''_{2\varepsilon_1-2\varepsilon_n}\in dZ_0^{2,0}$ of weight $2\varepsilon_1-2\varepsilon_n$ such that

$$dv_{2\varepsilon_1-2\varepsilon_n}=v'_{2\varepsilon_1-2\varepsilon_n}+v''_{2\varepsilon_1-2\varepsilon_n}.$$

Since $e_n\in V_0$, then $v'_{2\varepsilon_1-2\varepsilon_n}(e_n,f_1,f_1)=0$. Hence $v''_{2\varepsilon_1-2\varepsilon_n}\neq 0$ and therefore, $dZ_0^{2,0}$ has an irreducible $\mathfrak{gl}(n)$-submodule of highest weight $2\varepsilon_1-2\varepsilon_n$.

Similarly, having assumed that $dv_0\in Z_0^{3,0}+dZ_0^{2,0}$, we deduce that $dZ_0^{2,0}$ has an irreducible $\mathfrak{gl}(n)$-submodule of weight 0. Note that according to Table 6, $E_0^{2,0}$ has no submodules of highest weight $2\varepsilon_1-2\varepsilon_n$ or 0. Since $E_0^{2,0}=Z_0^{2,0}$, then $Z_0^{2,0}$ and $dZ_0^{2,0}$ have no such components either. Therefore, $dv_{2\varepsilon_1-2\varepsilon_n}$ and dv_0 do not belong to $Z_0^{3,0}+dZ_0^{2,0}$. Thanks to (3.8.1) this implies that

$$d_1^{1,1}v_{2\varepsilon_1-2\varepsilon_n}\neq 0 \text{ and } d_1^{1,1}v_0\neq 0.$$

Hence, $\mathrm{Ker} d_1^{1,1}$, and therefore, $E_2^{1,1}$, have no irreducible $\mathfrak{gl}(n)$-submodules of highest weight $2\varepsilon_1-2\varepsilon_n$ and 0.

Let us prove now that $E_\infty^{1,1}=E_3^{1,1}$ has no irreducible $\mathfrak{gl}(n)$-submodule of highest weight $3\varepsilon_1-\varepsilon_n$. Notice that $E_2^{1,1}$ has such a submodule, since by Table 4 it is contained in $E_1^{1,1}$ and is not contained in either $E_1^{0,1}$ or $E_1^{2,1}$. Tables 2 and 4 imply that the $\mathfrak{gl}(n)$-highest vector in $E_1^{1,1}$ of weight $3\varepsilon_1-\varepsilon_n$ is $v_{3\varepsilon_1-\varepsilon_n}=e_1\tilde e_1\otimes\tilde f_n\wedge\tilde e_1$.

Recall that the corresponding differentials act as follows:

$$0\xrightarrow{d_2^{-1,2}}E_2^{1,1}\xrightarrow{d_2^{1,1}}E_2^{3,0}.$$

By formula (3.2.2) we have

$$\begin{aligned} E_2^{1,1} &= Z_2^{1,1}/(Z_1^{2,0}+dZ_1^{0,1}),\\ E_2^{3,0} &= Z_2^{3,0}/dZ_1^{2,0}. \end{aligned} \tag{3.8.2}$$

Thanks to formulas (3.8.1) we see that the $\mathfrak{gl}(n)$-highest vector in $E_2^{1,1}$ of weight $3\varepsilon_1-\varepsilon_n$ is

$$w_{3\varepsilon_1-\varepsilon_n}=v_{3\varepsilon_1-\varepsilon_n}+v'_{3\varepsilon_1-\varepsilon_n}+v''_{3\varepsilon_1-\varepsilon_n},$$

where $v'_{3\varepsilon_1-\varepsilon_n}$ and $v''_{3\varepsilon_1-\varepsilon_n}$ are $\mathfrak{gl}(n)$-highest vectors in $Z_0^{2,0}$ and $dZ_0^{1,0}$, respectively.

Since by (3.2.1) $dw_{3\varepsilon_1-\varepsilon_n}\in Z_2^{3,0}$, then $dw_{3\varepsilon_1-\varepsilon_n}(e_n,f_1,f_1)=0$. Since $v''_{3\varepsilon_1-\varepsilon_n}\in dZ_0^{1,0}$, then $dv''_{3\varepsilon_1-\varepsilon_n}=0$. We have

$$dv_{3\varepsilon_1-\varepsilon_n}(e_n,f_1,f_1)=-2v_{3\varepsilon_1-\varepsilon_n}(e_n,f_1)(f_1)=-e_1\neq 0.$$

Therefore, $v'_{3\varepsilon_1-\varepsilon_n} \neq 0$. Looking at Table 6 we see that the unique highest vector of weight $3\varepsilon_1 - \varepsilon_n$ in $Z_0^{2,0}$ is $e_1 \wedge \tilde{f}_n \otimes \tilde{e}_1^2$. Hence, $v'_{3\varepsilon_1-\varepsilon_n} = ke_1 \wedge \tilde{f}_n \otimes \tilde{e}_1^2$, where $k \in \mathbb{C}^*$. Note that since $v_{3\varepsilon_1-\varepsilon_n}(f_1, f_1) = 0$, then $dv_{3\varepsilon_1-\varepsilon_n}(f_1, f_1, f_1) = 0$. Therefore,

$$dw_{3\varepsilon_1-\varepsilon_n}(f_1, f_1, f_1) = dv'_{3\varepsilon_1-\varepsilon_n}(f_1, f_1, f_1) =$$
$$- 3v'_{3\varepsilon_1-\varepsilon_n}(f_1, f_1)(f_1) = (-3/2)kf_n \neq 0.$$

Note that $dw_{3\varepsilon_1-\varepsilon_n} \notin dZ_1^{2,0}$. In fact, by Table 3 $E_1^{2,0}$ has no irreducible $\mathfrak{gl}(n)$-component with highest weight $3\varepsilon_1-\varepsilon_n$. Since $E_1^{2,0} = Z_1^{2,0}$, then $Z_1^{2,0}$ and $dZ_1^{2,0}$ have no such component either. Therefore, by (3.8.2) we have $d_2^{1,1}w_{3\varepsilon_1-\varepsilon_n} \neq 0$. Hence, $\mathrm{Ker} d_2^{1,1} = E_3^{1,1}$ has no components with highest weight $3\varepsilon_1 - \varepsilon_n$.

Let us prove that the irreducible component with highest weight $\varepsilon_1 - \varepsilon_n$ cannot be contained in $E_\infty^{1,1}$ with multiplicity greater than 1. Note that $E_1^{1,1}$ has two components of weight $\varepsilon_1-\varepsilon_n$. According to Tables 2 and 4, one of the $\mathfrak{gl}(n)$-highest vectors of weight $\varepsilon_1-\varepsilon_n$ in $E_1^{1,1}$ is $v_{\varepsilon_1-\varepsilon_n} = (\tau - z) \otimes \tilde{f}_n \wedge \tilde{e}_1$. We see that

$$dv_{\varepsilon_1-\varepsilon_n}(e_n, f_1, f_1) = -2v_{\varepsilon_1-\varepsilon_n}(e_n, f_1)(f_1) = -(\tau - z)(f_1) = 2f_1 \neq 0.$$

Suppose that $dv_{\varepsilon_1-\varepsilon_n} \in Z_0^{3,0} + dZ_0^{2,0}$. Then there exist $\mathfrak{gl}(n)$-highest vectors $v'_{\varepsilon_1-\varepsilon_n} \in Z_0^{3,0}$ and $v''_{\varepsilon_1-\varepsilon_n} \in dZ_0^{2,0}$ of weight $\varepsilon_1 - \varepsilon_n$ such that

$$dv_{\varepsilon_1-\varepsilon_n} = v'_{\varepsilon_1-\varepsilon_n} + v''_{\varepsilon_1-\varepsilon_n}.$$

Since $e_n \in V_0$, then $v'_{\varepsilon_1-\varepsilon_n}(e_n, f_1, f_1) = 0$. Therefore, $v''_{\varepsilon_1-\varepsilon_n} \neq 0$. Now note that

$$dv_{\varepsilon_1-\varepsilon_n}(e_1, f_1, f_1) = -v_{\varepsilon_1-\varepsilon_n}(f_1, f_1)(e_1) - 2v_{\varepsilon_1-\varepsilon_n}(e_1, f_1)(f_1) = 0.$$

Since $e_1 \in V_0$, then $v'_{\varepsilon_1-\varepsilon_n}(e_1, f_1, f_1) = 0$. Therefore,

$$v''_{\varepsilon_1-\varepsilon_n}(e_1, f_1, f_1) = 0. \tag{3.8.3}$$

By Table 6 $E_0^{2,0} = Z_0^{2,0}$ contains a unique highest vector of weight $\varepsilon_1 - \varepsilon_n$, namely,

$$\sum_{i=1}^{n} f_n \wedge \tilde{f}_i \otimes \tilde{e}_i\tilde{e}_1.$$

Then

$$v''_{\varepsilon_1-\varepsilon_n} = kd(\sum_{i=1}^{n} f_n \wedge \tilde{f}_i \otimes \tilde{e}_i\tilde{e}_1), \text{ where } k \in \mathbb{C}^*.$$

Note that in this case

$$v''_{\varepsilon_1-\varepsilon_n}(e_1, f_1, f_1) = kd(\sum_{i=1}^{n} f_n \wedge \tilde{f}_i \otimes \tilde{e}_i\tilde{e}_1)(e_1, f_1, f_1) =$$
$$- k(\sum_{i=1}^{n} f_n \wedge \tilde{f}_i \otimes \tilde{e}_i\tilde{e}_1)(f_1, f_1)(e_1) = (k/2)f_n \neq 0,$$

which contradicts (3.8.3). Thus, $dv_{3\varepsilon_1-\varepsilon_n} \notin Z_0^{3,0} + dZ_0^{2,0}$. Then (3.8.1) yields $d_1^{1,1}v_{\varepsilon_1-\varepsilon_n} \neq 0$. Therefore, the component of highest weight $\varepsilon_1 - \varepsilon_n$ can not be contained in $\mathrm{Ker} d_1^{1,1}$ and hence, in $E_\infty^{1,1}$, with multiplicity exceeding 1. So part a) of Lemma 3.8 is proved.

Let us prove that $E_\infty^{1,1} = E_3^{1,1}$ contains an irreducible $\mathfrak{gl}(n)$-submodule of highest weight $2\varepsilon_1$. Note that $E_2^{1,1}$ does contain such a submodule, since by Table 4 it is contained in $E_1^{1,1}$ and is not contained in either $E_1^{0,1}$ or $E_1^{2,1}$.

Let $u_{2\varepsilon_1}$ be the $\mathfrak{gl}(n)$-highest vector of weight $2\varepsilon_1$ in $E_1^{1,1}$. By (3.8.1) $u_{2\varepsilon_1}$ can be chosen so that

$$u_{2\varepsilon_1}(v_1, v_2) = 0 \text{ for any } v_1, v_2 \in V_1.$$

According to (3.8.1), the $\mathfrak{gl}(n)$-highest vector of weight $2\varepsilon_1$ in $E_2^{1,1}$ is

$$w_{2\varepsilon_1} = u_{2\varepsilon_1} + t_{2\varepsilon_1} + s_{2\varepsilon_1}, \tag{3.8.4}$$

where $t_{2\varepsilon_1} \in Z_0^{2,0}$ and $s_{2\varepsilon_1} \in dZ_0^{1,0}$ are $\mathfrak{gl}(n)$-highest vectors. If $dw_{2\varepsilon_1} = 0$, then

$$w_{2\varepsilon_1} \in \mathrm{Ker} d_2^{1,1} = E_3^{1,1}$$

and therefore, $E_\infty^{1,1}$ has an irreducible $\mathfrak{gl}(n)$-submodule with highest weight $2\varepsilon_1$.

Suppose that $dw_{2\varepsilon_1} \neq 0$. Let us prove that then

$$dw_{2\varepsilon_1} \in dZ_1^{2,0}. \tag{3.8.5}$$

Recall that $Z_1^{2,0} = E_1^{2,0}$ and by Table 3 $E_1^{2,0}$ has one highest vector of weight $2\varepsilon_1$, namely, $v_{2\varepsilon_1}$. Let us show that

$$dw_{2\varepsilon_1} = k dv_{2\varepsilon_1}, \text{ where } k \in \mathbb{C}^*. \tag{3.8.6}$$

Note that by (3.2.1) $dw_{2\varepsilon_1} \in Z_2^{3,0}$ and $dZ_1^{2,0} \subset Z_0^{3,0}$. Therefore, in order to prove (3.8.6) it suffices to show that

$$dw_{2\varepsilon_1}(v_1, v_2, v_3) = k dv_{2\varepsilon_1}(v_1, v_2, v_3), \text{ where } k \in \mathbb{C}^*, \text{ for any } v_1, v_2, v_3 \in V_1.$$

We have

$$du_{2\varepsilon_1}(v_1, v_2, v_3) = 0 \text{ for any } v_1, v_2, v_3 \in V_1.$$

Since $s_{2\varepsilon_1} \in dZ_0^{1,0}$, then $ds_{2\varepsilon_1} = 0$. Therefore,

$$dw_{2\varepsilon_1}(v_1, v_2, v_3) = dt_{2\varepsilon_1}(v_1, v_2, v_3) \text{ for any } v_1, v_2, v_3 \in V_1. \tag{3.8.7}$$

Since $t_{2\varepsilon_1} \in Z_0^{2,0}$, then $dt_{2\varepsilon_1} \in Z_0^{2,1}$. Hence $dt_{2\varepsilon_1} = t'_{2\varepsilon_1} + t''_{2\varepsilon_1}$,where $t'_{2\varepsilon_1}, t''_{2\varepsilon_1}$ are $\mathfrak{gl}(n)$-highest vectors from $Z_0^{2,1}$ such that

$$t'_{2\varepsilon_1}(v_1, v_2, v_3) = 0 \text{ for all } v_1, v_2, v_3 \in V_1 \text{ and } t''_{2\varepsilon_1} \in Z_0^{3,0}.$$

Since by hypothesis $dw_{2\varepsilon_1} \neq 0$, then $t''_{2\varepsilon_1} \neq 0$. Since $v_{2\varepsilon_1} \in Z_1^{2,0}$, then $dv_{2\varepsilon_1} \in Z_0^{3,0}$.

In Lemma 3.7 we have proved that $dv_{2\varepsilon_1} \neq 0$ (see (3.7.1)). By Table 6 $Z_0^{3,0} = E_0^{3,0}$ has only one irreducible $\mathfrak{gl}(n)$-submodule with highest weight $2\varepsilon_1$. Hence $t''_{2\varepsilon_1} = k dv_{2\varepsilon_1}$, where $k \in \mathbb{C}^*$. Therefore,

$$dt_{2\varepsilon_1}(v_1, v_2, v_3) = k dv_{2\varepsilon_1}(v_1, v_2, v_3) \text{ for any } v_1, v_2, v_3 \in V_1.$$

Thus, by (3.8.7)

$$dw_{2\varepsilon_1}(v_1, v_2, v_3) = k dv_{2\varepsilon_1}(v_1, v_2, v_3) \text{ for any } v_1, v_2, v_3 \in V_1$$

and formula (3.8.6) is proved. Then by (3.8.2) $d_2^{1,1} w_{2\varepsilon_1} = 0$. Therefore, $w_{2\varepsilon_1} \in \mathrm{Ker} d_2^{1,1} = E_3^{1,1}$. Thus, $E_\infty^{1,1}$ contains an irreducible $\mathfrak{gl}(n)$-submodule with highest weight $2\varepsilon_1$. This proves part b) of Lemma 3.8. $\square$

3.9. Lemma. *Only the following highest weights of irreducible $\mathfrak{gl}(n)$-submodules can be encountered among those in $E_\infty^{0,2}$:*

$$-2\varepsilon_{n-1} - 2\varepsilon_n, \; 2\varepsilon_1 - \varepsilon_{n-1} - \varepsilon_n, \; \varepsilon_1 - \varepsilon_{n-1} - 2\varepsilon_n, \; -\varepsilon_{n-1} - \varepsilon_n.$$

Proof. By Table 5 $E_1^{0,2}$ is a direct sum of irreducible $\mathfrak{gl}(n)$-components with the indicated highest weights.

Note that due to Table 2 $H^2(V,\ \mathfrak{g}_*)$ can only possess first and second order SFs. By the statement of Theorem 2.1 for SFs of order 1, $H^{1,2}_{\mathfrak{cpe}(n)}=0$. Therefore, by Lemmas 7, 8, and 9 $H^{2,2}_{\mathfrak{cpe}(n)}$ can only contain irreducible $\mathfrak{gl}(n)$-submodules with the following highest weights:

$$\begin{gathered}2\varepsilon_1+2\varepsilon_2,\ 2\varepsilon_1,\ 2\varepsilon_1+\varepsilon_2-\varepsilon_n,\ \varepsilon_1+\varepsilon_2-2\varepsilon_n,\ \varepsilon_1-\varepsilon_n,\\ -2\varepsilon_{n-1}-2\varepsilon_n,\ 2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n,\ \varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n,\ \ -\varepsilon_{n-1}-\varepsilon_n\end{gathered}$$

each with multiplicity not greater than one, and the components with highest weights $2\varepsilon_1+2\varepsilon_2$ and $2\varepsilon_1$ are contained in $H^{2,2}_{\mathfrak{cpe}(n)}$ with multiplicity one each.

Recall that by definition $H^{2,2}_{\mathfrak{g}_0}$ is determined by the sequence

$$\mathfrak{g}_1\otimes\mathfrak{g}_{-1}^*\xrightarrow{\partial^{3,1}_{\mathfrak{g}_0}}\mathfrak{g}_0\otimes E^2\mathfrak{g}_{-1}^*\xrightarrow{\partial^{2,2}_{\mathfrak{g}_0}}\mathfrak{g}_{-1}\otimes E^3\mathfrak{g}_{-1}^*.$$

Thus, $\mathfrak{g}_0\otimes E^2\mathfrak{g}_{-1}^*$ for

$$\mathfrak{g}_0=\mathfrak{cpe}(n)\text{ or }\mathfrak{spe}(n)\Subset\langle\tau+nz\rangle$$

is equal to

$$\mathfrak{cpe}(n)\otimes E^2\mathfrak{g}_{-1}^*\text{ or }(\mathfrak{spe}(n)\Subset\langle\tau+nz\rangle)\otimes E^2\mathfrak{g}_{-1}^*,$$

respectively. Note that $\mathrm{Im}\partial^{3,1}_{\mathfrak{cpe}(n)}$ and $\mathrm{Im}\partial^{3,1}_{\mathfrak{spe}(n)\Subset\langle\tau+nz\rangle}$ are isomorphic $\mathfrak{spe}(n)$-modules by part b) of Theorem 1.2 and (3.1.1).

Note also that by Lemma 1.4 the Jordan-Hölder series for E^2V^* contains $\mathfrak{spe}(n)$-modules with highest weights $2\varepsilon_1$ and 0. Therefore, the Jordan-Hölder series of the $\mathfrak{spe}(n)$-module $H^{2,2}_{\mathfrak{cpe}(n)}$, as compared with that of $H^{2,2}_{\mathfrak{spe}(n)\Subset\langle\tau+nz\rangle}$, can additionally contain only the irreducible $\mathfrak{spe}(n)$-modules with highest weights $2\varepsilon_1$ and 0. But we have shown that $H^{2,2}_{\mathfrak{cpe}(n)}$ has no trivial $\mathfrak{gl}(n)$-submodule and therefore, $H^{2,2}_{\mathfrak{cpe}(n)}$ and $H^{2,2}_{\mathfrak{spe}(n)\Subset\langle\tau+nz\rangle}$ can only differ by an irreducible $\mathfrak{spe}(n)$-submodule with highest weight $2\varepsilon_1$, which, being considered as $\mathfrak{sl}(n)$-module, is the sum of irreducible $\mathfrak{sl}(n)$-submodules with highest weights

$$2\varepsilon_1,\ \varepsilon_1-\varepsilon_n,\text{ and }-\varepsilon_{n-1}-\varepsilon_n\text{ (see (1.4.4)).}$$

□

3.10. Lemma. *$H^{2,2}_{\mathfrak{spe}(n)\Subset\langle\tau+nz\rangle}$ has no irreducible $\mathfrak{sl}(n)$-submodule with highest weight $2\varepsilon_1$.*

Proof. Making use of Lemma 3.8, let us prove that $E^{1,1}_\infty$ considered as an $\mathfrak{sl}(n)$-module has no irreducible component with highest weight $2\varepsilon_1$ in the case where $\mathfrak{g}_0=\mathfrak{spe}(n)\Subset\langle\tau+nz\rangle$.

Indeed, in Lemma 3.8 we have shown that if $w_{2\varepsilon_1}$ is the $\mathfrak{sl}(n)$-highest vector of weight $2\varepsilon_1$ in $E_2^{1,1}$ such that $dw_{2\varepsilon_1}\neq0$ then $dw_{2\varepsilon_1}\in dZ_1^{2,0}$ (see (3.8.5)).

According to Table 2, $H^0(V_0,\mathfrak{g}_*(\mathfrak{g}_{-1},\mathfrak{g}_0))$ has no irreducible $\mathfrak{sl}(n)$-submodule with highest weight 0 when $\mathfrak{g}_0=\mathfrak{spe}(n)\Subset\langle\tau+nz\rangle$ Therefore, by Table 3 $E_1^{2,0}$ has no irreducible $\mathfrak{sl}(n)$-submodule with highest weight $2\varepsilon_1$. Since $Z_1^{2,0}=E_1^{2,0}$, then $Z_1^{2,0}$ and $dZ_1^{2,0}$ have no such component either. Thus, by (3.8.2) $d_2^{1,1}w_{2\varepsilon_1}\neq0$ and therefore, $w_{2\varepsilon_1}\notin E_3^{1,1}=E_\infty^{1,1}$. It remains to show that $dw_{2\varepsilon_1}\neq0$. Recall that $w_{2\varepsilon_1}=u_{2\varepsilon_1}+t_{2\varepsilon_1}+s_{2\varepsilon_1}$, where $u_{2\varepsilon_1}\in E_1^{1,1}, t_{2\varepsilon_1}\in Z_0^{2,0}$ and $s_{2\varepsilon_1}\in dZ_0^{1,0}$ are $\mathfrak{sl}(n)$-highest vectors (see (3.8.4)). By Tables 2 and 4

$$u_{2\varepsilon_1}=2\sum_{i=1}^n e_1\tilde e_i\otimes\tilde f_i\wedge\tilde e_1-(n+1)\sum_{i=1}^n e_1\tilde e_1\otimes\tilde f_i\wedge\tilde e_i.\tag{3.10.1}$$

Therefore,

$$\begin{aligned}&du_{2\varepsilon_1}(e_1,f_1,f_1)=-u_{2\varepsilon_1}(f_1,f_1)(e_1)-2u_{2\varepsilon_1}(e_1,f_1)(f_1)=\\&-2(e_1\tilde{e}_1-(n+1)e_1\tilde{e}_1/2)(f_1)=(n-1)e_1\neq 0.\end{aligned}$$

Since $s_{2\varepsilon_1}\in dZ_0^{1,0}$, then $ds_{2\varepsilon_1}=0$. Hence, if $dw_{2\varepsilon_1}=0$ then $t_{2\varepsilon_1}\neq 0$ and since $u_{2\varepsilon_1}$ is an even vector, then vector $t_{2\varepsilon_1}$ must be even.

By Table 6 $Z_0^{2,0}=E_0^{2,0}$ has 4 irreducible $\mathfrak{sl}(n)$-submodules with highest weight $2\varepsilon_1$. The corresponding highest vectors are

$$\begin{aligned}&(\sum_{i=2}^n e_i\wedge\tilde{f}_i-(n-1)e_1\wedge\tilde{f}_1)\otimes\tilde{e}_1^2-n\sum_{i=2}^n e_1\wedge\tilde{f}_i\otimes\tilde{e}_1\tilde{e}_i,\\&\tau\otimes\tilde{e}_1^2, z\otimes\tilde{e}_1^2,\ \text{and}\ \mathfrak{g}_2\otimes\tilde{e}_1^2.\end{aligned}.$$

Only the first three of these vectors are even. Therefore, if we confine ourselves to the case $\mathfrak{g}_0=\mathfrak{spe}(n)\Subset\langle\tau+nz\rangle$, we see that there should be two $\mathfrak{sl}(n)$-highest vectors of weight $2\varepsilon_1$ in $E_0^{2,0}$. Let

$$\begin{aligned}t_{2\varepsilon_1}&=k_1((\sum_{i=2}^n e_i\wedge\tilde{f}_i-(n-1)e_1\wedge\tilde{f}_1)\otimes\tilde{e}_1^2-n\sum_{i=2}^n e_1\wedge\tilde{f}_i\otimes\tilde{e}_1\tilde{e}_i)\\&+k_2((\tau+nz)\otimes\tilde{e}_1^2),\ \text{where}\ k_1,\ k_2\in\mathbb{C},\end{aligned}$$

be a linear combination of these vectors. Note that

$$\begin{aligned}&du_{2\varepsilon_1}(e_2,f_2,f_1)=-u_{2\varepsilon_1}(f_2,f_1)(e_2)-u_{2\varepsilon_1}(e_2,f_1)(f_2)-u_{2\varepsilon_1}(e_2,f_2)(f_1)=\\&-2(e_1\tilde{e}_2/2)(f_2)+(n+1)(e_1\tilde{e}_1/2)(f_1)=-e_1/2+(n+1)e_1/2=ne_1/2,\end{aligned}$$

$$\begin{aligned}&dt_{2\varepsilon_1}(e_2,f_2,f_1)=-t_{2\varepsilon_1}(f_2,f_1)(e_2)-t_{2\varepsilon_1}(e_2,f_1)(f_2)-t_{2\varepsilon_1}(e_2,f_2)(f_1)=\\&(-1/2)k_1ne_1\wedge\tilde{f}_2(e_2)=(-1/4)k_1ne_1.\end{aligned}$$

Therefore, if $dw_{2\varepsilon_1}=0$, then $k_1=2$. Observe that

$$\begin{aligned}&dt_{2\varepsilon_1}(e_1,f_1,f_1)=-t_{2\varepsilon_1}(f_1,f_1)(e_1)-2t_{2\varepsilon_1}(e_1,f_1)(f_1)=\\&-k_1(n-1)(e_1\wedge\tilde{f}_1)(e_1)+k_2(\tau+nz)(e_1)=(-k_1(n-1)/2+k_2(n+1))e_1.\end{aligned}$$

Since $du_{2\varepsilon_1}(e_1,f_1,f_1)=(n-1)e_1$, then

$$-(n-1)+k_2(n+1)+(n-1)=0.$$

Hence, $k_2=0$. But then

$$\begin{aligned}&dw_{2\varepsilon_1}(f_1,f_1,f_1)=dt_{2\varepsilon_1}(f_1,f_1,f_1)=-3t_{2\varepsilon_1}(f_1,f_1)(f_1)=\\&-3k_1(n-1)e_1\wedge\tilde{f}_1(f_1)=(3/2)k_1(n-1)f_1=3(n-1)f_1\neq 0.\end{aligned}$$

This proves Lemma 3.10. □

Lemma 3.10 implies that $H^{2,2}_{\mathfrak{cpe}(n)}$ and $H^{2,2}_{\mathfrak{spe}(n)\Subset\langle\tau+nz\rangle}$ differ by an irreducible $\mathfrak{spe}(n)$-module with highest weight $2\varepsilon_1$. Thus, $H^{2,2}_{\mathfrak{spe}(n)\Subset\langle\tau+nz\rangle}$ can only contain irreducible $\mathfrak{sl}(n)$-submodules with highest weights

$$\begin{aligned}&2\varepsilon_1+2\varepsilon_2,\ 2\varepsilon_1+\varepsilon_2-\varepsilon_n,\ \varepsilon_1+\varepsilon_2-2\varepsilon_n,\\&-2\varepsilon_{n-1}-2\varepsilon_n,\ 2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n,\ \varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n\end{aligned}$$

each with multiplicity not greater than 1, and the multiplicity of the submodule with highest weight $2\varepsilon_1+2\varepsilon_2$ is precisely 1.

3.11. LEMMA. *The irreducible $\mathfrak{pe}(n)$-module with highest weight $2\varepsilon_1+2\varepsilon_2$ is the direct sum of irreducible $\mathfrak{gl}(n)$-modules with the following highest weights:*

a) *for $n>3$:*

$$2\varepsilon_1+2\varepsilon_2,\ 2\varepsilon_1+\varepsilon_2-\varepsilon_n,\ \varepsilon_1+\varepsilon_2-2\varepsilon_n,$$
$$-2\varepsilon_{n-1}-2\varepsilon_n,\ 2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n,\ \varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n;$$

b) *for $n=3$:*

$$2\varepsilon_1+2\varepsilon_2,\ 2\varepsilon_1+\varepsilon_2-\varepsilon_3,\ \varepsilon_1+\varepsilon_2-2\varepsilon_3.$$

PROOF. Let us consider the $\mathfrak{pe}(n)$-module $S^2(S^2V)$. Note that $v_{2\varepsilon_1+2\varepsilon_2}=(e_1e_2)^2-(e_1^2)(e_2^2)$ is a $\mathfrak{pe}(n)$-highest vector. Indeed, $B_{i,j}v_{2\varepsilon_1+2\varepsilon_2}=0$ for any i and j. Set

$$\begin{aligned}
v_{2\varepsilon_1+\varepsilon_2-\varepsilon_n}&=(e_1e_2)(e_1f_n)-(e_1^2)(e_2f_n),\\
v_{\varepsilon_1+\varepsilon_2-2\varepsilon_n}&=(e_1f_n)\wedge(e_2f_n),\\
v_{-2\varepsilon_{n-1}-2\varepsilon_n}&=(f_{n-1}\wedge f_n)(f_{n-1}\wedge f_n),\\
v_{2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n}&=(e_1f_{n-1})\wedge(e_1f_n)-(e_1^2)(f_{n-1}\wedge f_n),\\
v_{\varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n}&=(f_{n-1}\wedge f_n)(e_1f_n).
\end{aligned}$$

Notice that these vectors are the $\mathfrak{gl}(n)$-highest ones. Moreover,

$$\begin{aligned}
C_{2,n}v_{2\varepsilon_1+2\varepsilon_2}&=2v_{2\varepsilon_1+\varepsilon_2-\varepsilon_n}, &&(3.11.1)\\
B_{2,n}v_{2\varepsilon_1+\varepsilon_2-\varepsilon_n}&=v_{2\varepsilon_1+2\varepsilon_2}, &&(3.11.2)\\
C_{1,n}v_{2\varepsilon_1+\varepsilon_2-\varepsilon_n}&=-3v_{\varepsilon_1+\varepsilon_2-2\varepsilon_n}, &&(3.11.3)\\
B_{1,n}v_{\varepsilon_1+\varepsilon_2-2\varepsilon_n}&=-v_{2\varepsilon_1+\varepsilon_2-\varepsilon_n}. &&(3.11.4)
\end{aligned}$$

If $n>3$, then additionally

$$\begin{aligned}
C_{2,n-1}v_{2\varepsilon_1+\varepsilon_2-\varepsilon_n}&=v_{2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n},\\
B_{2,n-1}v_{2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n}&=v_{2\varepsilon_1+\varepsilon_2-\varepsilon_n},\\
C_{1,n}v_{2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n}&=-3v_{\varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n},\\
B_{1,n}v_{\varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n}&=-v_{2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n},\\
C_{1,n-1}v_{\varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n}&=v_{-2\varepsilon_{n-1}-2\varepsilon_n},\\
B_{1,n-1}v_{-2\varepsilon_{n-1}-2\varepsilon_n}&=2v_{\varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n}.
\end{aligned}$$

Therefore, if $n>3$, then the irreducible $\mathfrak{pe}(n)$-module with highest weight $2\varepsilon_1+2\varepsilon_2$ contains irreducible $\mathfrak{gl}(n)$-modules with highest weights

$$2\varepsilon_1+2\varepsilon_2,\ 2\varepsilon_1+\varepsilon_2-\varepsilon_n,\ \varepsilon_1+\varepsilon_2-2\varepsilon_n,$$
$$-2\varepsilon_{n-1}-2\varepsilon_n,\ 2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n,\ \varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n.$$

We have already shown that the $\mathfrak{spe}(n)$-module $H^{2,2}_{\mathfrak{spe}(n)\Subset\langle\tau+nz\rangle}$ does contain irreducible $\mathfrak{sl}(n)$-submodules with these highest weights exactly, their multiplicities are not greater than 1, and the multiplicity of $\mathfrak{sl}(n)$-submodule with highest weight $2\varepsilon_1+2\varepsilon_2$ is precisely one.

From Tables 2 and 3 we see that the corresponding $\mathfrak{sl}(n)$-highest vector is

$$v_{2\varepsilon_1+2\varepsilon_2}=(e_1\tilde e_1)\otimes(\tilde e_2\tilde e_2)+(e_2\tilde e_2)\otimes(\tilde e_1\tilde e_1)-2(e_1\tilde e_2)\otimes(\tilde e_1\tilde e_2). \qquad (3.11.5)$$

Hence, $v_{2\varepsilon_1+2\varepsilon_2}$ is an odd $\mathfrak{spe}(n)$-highest vector. So part a) of Lemma 3.11 and Theorem 2.1 for the case where $\mathfrak{g}_0=\mathfrak{spe}(n)\Subset\langle\tau+nz\rangle, n>3$, are proved.

Let $n=3$. Consider the $\mathfrak{spe}(3)$-module E^3V. As an $\mathfrak{sl}(3)$-module, this module is isomorphic to

$$E^3(V_0\oplus V_0^*)=E^3V_0\oplus(E^2V_0)(V_0^*)\oplus V_0(S^2V_0^*)\oplus S^3V_0^*.$$

Therefore, E^3V is the direct sum of irreducible $\mathfrak{sl}(3)$-modules with the highest weights and highest vectors listed in Table 7. Note that the vectors of weights 0 and ε_1 are the $\mathfrak{spe}(3)$-highest ones. Therefore, the Jordan-Hölder series of the $\mathfrak{spe}(3)$-module E^3V contains as quotient modules the trivial and the standard ones. Notice that the vector of weight $-2\varepsilon_3$ is the $\mathfrak{spe}(3)$-highest one in the corresponding quotient module which can only contain $\mathfrak{sl}(3)$-submodules with highest weights $-2\varepsilon_3,\ \varepsilon_1-2\varepsilon_3,$ and $-3\varepsilon_3$.

Since $v_{2\varepsilon_1+2\varepsilon_2}$ is the $\mathfrak{sl}(3)$-highest vector of weight $-2\varepsilon_3$, then the relations (3.11.1)–(3.11.4) imply part b) of Lemma 3.11. □

3.12. LEMMA. *For $n=3$ we have the following nonsplit exact sequence of $\mathfrak{spe}(3)$-modules*

$$0\longrightarrow \Pi(X_{2\varepsilon_1+2\varepsilon_2})\longrightarrow H^{2,2}_{\mathfrak{spe}(3)\subset\!\!\!+\langle\tau+3z\rangle}\longrightarrow \Pi(X_{3\varepsilon_1})\longrightarrow 0. \tag{3.12.1}$$

PROOF. By part b) of Lemma 3.11 and (3.11.5) we see that $H^{2,2}_{\mathfrak{spe}(3)\subset\!\!\!+\langle\tau+3z\rangle}$ contains an irreducible $\mathfrak{spe}(3)$-module with highest weight $-2\varepsilon_3$, which being considered as an $\mathfrak{sl}(3)$-module, is the sum of irreducible $\mathfrak{sl}(3)$-components with highest weights $-2\varepsilon_3$, $\varepsilon_1-2\varepsilon_3$, and $-3\varepsilon_3$.

In addition to these $\mathfrak{sl}(3)$-components, $H^{2,2}_{\mathfrak{spe}(3)\subset\!\!\!+\langle\tau+3z\rangle}$ can only contain irreducible $\mathfrak{sl}(3)$-components with the following highest weights:

$$3\varepsilon_1,\ 2\varepsilon_1,\ \text{and } 2\varepsilon_1-\varepsilon_3.$$

Let us show that these components are indeed contained in $H^{2,2}_{\mathfrak{spe}(3)\subset\!\!\!+\langle\tau+3z\rangle}$, and that their sum is an irreducible $\mathfrak{spe}(3)$-quotient module with highest weight $3\varepsilon_1$.

First, note that $E^{0,2}_\infty=E^{0,2}_4$ has an irreducible $\mathfrak{gl}(n)$-module with highest weight $2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n$. In fact, by Table 5 such a submodule is contained in $E^{0,2}_1$ but is not contained in $E^{1,2}_1$. Therefore, by (3.2.6) it is contained in $E^{0,2}_2$.

According to Table 4, in $E^{2,1}_1$ there is no submodule with highest weight $2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n$, hence such a submodule is not contained in $E^{2,1}_2$ either. Therefore, by (3.2.6) the submodule with this highest weight is contained in $E^{0,2}_3$.

According to Table 6, $E^{3,0}_0$ has no submodule with highest weight $2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n$, hence it is not contained in $E^{3,0}_3$ either. Therefore, by (3.2.6) it is contained in $E^{0,2}_\infty=E^{0,2}_4$.

By Tables 2 and 5 the $\mathfrak{gl}(n)$-highest vectors in $E^{0,2}_1$ of weights

$$-2\varepsilon_{n-1}-2\varepsilon_n,\ 2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n,\ \text{and } \varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n$$

are, respectively,

$$\begin{aligned}
v_{-2\varepsilon_{n-1}-2\varepsilon_n}&=(f_{n-1}\wedge\tilde f_n)\otimes(\tilde f_{n-1}\wedge\tilde f_n),\\
v_{2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n}&=(e_1\tilde e_1)\otimes(\tilde f_{n-1}\wedge\tilde f_n),\ \text{and}\\
v_{\varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n}&=(e_1\wedge\tilde f_n)\otimes(\tilde f_{n-1}\wedge\tilde f_n).
\end{aligned}$$

Note that

$$\begin{aligned}
C_{1,n}v_{2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n}&=-2v_{\varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n},\\
B_{1,n}v_{\varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n}&=-v_{2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n}-(e_1\wedge\tilde f_n)\otimes(\tilde f_{n-1}\wedge\tilde e_1),\\
C_{1,n-1}v_{\varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n}&=v_{-2\varepsilon_{n-1}-2\varepsilon_n},\\
B_{1,n-1}v_{-2\varepsilon_{n-1}-2\varepsilon_n}&=v_{\varepsilon_1-\varepsilon_{n-1}-2\varepsilon_n}+(f_{n-1}\wedge\tilde f_n)\otimes(\tilde e_1\wedge\tilde f_n).
\end{aligned}$$

Therefore, for $n=3$ the components with highest weights $3\varepsilon_1,\ 2\varepsilon_1,$ and $2\varepsilon_1-\varepsilon_3$ constitute an irreducible quotient module with highest weight $3\varepsilon_1$.

From Tables 2 and 4 we see that the $\mathfrak{gl}(n)$-highest vector in $E^{1,1}_1$ of weight $2\varepsilon_1+\varepsilon_2-\varepsilon_n$ is

$$v_{2\varepsilon_1+\varepsilon_2-\varepsilon_n}=(e_1\tilde e_2)\otimes(\tilde e_1\wedge\tilde f_n)-(e_1\tilde e_1)\otimes(\tilde e_2\wedge\tilde f_n).$$

Observe that

$$(B_{1,2}A_{2,1} - (1/2)B_{2,2})(v_{2\varepsilon_1-\varepsilon_2-\varepsilon_3}) = 2v_{2\varepsilon_1+\varepsilon_2-\varepsilon_3}.$$

Therefore, the sequence (3.12.1) is nonsplit. This proves Lemma 3.12, and Theorem 2.1 in the case where $\mathfrak{g}_0 = \mathfrak{spe}(3) \Subset \langle \tau + 3z \rangle$. $\square$

Recall that the Jordan-Hölder series of $\mathfrak{spe}(n)$-module $H^{2,2}_{\mathfrak{cpe}(n)}$, as compared to that of $H^{2,2}_{\mathfrak{spe}(n) \Subset \langle \tau+nz \rangle}$, contains in addition the $\mathfrak{spe}(n)$-component with highest weight $2\varepsilon_1$. Recall that by (3.10.1) the highest $\mathfrak{gl}(n)$-vector with weight $2\varepsilon_1$ in $E_1^{1,1}$ is

$$u_{2\varepsilon_1} = 2\sum_{i=1}^{n} e_1\tilde{e}_i \otimes \tilde{f}_i \wedge \tilde{e}_1 - (n+1)\sum_{i=1}^{n} e_1\tilde{e}_1 \otimes \tilde{f}_i \wedge \tilde{e}_i.$$

Note that

$$C_{n-1,n}(u_{2\varepsilon_1}) = -2(n+1)v_{2\varepsilon_1-\varepsilon_{n-1}-\varepsilon_n} + 2(e_1 \wedge \tilde{f}_{n-1} \otimes \tilde{f}_n \wedge \tilde{e}_1 - e_1 \wedge \tilde{f}_n \otimes \tilde{f}_{n-1} \wedge \tilde{e}_1).$$

This proves Theorem 2.1 in the case where $\mathfrak{g}_0 = \mathfrak{cpe}(n)$.

The proof of Theorem 2.1 in the case where $\mathfrak{g}_0 = \mathfrak{spe}(n)$ follows from the fact that the Jordan-Hölder series of $\mathfrak{spe}(n)$-module $H^{2,2}_{\mathfrak{spe}(n)}$, as compared to that of $H^{2,2}_{\mathfrak{spe}(n) \Subset \langle \tau+nz \rangle}$, contains in addition the $\mathfrak{spe}(n)$-component with highest weight $\varepsilon_1 + \varepsilon_2$.

Finally, the proof of Theorem 2.1 in the case when $\mathfrak{g}_0 = \mathfrak{spe}(n) \Subset \langle a\tau + bz \rangle$, where a, $b \in \mathbb{C}$ are such that $a = 0$, $b \neq 0$, or $a \neq 0$, $b/a \neq n$, follows from the fact that the Jordan-Hölder series of $\mathfrak{spe}(n)$-module $\mathrm{Ker}\partial^{2,2}_{\mathfrak{spe}(n) \Subset \langle a\tau+bz \rangle}$, as compared to that of $\mathrm{Ker}\partial^{2,2}_{\mathfrak{spe}(n)}$, contains in addition the $\mathfrak{spe}(n)$-component with highest weight $2\varepsilon_1$.

CHAPTER 2

The analogues of Penrose's tensors

In this chapter $\Lambda^i V$ is the i-th exterior power of a vector (super)space V.

Standard $\mathbb{Z}$-grading of $\mathfrak{sl}(m|n)$ and the corresponding Cartan prolongations

Let $V = V(m|0)$ and $U = U(0|n)$ be the standard (identity) $\mathfrak{gl}(m)$- and $\mathfrak{gl}(n)$-modules. (Hereafter $\mathfrak{gl}(m) = \mathfrak{gl}(m|0)$, $\mathfrak{gl}(n) = \mathfrak{gl}(0|n)$, etc.)

In what follows we will consider the standard (compatible) $\mathbb{Z}$-grading of $\mathfrak{g} = \mathfrak{sl}(m|n)$ with $m \leq n$ and let the degrees of all even roots be zero. This yields the $\mathbb{Z}$-grading of the form:

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1, \text{ where } \mathfrak{g}_0 = \mathfrak{sl}(m) \oplus \mathfrak{sl}(n) \oplus \mathbb{C}, \ \mathfrak{g}_{-1} = \mathfrak{g}_1^* = U \otimes V^*.$$

Let $\hat{\mathfrak{g}_0}$ be the Levi subalgebra of $\mathfrak{g}_0$, i.e., $\hat{\mathfrak{g}_0} = \mathfrak{sl}(m) \oplus \mathfrak{sl}(n)$. The weights are given with respect to the bases $\varepsilon_1, \dots, \varepsilon_m$ and $\delta_1, \dots, \delta_n$ of the dual spaces to the maximal tori of $\mathfrak{gl}(m|n)$. Let $e_1, \dots, e_m$ be the weight basis of V and $f_1, \dots, f_n$ be the weight basis of U. Let $\tilde{e}_1, \dots, \tilde{e}_m$ and $\tilde{f}_1, \dots, \tilde{f}_n$ be the bases of the dual spaces to V and U, respectively, normed so that $\tilde{e}_i(e_j) = \tilde{f}_i(f_j) = \delta_{ij}$. If $\oplus_\lambda k_\lambda V_\lambda$ is a direct sum of irreducible $\mathfrak{g}_0$-modules (here k_λ is the multiplicity of V_λ) with highest weight λ, denote by v^i_λ the highest weight vectors of the corresponding components: $i = 1, \dots, k_\lambda$. We will often represent the elements of $\mathfrak{gl}(m|n)$ by the matrices

$$X = \operatorname{diag}(A,\ D) + \operatorname{antidiag}(B,\ C)$$

where the dimensions of the matrices A, B, C, and D are $m \times m$, $m \times n$, $n \times m$ and $n \times n$, respectively. Denote by $A_{i,j}$ the matrix X whose components B, C, and D are zero and all the entries of A are also zero except for the $(i,\ j)$-th. The matrices $B_{i,j}$, $C_{i,j}$, and $D_{i,j}$ are defined similarly.

1.1. Theorem. a) *If $m = 1$, $n > 1$, then $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0) = \mathfrak{vect}(0|n)$, $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0}) = \mathfrak{svect}(0|n)$;*

b) *if $m,\ n > 1$ and $m \neq n$, then $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0) = \mathfrak{g}$, $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0}) = \mathfrak{g}_{-1} \oplus \hat{\mathfrak{g}_0}$;*

c) *if $m = n = 2$, then $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0}) = \mathfrak{h}(0|4)$, $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0) = S^*(\mathfrak{g}_{-1}^*) \Subset \mathfrak{h}(0|4)$;*

d) *if $m = n > 2$, then $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0}) = \mathfrak{psl}(n|n)$, $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0) = S^*(\mathfrak{g}_{-1}^*) \Subset \mathfrak{psl}(n|n)$.*

Proof. Consider all cases mentioned in Theorem 1.1.

1.2. $\mathbf{m = 1}$, $\mathbf{n \geq 2}$. Then $\mathfrak{g}_0 = \mathfrak{sl}(n) \oplus \mathbb{C} = \mathfrak{gl}(n)$ and $\hat{\mathfrak{g}_0} = \mathfrak{sl}(n)$, where $\mathfrak{g}_{-1}$ is the standard $\mathfrak{g}_0$ (or $\hat{\mathfrak{g}_0}$) module. Therefore, $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0) = \mathfrak{vect}(0|n)$, $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0}) = \mathfrak{svect}(0|n)$.

Notice that if $m \neq n$, then

$$\mathfrak{sl}(m|n) \subset \mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0) \tag{1.2.1}$$

and if $m = n$, then

$$\mathfrak{psl}(n|n) \subset \mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0}). \tag{1.2.2}$$

Indeed, the Lie superalgebras $\mathfrak{sl}(m|n)$, where $m \neq n$, and $\mathfrak{psl}(n|n)$ are simple and therefore, they are transitive (i.e., if there exists $g \in \mathfrak{g}_i$ $(i \geq 0)$ such that $[\mathfrak{g}_{-1},\ g] = 0$, then $g = 0$. It follows that $\mathfrak{g}_1$ is embedded into $\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^*$ (or $\hat{\mathfrak{g}_0} \otimes \mathfrak{g}_{-1}^*$). The Jacobi identity implies $\mathfrak{g}_1 \subset \mathfrak{g}_{-1} \otimes S^2\mathfrak{g}_{-1}^*$.

1.3. Calculation of the first term of the Cartan prolongation for $m,\ n \geq 2,\ m \neq n$**.** Let $\mathfrak{g}_1'$ be the first term of the Cartan prolongation of the pair $(\mathfrak{g}_{-1},\ \mathfrak{g}_0)$. Let us show that $\mathfrak{g}_1' = \mathfrak{g}_1$. By definition,

$$\mathfrak{g}_1' = (\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^*) \cap (\mathfrak{g}_{-1} \otimes S^2\mathfrak{g}_{-1}^*), \text{ where, as } \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)\text{-modules },$$
$$\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^* \cong [(V \otimes V^*)/\mathbb{C} \oplus (U \otimes U^*)/\mathbb{C} \oplus \mathbb{C}] \otimes (U^* \otimes V).$$

Note that if $g \in \mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^*$, then

$$g \in \mathfrak{g}_{-1} \otimes S^2\mathfrak{g}_{-1}^* \text{ if and only if } g(g_1)(g_2) = -g(g_2)(g_1) \text{ for any } g_1, g_2 \in \mathfrak{g}_{-1},$$

since $\mathfrak{g}_{-1}$ is purely odd.

Lemma. *The $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-module $\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^*$ is the direct sum of irreducible submodules whose highest weights and highest vectors are listed in Table 8.*

Proof. The proof of the Lemma consists of: *a)* a verification of the fact that vectors v_λ from Table 8 are indeed highest with respect to $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$, i.e. $A_{i,j}v_\lambda = D_{i,j}v_\lambda = 0$ for $i < j$;

b) a calculation of dimension of $\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^*$ and of dimensions of the irreducible submodules of $\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^*$ by the formula from the Appendix.

Let us show that if

$$\lambda = 2\varepsilon_1 - \varepsilon_m - \delta_n,\ \varepsilon_1 + \delta_1 - 2\delta_n,\ \varepsilon_1 + \varepsilon_2 - \varepsilon_m - \delta_n \text{ (if } m \geq 3),$$

or

$$\lambda = \varepsilon_1 + \delta_1 - \delta_{n-1} - \delta_n \text{ (if } n \geq 3),$$

then $v_\lambda \notin \mathfrak{g}_1'$. For this it suffices to indicate $g_1, g_2 \in \mathfrak{g}_{-1}$ such that

$$v_\lambda(g_1)(g_2) \neq -v_\lambda(g_2)(g_1) \tag{1.3.1}$$

or, perhaps, there exists just one $g \in \mathfrak{g}_{-1}$ such that

$$v_\lambda(g)(g) \neq 0. \tag{1.3.2}$$

Let $\lambda = 2\varepsilon_1 - \varepsilon_m - \delta_n$. Then

$$v_\lambda(f_n \otimes \tilde{e}_1)(f_n \otimes \tilde{e}_1) = A_{1,m}(f_n \otimes \tilde{e}_1) = -f_n \otimes \tilde{e}_m \neq 0.$$

If $\lambda = \varepsilon_1 + \delta_1 - 2\delta_n$, then

$$v_\lambda(f_n \otimes \tilde{e}_1)(f_n \otimes \tilde{e}_1) = D_{1,n}(f_n \otimes \tilde{e}_1) = f_1 \otimes \tilde{e}_1 \neq 0.$$

If $\lambda = \varepsilon_1 + \varepsilon_2 - \varepsilon_m - \delta_n$ (for $m \geq 3$), then

$$v_\lambda(f_n \otimes \tilde{e}_2)(f_{n-1} \otimes \tilde{e}_1) = A_{1,m}(f_{n-1} \otimes \tilde{e}_1) = -f_{n-1} \otimes \tilde{e}_m,$$
$$\text{but } v_\lambda(f_{n-1} \otimes \tilde{e}_1)(f_n \otimes \tilde{e}_2) = 0.$$

Finally, if $\lambda = \varepsilon_1 + \delta_1 - \delta_{n-1} - \delta_n$ (for $n \geq 3$), then

$$v_\lambda(f_n \otimes \tilde{e}_1)(f_{n-1} \otimes \tilde{e}_2) = D_{1,n-1}(f_{n-1} \otimes \tilde{e}_2) = f_1 \otimes \tilde{e}_2,$$
$$\text{but } v_\lambda(f_{n-1} \otimes \tilde{e}_2)(f_n \otimes \tilde{e}_1) = 0.$$

Now, let us show that if $\lambda = \varepsilon_1 - \delta_n$, then $\mathfrak{g}_1'$ contains precisely one irreducible $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-module with highest weight λ. Notice that by (1.2.1) $\mathfrak{g}_1'$ contains at least one such module. Let

$$v_\lambda = k_1 v_\lambda^1 + k_2 v_\lambda^2 + k_3 v_\lambda^3, \text{ where } k_1, k_2, k_3 \in \mathbb{C},$$

be a linear combination of highest vectors of weight λ. Then the condition

$$v_\lambda(f_n \otimes \tilde{e}_2)(f_{n-1} \otimes \tilde{e}_1) = -v_\lambda(f_{n-1} \otimes \tilde{e}_1)(f_n \otimes \tilde{e}_2)$$

implies

$$mk_1 = nk_2, \tag{1.3.3}$$

whereas the condition

$$v_\lambda(f_n \otimes \tilde{e}_1)(f_n \otimes \tilde{e}_1) = 0$$

implies

$$k_1(m-1) + k_2(1-n) + k_3(m-n) = 0.$$

Hence,

$$k_2 = mk_1/n \text{ and } k_3 = -k_1/n. \tag{1.3.4}$$

Thus, $\mathfrak{g}_1' = V_{\varepsilon_1 - \delta_n}$ and $\mathfrak{g}_1' = \mathfrak{g}_1$. □

1.4. Calculation of the second term of the Cartan prolongation for $m,\ n \geq 2,\ m \neq n$**.** Let $\mathfrak{g}_2$ be the second term of the Cartan prolongation of $(\mathfrak{g}_{-1},\ \mathfrak{g}_0)$.

Let us show that $\mathfrak{g}_2 = 0$. Indeed, by definition, $\mathfrak{g}_2 = (\mathfrak{g}_1 \otimes \mathfrak{g}_{-1}^*) \cap (\mathfrak{g}_0 \otimes S^2\mathfrak{g}_{-1}^*)$. Notice that, as $\mathfrak{g}_0$-module,

$$\begin{gathered}\mathfrak{g}_1 \otimes \mathfrak{g}_{-1}^* \cong (U^* \otimes V) \otimes (U^* \otimes V) = \\ S^2U^* \otimes S^2V \oplus \Lambda^2U^* \otimes \Lambda^2V \oplus \Lambda^2U^* \otimes S^2V \oplus S^2U^* \otimes \Lambda^2V.\end{gathered}$$

This decomposition and Table 5 of [OV] imply the following

LEMMA. *The $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-module $(U^* \otimes V) \otimes (U^* \otimes V)$ is the direct sum of irreducible submodules whose highest weights and the corresponding highest vectors are listed in Table 9.*

Let us show that $v_\lambda \notin \mathfrak{g}_2$, where v_λ is any of the highest vectors listed in Table 9. Let us indicate $g_1, g_2 \in \mathfrak{g}_{-1}$ for which either (1.3.1) or (1.3.2) holds.

Let $\lambda = 2\varepsilon_1 - 2\delta_n$. Then

$$v_\lambda(f_n \otimes \tilde{e}_1)(f_n \otimes \tilde{e}_2) = B_{1,n}(f_n \otimes \tilde{e}_2) = e_1 \otimes \tilde{e}_2, \text{ but } v_\lambda(f_n \otimes \tilde{e}_2)(f_n \otimes \tilde{e}_1) = 0.$$

If $\lambda = \varepsilon_1 + \varepsilon_2 - 2\delta_n$, then

$$v_\lambda(f_n \otimes \tilde{e}_2)(f_n \otimes \tilde{e}_2) = B_{1,n}(f_n \otimes \tilde{e}_2) = e_1 \otimes \tilde{e}_2 \neq 0.$$

If $\lambda = 2\varepsilon_1 - \delta_{n-1} - \delta_n$, then

$$v_\lambda(f_{n-1} \otimes \tilde{e}_1)(f_n \otimes \tilde{e}_2) = -B_{1,n}(f_n \otimes \tilde{e}_2) = -e_1 \otimes \tilde{e}_2, \text{ but } v_\lambda(f_n \otimes \tilde{e}_2)(f_{n-1} \otimes \tilde{e}_1) = 0.$$

Let $\lambda = \varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n$. Then if $n > 2$, we have

$$v_\lambda(f_{n-1} \otimes \tilde{e}_2)(f_1 \otimes \tilde{e}_1) = B_{1,n}(f_1 \otimes \tilde{e}_1) = f_1 \otimes \tilde{f}_n,$$

but $v_\lambda(f_1 \otimes \tilde{e}_1)(f_{n-1} \otimes \tilde{e}_2) = 0$. If $m > 2$, then

$$v_\lambda(f_{n-1} \otimes \tilde{e}_2)(f_n \otimes \tilde{e}_m) = B_{1,n}(f_n \otimes \tilde{e}_m) = e_1 \otimes \tilde{e}_m,$$

but $v_\lambda(f_n \otimes \tilde{e}_m)(f_{n-1} \otimes \tilde{e}_2) = 0$. Therefore, $\mathfrak{g}_2 = 0$ and $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0) = \mathfrak{g}$. Note that by (1.3.4) we have $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0}) = \mathfrak{g}_{-1} \oplus \hat{\mathfrak{g}_0}$. This proves part b) of Theorem 1.1.

1.5. m = n. Let $m = n = 2$ Since $\hat{\mathfrak{g}_0} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) = \mathfrak{o}(4)$ and $\mathfrak{g}_{-1}$ is the standard $\mathfrak{o}(4)$-module (considered as purely odd superspace), then $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0}) = \mathfrak{h}(0|4)$.

Let $m = n > 2$ and $\mathfrak{g}_1'$ be the first term of the Cartan prolongation of the pair $(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0})$. Let us show that $\mathfrak{g}_1' = \mathfrak{g}_1$. Indeed, by (1.2.2) $\mathfrak{g}_1 \subset \mathfrak{g}_1'$. By sec.1.3 and Table 8 we see that the only highest weights of $\mathfrak{g}_1'$ are all equal to $\varepsilon_1 - \delta_n$. Then formula (1.3.3) implies that the highest vector of such weight in $\mathfrak{g}_1'$ is precisely one and therefore, $\mathfrak{g}_1' = \mathfrak{g}_1$. By sec.1.4 the second term of the Cartan prolongation of the pair $(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0})$ is zero. Hence, for $m = n > 2$ we have $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0}) = \mathfrak{psl}(n|n)$.

Let $m = n > 1$ and let $\mathfrak{g}_k$ be the k-th term of the Cartan prolongation of the pair $(\mathfrak{g}_{-1},\ \mathfrak{g}_0)$. Recall that $\mathfrak{g}_k = (\mathfrak{g}_0 \otimes S^k\mathfrak{g}_{-1}^*) \cap (\mathfrak{g}_{-1} \otimes S^{k+1}\mathfrak{g}_{-1}^*)$ $(k \geq 1)$. Observe that

$$\mathfrak{g}_0 \otimes S^k\mathfrak{g}_{-1}^* = (\hat{\mathfrak{g}_0} \oplus \langle z\rangle) \otimes S^k\mathfrak{g}_{-1}^*, \text{ where } z = 1_{2n} \text{ is the center of } \mathfrak{sl}(n|n).$$

Note that

$$\langle z\rangle \otimes S^k\mathfrak{g}_{-1}^* \subset \mathfrak{g}_{-1} \otimes S^{k+1}\mathfrak{g}_{-1}^*.$$

Therefore,

$$\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0) = S^*(\mathfrak{g}_{-1}^*) \Subset \mathfrak{g}_*(\mathfrak{g}_{-1}, \hat{\mathfrak{g}_0}).$$

□

Structure functions of Lie superalgebras $\mathfrak{vect}(0|n)$ and $\mathfrak{svect}(0|n)$

Recall that U is the purely odd standard $\mathfrak{gl}(n)$-module.

2.1. THEOREM. *If $m = 1, n > 1$, then*

a) $H^{k,2}_{\mathfrak{g}_0} = 0$ for any $k > 0$;

b) $H^{k,2}_{\hat{\mathfrak{g}_0}} = \Pi^n(\mathbb{C})\delta_{kn}$.

2.2. Proof of part a). Since $\mathfrak{g}_*(\mathfrak{g}_{-1}, \mathfrak{g}_0) = \mathfrak{vect}(0|n) = \sum_{i=-1}^{n-1} \mathfrak{g}_i$, where $\mathfrak{g}_0 = \mathfrak{gl}(n)$, and the $\mathfrak{gl}(n)$-module $\mathfrak{g}_i$ is isomorphic to $U \otimes S^{i+1}U^*$, then for $k \geq n+2$ we have $H^{k,2}_{\mathfrak{g}_0} = 0$, and for $k \leq n+1$ there exist the following Spencer cochain sequences:

$$\begin{array}{ll}
\mathfrak{gl}(n) \otimes U^* \xrightarrow{\partial^{2,1}_{\mathfrak{gl}(n)}} U \otimes \Lambda^2 U^* \xrightarrow{\partial^{1,2}_{\mathfrak{gl}(n)}} 0 & (k = 1),\\
C^{k+1,1}_{\mathfrak{gl}(n)} \xrightarrow{\partial^{k+1,1}_{\mathfrak{gl}(n)}} C^{k,2}_{\mathfrak{gl}(n)} \xrightarrow{\partial^{k,2}_{\mathfrak{gl}(n)}} C^{k-1,3}_{\mathfrak{gl}(n)} & (2 \leq k \leq n),\\
0 \xrightarrow{\partial^{n+2,1}_{\mathfrak{gl}(n)}} C^{n+1,2}_{\mathfrak{gl}(n)} \xrightarrow{\partial^{n+1,2}_{\mathfrak{gl}(n)}} C^{n,3}_{\mathfrak{gl}(n)} & (k = n+1),
\end{array}$$

where

$$\begin{aligned}
C^{k+1,1}_{\mathfrak{gl}(n)} &= \mathfrak{g}_{k-1} \otimes \mathfrak{g}_{-1}^* \cong (U \otimes S^kU^*) \otimes U^*,\\
C^{k,2}_{\mathfrak{gl}(n)} &= \mathfrak{g}_{k-2} \otimes \Lambda^2\mathfrak{g}_{-1}^* \cong (U \otimes S^{k-1}U^*) \otimes \Lambda^2U^*,\\
C^{k-1,3}_{\mathfrak{gl}(n)} &= \mathfrak{g}_{k-3} \otimes \Lambda^3\mathfrak{g}_{-1}^* \cong (U \otimes S^{k-2}U^*) \otimes \Lambda^3U^*.
\end{aligned}$$

Recall (see sec.3.1 of Chapter 1) that if $\mathfrak{g}_{-1}$ is a faithful module over a Lie superalgebra $\mathfrak{g}_0$ and

$$\mathfrak{g}_{k-1} \otimes \mathfrak{g}_{-1}^* \xrightarrow{\partial^{k+1,1}_{\mathfrak{g}_0}} \mathfrak{g}_{k-2} \otimes \Lambda^2\mathfrak{g}_{-1}^* \xrightarrow{\partial^{k,2}_{\mathfrak{g}_0}} \mathfrak{g}_{k-3} \otimes \Lambda^3\mathfrak{g}_{-1}^* \qquad (k \geq 1)$$

is the Spencer cochain sequence , which corresponds to the pair $(\mathfrak{g}_{-1},\ \mathfrak{g}_0)$, then

$$\mathrm{Im}\partial^{k+1,1}_{\mathfrak{g}_0} \cong (\mathfrak{g}_{k-1} \otimes \mathfrak{g}_{-1}^*)/\mathfrak{g}_k. \tag{2.2.1}$$

Let us show that $H^{1,2}_{\mathfrak{g}_0} = 0$. Indeed, by (2.2.1)

$$\mathrm{Im}\partial^{2,1}_{\mathfrak{g}_0} \cong (\mathfrak{gl}(n) \otimes U^*)/(U \otimes S^2U^*) \cong U \otimes \Lambda^2U^* = \mathrm{Ker}\partial^{1,2}_{\mathfrak{g}_0}.$$

We will prove that for $2 \leq k \leq n+1$ we have $H^{k,2}_{\mathfrak{gl}(n)} = 0$, using the following

LEMMA. *As $\mathfrak{gl}(n)$-modules $C^{k,2}_{\mathfrak{gl}(n)}$, $C^{k+1,1}_{\mathfrak{gl}(n)}$, and $\mathfrak{g}_k$, where $2 \leq k \leq n+1$, are the direct sums of the irreducible submodules whose highest weights and highest vectors are listed in Tables 10, 11, and 12, respectively. [r, s, and t denote the cyclic permutations of $(n-k,\dots,n)$, $(n-k+1,\dots,n)$, and $(n-k+2,\dots,n)$, respectively.]*

PROOF. The proof of the Lemma consists of

a) a verification of the fact that vectors $v_{\lambda_l}, v_{\beta_l}, v_{\gamma_l}$ from the Tables 10, 11, and 12 are indeed highest with respect to $\mathfrak{gl}(n)$, i.e., $D_{i,j}v_{\lambda_l} = D_{i,j}v_{\beta_l} = D_{i,j}v_{\gamma_l} = 0$ for $i < j$;

b) a calculation of dimensions of given modules and dimensions of their irreducible submodules by the formula from the Appendix.

Notice that $D_{i,j}f_p = \delta_{jp}f_i,\ D_{i,j}\tilde{f}_p = -\delta_{ip}\tilde{f}_j$. According to Table 10,

$$\lambda_1 = \delta_1 - \delta_{n-k+2} - \ldots - \delta_{n-1} - 3\delta_n,\ v_{\lambda_1} = (f_1 \otimes \tilde{f}_{n-k+2} \wedge \tilde{f}_{n-k+3} \wedge \ldots \wedge \tilde{f}_n) \otimes \tilde{f}_n^{\,2}.$$

Then for $i < n-k+2$ $D_{i,j}v_{\lambda_1} = 0$. For $j > i \geq n-k+2$

$$D_{i,j}(v_{\lambda_1}) = D_{i,j}(f_1 \otimes \tilde{f}_{n-k+2} \wedge \ldots \wedge \tilde{f}_i \wedge \ldots \wedge \tilde{f}_j \wedge \ldots \wedge \tilde{f}_n) \otimes \tilde{f}_n^{\,2} =$$
$$(f_1 \otimes \tilde{f}_{n-k+2} \wedge \ldots \wedge (-\tilde{f}_j) \wedge \ldots \wedge \tilde{f}_j \wedge \ldots \wedge \tilde{f}_n) \otimes \tilde{f}_n^{\,2} = 0.$$

Thus, v_{λ_1} is highest vector. The proof of the fact that the other vectors from Tables 10, 11, and 12 are highest with respect to $\mathfrak{gl}(n)$ is similar.

Using the formula from the Appendix we find the dimensions of the $\mathfrak{gl}(n)$-modules given in Table 10:

if $2 \leq k \leq n$, then
$$\dim V_{\lambda_1} = \frac{n(n+3)(n+1)!}{2(n-k+2)(k+1)(k-2)!(n-k)!},$$
if $3 \leq k = n+1$, then
$$\dim V_{\lambda_1} = \frac{(n-1)n(n+2)}{2},$$
if $2 \leq k \leq n+1$, then
$$\dim V_{\lambda_2} = \frac{(n+1)!}{(k-2)!((n-k+1)!k},$$
if $k = 2 \leq n$, then
$$\dim V_{\lambda_3} = \dim V_{\lambda_2},$$
if $3 \leq k \leq n$, then
$$\dim V_{\lambda_3} = \frac{(n+2)!}{2(k-3)!(n-k+2)!k},$$
if $3 \leq k = n$, then
$$\dim V_{\lambda_4} = n^2 - 1,$$
if $2 \leq k \leq n-1$, then
$$\dim V_{\lambda_4} = \frac{n(n+2)n!}{(n-k+1)(k-1)!(n-k-1)!(k+1)}, \tag{2.2.2}$$
if $2 \leq k \leq n$, then
$$\dim V_{\lambda_5} = \frac{n!}{k!(n-k)!}, \tag{2.2.3}$$
if $3 \leq k \leq n-1$, then
$$\dim V_{\lambda_6} = \dim V_{\lambda_2}.$$

Therefore, if $2 = k = n$, then

$$\dim V_{\lambda_1} + 2dimV_{\lambda_2} + dimV_{\lambda_5} = \frac{n^3(n+1)}{2} = \dim(U \otimes U^*) \otimes \Lambda^2 U^*,$$

if $2 = k \leq n-1$, then

$$\sum_{l=1}^{5} \dim V_{\lambda_l} = \frac{n^3(n+1)}{2} = \dim(U \otimes U^*) \otimes \Lambda^2 U^*,$$

if $3 \leq k = n+1$, then

$$\sum_{l=1}^{2} \dim V_{\lambda_l} = \frac{n^2(n+1)}{2} = \dim U \otimes \Lambda^2 U^*,$$

if $3 \leq k = n$, then

$$\sum_{l=1}^{5} \dim V_{\lambda_l} = \frac{n^3(n+1)}{2} = \dim(U \otimes S^{n-1}U^*) \otimes \Lambda^2 U^*,$$

if $3 \leq k \leq n-1$, then

$$\sum_{l=1}^{6} \dim V_{\lambda_l} = \frac{n^2(n+1)!}{2(n-k+1)!(k-1)!} = \dim(U \otimes S^{k-1}U^*) \otimes \Lambda^2 U^*.$$

In order to find the dimensions of the $\mathfrak{gl}(n)$-modules given in Table 11, note that if $k \geq 2$, then $\beta_1 = \lambda_4, \beta_2 = \beta_4 = \lambda_5, \beta_3 = \lambda_2$. Using the formula from the Appendix we get

$$\dim V_{\beta_5} = \frac{(n+1)!}{(n-k)(k+1)!(n-k-2)!} \quad \text{for } 2 \leq k \leq n-2. \tag{2.2.4}$$

Therefore, if $2 \leq k = n$, then

$$\sum_{l=1}^{2} \dim V_{\beta_l} = n^2 = \dim U \otimes U^*,$$

if $2 \leq k = n-1$, then

$$\sum_{l=1}^{4} \dim V_{\beta_l} = n^3 = \dim(U \otimes S^{n-1}U^*) \otimes U^*,$$

if $2 \leq k \leq n-2$, then

$$\sum_{l=1}^{5} \dim V_{\beta_l} = \frac{n^2 n!}{(n-k)!k!} = \dim(U \otimes S^k U^*) \otimes U^*.$$

Finally, in order to find the dimensions of the $\mathfrak{gl}(n)$-modules given in Table 12, note that if $2 \leq k \leq n-2$, then $\gamma_1 = \beta_5$, and $\gamma_2 = \lambda_5$. Therefore, if $2 \leq k \leq n-2$, then

$$\sum_{l=1}^{2} \dim V_{\gamma_l} = \frac{nn!}{(n-k-1)!(k+1)!} = \dim(U \otimes S^{k+1}U^*) = \dim \mathfrak{g}_k,$$

and if $k = n-1$, then $\dim V_{\gamma_1} = n = \dim U = \dim \mathfrak{g}_k$. This proves the Lemma. $\square$

Let $k \geq 2$ and $\lambda = \lambda_1,\ \lambda_2,\ \lambda_3$. Then

$$v_\lambda \notin \mathrm{Ker}\partial^{k,2}_{\mathfrak{gl}(n)}. \tag{2.2.5}$$

Indeed,

$$\partial^{k,2}_{\mathfrak{gl}(n)} v_\lambda(f_n, f_n, f_n) = -3v_\lambda(f_n, f_n)(f_n),$$

and according to Table 10, $v_\lambda(f_n, f_n)(f_n) \neq 0$. Note that if $2 \leq k = n$, then

$$\mathrm{Im}\partial^{k+1,1}_{\mathfrak{gl}(n)} = V_{\lambda_2} \oplus V_{\lambda_5}. \tag{2.2.6}$$

In fact, according to Table 11,

$$C^{k+1,1}_{\mathfrak{gl}(n)} = V_{\lambda_2} \oplus V_{\lambda_5},$$

and we get (2.2.6) by (2.2.1), since $\mathfrak{g}_k = 0$. Note that if $2 \leq k \leq n-1$, then

$$\mathrm{Im}\partial^{k+1,1}_{\mathfrak{gl}(n)} = V_{\lambda_2} \oplus V_{\lambda_4} \oplus V_{\lambda_5}. \tag{2.2.7}$$

Indeed, according to Table 11, if $2 \leq k = n-1$, then

$$C^{k+1,1}_{\mathfrak{gl}(n)} = V_{\lambda_2} \oplus V_{\lambda_4} \oplus 2V_{\lambda_5}, \tag{2.2.8}$$

and if $2 \leq k \leq n-2$, then

$$C^{k+1,1}_{\mathfrak{gl}(n)} = V_{\lambda_2} \oplus V_{\lambda_4} \oplus 2V_{\lambda_5} \oplus V_{\beta_5}. \tag{2.2.9}$$

Since by Table 12

$$\mathfrak{g}_k = V_{\lambda_5} \text{ for } 2 \leq k = n-1,$$
$$\mathfrak{g}_k = V_{\lambda_5} \oplus V_{\beta_5} \text{ for } 2 \leq k \leq n-2,$$

we get (2.2.7) by (2.2.1).

We will show now that for $2 \leq k \leq n+1$

$$\mathrm{Ker}\partial^{k,2}_{\mathfrak{gl}(n)} = \mathrm{Im}\partial^{k+1,1}_{\mathfrak{gl}(n)}. \tag{2.2.10}$$

Let

$$\mathrm{Ker}\partial^{k,2}_{\mathfrak{gl}(n)} = \oplus_\lambda k_\lambda V_\lambda. \tag{2.2.11}$$

Let $k = 2$. According to Table 10, if $n = 2$, then

$$C^{k,2}_{\mathfrak{gl}(n)} = V_{\lambda_1} \oplus 2V_{\lambda_2} \oplus V_{\lambda_5}, \tag{2.2.12}$$

and if $n \geq 3$, then

$$C^{k,2}_{\mathfrak{gl}(n)} = V_{\lambda_1} \oplus 2V_{\lambda_2} \oplus V_{\lambda_4} \oplus V_{\lambda_5}.$$

Therefore, in (2.2.11) $k_{\lambda_5} \leq 1$, and by (2.2.5) $k_{\lambda_1} = 0, k_{\lambda_2} \leq 1$. Note that if $n = 2$, then $k_{\lambda_4} = 0$, and if $n \geq 3$, then $k_{\lambda_4} \leq 1$. Thus, by (2.2.6) and (2.2.7) we get (2.2.10).

Let $k \geq 3$. Then according to Table 10, if $k = n+1$, then

$$C^{k,2}_{\mathfrak{gl}(n)} = V_{\lambda_1} \oplus V_{\lambda_2}.$$

Hence, by (2.2.5) $\mathrm{Ker}\partial^{k,2}_{\mathfrak{gl}(n)} = 0$. If $k = n$, then

$$C^{k,2}_{\mathfrak{gl}(n)} = V_{\lambda_1} \oplus 2V_{\lambda_2} \oplus V_{\lambda_3} \oplus V_{\lambda_5}. \tag{2.2.13}$$

Therefore, in (2.2.11) $k_{\lambda_5} \leq 1$ and by (2.2.5) $k_{\lambda_1} = k_{\lambda_3} = 0, k_{\lambda_2} \leq 1$. So from (2.2.6) we get (2.2.10).

Finally, if $k \leq n-1$, then

$$C^{k,2}_{\mathfrak{gl}(n)} = V_{\lambda_1} \oplus 2V_{\lambda_2} \oplus V_{\lambda_3} \oplus V_{\lambda_4} \oplus V_{\lambda_5}.$$

Therefore, in (2.2.11) $k_{\lambda_4} \leq 1,\ k_{\lambda_5} \leq 1$ and by (2.2.5) $k_{\lambda_1} = k_{\lambda_3} = 0, k_{\lambda_2} \leq 1$. Thus, by (2.2.7) we get (2.2.10). This proves part a) of Theorem 2.1.

2.3. Proof of part b) of Theorem 2.1. Note that $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}}_0) = \mathfrak{svect}(0|n) = \oplus_{i=-1}^{n-2}\mathfrak{g}_i$, where $\hat{\mathfrak{g}}_0 = \mathfrak{sl}(n)$, and the $\mathfrak{sl}(n)$-module $\mathfrak{g}_i$ is isomorphic to U if $i=-1$ and to $V_{\delta_1-\delta_{n-k}-\delta_{n-k+1}-\ldots-\delta_n}$ if $0\leq k\leq n-2$.

Hence, for $k\geq n+1$ we have $H^{k,2}_{\mathfrak{sl}(n)}=0$ and for $1\leq k\leq n$ there exist the following Spencer cochain sequences:

$$\begin{array}{ll}
\mathfrak{sl}(n)\otimes U^* \xrightarrow{\partial^{2,1}_{\mathfrak{sl}(n)}} U\otimes\Lambda^2U^* \xrightarrow{\partial^{1,2}_{\mathfrak{sl}(n)}} 0 & (k=1),\\
C^{k+1,1}_{\mathfrak{sl}(n)} \xrightarrow{\partial^{k+1,1}_{\mathfrak{sl}(n)}} C^{k,2}_{\mathfrak{sl}(n)} \xrightarrow{\partial^{k,2}_{\mathfrak{sl}(n)}} C^{k-1,3}_{\mathfrak{sl}(n)} & (2\leq k\leq n-1),\\
0 \xrightarrow{\partial^{n+1,1}_{\mathfrak{sl}(n)}} C^{n,2}_{\mathfrak{sl}(n)} \xrightarrow{\partial^{n,2}_{\mathfrak{sl}(n)}} C^{n-1,3}_{\mathfrak{sl}(n)} & (k=n).
\end{array}$$

First, we will show that $H^{1,2}_{\mathfrak{sl}(n)}=0$. In fact, since $\mathfrak{g}_1\cong V_{\delta_1-\delta_{n-1}-\delta_n}$, then by (2.2.1)

$$\mathrm{Im}\partial^{2,1}_{\mathfrak{sl}(n)} \cong (\mathfrak{sl}(n)\otimes U^*)/V_{\delta_1-\delta_{n-1}-\delta_n}.$$

Since $\dim\mathfrak{sl}(n)\otimes U^* = (n^2-1)n$ and by the formula from the Appendix

$$\dim V_{\delta_1-\delta_{n-1}-\delta_n} = \frac{n(n+1)(n-2)}{2},$$

then

$$\begin{aligned}
&\dim\mathrm{Im}\partial^{2,1}_{\mathfrak{sl}(n)} = (n^2-1)n - \frac{n(n+1)(n-2)}{2} = \frac{n^3+n^2}{2} =\\
&\dim U\otimes\Lambda^2U^* = \dim\mathrm{Ker}\partial^{1,2}_{\mathfrak{sl}(n)}.
\end{aligned}$$

Next, we will prove that for $2\leq k\leq n-1$ we have $H^{k,2}_{\mathfrak{sl}(n)}=0$.

Lemma. *If $2\leq k=n-1$, then*

$$C^{k+1,1}_{\mathfrak{sl}(n)} = V_{\lambda_4}\oplus V_{\lambda_5}. \tag{2.3.1}$$

If $2\leq k\leq n-2$, then

$$C^{k+1,1}_{\mathfrak{sl}(n)} = V_{\lambda_4}\oplus V_{\lambda_5}\oplus V_{\beta_5}. \tag{2.3.2}$$

Proof. Let $c\in C^{k+1,1}_{\mathfrak{gl}(n)}$. Then $c\in C^{k+1,1}_{\mathfrak{sl}(n)}$ if and only if

$$c\in(\mathfrak{sl}(n)\otimes S^{k-1}U^*)\otimes U^*. \tag{2.3.3}$$

Let

$$C^{k+1,1}_{\mathfrak{sl}(n)} = \oplus_\beta k_\beta V_\beta. \tag{2.3.4}$$

Using the decomposition of $C^{k+1,1}_{\mathfrak{gl}(n)}$ into direct sum of irreducible $\mathfrak{gl}(n)$-modules given in (2.2.8) and (2.2.9), we check condition (2.3.3) for the corresponding highest vectors: let $\beta=\lambda_4=\beta_1$, then up to a complex constant,

$$v_{\beta_1}(f_n)(f_{j_1}\wedge f_{j_2}\wedge\ldots\wedge f_{j_{k-1}}) = f_1\otimes\tilde{f}_{j_k},\ \text{where } 2\leq j_k\leq n.$$

Since $f_1\otimes\tilde{f}_{j_k}\in\mathfrak{sl}(n)$, then in (2.3.4) $k_{\lambda_4}=1$. Let $\beta=\lambda_2=\beta_3$, then up to a nonzero constant,

$$v_{\beta_3}(f_n)(f_{n-k+2}\wedge f_{n-k+3}\wedge\ldots\wedge f_n) = \sum_{j=1}^n f_j\otimes\tilde{f}_j.$$

Since $\sum_{j=1}^n f_j\otimes\tilde{f}_j\notin\mathfrak{sl}(n)$, then $k_{\lambda_2}=0$. Let $\beta=\beta_5$. Then up to a constant,

$$v_{\beta_5}(f_{j_1})(f_{j_2}\wedge\ldots\wedge f_{j_k}) = f_1\otimes\tilde{f}_{j_{k+1}},\ \text{where } 2\leq j_{k+1}\leq n.$$

Since $f_1 \otimes \tilde{f}_{j_{k+1}} \in \mathfrak{sl}(n)$, then $k_{\beta_5} = 1$. Let us show that $k_{\lambda_5} = 1$. Indeed, by the formula from the Appendix

$$\dim V_{\delta_1 - \delta n - k + 1 - \ldots - \delta_n} = \frac{(n+1)!}{(n-k+1)k!(n-k-1)!}.$$

Therefore,

$$\dim C^{k+1,1}_{\mathfrak{sl}(n)} = \dim(V_{\delta_1 - \delta n - k + 1 - \ldots - \delta_n} \otimes U^*) = \frac{n(n+1)!}{(n-k+1)k!(n-k-1)!}.$$

Then by (2.2.2) and (2.2.3) if $2 \le k = n-1$, then

$$\dim C^{k+1,1}_{\mathfrak{sl}(n)} = \dim V_{\lambda_4} + dim V_{\lambda_5},$$

if $2 \le k \le n-2$, then by (2.2.2), (2.2.3), and (2.2.4)

$$\dim C^{k+1,1}_{\mathfrak{sl}(n)} = \dim V_{\lambda_4} + dim V_{\lambda_5} + \dim V_{\beta_5},$$

Thus, $k_{\lambda_5} = 1$. This proves the Lemma. □

2.4. Lemma. $\mathrm{Ker}\partial^{k,2}_{\mathfrak{sl}(n)}$ *doesn't contain irreducible* $\mathfrak{sl}(n)$*-submodules with highest weight* λ_2.

Proof. Let $c \in C^{k,2}_{\mathfrak{gl}(n)}$. Then $c \in C^{k,2}_{\mathfrak{sl}(n)}$ if and only if

$$c \in (\mathfrak{sl}(n) \otimes S^{k-2}U^*) \otimes \lambda^2 U^*.$$

According to Table 10, each highest vector with weight λ_2 in $C^{k,2}_{\mathfrak{gl}(n)}$ is

$$\begin{aligned} &v = av_{\lambda_2} + bv_{\lambda_3}, \text{ if } k = 2 \le n,\\ &v = av_{\lambda_2} + bv_{\lambda_6}, \text{ if } 3 \le k \le n-1, \text{ where } a,\ b \in \mathbb{C}. \end{aligned}$$

Therefore, up to a nonzero constant,

$$v(f_n, f_n)(f_{n-k+2} \wedge \ldots \wedge f_{n-1}) = (-1)^k a f_n \otimes \tilde{f}_n + b \sum_{j=1}^{n} f_j \otimes \tilde{f}_j.$$

Note that

$$(-1)^k a f_n \otimes \tilde{f}_n + b \sum_{j=1}^{n} f_j \otimes \tilde{f}_j \in \mathfrak{sl}(n)$$

if and only if $(-1)^k a + nb = 0$. But in this case $v \notin \mathrm{Ker}^{k,2}_{\mathfrak{sl}(n)}$. In fact,

$$\partial^{k,2}_{\mathfrak{sl}(n)}(f_n, f_n, f_n) = -3v(f_n, f_n)(f_n) \ne 0.$$

This proves Lemma 2.4. □

In order to prove that for $2 \le k \le n-1$

$$\mathrm{Im}\partial^{k+1,1}_{\mathfrak{sl}(n)} = \mathrm{Ker}\partial^{k,2}_{\mathfrak{sl}(n)}, \tag{2.4.1}$$

observe that if $2 \le k = n-1$, then $\mathfrak{g}_k = 0$ and if $2 \le k \le n-2$, then $\mathfrak{g}_k = V_{\beta_5}$. Thus, by (2.2.1), (2.3.1), and (2.3.2)

$$\mathrm{Im}\partial^{k+1,1}_{\mathfrak{sl}(n)} = V_{\lambda_4} \oplus V_{\lambda_5}.$$

Therefore, by (2.2.7), (2.2.10), and Lemma 2.4, we get (2.4.1).

Finally, let $2 \le k = n$. Notice that by (2.2.6) and (2.2.10)

$$\mathrm{Ker}\partial^{n,2}_{\mathfrak{gl}(n)} = V_{\lambda_2} \oplus V_{\lambda_5}, \tag{2.4.2}$$

where V_{λ_5} is a trivial $\mathfrak{sl}(n)$-module. By (2.2.12) and (2.2.13) the multiplicity of λ_5 in $C^{n,2}_{\mathfrak{gl}(n)}$ is 1. Moreover, this trivial submodule is contained in $C^{n,2}_{\mathfrak{sl}(n)}$, because the $\mathfrak{sl}(n)$-module $C^{n,2}_{\mathfrak{sl}(n)}$ is isomorphic to $\Lambda^2 U \otimes \Lambda^2 U^*$, which contains a trivial $\mathfrak{sl}(n)$-submodule (generated by $\sum_{i,j} f_i f_j \otimes \tilde{f}_i \tilde{f}_j$). Thus, $\mathrm{Ker}\partial^{n,2}_{\mathfrak{sl}(n)}$ must contain a trivial submodule. According to (2.4.2) and Lemma 2.4, $\mathrm{Ker}\partial^{n,2}_{\mathfrak{sl}(n)}$ is a trivial $\mathfrak{sl}(n)$-submodule, generated by

$$v_{\lambda_5} = \sum_{j=0}^{n-1} (-1)^{(n-1)j} \sum_{i=1}^{n} f_i \otimes \tilde{f}_{s^j(1)} \wedge \tilde{f}_{s^j(2)} \wedge \ldots \wedge \tilde{f}_{s^j(n-1)} \otimes \tilde{f}_{s^j(n)} \tilde{f}_i,$$

where s is a cyclic permutation of $(1,\ 2, \ldots,\ n)$. This proves Theorem 2.1.

Penrose's tensors

3.1. Theorem. *If $m, n > 1$, then $H^{k,2}_{\mathfrak{g}_0} = 0$ for $k > 2$ and the $\mathfrak{g}_0$-modules $H^{1,2}_{\mathfrak{g}_0}$ and $H^{2,2}_{\mathfrak{g}_0}$ are the direct sums of irreducible submodules whose highest weights are given in Table 13.*

If $m = n$, then $H^{k,2}_{\hat{\mathfrak{g}}_0} = H^{k,2}_{\mathfrak{g}_0}$ for any k and if $m \neq n$, then $H^{1,2}_{\hat{\mathfrak{g}}_0} = H^{1,2}_{\mathfrak{g}_0}$ whereas $H^{2,2}_{\hat{\mathfrak{g}}_0} = H^{2,2}_{\mathfrak{g}_0} \oplus V_{\varepsilon_1+\varepsilon_2-2\delta_n} \oplus V_{2\varepsilon_1-\delta_{n-1}-\delta_n}$ if either $m = 2$ or $n = 2$;

$H^{2,2}_{\hat{\mathfrak{g}}_0} = H^{2,2}_{\mathfrak{g}_0} \oplus V_{\varepsilon_1+\varepsilon_2-2\delta_n} \oplus V_{2\varepsilon_1-\delta_{n-1}-\delta_n} \oplus V_{\varepsilon_1+\varepsilon_2-\delta_{n-1}-\delta_n}$ if $m,\ n > 2$.

3.2. Calculation of $H^{1,2}_{\mathfrak{g}_0}$ and $H^{1,2}_{\hat{\mathfrak{g}}_0}$ for $m,\ n \geq 2,\ m \neq n$. For $k = 1$ the Spencer cochain sequence is of the form

$$\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^* \xrightarrow{\partial^{2,1}_{\mathfrak{g}_0}} \mathfrak{g}_{-1} \otimes \Lambda^2 \mathfrak{g}_{-1}^* \xrightarrow{\partial^{1,2}_{\mathfrak{g}_0}} 0.$$

Observe that

$$\begin{aligned}
&\mathfrak{g}_{-1} \otimes \Lambda^2 \mathfrak{g}_{-1}^* = (U \otimes V^*) \otimes \Lambda^2(U^* \otimes V) \cong \\
&(U \otimes V^*) \otimes (\Lambda^2 U^* \otimes S^2 V \oplus S^2 U^* \otimes \Lambda^2 V) \cong \\
&(\Lambda^2 U^* \otimes U) \otimes (S^2 V \otimes V^*) \oplus (S^2 U^* \otimes U) \otimes (\Lambda^2 V \otimes V^*), \\
&\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^* = (V \otimes V^*/\mathbb{C} \oplus U \otimes U^*/\mathbb{C} \oplus \mathbb{C}) \otimes (U^* \otimes V), \\
&\mathfrak{g}_1 = U^* \otimes V.
\end{aligned}$$

Therefore, as $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-modules,

$$\mathrm{Im}\partial^{2,1}_{\mathfrak{g}_0} \cong (V \otimes V^*/\mathbb{C} \oplus U \otimes U^*/\mathbb{C}) \otimes (U^* \otimes V) \tag{3.2.1}$$

and

$$H^{1,2}_{\mathfrak{g}_0} \cong (\Lambda^2 U^* \otimes U/U^*) \otimes (S^2 V \otimes V^*/V) \oplus (S^2 U^* \otimes U/U^*) \otimes (\Lambda^2 V \otimes V^*/V). \tag{3.2.2}$$

Note that

$$\begin{aligned}
&S^2 V \otimes V^*/V = V_{2\varepsilon_1 - \varepsilon_m}, \\
&\Lambda^2 V \otimes V^*/V = V_{\varepsilon_1+\varepsilon_2-\varepsilon_m} \text{ for } m > 2, \\
&\Lambda^2 V \otimes V^*/V = 0 \text{ for } m = 2.
\end{aligned}$$

Since U is purely odd, we deduce with the help of Table 5 of [OV] that

$$\begin{aligned}
&\Lambda^2 U^* \otimes U/U^* = V_{\delta_1 - 2\delta_n}, \\
&S^2 U^* \otimes U/U^* = V_{\delta_1-\delta_{n-1}-\delta_n} \text{ for } n > 2, \\
&S^2 U^* \otimes U/U^* = 0 \text{ for } n = 2.
\end{aligned}$$

Therefore, we have

$$H^{1,2}_{\mathfrak{g}_0} = V_{2\varepsilon_1-\varepsilon_m+\delta_1-2\delta_n} \text{ if } m = 2,\ n > 2$$

and

$$H^{1,2}_{\mathfrak{g}_0} = V_{2\varepsilon_1-\varepsilon_m+\delta_1-2\delta_n} \oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_m+\delta_1-\delta_{n-1}-\delta_n} \text{ if } m,\ n > 2.$$

By part b) of Theorem 1.1 $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0}) = \mathfrak{g}_{-1} \oplus \hat{\mathfrak{g}_0}$. Therefore, by (2.2.1), we have

$$\mathrm{Im}\partial^{2,1}_{\hat{\mathfrak{g}_0}} = \hat{\mathfrak{g}_0} \otimes \mathfrak{g}^*_{-1} = \mathrm{Im}\partial^{2,1}_{\mathfrak{g}_0}.$$

Hence, $H^{1,2}_{\hat{\mathfrak{g}_0}} = H^{1,2}_{\mathfrak{g}_0}$.

3.3. Calculation of $H^{1,2}_{\hat{\mathfrak{g}_0}}$ for $m = n > 1$. Since by parts c) and d) of Theorem 1.1 the first term of the Cartan prolongation $\mathfrak{g}_*(\mathfrak{g}_{-1}, \hat{\mathfrak{g}_0})$ is $U^* \otimes V$, then by (2.2.1)

$$\mathrm{Im}\partial^{2,1}_{\hat{\mathfrak{g}_0}} = [(V \otimes V^*/\mathbb{C} \oplus U \otimes U^*/\mathbb{C}) \otimes (U^* \otimes V)]/(U^* \otimes V).$$

Therefore, by (3.2.1) and (3.2.2),

$$H^{1,2}_{\hat{\mathfrak{g}_0}} \cong (\Lambda^2 U^* \otimes U/U^*) \otimes (S^2 V \otimes V^*/V) \oplus (S^2 U^* \otimes U/U^*) \otimes (\Lambda^2 V \otimes V^*/V) \oplus (U^* \otimes V).$$

Hence,

$$H^{1,2}_{\hat{\mathfrak{g}_0}} = V_{2\varepsilon_1-\varepsilon_2+\delta_1-2\delta_2} \oplus V_{\varepsilon_1-\delta_2} \text{ for } n = 2$$

and

$$H^{1,2}_{\hat{\mathfrak{g}_0}} = V_{2\varepsilon_1-\varepsilon_n+\delta_1-2\delta_n} \oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_n+\delta_1-\delta_{n-1}-\delta_n} \oplus V_{\varepsilon_1-\delta_n} \text{ for } n > 2.$$

3.4. Calculation of $H^{2,2}_{\mathfrak{g}_0}$ for $m,\ n > 1,\ m \neq n$. For $k = 2$ the Spencer cochain sequence is of the form

$$\mathfrak{g}_1 \otimes \mathfrak{g}^*_{-1} \xrightarrow{\partial^{3,1}_{\mathfrak{g}_0}} \mathfrak{g}_0 \otimes \Lambda^2\mathfrak{g}^*_{-1} \xrightarrow{\partial^{2,2}_{\mathfrak{g}_0}} \mathfrak{g}_{-1} \otimes \Lambda^3\mathfrak{g}^*_{-1}.$$

Observe that

$$\mathfrak{g}_0 \otimes \Lambda^2\mathfrak{g}^*_{-1} = (V \otimes V^*/\mathbb{C} \oplus U \otimes U^*/\mathbb{C} \oplus \mathbb{C}) \otimes (\Lambda^2 U^* \otimes S^2 V \oplus S^2 U^* \otimes \Lambda^2 V),$$
$$\mathfrak{g}_1 \otimes \mathfrak{g}^*_{-1} = (U^* \otimes V) \otimes (U^* \otimes V),$$
$$\mathfrak{g}_2 = 0.$$

LEMMA. *As $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-module, $\mathfrak{g}_0 \otimes \Lambda^2\mathfrak{g}^*_{-1}$ is the direct sum of the irreducible submodules whose highest weights and highest vectors are listed in Table 14. [s and t denote the cyclic permutations of* $(1,\ 2,\ 3)$ *and* $(n-2,\ n-1,\ n)$*, respectively.]*

The proof follows from the formula given in the Appendix.

Let us show that if

$$\lambda = 3\varepsilon_1 - \varepsilon_m - 2\delta_n,\ 2\varepsilon_1 + \varepsilon_2 - \varepsilon_m - 2\delta_n\ (m > 2),\ 2\varepsilon_1 + \varepsilon_2 - \varepsilon_m - \delta_{n-1} - \delta_n,$$
$$2\varepsilon_1 + \delta_1 - 3\delta_n,\ 2\varepsilon_1 + \delta_1 - \delta_{n-1} - 2\delta_n\ (n > 2),\ \text{or } \varepsilon_1 + \varepsilon_2 + \delta_1 - \delta_{n-1} - 2\delta_n,$$

then $v_\lambda \notin \mathrm{Ker}\partial^{2,2}_{\mathfrak{g}_0}$. Recall that if $v \in \mathfrak{g}_0 \otimes \Lambda^2\mathfrak{g}^*_{-1}$, then

$$\partial^{2,2}_{\mathfrak{g}_0} v(g_1, g_2, g_3) = -v(g_1, g_2)g_3 - v(g_1, g_3)g_2 - v(g_2, g_3)g_1 \tag{3.4.1}$$

for any $g_1, g_2, g_3 \in \mathfrak{g}_{-1}$.

Let $\lambda = 3\varepsilon_1 - \varepsilon_m - 2\delta_n$. Then

$$\partial^{2,2}_{\mathfrak{g}_0} v_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1) = -3v_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1)(f_n \otimes \tilde{e}_1) =$$
$$= 3A_{1,m}(f_n \otimes \tilde{e}_1) = -3f_n \otimes \tilde{e}_m \neq 0.$$

Let $\lambda = 2\varepsilon_1 + \varepsilon_2 - \varepsilon_m - 2\delta_n$ $(m > 2)$. Then

$$\partial^{2,2}_{\mathfrak{g}_0} v_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1, f_{n-1} \otimes \tilde{e}_2) = -v_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1)(f_{n-1} \otimes \tilde{e}_2) =$$
$$= -A_{2,m}(f_{n-1} \otimes \tilde{e}_2) = f_{n-1} \otimes \tilde{e}_m \neq 0.$$

Let $\lambda = 2\varepsilon_1 + \varepsilon_2 - \varepsilon_m - \delta_{n-1} - \delta_n$. Then

$$\begin{aligned}&\partial^{2,2}_{\mathfrak{g}_0} v_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1, f_{n-1} \otimes \tilde{e}_2) = -2v_\lambda(f_n \otimes \tilde{e}_1, f_{n-1} \otimes \tilde{e}_2)(f_n \otimes \tilde{e}_1) = \\ &= -A_{1,m}(f_n \otimes \tilde{e}_1) = f_n \otimes \tilde{e}_m \neq 0.\end{aligned}$$

The proof of the fact that $v_\lambda \notin \mathrm{Ker}\partial^{2,2}_{\mathfrak{g}_0}$ for $\lambda = 2\varepsilon_1 + \delta_1 - 3\delta_n,\ 2\varepsilon_1 + \delta_1 - \delta_{n-1} - 2\delta_n\ (n > 2)$, and $\varepsilon_1 + \varepsilon_2 + \delta_1 - \delta_{n-1} - 2\delta_n$ is similar.

Let $\lambda = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_m - \delta_{n-1} - \delta_n$ $(m > 3)$. Let us show that if $n = 2$, then $v_\lambda \in \mathrm{Ker}\partial^{2,2}_{\mathfrak{g}_0}$ and if $n > 2$, then $v_\lambda \notin \mathrm{Ker}\partial^{2,2}_{\mathfrak{g}_0}$. Indeed, if $n = 2$, then for $j = 0,\ 1,\ 2$ we have

$$\begin{aligned}&\partial^{2,2}_{\mathfrak{g}_0} v_\lambda(f_1 \otimes \tilde{e}_{s^j(2)}, f_2 \otimes \tilde{e}_{s^j(3)}, f_1 \otimes \tilde{e}_{s^j(1)}) = \\ &- v_\lambda(f_1 \otimes \tilde{e}_{s^j(2)}, f_2 \otimes \tilde{e}_{s^j(3)})(f_1 \otimes \tilde{e}_{s^j(1)}) \\ &- v_\lambda(f_2 \otimes \tilde{e}_{s^j(3)}, f_1 \otimes \tilde{e}_{s^j(1)})(f_1 \otimes \tilde{e}_{s^j(2)}) = \\ &A_{s^j(1),m}(f_1 \otimes \tilde{e}_{s^j(1)})/2 - A_{s^j(2),m}(f_1 \otimes \tilde{e}_{s^j(2)})/2 = \\ &- f_1 \otimes \tilde{e}_m/2 + f_1 \otimes \tilde{e}_m/2 = 0,\end{aligned}$$

$$\begin{aligned}&\partial^{2,2}_{\mathfrak{g}_0} v_\lambda(f_1 \otimes \tilde{e}_{s^j(2)}, f_2 \otimes \tilde{e}_{s^j(3)}, f_2 \otimes \tilde{e}_{s^j(1)}) = \\ &- v_\lambda(f_1 \otimes \tilde{e}_{s^j(2)}, f_2 \otimes \tilde{e}_{s^j(3)})(f_2 \otimes \tilde{e}_{s^j(1)}) - \\ &- v_\lambda(f_2 \otimes \tilde{e}_{s^j(1)}, f_1 \otimes \tilde{e}_{s^j(2)})(f_2 \otimes \tilde{e}_{s^j(3)}) = \\ &A_{s^j(1),m}(f_2 \otimes \tilde{e}_{s^j(1)})/2 - A_{s^j(3),m}(f_2 \otimes \tilde{e}_{s^j(3)})/2 = \\ &- f_2 \otimes \tilde{e}_m/2 + f_2 \otimes \tilde{e}_m/2 = 0.\end{aligned}$$

Therefore, $v_\lambda \in \mathrm{Ker}\partial^{2,2}_{\mathfrak{g}_0}$. If $n > 2$, then

$$\begin{aligned}&\partial^{2,2}_{\mathfrak{g}_0} v_\lambda(f_{n-1} \otimes \tilde{e}_2, f_n \otimes \tilde{e}_3, f_1 \otimes \tilde{e}_1) = -v_\lambda(f_{n-1} \otimes \tilde{e}_2, f_n \otimes \tilde{e}_3)(f_1 \otimes \tilde{e}_1) = \\ &= A_{1,m}(f_1 \otimes \tilde{e}_1)/2 = -f_1 \otimes \tilde{e}_m/2 \neq 0.\end{aligned}$$

The proof of the fact that if $\lambda = \varepsilon_1 + \varepsilon_2 + \delta_1 - \delta_{n-2} - \delta_{n-1} - \delta_n$ $(n \geq 4)$, then $v_\lambda \in \mathrm{Ker}\partial^{2,2}_{\mathfrak{g}_0}$ for $m = 2$ and $v_\lambda \notin \mathrm{Ker}\partial^{2,2}_{\mathfrak{g}_0}$ for $m > 2$ is similar.

Finally, let us show that if

$$\lambda = 2\varepsilon_1 - 2\delta_n,\ \varepsilon_1 + \varepsilon_2 - 2\delta_n,\ 2\varepsilon_1 - \delta_{n-1} - \delta_n,\ \text{or } \varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n$$

and $v_\lambda \in \mathrm{Ker}\partial^{2,2}_{\mathfrak{g}_0}$, then $v_\lambda \in \mathrm{Im}\partial^{3,1}_{\mathfrak{g}_0}$. Note that since $\mathfrak{g}_2 = 0$, then, as $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-modules,

$$\mathrm{Im}\partial^{3,1}_{\mathfrak{g}_0} \cong \mathfrak{g}_1 \otimes \mathfrak{g}^*_{-1} = (U^* \otimes V) \otimes (U^* \otimes V).$$

Therefore, by Table 9,

$$\mathrm{Im}\partial^{3,1}_{\mathfrak{g}_0} = V_{2\varepsilon_1 - 2\delta_n} \oplus V_{\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n} \oplus V_{\varepsilon_1 + \varepsilon_2 - 2\delta_n} \oplus V_{2\varepsilon_1 - \delta_{n-1} - \delta_n}.$$

Let $\lambda = \varepsilon_1 + \varepsilon_2 - 2\delta_n$. By Table 14 $\mathfrak{g}_0 \otimes \Lambda^2\mathfrak{g}^*_{-1}$ contains two irreducible components with the indicated highest weight, and one of the corresponding highest vectors is v^1_λ. Observe that

$$\begin{aligned}&\partial^{2,2}_{\mathfrak{g}_0} v^1_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1, f_{n-1} \otimes \tilde{e}_2) = -v^1_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1)(f_{n-1} \otimes \tilde{e}_2) = \\ &= -A_{2,1}(f_{n-1} \otimes \tilde{e}_2) = f_{n-1} \otimes \tilde{e}_1 \neq 0.\end{aligned}$$

Therefore, $\mathrm{Ker}\partial^{2,2}_{\mathfrak{g}_0}$ contains precisely one irreducible submodule with highest weight $\varepsilon_1 + \varepsilon_2 - 2\delta_n$ and this submodule belongs to $\mathrm{Im}\partial^{3,1}_{\mathfrak{g}_0}$. Similarly, $\mathfrak{g}_0 \otimes \Lambda^2\mathfrak{g}^*_{-1}$ contains two irreducible submodules with highest weight $2\varepsilon_1 - \delta_{n-1} - \delta_n$, one of which belongs to $\mathrm{Ker}\partial^{2,2}_{\mathfrak{g}_0}$ and, therefore, to $\mathrm{Im}\partial^{3,1}_{\mathfrak{g}_0}$.

Let $\lambda = 2\varepsilon_1 - 2\delta_n$. Then by Table 14 any $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-highest vector of weight λ, which belongs to $\mathfrak{g}_0 \otimes \Lambda^2\mathfrak{g}^*_{-1}$, is

$$v_\lambda = k_1 v^1_\lambda + k_2 v^2_\lambda + k_3 v^3_\lambda, \text{ where } k_1, k_2, k_3 \in \mathbb{C}.$$

If $v_\lambda \in \mathrm{Ker}\partial^{2,2}_{\mathfrak{g}_0}$, then the condition $\partial^{2,2}_{\mathfrak{g}_0} v_\lambda(f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1, f_n \otimes \tilde{e}_1) = 0$ implies

$$k_1(m-1) - k_2(n-1) + k_3(m-n) = 0, \tag{3.4.2}$$

and the condition $\partial^{2,2}_{\mathfrak{g}_0} v_\lambda(f_n \otimes \tilde{e}_2, f_n \otimes \tilde{e}_1, f_1 \otimes \tilde{e}_1) = 0$ implies that

$$k_1 m - k_2 n = 0. \tag{3.4.3}$$

Thus, for $m \neq n$ we have

$$k_2 = mk_1/n, k_3 = -k_1/n. \tag{3.4.4}$$

Therefore, $\mathrm{Ker}\partial^{2,2}_{\mathfrak{g}_0}$ contains precisely one irreducible submodule with highest weight $2\varepsilon_1 - 2\delta_n$ and this submodule belongs to $\mathrm{Im}\partial^{3,1}_{\mathfrak{g}_0}$.

Finally, let $\lambda = \varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n$. Then by Table 14 any highest vector with weight λ, which belongs to $\mathfrak{g}_0 \otimes \Lambda^2\mathfrak{g}^*_{-1}$, is

$$v_\lambda = k_1 v^1_\lambda + k_2 v^2_\lambda + k_3 v^3_\lambda, \text{ where } k_1, k_2, k_3 \in \mathbb{C},$$

and if $m = 2$, then $k_1 = 0$. Note that if $v_\lambda \in \mathrm{Ker}\partial^{2,2}_{\mathfrak{g}_0}$, then

$$\partial^{2,2}_{\mathfrak{g}_0} v_\lambda(f_{n-1} \otimes \tilde{e}_1, f_n \otimes \tilde{e}_2, f_n \otimes \tilde{e}_1) = 0$$

implies that

$$k_1 + k_2 + k_3(n-m) = 0. \tag{3.4.5}$$

Thus, if $m = 2$, then

$$k_2 = (2-n)k_3. \tag{3.4.6}$$

If $m,\ n > 2$, then the condition

$$\partial^{2,2}_{\mathfrak{g}_0} v_\lambda(f_{n-1} \otimes \tilde{e}_m, f_n \otimes \tilde{e}_2, f_1 \otimes \tilde{e}_1) = 0 \tag{3.4.7}$$

implies that $k_1 + k_2 = 0$. Hence,

$$k_2 = -k_1,\ k_3 = 0. \tag{3.4.8}$$

Therefore, $\mathrm{Ker}\partial^{2,2}_{\mathfrak{g}_0}$ contains precisely one highest vector of weight $\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n$ which belongs to $\mathrm{Im}\partial^{3,1}_{\mathfrak{g}_0}$. Thus, we have the description of $H^{2,2}_{\mathfrak{g}_0}$ given in Table 13.

3.5. Calculation of $H^{2,2}_{\hat{\mathfrak{g}_0}}$ for $m,\ n > 1,\ m \neq n$. By part b) of Theorem 1.1

$$\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0}) = \mathfrak{g}_{-1} \oplus \hat{\mathfrak{g}_0}.$$

Therefore, the Spencer cochain sequence for $k = 2$ takes the form

$$0 \xrightarrow{\partial^{3,1}_{\hat{\mathfrak{g}_0}}} \hat{\mathfrak{g}_0} \otimes \Lambda^2\mathfrak{g}^*_{-1} \xrightarrow{\partial^{2,2}_{\hat{\mathfrak{g}_0}}} \mathfrak{g}_{-1} \otimes \Lambda^3\mathfrak{g}^*_{-1}.$$

Note that since $\mathfrak{g}_0 = \hat{\mathfrak{g}_0} \oplus \mathbb{C}$, then

$$\mathfrak{g}_0 \otimes \Lambda^2\mathfrak{g}^*_{-1} = \hat{\mathfrak{g}_0} \otimes \Lambda^2\mathfrak{g}^*_{-1} \oplus V_{2\varepsilon_1 - 2\delta_n} \oplus V_{\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n}. \tag{3.5.1}$$

As we have shown in sec.3.4, if λ is one of the weights from Table 14, then an irreducible module with highest weight λ is contained in the decomposition of $\mathrm{Ker}\partial^{2,2}_{\mathfrak{g}_0}$ into irreducible $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-modules if and only if

$$\begin{aligned}&\lambda = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_m - \delta_{n-1} - \delta_n\ (m > 3),\\ &\varepsilon_1 + \varepsilon_2 + \delta_1 - \delta_{n-2} - \delta_{n-1} - \delta_n\ (n > 3),\\ &\varepsilon_1 + \varepsilon_2 - 2\delta_n,\ 2\varepsilon_1 - \delta_{n-1} - \delta_n,\ 2\varepsilon_1 - 2\delta_n \text{ or } \varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n\end{aligned} \tag{3.5.2}$$

and its multiplicity is 1. Therefore, by (3.5.1), if

$$\begin{aligned}\lambda &= \varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_m-\delta_{n-1}-\delta_n \ (m>3),\\ &\varepsilon_1+\varepsilon_2+\delta_1-\delta_{n-2}-\delta_{n-1}-\delta_n \ (n>3),\\ &\varepsilon_1+\varepsilon_2-2\delta_n, \text{ or } 2\varepsilon_1-\delta_{n-1}-\delta_n,\end{aligned}$$

then the corresponding submodule is contained in $\mathrm{Ker}\partial^{2,2}_{\hat{\mathfrak{g}}_0}$ as well.

Let $\lambda = 2\varepsilon_1-2\delta_n$ and $v_\lambda \in \mathrm{Ker}\partial^{2,2}_{\mathfrak{g}_0}$. Then (3.4.4) where $k_3=0$, implies that $v_\lambda \notin \mathrm{Ker}\partial^{2,2}_{\hat{\mathfrak{g}}_0}$.

Let $\lambda = \varepsilon_1+\varepsilon_2-\delta_{n-1}-\delta_n$, $v_\lambda \in \mathrm{Ker}\partial^{2,2}_{\mathfrak{g}_0}$. Then (3.4.6) implies that $v_\lambda \notin \mathrm{Ker}\partial^{2,2}_{\hat{\mathfrak{g}}_0}$ for $m=2$, and (3.4.8) implies that $v_\lambda \in \mathrm{Ker}\partial^{2,2}_{\hat{\mathfrak{g}}_0}$ for $m,\ n>2$. Thus, we have

$$H^{2,2}_{\hat{\mathfrak{g}}_0} = H^{2,2}_{\mathfrak{g}_0} \oplus V_{\varepsilon_1+\varepsilon_2-2\delta_n} \oplus V_{2\varepsilon_1-\delta_{n-1}-\delta_n} \text{ if either } m=2 \text{ or } n=2$$

and

$$H^{2,2}_{\hat{\mathfrak{g}}_0} = H^{2,2}_{\mathfrak{g}_0} \oplus V_{\varepsilon_1+\varepsilon_2-2\delta_n} \oplus V_{2\varepsilon_1-\delta_{n-1}-\delta_n} \oplus V_{\varepsilon_1+\varepsilon_2-\delta_{n-1}-\delta_n} \text{ if } m,\ n>2.$$

3.6. Calculation of $H^{2,2}_{\hat{\mathfrak{g}}_0}$ for $m=n>1$. By parts c) and d) of Theorem 1.1 the first term of $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}}_0)$ is $U^*\otimes V$ and the second one is $\mathbb{C}$ for $n=2$ and zero for $n>2$. By formula (2.2.1) we have

$$\mathrm{Im}\partial^{3,1}_{\hat{\mathfrak{g}}_0} = (U^*\otimes V)\otimes(U^*\otimes V) \text{ for } n>2 \tag{3.6.1}$$

and

$$\mathrm{Im}\partial^{3,1}_{\hat{\mathfrak{g}}_0} = (U^*\otimes V)\otimes(U^*\otimes V)/\mathbb{C} \text{ for } n=2. \tag{3.6.2}$$

Therefore, by Table 9,

$$\mathrm{Im}\partial^{3,1}_{\mathfrak{g}_0} = V_{2\varepsilon_1-2\delta_n} \oplus V_{2\varepsilon_1-\delta_{n-1}-\delta_n} \oplus V_{\varepsilon_1+\varepsilon_2-2\delta_n} \text{ for } n=2$$

and

$$\mathrm{Im}\partial^{3,1}_{\hat{\mathfrak{g}}_0} = V_{2\varepsilon_1-2\delta_n} \oplus V_{2\varepsilon_1-\delta_{n-1}-\delta_n} \oplus V_{\varepsilon_1+\varepsilon_2-2\delta_n} \oplus V_{\varepsilon_1+\varepsilon_2-\delta_{n-1}-\delta_n} \text{ for } n>2.$$

Therefore, by (3.4.6) and (3.5.2),

$$H^{2,2}_{\hat{\mathfrak{g}}_0} = 0 \text{ for } n=2,\ 3$$

and

$$H^{2,2}_{\hat{\mathfrak{g}}_0} = V_{\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_n-\delta_{n-1}-\delta_n} \oplus V_{\varepsilon_1+\varepsilon_2+\delta_1-\delta_{n-2}-\delta_{n-1}-\delta_n} \text{ for } n>3.$$

3.7. Computation of $H^{3,2}_{\mathfrak{g}_0}$ for $m,\ n>1,\ m\neq n$. For $k=3$ the Spencer cochain sequence is of the form

$$\mathfrak{g}_2\otimes\mathfrak{g}^*_{-1} \xrightarrow{\partial^{4,1}_{\mathfrak{g}_0}} \mathfrak{g}_1\otimes\Lambda^2\mathfrak{g}^*_{-1} \xrightarrow{\partial^{3,2}_{\mathfrak{g}_0}} \mathfrak{g}_0\otimes\Lambda^3\mathfrak{g}^*_{-1}.$$

Observe that

$$\begin{aligned}&\mathfrak{g}_1\otimes\Lambda^2\mathfrak{g}^*_{-1} = (U^*\otimes V)\otimes\Lambda^2(U^*\otimes V)\cong\\ &(\Lambda^2U^*\otimes U^*)\otimes(S^2V\otimes V)\oplus(S^2U^*\otimes U^*)\otimes(\Lambda^2V\otimes V),\\ &\mathfrak{g}_2=0.\end{aligned}$$

By Table 5 from [OV]

$$\begin{aligned}&S^2V\otimes V = V_{3\varepsilon_1}\oplus V_{2\varepsilon_1+\varepsilon_2},\\ &\Lambda^2V\otimes V = V_{2\varepsilon_1+\varepsilon_2}\oplus V_{\varepsilon_1+\varepsilon_2+\varepsilon_3} \text{ for } m>2,\\ &\Lambda^2V\otimes V = V_{2\varepsilon_1+\varepsilon_2} \text{ for } m=2.\end{aligned}$$

Since U is purely odd,

$$\Lambda^2U^*\otimes U^* = V_{-3\delta_n}\oplus V_{-\delta_{n-1}-2\delta_n},$$
$$S^2U^*\otimes U^* = V_{-\delta_{n-1}-2\delta_n}\oplus V_{-\delta_{n-2}-\delta_{n-1}-\delta_n} \text{ for } n>2,$$
$$S^2U^*\otimes U^* = V_{-\delta_{n-1}-2\delta_n} \text{ for } n=2.$$

The above decompositions imply the following

Lemma. *The $\mathfrak{gl}(m)\oplus\mathfrak{gl}(n)$-module $\mathfrak{g}_1\otimes\Lambda^2\mathfrak{g}_{-1}^*$ is the direct sum of irreducible submodules whose highest weights and highest vectors are listed in Table 15. [s and t denote the cyclic permutations of* $(1,\ 2,\ 3)$ *and* $(n-2,\ n-1,\ n)$*, respectively.]*

Let us show that $\mathrm{Ker}\partial^{3,2}_{\mathfrak{g}_0}=0$. Let $\lambda=3\varepsilon_1-3\delta_n$. Then

$$\partial^{3,2}_{\mathfrak{g}_0}v_\lambda(f_n\otimes\tilde e_1,f_n\otimes\tilde e_1,f_n\otimes\tilde e_1)=3B_{1,n}(f_n\otimes\tilde e_1)=3(e_1\otimes\tilde e_1+f_n\otimes\tilde f_n)\neq 0.$$

Let $\lambda=2\varepsilon_1+\varepsilon_2-3\delta_n$. Then

$$\partial^{3,2}_{\mathfrak{g}_0}v_\lambda(f_n\otimes\tilde e_1,f_n\otimes\tilde e_1,f_n\otimes\tilde e_1)=-3B_{2,n}(f_n\otimes\tilde e_1)=-3e_2\otimes\tilde e_1\neq 0.$$

Let $\lambda=3\varepsilon_1-\delta_{n-1}-2\delta_n$. Then

$$\partial^{3,2}_{\mathfrak{g}_0}v_\lambda(f_n\otimes\tilde e_1,f_n\otimes\tilde e_1,f_n\otimes\tilde e_1)=3B_{1,n-1}(f_n\otimes\tilde e_1)=3f_n\otimes\tilde f_{n-1}\neq 0.$$

Let $\lambda=2\varepsilon_1+\varepsilon_2-\delta_{n-1}-2\delta_n$. Since by Table 15 $\mathfrak{g}_1\otimes\Lambda^2\mathfrak{g}_{-1}^*$ contains two irreducible submodules with highest weight λ, then any highest vector of weight λ in $\mathfrak{g}_1\otimes\Lambda^2\mathfrak{g}_{-1}^*$ is of the form

$$v_\lambda=k_1v_\lambda^1+k_2v_\lambda^2,\ \text{where } k_1,k_2\in\mathbb{C}.$$

Let $v_\lambda\in\mathrm{Ker}\partial^{3,2}_{\mathfrak{g}_0}$. If $m>2$, then

$$\partial^{3,2}_{\mathfrak{g}_0}v_\lambda(f_{n-1}\otimes\tilde e_2,f_n\otimes\tilde e_1,f_n\otimes\tilde e_m)=-(1/2)k_2B_{1,n}(f_n\otimes\tilde e_m)=-(1/2)k_2e_1\otimes\tilde e_m=0.$$

Therefore, $k_2=0$. Moreover,

$$\partial^{3,2}_{\mathfrak{g}_0}v_\lambda(f_n\otimes\tilde e_1,f_n\otimes\tilde e_1,f_{n-1}\otimes\tilde e_m)=k_1B_{2,n-1}(f_{n-1}\otimes\tilde e_m)=k_1e_2\otimes\tilde e_m=0.$$

Hence, $k_1=0$. If $n>2$, then

$$\partial^{3,2}_{\mathfrak{g}_0}v_\lambda(f_{n-1}\otimes\tilde e_2,f_n\otimes\tilde e_1,f_1\otimes\tilde e_1)=-(1/2)k_2B_{1,n}(f_1\otimes\tilde e_1)=-(1/2)k_2f_1\otimes\tilde f_n=0.$$

Therefore, $k_2=0$. Moreover,

$$\partial^{3,2}_{\mathfrak{g}_0}v_\lambda(f_n\otimes\tilde e_1,f_n\otimes\tilde e_1,f_1\otimes\tilde e_2)=k_1B_{2,n-1}(f_1\otimes\tilde e_2)=k_1f_1\otimes\tilde f_{n-1}=0.$$

Hence, $k_1=0$.

Let $\lambda=\varepsilon_1+\varepsilon_2+\varepsilon_3-\delta_{n-1}-2\delta_n$. Then

$$\partial^{3,2}_{\mathfrak{g}_0}v_\lambda(f_{n-1}\otimes\tilde e_2,f_n\otimes\tilde e_3,f_n\otimes\tilde e_2)=(1/2)B_{1,n}(f_n\otimes\tilde e_2)=(1/2)e_1\otimes\tilde e_2\neq 0.$$

Let $\lambda=2\varepsilon_1+\varepsilon_2-\delta_{n-2}-\delta_{n-1}-\delta_n$. Then

$$\partial^{3,2}_{\mathfrak{g}_0}v_\lambda(f_{n-1}\otimes\tilde e_2,f_n\otimes\tilde e_1,f_{n-1}\otimes\tilde e_1)=$$
$$(1/2)B_{1,n-2}(f_{n-1}\otimes\tilde e_1)=(1/2)f_{n-1}\otimes\tilde f_{n-2}\neq 0.$$

Finally, let $\lambda=\varepsilon_1+\varepsilon_2+\varepsilon_3-\delta_{n-2}-\delta_{n-1}-\delta_n$. Then

$$\partial^{3,2}_{\mathfrak{g}_0}v_\lambda(f_{n-1}\otimes\tilde e_2,f_n\otimes\tilde e_3,f_{n-2}\otimes\tilde e_1)=$$
$$(1/2)(B_{1,n-2}(f_{n-2}\otimes\tilde e_1)+B_{3,n}(f_n\otimes\tilde e_3)+B_{2,n-1}(f_{n-1}\otimes\tilde e_2))=$$
$$(1/2)(e_1\otimes\tilde e_1+f_{n-2}\otimes\tilde f_{n-2}+e_3\otimes\tilde e_3+f_n\otimes\tilde f_n+e_2\otimes\tilde e_2+f_{n-1}\otimes\tilde f_{n-1})\neq 0.$$

Thus, $H^{3,2}_{\mathfrak{g}_0}=0$.

3.8. Calculation of $H^{3,2}_{\hat{\mathfrak{g}_0}}$ for $m = n > 1$. By part d) of Theorem 1.1 for $n > 2$ the first term of the Cartan prolongation of the pair $(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0})$ is $U^* \otimes V$ and the second one is zero. Therefore, by arguments similar to those from sec.3.7 we get $H^{3,2}_{\hat{\mathfrak{g}_0}} = 0$.

If $n = 2$, then by part c) of Theorem 1.1 the first term of $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0})$ is $U^* \otimes V$, the second one is the 1-dimensional $\mathfrak{gl}(2) \oplus \mathfrak{gl}(2)$-module with highest weight $\varepsilon_1 + \varepsilon_2 - \delta_1 - \delta_2$, and the third one is zero. Thus, by (2.2.1),

$$\mathrm{Im}\partial^{4,1}_{\hat{\mathfrak{g}_0}} = V_{2\varepsilon_1+\varepsilon_2-\delta_1-2\delta_2}.$$

By Table 15 $(U^* \otimes V) \otimes \Lambda^2(U^* \otimes V)$ contains two irreducible $\mathfrak{gl}(2) \oplus \mathfrak{gl}(2)$-modules with highest weight $\lambda = 2\varepsilon_1 + \varepsilon_2 - \delta_1 - 2\delta_2$ and one of the corresponding highest vectors is v^2_λ. Since

$$\partial^{3,2}_{\hat{\mathfrak{g}_0}} v^2_\lambda(f_1 \otimes \tilde{e}_1, f_2 \otimes \tilde{e}_2, f_2 \otimes \tilde{e}_2) = B_{1,2}(f_2 \otimes \tilde{e}_2) = e_1 \otimes \tilde{e}_2 \neq 0,$$

then $\mathrm{Ker}\partial^{3,2}_{\hat{\mathfrak{g}_0}} = \mathrm{Im}\partial^{4,1}_{\hat{\mathfrak{g}_0}}$. Thus, $H^{3,2}_{\hat{\mathfrak{g}_0}} = 0$.

3.9. Calculation of $H^{4,2}_{\hat{\mathfrak{g}_0}}$ for $m = n = 2$. For $k = 4$ the Spencer cochain sequence is of the form

$$\mathfrak{g}_3 \otimes \mathfrak{g}^*_{-1} \xrightarrow{\partial^{5,1}_{\hat{\mathfrak{g}_0}}} \mathfrak{g}_2 \otimes \Lambda^2\mathfrak{g}^*_{-1} \xrightarrow{\partial^{4,2}_{\hat{\mathfrak{g}_0}}} \mathfrak{g}_1 \otimes \Lambda^3\mathfrak{g}^*_{-1}.$$

By part c) of Theorem 1.1 the second term of $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}_0})$ is $\mathfrak{g}_2 = V_{\varepsilon_1+\varepsilon_2-\delta_1-\delta_2} = \langle g\rangle$, the 1-dimensional $\mathfrak{gl}(2) \oplus \mathfrak{gl}(2)$-module, and the third one is zero. Since by Table 9

$$\Lambda^2\mathfrak{g}^*_{-1} = V_{2\varepsilon_1-2\delta_2} \oplus V_{\varepsilon_1+\varepsilon_2-\delta_1-\delta_2},$$

then

$$\mathfrak{g}_2 \otimes \Lambda^2\mathfrak{g}^*_{-1} = V_{3\varepsilon_1+\varepsilon_2-\delta_1-3\delta_2} \oplus V_{2\varepsilon_1+2\varepsilon_2-2\delta_1-2\delta_2}.$$

Let $\lambda = 3\varepsilon_1 + \varepsilon_2 - \delta_1 - 3\delta_2$. Then by Table 9 $v_\lambda = g \otimes (\tilde{f}_2 \otimes e_1) \wedge (\tilde{f}_2 \otimes e_1)$. Let v be an element from the basis of $\mathfrak{g}_{-1}$ such that $g(v) \neq 0$. If $v = f_2 \otimes \tilde{e}_1$, then

$$\partial^{4,2}_{\hat{\mathfrak{g}_0}}(f_2 \otimes \tilde{e}_1, f_2 \otimes \tilde{e}_1, v) = -3v_\lambda(f_2 \otimes \tilde{e}_1, f_2 \otimes \tilde{e}_1)(v) = 3g(v) \neq 0,$$

and if $v \neq f_2 \otimes \tilde{e}_1$, then

$$\partial^{4,2}_{\hat{\mathfrak{g}_0}} v_\lambda(f_2 \otimes \tilde{e}_1, f_2 \otimes \tilde{e}_1, v) = -v_\lambda(f_2 \otimes \tilde{e}_1, f_2 \otimes \tilde{e}_1)(v) = g(v) \neq 0.$$

Let $\lambda = 2\varepsilon_1 + 2\varepsilon_2 - 2\delta_1 - 2\delta_2$. Then by Table 9

$$\begin{aligned} v_\lambda = g \otimes ((\tilde{f}_2 \otimes e_1) \wedge (\tilde{f}_1 \otimes e_2) - (\tilde{f}_2 \otimes e_2) \wedge (\tilde{f}_1 \otimes e_1) - \\ - (\tilde{f}_1 \otimes e_1) \wedge (\tilde{f}_2 \otimes e_2) + (\tilde{f}_1 \otimes e_2) \wedge (\tilde{f}_2 \otimes e_1)). \end{aligned}$$

Let v be an element of the basis of $\mathfrak{g}_{-1}$ such that $g(v) \neq 0$. Then

$$\partial^{4,2}_{\hat{\mathfrak{g}_0}} v_\lambda(f_2 \otimes \tilde{e}_1, f_1 \otimes \tilde{e}_2, v) = -2v_\lambda(f_2 \otimes \tilde{e}_1, f_1 \otimes \tilde{e}_2)(v) = g(v) \neq 0$$
$$\text{if either } v = f_2 \otimes \tilde{e}_1 \text{ or } v = f_1 \otimes \tilde{e}_2,$$

and

$$\partial^{4,2}_{\hat{\mathfrak{g}_0}} v_\lambda(f_1 \otimes \tilde{e}_1, f_2 \otimes \tilde{e}_2, v) = -2v_\lambda(f_1 \otimes \tilde{e}_1, f_2 \otimes \tilde{e}_2)(v) = -g(v) \neq 0$$
$$\text{if either } v = f_2 \otimes \tilde{e}_2 \text{ or } v = f_1 \otimes \tilde{e}_1.$$

Therefore, $H^{4,2}_{\hat{\mathfrak{g}_0}} = 0$.

3.10. Calculation of $H^{k,2}_{\mathfrak{g}_0}$ for $m = n > 1,\ k > 0$.

LEMMA. $H^{k,2}_{\mathfrak{g}_0} = H^{k,2}_{\hat{\mathfrak{g}}_0}$.

PROOF. Note that if $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}}_0) = \mathfrak{g}_{-1} \oplus (\oplus_{k\geq 0}\hat{\mathfrak{g}}_k)$ is the Cartan prolongation of the pair $(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}}_0)$, then, since $\mathfrak{g}_k = \hat{\mathfrak{g}}_k \oplus S^k(\mathfrak{g}^*_{-1})$ $(k \geq 0)$, the Spencer cochain sequence is of the form

$$\begin{aligned}
&(\hat{\mathfrak{g}}_0 \oplus \mathbb{C}) \otimes \mathfrak{g}^*_{-1} \xrightarrow{\partial^{2,1}_{\mathfrak{g}_0}} \mathfrak{g}_{-1} \otimes \Lambda^2\mathfrak{g}^*_{-1} \xrightarrow{\partial^{1,2}_{\mathfrak{g}_0}} 0 && \text{for } k = 1,\\
&(\hat{\mathfrak{g}}_{k-1} \oplus S^{k-1}(\mathfrak{g}^*_{-1})) \otimes \mathfrak{g}^*_{-1} \xrightarrow{\partial^{k+1,1}_{\mathfrak{g}_0}} (\hat{\mathfrak{g}}_{k-2} \oplus S^{k-2}(\mathfrak{g}^*_{-1})) \otimes \Lambda^2\mathfrak{g}^*_{-1} \xrightarrow{\partial^{k,2}_{\mathfrak{g}_0}} \\
&(\hat{\mathfrak{g}}_{k-3} \oplus S^{k-3}(\mathfrak{g}^*_{-1})) \otimes \Lambda^3\mathfrak{g}^*_{-1} && \text{for } k > 1.
\end{aligned}$$

Note that since $\mathfrak{g}_*(\mathfrak{g}_{-1}, \mathfrak{g}_0) = S^*(\mathfrak{g}^*_{-1}) \Subset \mathfrak{g}_*(\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)$, then the sequence

$$S^{k-1}(\mathfrak{g}^*_{-1}) \otimes \mathfrak{g}^*_{-1} \xrightarrow{\bar{\partial}^{k+1,1}_{\mathfrak{g}_0}} S^{k-2}(\mathfrak{g}^*_{-1}) \otimes \Lambda^2\mathfrak{g}^*_{-1} \xrightarrow{\bar{\partial}^{k,2}_{\mathfrak{g}_0}} S^{k-3}(\mathfrak{g}^*_{-1}) \otimes \Lambda^3\mathfrak{g}^*_{-1} \text{ for } k \geq 1,$$

where $\bar{\partial}^{k+1,1}_{\mathfrak{g}_0}$ and $\bar{\partial}^{k,2}_{\mathfrak{g}_0}$ are the restrictions of the operators $\partial^{k+1,1}_{\mathfrak{g}_0}$ and $\partial^{k,2}_{\mathfrak{g}_0}$ to $S^{k-1}(\mathfrak{g}^*_{-1}) \otimes \mathfrak{g}^*_{-1}$ and $S^{k-2}(\mathfrak{g}^*_{-1}) \otimes \Lambda^2\mathfrak{g}^*_{-1}$, respectively, and $S^k(\mathfrak{g}^*_{-1}) = 0$ for $k < 0$, is well-defined. Hence the corresponding cohomology groups

$$\bar{H}^{k,2}_{\mathfrak{g}_0} = \mathrm{Ker}\bar{\partial}^{k,2}_{\mathfrak{g}_0}/\mathrm{Im}\bar{\partial}^{k+1,1}_{\mathfrak{g}_0}$$

are well-defined and $H^{k,2}_{\mathfrak{g}_0} = H^{k,2}_{\hat{\mathfrak{g}}_0} \oplus \bar{H}^{k,2}_{\mathfrak{g}_0}$.

Let us show that $\bar{H}^{k,2}_{\mathfrak{g}_0} = 0$ for $k > 0$. For $k = 1$ this is obvious. Let $k = 2$. Since $S^{k-2}(\mathfrak{g}^*_{-1}) \otimes \Lambda^2\mathfrak{g}^*_{-1} = \langle z\rangle \otimes \Lambda^2\mathfrak{g}^*_{-1}$, where z is a generator of the center of $\mathfrak{gl}(n|n)$, then

$$\mathrm{Ker}\bar{\partial}^{k,2}_{\mathfrak{g}_0} \cong \Lambda^2\mathfrak{g}^*_{-1}.$$

By formula (2.2.1)

$$\mathrm{Im}\bar{\partial}^{k+1,1}_{\mathfrak{g}_0} \cong \mathfrak{g}^*_{-1} \otimes \mathfrak{g}^*_{-1}/S^2\mathfrak{g}^*_{-1} = \Lambda^2\mathfrak{g}^*_{-1}.$$

Therefore, $\bar{H}^{2,2}_{\mathfrak{g}_0} = 0$. Let $k = 3$. Observe that

$$S^2(U^* \otimes V) \otimes (U^* \otimes V) = (S^2U^* \otimes U^*) \otimes (S^2V \otimes V) \oplus (\Lambda^2U^* \otimes U^*) \otimes (\Lambda^2V \otimes V).$$

By Table 5 from [OV] we get:

$$\begin{aligned}
&S^2V \otimes V = V_{3\varepsilon_1} \oplus V_{2\varepsilon_1+\varepsilon_2},\ \Lambda^2V \otimes V = V_{2\varepsilon_1+\varepsilon_2} \text{ if } n = 2,\\
&\Lambda^2V \otimes V = V_{2\varepsilon_1+\varepsilon_2} \oplus V_{\varepsilon_1+\varepsilon_2+\varepsilon_3} \text{ if } n > 2.
\end{aligned}$$

Since U is odd,

$$\begin{aligned}
&\Lambda^2U^* \otimes U^* = V_{-3\delta_n} \oplus V_{-\delta_{n-1}-2\delta_n},\\
&S^2U^* \otimes U^* = V_{-\delta_{n-1}-2\delta_n} \text{ if } n = 2,\\
&S^2U^* \otimes U^* = V_{-\delta_{n-1}-2\delta_n} \oplus V_{-\delta_{n-2}-\delta_{n-1}-\delta_n} \text{ if } n > 2.
\end{aligned}$$

Therefore,

$$S^2(U^* \otimes V) \otimes (U^* \otimes V) = V_{3\varepsilon_1-\delta_{n-1}-2\delta_n} \oplus V_{2\varepsilon_1+\varepsilon_2-3\delta_n} \oplus 2V_{2\varepsilon_1+\varepsilon_2-\delta_{n-1}-2\delta_n} \text{ if } n = 2$$

and

$$\begin{aligned}
&S^2(U^* \otimes V) \otimes (U^* \otimes V) =\\
&V_{3\varepsilon_1-\delta_{n-1}-2\delta_n} \oplus V_{2\varepsilon_1+\varepsilon_2-3\delta_n} \oplus 2V_{2\varepsilon_1+\varepsilon_2-\delta_{n-1}-2\delta_n} \oplus V_{3\varepsilon_1-\delta_{n-2}-\delta_{n-1}-\delta_n} \oplus\\
&V_{\varepsilon_1+\varepsilon_2+\varepsilon_3-3\delta_n} \oplus V_{2\varepsilon_1+\varepsilon_2-\delta_{n-2}-\delta_{n-1}-\delta_n} \oplus V_{\varepsilon_1+\varepsilon_2+\varepsilon_3-\delta_{n-1}-2\delta_n} && \text{if } n > 2.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&S^3(U^* \otimes V) = V_{2\varepsilon_1+\varepsilon_2-\delta_1-2\delta_2} && \text{if } n = 2 \text{ and}\\
&S^3(U^* \otimes V) = V_{\varepsilon_1+\varepsilon_2+\varepsilon_3-3\delta_n} \oplus V_{3\varepsilon_1-\delta_{n-2}-\delta_{n-1}-\delta_n} \oplus V_{2\varepsilon_1+\varepsilon_2-\delta_{n-1}-2\delta_n} && \text{if } n > 2.
\end{aligned}$$

Thus, by (2.2.1)

$$\begin{array}{ll}
\mathrm{Im}\bar{\partial}^{4,1}_{\mathfrak{g}_0} = V_{3\varepsilon_1-\delta_{n-1}-2\delta_n} \oplus V_{2\varepsilon_1+\varepsilon_2-3\delta_n} \oplus V_{2\varepsilon_1+\varepsilon_2-\delta_{n-1}-2\delta_n} & \text{if } n=2,\\
\mathrm{Im}\bar{\partial}^{4,1}_{\mathfrak{g}_0} = V_{3\varepsilon_1-\delta_{n-1}-2\delta_n} \oplus V_{2\varepsilon_1+\varepsilon_2-3\delta_n} \oplus V_{2\varepsilon_1+\varepsilon_2-\delta_{n-1}-2\delta_n} \oplus & \\
\oplus V_{2\varepsilon_1+\varepsilon_2-\delta_{n-2}-\delta_{n-1}-\delta_n} \oplus V_{\varepsilon_1+\varepsilon_2+\varepsilon_3-\delta_{n-1}-2\delta_n} & \text{if } n>2.
\end{array}$$

Finally, the decomposition of the $\mathfrak{gl}(n) \oplus \mathfrak{gl}(n)$-module $\mathfrak{g}^*_{-1} \otimes \Lambda^2\mathfrak{g}^*_{-1}$ into the direct sum of irreducible components is given in Table 15. Checking the action of $\bar{\partial}^{3,2}_{\mathfrak{g}_0}$ on the highest vectors we get:

$$\mathrm{Im}\bar{\partial}^{4,1}_{\mathfrak{g}_0} = \mathrm{Ker}\bar{\partial}^{3,2}_{\mathfrak{g}_0}.$$

Note that for $k>3$ the cohomology groups $\bar{H}^{k,2}_{\mathfrak{g}_0}$ coincide with the Spencer cohomology groups $H^{k-2,2}_{\mathfrak{o}(n^2)}$ corresponding to the Cartan prolongation $\mathfrak{g}_*(V(0|n^2),\ \mathfrak{o}(n^2)) = \mathfrak{h}(0|n^2)$, where $\mathfrak{o}(n^2)$ is the orthogonal Lie algebra and $V(0|n^2)$ is the standard odd $\mathfrak{o}(n^2)$-module. These groups are vanishing for $k>3$ (see Theorem 1.3 of Chapter 3).

□

CHAPTER 3

The analogues of the Riemann–Weyl tensors for classical superspaces

Recall that $\mathbb{Z}$-grading of depth 1 of a Lie (super)algebra $\mathfrak{g}$ is the $\mathbb{Z}$-grading of the form $\mathfrak{g} = \oplus_{i\geq -1}\mathfrak{g}_i$. All such $\mathbb{Z}$-gradings of simple finite-dimensional complex Lie superalgebras are listed in [S2]. Denote by V_λ the irreducible module over a Lie superalgebra with highest weight λ and an even highest vector.

Spencer cohomology of $\mathfrak{sl}(m|n)$ and $\mathfrak{psl}(n|n)$

1.1. Description of the $\mathbb{Z}$-gradings of depth 1. Let $V(m-p|q)$ and $U(p|n-q)$ be the standard $\mathfrak{sl}(m-p|q)$ and $\mathfrak{sl}(p|n-q)$- modules, respectively.

All $\mathbb{Z}$-gradings of depth 1 of $\mathfrak{g} = \mathfrak{sl}(m|n)$ and $\mathfrak{psl}(n|n)$ are of the form $\mathfrak{g} = \mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$, where $\mathfrak{g}_1 = \mathfrak{g}_{-1}^* = V(m-p|q)\otimes U(p|n-q)^*$.

A) For $\mathfrak{sl}(m|n)$, where $m\neq n$, there are the following possible values of $\mathfrak{g}_0$ for the $\mathbb{Z}$-gradings of depth 1:

a) $\mathfrak{c}(\mathfrak{sl}(m)\oplus\mathfrak{sl}(n))$;

b) $\mathfrak{c}(\mathfrak{sl}(m|q)\oplus\mathfrak{sl}(n-q))$, if $p=0, q\neq 0, n-q\neq 0$;

c) $\mathfrak{c}(\mathfrak{sl}(m-p)\oplus\mathfrak{sl}(p|n))$, if $q=0, p\neq 0, m-p\neq 0$;

d) $\mathfrak{c}(\mathfrak{sl}(m-p|q)\oplus\mathfrak{sl}(p|n-q))$, if $p\neq 0, q\neq 0$.

B) For $\mathfrak{sl}(n|n)$ there are the following possible values of $\mathfrak{g}_0$ for the $\mathbb{Z}$-gradings of depth 1:

a) $\mathfrak{c}(\mathfrak{sl}(n)\oplus\mathfrak{sl}(n))$;

b) $\mathfrak{c}(\mathfrak{sl}(n|q)\oplus\mathfrak{sl}(n-q))$, if $p=0, q\neq 0, n-q\neq 0$;

c) $\mathfrak{c}(\mathfrak{sl}(n-p)\oplus\mathfrak{sl}(p|n))$, if $q=0, p\neq 0, n-p\neq 0$;

d) $\mathfrak{c}(\mathfrak{sl}(n-p|q)\oplus\mathfrak{sl}(p|n-q))$, if $p\neq 0, q\neq 0$.

C) The $\mathbb{Z}$-gradings of $\mathfrak{psl}(n|n)$ are similar to those of $\mathfrak{sl}(n|n)$, only $\mathfrak{g}_0$ is centerless.

1.2. THEOREM (Cartan prolongations). *For the cases of sec.1.1. we have:*

A) $\mathfrak{g} = \mathfrak{sl}(m|n)$, *where* $m\neq n$. *Then* $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0) = \mathfrak{g}$, *except for the following cases:*

a) if $n=1$, *then* $\mathfrak{g}_* = \mathfrak{vect}(0|m)$, *if* $m=1$, *then* $\mathfrak{g}_* = \mathfrak{vect}(0|n)$;

b) if $n-q=1$, *then* $\mathfrak{g}_* = \mathfrak{vect}(q|m)$, *if* $m=0, q=1$, *then* $\mathfrak{g}_* = \mathfrak{vect}(n-1|0)$;

c) if $m-p=1$, *then* $\mathfrak{g}_* = \mathfrak{vect}(p|n)$, *if* $n=0, p=1$, *then* $\mathfrak{g}_* = \mathfrak{vect}(m-1|0)$;

d) if $n-q=0, p=1$, *then* $\mathfrak{g}_* = \mathfrak{vect}(m-1|n)$, *if* $m-p=0, q=1$, *then* $\mathfrak{g}_* = \mathfrak{vect}(n-1|m)$.

B) $\mathfrak{g} = \mathfrak{sl}(n|n)$. *Then* $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0) = S^*(\mathfrak{g}_{-1}^*)\Subset\mathfrak{psl}(n|n)$, *except for the following cases:*

a) if $n=2$, *then* $\mathfrak{g}_* = S^*(\mathfrak{g}_{-1}^*)\Subset\mathfrak{h}(0|4)$;

b) if $n-q=1$, *then* $\mathfrak{g}_* = \mathfrak{vect}(q|n)$;

c) if $n-p=1$, *then* $\mathfrak{g}_* = \mathfrak{vect}(p|n)$;

d) if $n-q=0, p=1$ *or* $n-p=0, q=1$ *then* $\mathfrak{g}_* = \mathfrak{vect}(n-1|n)$.

C) $\mathfrak{g} = \mathfrak{psl}(n|n)$. *Then* $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0) = \mathfrak{g}$, *except for the following cases:*

a) if $n=2$, *then* $\mathfrak{g}_* = \mathfrak{h}(0|4)$;

b) if $n-q=1$, *then* $\mathfrak{g}_* = \mathfrak{svect}(q|n)$;

c) if $n-p=1$, *then* $\mathfrak{g}_* = \mathfrak{svect}(p|n)$;

d) if $n-q=0, p=1$ *or* $n-p=0, q=1$, *then* $\mathfrak{g}_* = \mathfrak{svect}(n-1|n)$.

Let $\mathfrak{g}=\mathfrak{sl}(m|n)$, where $m\neq n$, or $\mathfrak{psl}(n|n)$. We will describe the Spencer cohomology groups for all $\mathbb{Z}$-gradings of depth 1 listed in sec.1.1.

First consider the cases easiest to formulate. Let $\langle\pi_i\rangle$ be the i-th fundamental weight of $\mathfrak{g}_0$.

1.3. THEOREM. 1) *For* $\mathfrak{g}_*=\mathfrak{vect}(m|n)$, $\mathfrak{svect}(m|n)$ *SFs vanish except for* $\mathfrak{svect}(0|n)$, *when SFs are of order* n *and constitute the* $\mathfrak{g}_0$*-module* $\Pi^n(\langle 1\rangle)$.

2) *For* $\mathfrak{g}_*$ *of series* $\mathfrak{h}(0|n)$, *nonzero SFs are of order 1. For* $n>3$ *SFs constitute* $\mathfrak{g}_0$*-module* $\Pi(V_{3\pi_1}\oplus V_{\pi_1})$.

3) *For* $\mathfrak{g}_*=\mathfrak{sh}(0|n)$, *nonzero SFs are the same as for* $\mathfrak{h}(0|n)$ *and additionally* $\Pi^{n-1}(V_{\pi_1})$ *of order* $n-1$.

Consider the $\mathbb{Z}$-gradings of depth 1 of $\mathfrak{g}=\mathfrak{sl}(m|n)$ $(m\neq n)$ and $\mathfrak{psl}(n|n)$ listed in sec.1.1 for which $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0)=\mathfrak{g}$. Describe the corresponding SFs.

Case a) was discussed in Chapter 2. Consider case b).

1.4. THEOREM. *The nonzero SFs are of orders 1 and 2. The* $\mathfrak{g}_0$*-module* $H^{2,2}_{\mathfrak{g}_0}$ *splits into the direct sum of irreducible components whose weights are given in Table 16. Table 16 also contains the highest weights (with respect to the bases* $\varepsilon_1,\dots,\varepsilon_{m+q}$ *and* $\delta_1,\dots,\delta_{n-q}$ *of the dual spaces to the maximal tori of* $\mathfrak{sl}(m|q)$ *and* $\mathfrak{sl}(n-q)$, *respectively) of irreducible components of* $H^{1,2}_{\mathfrak{g}_0}$ *for the cases when* $H^{1,2}_{\mathfrak{g}_0}$ *does split into the direct sum of irreducible* $\mathfrak{g}_0$*-modules.*

Exceptional cases are as follows: if $m=q-1, m>1, n-q\geq 3$, then $H^{1,2}_{\mathfrak{g}_0}=V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m+q}+\delta_1-\delta_{n-q-1}-\delta_{n-q}}\oplus X$, where X is given by the nonsplit exact sequence of $\mathfrak{g}_0$-modules

$$0\longrightarrow V_{2\varepsilon_1-\varepsilon_{m+q}+\delta_1-2\delta{n-q}}\longrightarrow X\longrightarrow \Pi(V_{\varepsilon_1+\delta_1-2\delta_{n-q}})\longrightarrow 0; \tag{1}$$

if $m=q-1, m>1, n-q=2$, then $H^{1,2}_{\mathfrak{g}_0}=X$, where X is given by (1);

if $m=1, q=2, n-q\geq 3$, then $H^{1,2}_{\mathfrak{g}_0}=\Pi(V_{\varepsilon_1+\varepsilon_2-\varepsilon_3+\delta_1-\delta_{n-q-1}-\delta_{n-q}})\oplus X$, where X is given by the nonsplit exact sequence of $\mathfrak{g}_0$-modules

$$\begin{gathered}0\longrightarrow V_{2\varepsilon_1-\varepsilon_3+\delta_1-2\delta_{n-q}}\oplus\Pi(V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta_1-2\delta_{n-q}})\\ \longrightarrow X\longrightarrow\Pi(V_{\varepsilon_1+\delta_1-2\delta_{n-q}})\longrightarrow 0;\end{gathered} \tag{2}$$

if $m=1, q=2, n-q=2$, then $H^{1,2}_{\mathfrak{g}_0}=X$, where X is given by (2); if $m=q+1, n-q\geq 3$, then $H^{1,2}_{\mathfrak{g}_0}=V_{2\varepsilon_1-\varepsilon_{m+q}+\delta_1-2\delta_{n-q}}\oplus X$, where X is given by the nonsplit exact sequence of $\mathfrak{g}_0$-modules

$$\begin{gathered}0\longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m+q}+\delta_1-\delta_{n-q-1}-\delta_{n-q}}\longrightarrow X\\ \longrightarrow\Pi(V_{\varepsilon_1+\delta_1-\delta_{n-q-1}-\delta_{n-q}})\longrightarrow 0\qquad (q\geq 2),\end{gathered}$$

$$\begin{gathered}0\longrightarrow V_{\varepsilon_1-\varepsilon_2+\varepsilon_3+\delta_1-\delta_{n-2}-\delta_{n-1}}\oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_3+\delta_1-\delta_{n-2}-\delta_{n-1}}\longrightarrow X\\ \longrightarrow\Pi(V_{\varepsilon_1+\delta_1-\delta_{n-2}-\delta_{n-1}})\longrightarrow 0\qquad (q=1).\end{gathered}$$

Case c) is similar to case b). Consider case d).

1.5. THEOREM. *The nonzero SFs are of orders 1 and 2. The* $\mathfrak{g}_0$*-module* $H^{2,2}_{\mathfrak{g}_0}$ *splits into the direct sum of irreducible components whose weights are given in Table 17. Table 17 also contains the highest weights (with respect to the bases* $\varepsilon_1,\dots,\varepsilon_{m-p+q}$ *and* $\delta_1,\dots,\delta_{p+n-q}$ *of the dual spaces to the maximal tori of* $\mathfrak{sl}(m-p|q)$ *and* $\mathfrak{sl}(p|n-q)$, *respectively) of irreducible components of* $H^{1,2}_{\mathfrak{g}_0}$ *for the cases when* $H^{1,2}_{\mathfrak{g}_0}$ *does split into the direct sum of irreducible* $\mathfrak{g}_0$*-modules.*

Exceptional cases are $m=p+q\pm 1$ and $n=p+q\pm 1$. More precisely: if $m=p+q+1$, $n\neq p+q\pm 1,\ q$, then $H^{1,2}_{\mathfrak{g}_0}=V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-2\delta_{p+n-q}}\oplus Y$, where Y is given by the nonsplit exact sequence of $\mathfrak{g}_0$-modules

$$0 \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_3+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}} \oplus V_{\varepsilon_1-\varepsilon_2+\varepsilon_3+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}} \longrightarrow Y \longrightarrow V_{\varepsilon_1+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}} \longrightarrow 0 \qquad (q=1),$$

$$0 \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}} \longrightarrow Y \longrightarrow V_{\varepsilon_1+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}} \longrightarrow 0 \qquad (q\geq 2);$$

if $m=p+q+1$, $n=p+q-1$, then $H^{1,2}_{\mathfrak{g}_0}=X\oplus Y$, where X is given by the nonsplit exact sequence of $\mathfrak{g}_0$-modules

$$0 \longrightarrow V_{2\varepsilon_1-\varepsilon_{m-2+q}+\delta_1-2\delta_3} \oplus V_{2\varepsilon_1-\varepsilon_{m-2+q}-\delta_1-\delta_2+\delta_3} \longrightarrow X \longrightarrow V_{2\varepsilon_1-\varepsilon_{m-2+q}-\delta_3} \longrightarrow 0 \qquad (p=2),$$

$$0 \longrightarrow V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-2\delta_{p+n-q}} \longrightarrow X \longrightarrow V_{2\varepsilon_1-\varepsilon_{m-p+q}-\delta_{p+n-q}} \longrightarrow 0 \qquad (p\geq 3),$$

and Y is given by the nonsplit exact sequence of $\mathfrak{g}_0$-modules

$$0 \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_3+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}} \oplus V_{\varepsilon_1-\varepsilon_2+\varepsilon_3+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}} \longrightarrow Y \longrightarrow V_{\varepsilon_1+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}} \longrightarrow 0 \qquad (q=1),$$

$$0 \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}} \longrightarrow Y \longrightarrow V_{\varepsilon_1+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}} \longrightarrow 0 \qquad (q\geq 2);$$

if $m=p+q+1$, $n=q$, then $H^{1,2}_{\mathfrak{g}_0}=V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-\delta_{p-1}-\delta_p}\oplus Y$ $(p\geq 3)$ or $H^{1,2}_{\mathfrak{g}_0}=Y$ $(p=2)$, where Y is given by the nonsplit exact sequence of $\mathfrak{g}_0$-modules

$$0 \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_3+\delta_1-2\delta_p} \oplus V_{\varepsilon_1-\varepsilon_2+\varepsilon_3+\delta_1-2\delta_p} \longrightarrow Y \longrightarrow V_{\varepsilon_1+\delta_1-2\delta_p} \longrightarrow 0 \qquad (q=1),$$

$$0 \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-2\delta_p} \longrightarrow Y \longrightarrow V_{\varepsilon_1+\delta_1-2\delta_p} \longrightarrow 0 \qquad (q\geq 2);$$

if $n=p+q+1$, $m\neq p+q\pm 1$, p, then $H^{1,2}_{\mathfrak{g}_0}=V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-2\delta_{p+n-q}}\oplus Y$, where

$$0 \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}+\delta_1-\delta_2-\delta_3} \oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}-\delta_1+\delta_2-\delta_3} \longrightarrow Y \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}-\delta_3} \longrightarrow 0 \qquad (p=1),$$

$$0 \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}} \longrightarrow Y \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}-\delta_{p+n-q}} \longrightarrow 0 \qquad (p\geq 2);$$

if $n=p+q+1$, $m=p+q-1$, then $H^{1,2}_{\mathfrak{g}_0}=X\oplus Y$, where

$$0 \longrightarrow V_{2\varepsilon_1-\varepsilon_3+\delta_1-2\delta_{p+n-2}} \oplus V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta_1-2\delta_{p+n-2}} \longrightarrow X \longrightarrow V_{\varepsilon_1+\delta_1-2\delta_{p+n-2}} \longrightarrow 0 \qquad (q=2),$$

$$0 \longrightarrow V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-2\delta_{p+n-q}} \longrightarrow X \longrightarrow V_{\varepsilon_1+\delta_1-2\delta_{p+n-q}} \longrightarrow 0 \qquad (q\geq 3),$$

and Y is given by the nonsplit exact sequence of $\mathfrak{g}_0$-modules

$$0 \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}+\delta_1-\delta_2-\delta_3} \oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}-\delta_1+\delta_2-\delta_3} \longrightarrow Y \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}-\delta_3} \longrightarrow 0 \qquad (p=1),$$

$$0 \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}} \longrightarrow Y \longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}-\delta_{p+n-q}} \longrightarrow 0 \qquad (p\geq 2);$$

if $n=p+q+1$, $m=p$, then $H^{1,2}_{\mathfrak{g}_0}=V_{\varepsilon_1+\varepsilon_2-\varepsilon_q+\delta_1-2\delta_{p+n-q}}\oplus Y$ $(q\geq 3)$, and $H^{1,2}_{\mathfrak{g}_0}=Y$ $(q=2)$, where

$$0 \longrightarrow V_{2\varepsilon_1-\varepsilon_q+\delta_1-\delta_2-\delta_3} \oplus V_{2\varepsilon_1-\varepsilon_q-\delta_1+\delta_2-\delta_3} \longrightarrow Y \longrightarrow V_{2\varepsilon_1-\varepsilon_q-\delta_3} \longrightarrow 0 \qquad (p=1),$$

$$0 \longrightarrow V_{2\varepsilon_1-\varepsilon_q+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}} \longrightarrow Y \longrightarrow V_{2\varepsilon_1-\varepsilon_q-\delta_{p+n-q}} \longrightarrow 0 \qquad (p\geq 2);$$

if $m=p+q-1$, $n\neq p+q\pm 1$, q, then $H^{1,2}_{\mathfrak{g}_0}=X\oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}}$, where X is given by the nonsplit exact sequence of $\mathfrak{g}_0$-modules

$$0\longrightarrow V_{2\varepsilon_1-\varepsilon_3+\delta_1-2\delta_{p+n-2}}\oplus V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta_1-2\delta_{p+n-2}}\longrightarrow X\longrightarrow V_{\varepsilon_1+\delta_1-2\delta_{p+n-2}}\longrightarrow 0 \qquad (q=2),$$

$$0\longrightarrow V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-2\delta_{p+n-q}}\longrightarrow X\longrightarrow V_{\varepsilon_1+\delta_1-2\delta_{p+n-q}}\longrightarrow 0 \qquad (q\geq 3);$$

if $m=p+q-1$, $n=q$, then $H^{1,2}_{\mathfrak{g}_0}=X\oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-2\delta_p}$ $(p\geq 3)$, where X is given by the nonsplit exact sequence of $\mathfrak{g}_0$-modules

$$0\longrightarrow V_{2\varepsilon_1-\varepsilon_3+\delta_1-\delta_{p-1}-\delta_p}\oplus V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta_1-\delta_{p-1}-\delta_p}\longrightarrow X\longrightarrow V_{\varepsilon_1+\delta_1-\delta_{p-1}-\delta_p}\longrightarrow 0 \qquad (q=2),$$

$$0\longrightarrow V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-\delta_{p-1}-\delta_p}\longrightarrow X\longrightarrow V_{\varepsilon_1+\delta_1-\delta_{p-1}-\delta_p}\longrightarrow 0 \qquad (q\geq 3);$$

if $n=p+q-1$, $m\neq p+q\pm 1$, p, then $H^{1,2}_{\mathfrak{g}_0}=X\oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}}$, where X is given by the nonsplit exact sequence of $\mathfrak{g}_0$-modules

$$0\longrightarrow V_{2\varepsilon_1-\varepsilon_{m-2+q}+\delta_1-2\delta_3}\oplus V_{2\varepsilon_1-\varepsilon_{m-2+q}-\delta_1-\delta_2+\delta_3}\longrightarrow X\longrightarrow V_{2\varepsilon_1-\varepsilon_{m-2+q}-\delta_3}\longrightarrow 0 \qquad (p=2),$$

$$0\longrightarrow V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-2\delta_{p+n-q}}\longrightarrow X\longrightarrow V_{2\varepsilon_1-\varepsilon_{m-p+q}-\delta_{p+n-q}}\longrightarrow 0 \qquad (p\geq 3);$$

if $n=p+q-1$, $m=p$, then $H^{1,2}_{\mathfrak{g}_0}=X\oplus V_{2\varepsilon_1-\varepsilon_q+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}}$ $(q\geq 3)$, where X is given by the nonsplit exact sequence of $\mathfrak{g}_0$-modules

$$0\longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_q+\delta_1-2\delta_3}\oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_q-\delta_1-\delta_2+\delta_3}\longrightarrow X\longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_q-\delta_3}\longrightarrow 0 \qquad (p=2),$$

$$0\longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_q+\delta_1-2\delta_{p+n-q}}\longrightarrow X\longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_q-\delta_{p+n-q}}\longrightarrow 0 \qquad (p\geq 3);$$

if $m=n=p+q+1$, then $H^{1,2}_{\mathfrak{g}_0}=X\oplus V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-2\delta_{p+n-q}}\oplus V_{\varepsilon_1-\delta_{p+n-q}}$, where X is given by the nonsplit exact sequence of $\mathfrak{g}_0$-modules

$$0\longrightarrow Y\longrightarrow X\longrightarrow V_{\varepsilon_1-\delta_{p+n-q}}\longrightarrow 0,$$

and Y is given by the nonsplit exact sequence of $\mathfrak{g}_0$-modules

$$0\longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-\delta_{p+n-q-1}-\delta p+n-q}\longrightarrow Y\longrightarrow V_{\varepsilon_1+\delta_1-\delta_{p+n-q-1}-\delta p+n-q}\oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}-\delta p+n-q}\longrightarrow 0 \qquad (p\geq 2, q\geq 2),$$

$$0\longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}+\delta_1-\delta_2-\delta_3}\oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}-\delta_1+\delta_2-\delta_3}\longrightarrow Y\longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-1+q}-\delta_3}\oplus V_{\varepsilon_1+\delta_1-\delta_2-\delta_3}\oplus V_{\varepsilon_1-\delta_1+\delta_2-\delta_3}\longrightarrow 0 \qquad (p=1, q\geq 2),$$

$$0\longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_3+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}}\oplus V_{\varepsilon_1-\varepsilon_2+\varepsilon_3+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}}\longrightarrow Y\longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_3-\delta_{p+n-1}}\oplus V_{\varepsilon_1+\delta_1-\delta_{p+n-2}-\delta_{p+n-1}}\oplus V_{\varepsilon_1-\varepsilon_2+\varepsilon_3-\delta_{p+n-1}}\longrightarrow 0 \qquad (p\geq 2, q=1),$$

$$0\longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_3+\delta_1-\delta_2-\delta_3}\oplus V_{\varepsilon_1-\varepsilon_2+\varepsilon_3+\delta_1-\delta_2-\delta_3}\oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_3-\delta_1+\delta_2-\delta_3}\oplus V_{\varepsilon_1-\varepsilon_2+\varepsilon_3-\delta_1+\delta_2-\delta_3}\longrightarrow Y\longrightarrow V_{\varepsilon_1+\varepsilon_2-\varepsilon_3-\delta_3}\oplus V_{\varepsilon_1-\varepsilon_2+\varepsilon_3-\delta_3}\oplus V_{\varepsilon_1+\delta_1-\delta_2-\delta_3}\oplus V_{\varepsilon_1-\delta_1+\delta_2-\delta_3}\longrightarrow 0 \qquad (p=1, q=1)$$

if $m=n=p+q-1$, then $H^{1,2}_{\mathfrak{g}_0}=X\oplus V_{\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}}\oplus V_{\varepsilon_1-\delta_{p+n-q}}$, where X is given by the nonsplit exact sequence of $\mathfrak{g}_0$-modules

$$0\longrightarrow Y\longrightarrow X\longrightarrow V_{\varepsilon_1-\delta_{p+n-q}}\longrightarrow 0,$$

and Y is given by the nonsplit exact sequence of $\mathfrak{g}_0$-modules

$$0\longrightarrow V_{2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-2\delta_{p+n-q}}\longrightarrow Y\longrightarrow V_{2\varepsilon_1-\varepsilon_{m-p+q}-\delta_{p+n-q}}\oplus V_{\varepsilon_1+\delta_1-2\delta_{p+n-q}}\longrightarrow 0 \qquad (p\geq 3, q\geq 3),$$

$$\begin{aligned}&0\longrightarrow V_{2\varepsilon_1-\varepsilon_{m-2+q}+\delta_1-2\delta_3}\oplus V_{2\varepsilon_1-\varepsilon_{m-2+q}-\delta_1-\delta_2+\delta_3}\longrightarrow Y\\&\longrightarrow V_{2\varepsilon_1-\varepsilon_{m-2+q}-\delta_3}\oplus V_{\varepsilon_1+\delta_1-2\delta_3}\oplus\\&V_{\varepsilon_1-\delta_1-\delta_2+\delta_3}\longrightarrow 0 \qquad\qquad (p=2, q\geq 3),\end{aligned}$$

$$\begin{aligned}&0\longrightarrow V_{2\varepsilon_1-\varepsilon_3+\delta_1-2\delta_{p+n-2}}\oplus V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta_1-2\delta_{p+n-2}}\longrightarrow Y\\&\longrightarrow V_{\varepsilon_1+\delta_1-2\delta_{p+n-2}}\oplus V_{2\varepsilon_1-\varepsilon_3-\delta_{p+n-2}}\oplus\\&V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3-\delta_{p+n-2}}\longrightarrow 0 \qquad\qquad (p\geq 3, q=2),\end{aligned}$$

$$\begin{aligned}&0\longrightarrow V_{2\varepsilon_1-\varepsilon_3+\delta_1-2\delta_3}\oplus V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta_1-2\delta_3}\oplus V_{2\varepsilon_1-\varepsilon_3-\delta_1-\delta_2+\delta_3}\oplus\\&V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3-\delta_1-\delta_2+\delta_3}\longrightarrow Y\longrightarrow V_{2\varepsilon_1-\varepsilon_3-\delta_3}\oplus V_{-\varepsilon_1+\varepsilon_2+\varepsilon_3-\delta_3}\oplus\\&V_{\varepsilon_1+\delta_1-2\delta_3}\oplus V_{\varepsilon_1-\delta_1-\delta_2+\delta_3}\longrightarrow 0 \qquad\qquad (p=2, q=2).\end{aligned}$$

Remark. The irreducible $\mathfrak{g}_0$-modules in the above listed nonsplit exact sequences are given regardless of their parity, which can be easily recovered from the corresponding highest weights.

Spencer cohomology of $\mathfrak{psq}(n)$

2.1. Definition of $\mathfrak{psq}(n)$. Set $\mathfrak{q}(n)=\{X\in\mathfrak{gl}(n|n)\mid [X,J_{2n}]=0\}$ for an odd nondegenerate form J_{2n}, such that $J_{2n}^2=-1_{2n}$. The usual choice for J_{2n} is $J_{2n}=\text{antidiag}(1_n,\ -1_n)$. Then we have:

$\mathfrak{q}(n)=\{X\in\mathfrak{gl}(n|n)\mid X=\text{diag}(A,\ A)+\text{antidiag}(B,\ B),\text{ where }A,\ B\in\mathfrak{gl}(n)\}$. Let $otrX=trB$. Set $\mathfrak{sq}(n)=\{X\in\mathfrak{q}(n)\mid otrX=0\}$,

$\mathfrak{psq}(n)=\mathfrak{sq}(n)/\langle 1_{2n}\rangle$,

$\mathfrak{s}(\mathfrak{q}(p)\oplus\mathfrak{q}(n-p))=\{X\in\mathfrak{q}(p)\oplus\mathfrak{q}(n-p)\mid otr\mathfrak{q}(p)+otr\mathfrak{q}(n-p)=0\}$, $\mathfrak{ps}(\mathfrak{q}(p)\oplus\mathfrak{q}(n-p))=\mathfrak{s}(\mathfrak{q}(p)\oplus\mathfrak{q}(n-p))/\langle 1_p+1_{n-p}\rangle$

2.2. $\mathbb{Z}$-gradings of depth 1 of $\mathfrak{psq}(n)$. Let $V(n\mid n)$ be the standard $\mathfrak{q}(n)$-module. All $\mathbb{Z}$-gradings of depth 1 of $\mathfrak{g}=\mathfrak{psq}(n)$ are of the form $\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$, where $\mathfrak{g}_0=\mathfrak{ps}(\mathfrak{q}(p)\oplus\mathfrak{q}(n-p))$, $p>0$, and as $\mathfrak{g}_0$-modules $\mathfrak{g}_1\cong\mathfrak{g}_{-1}^*$, where $\mathfrak{g}_{-1}$ is either one of the two irreducible $\mathfrak{g}_0$-modules in $V(p\mid p)^*\otimes V(n-p\mid n-p)$. Explicitly:

$\mathfrak{g}_{-1}=\langle(x\pm\Pi(x))\otimes(y\pm\Pi(y))\rangle$, where $x\in V(p\mid p)^*, y\in V(n-p\mid n-p)$.

Let $\varepsilon_1,\dots,\varepsilon_p$ and $\delta_1,\dots,\delta_{n-p}$ be the standard bases of the dual spaces to the spaces of diagonal matrices in $\mathfrak{q}(p)$ and $\mathfrak{q}(n-p)$, respectively.

2.3. Theorem. 1) $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0)=\mathfrak{g}$,

2) *all SFs are of order 1 and split into the direct sum of two irreducible $\mathfrak{g}_0$-submodules with highest weights $2\varepsilon_1-\varepsilon_p+\delta_1-2\delta_{n-p}$ and $\varepsilon_1-\delta_{n-p}$.*

Spencer cohomology of $\mathfrak{osp}(m|2n)$

3.1. Definition of $\mathfrak{osp}(m|2n)$. . $\mathfrak{osp}(m|2n)$ is a Lie superalgebra, which preserves a nondegenerate supersymmetric even bilinear form on a superspace V, $\dim V=(m|2n)$.

3.2. Consider the $\mathbb{Z}$-grading of depth 1 of $\mathfrak{g}=\mathfrak{osp}(m|2n)$, which is defined as follows: $\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$, where $\mathfrak{g}_0=\mathfrak{cosp}(m-2|2n)$ is the central extension of $\hat{\mathfrak{g}}_0=\mathfrak{osp}(m-2|2n)$, $\mathfrak{g}_1\cong\mathfrak{g}_{-1}$ is the standard $\mathfrak{g}_0$-module.

Let $m=2r+2$ or $m=2r+3$, $n>0$. Let $\varepsilon_1,\dots,\varepsilon_r$ and $\delta_1,\dots,\delta_n$ be the standard bases of the dual spaces to the spaces of diagonal matrices in $\mathfrak{o}(m-2)$ and $\mathfrak{sp}(n)$, respectively.

3.3. Theorem. 1) $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0)=\mathfrak{g}$, $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \hat{\mathfrak{g}}_0)=\mathfrak{g}_{-1}\oplus\hat{\mathfrak{g}}_0$.

2) *If $k\neq 2$, then $H^{k,2}_{\mathfrak{g}_0}=H^{k,2}_{\hat{\mathfrak{g}}_0}=0$. As a $\hat{\mathfrak{g}}_0$-module $H^{2,2}_{\hat{\mathfrak{g}}_0}$ is isomorphic to $S^2(E^2(\mathfrak{g}_{-1}))/E^4(\mathfrak{g}_{-1})$ and splits into the direct sum of three irreducible components (analogues of the Weyl tensor,*

the traceless Ricci tensor, and the scalar curvature). The highest weights of these components are listed in Table 18. As $\hat{\mathfrak{g}}_0$-modules, $H^{2,2}_{\hat{\mathfrak{g}}_0} \cong H^{2,2}_{\mathfrak{g}_0} \oplus S^2(\mathfrak{g}_{-1})$. The $\mathfrak{g}_0$-module $H^{2,2}_{\mathfrak{g}_0}$ is irreducible.

Spencer cohomology of $D(\alpha)$

4.1. Definition of $D(\alpha)$. $D(\alpha)$, where $\alpha \in \mathbb{C}\backslash\{0,-1\}$, is a one-parameter family consisting of all simple Lie superalgebras for which $D(\alpha)_0 = \mathfrak{sl}(2)\oplus\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$ and its representation on $D(\alpha)_1$ is $\mathfrak{sl}(2)\otimes\mathfrak{sl}(2)\otimes\mathfrak{sl}(2)$.

4.2. $\mathbb{Z}$-gradings of depth 1 of $D(\alpha)$. Let $\varepsilon_1,\varepsilon_2,\varepsilon_3$ be the standard basis of the dual space to the space of diagonal matrices in $\mathfrak{gl}(1|2), V_\lambda$ be the irreducible $\mathfrak{sl}(1|2)$-module with highest weight λ and an even highest vector. All $\mathbb{Z}$-gradings of depth 1 of $D(\alpha)$ are of the form $\mathfrak{g} = \mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$, where $\mathfrak{g}_0 \cong \mathfrak{gl}(1|2)$. There are the following possible values of $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ for the $\mathbb{Z}$-gradings of depth 1:

a) $\mathfrak{g}_1 = V_{(1+\alpha)\varepsilon_1},\ \mathfrak{g}_{-1} = V_{-\alpha\varepsilon_1}$,

b) $\mathfrak{g}_1 = V_{(\frac{1+\alpha}{\alpha})\varepsilon_1},\ \mathfrak{g}_{-1} = V_{-\frac{1}{\alpha}\varepsilon_1}$,

c) $\mathfrak{g}_1 = V_{(\frac{\alpha}{1+\alpha})\varepsilon_1},\ \mathfrak{g}_{-1} = V_{\frac{1}{1+\alpha}\varepsilon_1}$.

More explicitly, let e^i_1, e^i_2 be the basis of the standard $\mathfrak{sl}(2)_i$-module V_i, where $i = 1,\ 2,\ 3$. Then the $\mathbb{Z}$-grading in case a) can be described as follows: $\mathfrak{g}_0 = (\mathfrak{g}_0)_0\oplus(\mathfrak{g}_0)_1$, where $(\mathfrak{g}_0)_0 = \mathfrak{sl}(2)_1\oplus\langle\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)_2\rangle\oplus\langle\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)_3\rangle$, $(\mathfrak{g}_0)_1 = V_1\otimes e^2_1\otimes e^3_2\oplus V_1\otimes e^2_2\otimes e^3_1$;

$\mathfrak{g}_1 = (\mathfrak{g}_1)_0\oplus(\mathfrak{g}_1)_1$, where $(\mathfrak{g}_1)_0 = \langle\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)_2\rangle\oplus\langle\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)_3\rangle$, $(\mathfrak{g}_1)_1 = V_1\otimes e^2_1\otimes e^3_1$; $\mathfrak{g}_{-1} = (\mathfrak{g}_{-1})_0\oplus(\mathfrak{g}_{-1})_1$, where $(\mathfrak{g}_{-1})_0 = \langle\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)_2\rangle\oplus\langle\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)_3\rangle$, $(\mathfrak{g}_{-1})_1 = V_1\otimes e^2_2\otimes e^3_2$; The $\mathbb{Z}$-gradings in cases b) and c) can be described similarly.

4.3. Theorem. *For all $\mathbb{Z}$-gradings of depth 1 of $\mathfrak{g} = D(\alpha)$ we have*

1) $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0) = \mathfrak{g}$.

2) *The nonzero SFs are of order 2, and for the cases considered in sec.4.2, the $\mathfrak{g}_0$-module $H^{2,2}_{\mathfrak{g}_0}$ is isomorphic to*

a) $\Pi(V_{(2\alpha+1)\varepsilon_1+\varepsilon_2})$,

b) $\Pi(V_{(\frac{\alpha+2}{\alpha})\varepsilon_1+\varepsilon_2})$,

c) $\Pi(V_{(\frac{\alpha-1}{\alpha+1})\varepsilon_1+\varepsilon_2})$, *respectively.*

Spencer cohomology of AB_3

Definition of AB_3. AB_3 is a simple Lie superalgebra for which $(AB_3)_0 = \mathfrak{sl}(2)\oplus\mathfrak{o}(7)$ and its representation on $(AB_3)_1$ is $\mathfrak{sl}(2)\otimes spin_7$.

5.2. $\mathbb{Z}$-grading of depth 1 of AB_3. Let $\varepsilon_1,\delta_1,\delta_2$ be the standard basis of the dual space to the space of diagonal matrices in $\mathfrak{osp}(2|4)$, V_λ be an irreducible $\mathfrak{osp}(2|4)$-module with highest weight λ and an even highest vector.

There is only one $\mathbb{Z}$-grading of depth 1 in $\mathfrak{g} = AB_3$, namely, $\mathfrak{g} = \mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$, where $\mathfrak{g}_{-1} = V_{-\varepsilon_1+\delta_1+\delta_2}$, $\mathfrak{g}_0 = \mathfrak{cosp}(2|4)$, $\mathfrak{g}_1 = V_{3\varepsilon_1}$.

Note that $\mathfrak{o}(7) = V_1\oplus\mathfrak{o}(5)\oplus\mathbb{C}\oplus V_2$, where V_1, V_2 are standard $\mathfrak{o}(5)$-modules. The space of the representation $spin_7$ after restriction of $\mathfrak{o}(7)$ to $\mathfrak{o}(5)$ decomposes into the direct sum of two irreducible subspaces, which we denote by U_1 and U_2. Let e_1, e_2 be the basis of the standard $\mathfrak{sl}(2)$-module V. Then $\mathfrak{g}_0 = (\mathfrak{g}_0)_0\oplus(\mathfrak{g}_0)_1$, where

$(\mathfrak{g}_0)_0 = \langle\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)\rangle\oplus\mathfrak{o}(5)\oplus\mathbb{C}$, $(\mathfrak{g}_0)_1 = e_1\otimes U_1\oplus e_2\otimes U_2$;

$\mathfrak{g}_1 = (\mathfrak{g}_1)_0\oplus(\mathfrak{g}_1)_1$, where $(\mathfrak{g}_1)_0 = \langle\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)\rangle\oplus V_2$, $(\mathfrak{g}_1)_1 = e_1\otimes U_2$;

$\mathfrak{g}_{-1} = (\mathfrak{g}_{-1})_0\oplus(\mathfrak{g}_{-1})_1$, where $(\mathfrak{g}_{-1})_0 = \langle\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)\rangle\oplus V_1$, $(\mathfrak{g}_{-1})_1 = e_2\otimes U_1$.

5.3. Theorem. 1) $\mathfrak{g}_*(\mathfrak{g}_{-1},\ \mathfrak{g}_0) = \mathfrak{g}$.

2) *The nonzero SFs are of order 1. The* $\mathfrak{g}_0$*- module* $H^{1,2}_{\mathfrak{g}_0}$ *is given by the nonsplit exact sequence of* $\mathfrak{g}_0$*- modules*

$$0 \longrightarrow X \longrightarrow H^{1,2}_{\mathfrak{g}_0} \longrightarrow V_{\varepsilon_1+2\delta_1} \longrightarrow 0,$$

where X *is given by the nonsplit exact sequence of* $\mathfrak{g}_0$*-modules*

$$0 \longrightarrow \Pi(V_{4\varepsilon_1+2\delta_1+\delta_2}) \longrightarrow X \longrightarrow V_{3\varepsilon_1+2\delta_1} \longrightarrow 0.$$

Appendix. The formula of dimensions of irreducible $\mathfrak{sl}(n)$-modules

Let $\varepsilon_1, \ldots, \varepsilon_n$ be the standard basis of the dual space to the space of diagonal matrices in $\mathfrak{gl}(n)$, V_λ be the irreducible $\mathfrak{sl}(n)$-module with highest weight $\lambda = k_1\varepsilon_1 + k_2\varepsilon_2 + \ldots + k_n\varepsilon_n$, where $k_i \in \mathbb{Z}$. Then

$$\dim V_\lambda = \Pi_{i=1}^{n-1}\Pi_{j=1}^{n-i}(1 + \frac{k_i - k_{i+j}}{j}).$$

Proof. A weight λ is the highest weight of an irreducible $\mathfrak{sl}(n)$-module if and only if λ is a dominant integer form, i.e., if

$$2(\lambda, \alpha_i)/(\alpha_i, \alpha_i) \in \mathbb{Z}_+.$$

It is known [GG] that the inner products of the weights ε_i and of weight ρ, where $\rho = (\sum_{\beta\in\Delta_+}\beta)/2$, with fundamental weights α_j are:

$$\begin{array}{ll}
(\varepsilon_1, \alpha_1) = 1/(2n),\ (\varepsilon_1, \alpha_j) = 0 & \text{for } 2 \le j \le n-1; \\
(\varepsilon_i, \alpha_{i-1}) = -1/(2n),\ (\varepsilon_i, \alpha_i) = 1/(2n),\ (\varepsilon_i, \alpha_j) = 0\ (j \ne i-1, i) & \\
 & \text{for } 2 \le i \le n-1; \\
(\varepsilon_n, \alpha_{n-1}) = -1/(2n),\ (\varepsilon_n, \alpha_j) = 0 & \text{for } 1 \le j \le n-2; \\
(\rho, \alpha_i) = 1/(2n) & \text{for } 1 \le i \le n-1.
\end{array}$$

Thus,

$$(\lambda, \alpha_i) = \frac{k_i - k_{i+1}}{2n} \text{ and } k_i \ge k_{i+1}.$$

By Weyl's character formula [GG]

$$\dim V_\lambda = \Pi_{\beta\in\Delta_+}(1 + \frac{(\lambda, \beta)}{(\rho, \beta)}).$$

For $\mathfrak{sl}(n)$ we have $\Delta_+ = \{\alpha_i + \alpha_{i+1} + \ldots + \alpha_j,\ \text{where } 1 \le i \le n-1, j \ge i\}$. Since

$$(\lambda, \alpha_i + \ldots + \alpha_j) = 1/(2n)((k_i - k_{i+1}) + (k_{i+1} - k_{i+2}) + \ldots + (k_j - k_{j+1})) = \frac{k_i - k_{j+1}}{2n},$$

we have

$$\Pi_{\beta\in\Delta_+}(1 + \frac{(\lambda, \beta)}{(\rho, \beta)}) = \Pi_{i=1}^{n-1}\Pi_{j=i}^{n-1}(1 + \frac{(\frac{k_i - k_{j+1}}{2n})}{(\frac{j-i+1}{2n})}) =$$

$$= \Pi_{i=1}^{n-1}\Pi_{j=1}^{n-i}(1 + \frac{k_i - k_{i+j}}{j}).$$

$\square$

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Tables

Table 1. Irreducible $\mathfrak{gl}(n)$-submodules of $\mathfrak{cpe}(n) \otimes V^*$

$\mathfrak{gl}(n)$-submodule	Highest weight	Highest vector
$E^2V_0^* \otimes V_0$	$\varepsilon_1 - \varepsilon_{n-1} - \varepsilon_n$ $-\varepsilon_n$	$f_{n-1} \wedge \tilde{f}_n \otimes \tilde{e}_1$ $\sum_{i=1}^n f_n \wedge \tilde{f}_i \otimes \tilde{e}_i$
$E^2V_0^* \otimes V_0^*$	$-\varepsilon_{n-1} - 2\varepsilon_n$ $-\varepsilon_{n-2} - \varepsilon_{n-1} - \varepsilon_n$	$f_{n-1} \wedge \tilde{f}_n \otimes \tilde{f}_n$ $f_{n-2} \wedge \tilde{f}_{n-1} \otimes \tilde{f}_n + f_{n-1} \wedge \tilde{f}_n \otimes \tilde{f}_{n-2}$ $+f_n \wedge \tilde{f}_{n-2} \otimes \tilde{f}_{n-1}$
$V_0^* \wedge V_0 \otimes V_0$	$2\varepsilon_1 - \varepsilon_n$ $\varepsilon_1 + \varepsilon_2 - \varepsilon_n$ ε_1 ε_1	$f_n \wedge \tilde{e}_1 \otimes \tilde{e}_1$ $f_n \wedge \tilde{e}_1 \otimes \tilde{e}_2 - f_n \wedge \tilde{e}_2 \otimes e_1$ $\sum_{i=1}^n f_i \wedge \tilde{e}_i \otimes \tilde{e}_1$ $\sum_{i=1}^n f_i \wedge \tilde{e}_1 \otimes \tilde{e}_i$
$V_0^* \wedge V_0 \otimes V_0^*$	$\varepsilon_1 - 2\varepsilon_n$ $\varepsilon_1 - \varepsilon_{n-1} - \varepsilon_n$ $-\varepsilon_n$ $-\varepsilon_n$	$f_n \wedge \tilde{e}_1 \otimes \tilde{f}_n$ $f_{n-1} \wedge \tilde{e}_1 \otimes \tilde{f}_n - f_n \wedge \tilde{e}_1 \otimes \tilde{f}_{n-1}$ $\sum_{i=1}^n f_i \wedge \tilde{e}_i \otimes \tilde{f}_n$ $\sum_{i=1}^n f_n \wedge \tilde{e}_i \otimes \tilde{f}_i$
$S^2V_0 \otimes V_0$	$3\varepsilon_1$ $2\varepsilon_1 + \varepsilon_2$	$e_1\tilde{e}_1 \otimes \tilde{e}_1$ $e_1\tilde{e}_2 \otimes \tilde{e}_1 - e_1\tilde{e}_1 \otimes \tilde{e}_2$
$S^2V_0 \otimes V_0^*$	$2\varepsilon_1 - \varepsilon_n$ ε_1	$e_1\tilde{e}_1 \otimes \tilde{f}_n$ $\sum_{i=1}^n e_1\tilde{e}_i \otimes \tilde{f}_i$
$z \otimes V_0$	ε_1	$\sum_{i=1}^n e_i\tilde{f}_i \otimes \tilde{e}_1$
$z \otimes V_0^*$	$-\varepsilon_n$	$\sum_{i=1}^n e_i\tilde{f}_i \otimes \tilde{f}_n$

Table 2. Irreducible $\mathfrak{gl}(n)$-submodules of $H^q(V_0, \mathfrak{g}_*)$

q	Highest weight	Highest vector
0	$-\varepsilon_n$ $2\varepsilon_1$ ε_1 0	f_n $e_1\tilde{e}_1$ e_1 $\tau - z$
1	$-2\varepsilon_n$ $2\varepsilon_1 - \varepsilon_n$ $\varepsilon_1 - 2\varepsilon_n$ $-\varepsilon_n$	$f_n \otimes \tilde{f}_n$ $(e_1\tilde{e}_1) \otimes \tilde{f}_n$ $(e_1 \wedge \tilde{f}_n) \otimes \tilde{f}_n$ $(\tau - z) \otimes \tilde{f}_n$
2	$-2\varepsilon_{n-1} - 2\varepsilon_n$ $2\varepsilon_1 - \varepsilon_{n-1} - \varepsilon_n$ $\varepsilon_1 - \varepsilon_{n-1} - 2\varepsilon_n$ $-\varepsilon_{n-1} - \varepsilon_n$	$(f_{n-1} \wedge \tilde{f}_n) \otimes \tilde{f}_{n-1} \wedge \tilde{f}_n$ $e_1\tilde{e}_1 \otimes \tilde{f}_{n-1} \wedge \tilde{f}_n$ $(e_1 \wedge \tilde{f}_n) \otimes \tilde{f}_{n-1} \wedge \tilde{f}_n$ $(\tau - z) \otimes \tilde{f}_{n-1} \wedge \tilde{f}_n$

Table 3. Irreducible $\mathfrak{gl}(n)$-submodules of $E_1^{p,0}$

weight λ	$\mathbf{E_1^{1,0}} = \oplus_\lambda \mathbf{V}_\lambda \otimes \mathbf{V_0}$ $\mathbf{V}_\lambda \otimes \mathbf{V_0}$	$\mathbf{E_1^{2,0}} = \oplus_\lambda \mathbf{V}_\lambda \otimes \mathbf{S^2V_0}$ $\mathbf{V}_\lambda \otimes \mathbf{S^2V_0}$	$\mathbf{E_1^{3,0}} = \oplus_\lambda \mathbf{V}_\lambda \otimes \mathbf{S^3V_0}$ $\mathbf{V}_\lambda \otimes \mathbf{S^3V_0}$
$-\varepsilon_n$	$\varepsilon_1 - \varepsilon_n$ 0	$2\varepsilon_1 - \varepsilon_n$ ε_1	$3\varepsilon_1 - \varepsilon_n$ $2\varepsilon_1$
$2\varepsilon_1$	$3\varepsilon_1$ $2\varepsilon_1 + \varepsilon_2$	$4\varepsilon_1$ $2\varepsilon_1 + 2\varepsilon_2$ $3\varepsilon_1 + \varepsilon_2$	$5\varepsilon_1$ $4\varepsilon_1 + \varepsilon_2$ $3\varepsilon_1 + 2\varepsilon_2$
ε_1	$2\varepsilon_1$ $\varepsilon_1 + \varepsilon_2$	$3\varepsilon_1$ $2\varepsilon_1 + \varepsilon_2$	$4\varepsilon_1$ $3\varepsilon_1 + \varepsilon_2$
0	ε_1	$2\varepsilon_1$	$3\varepsilon_1$

Table 4. Irreducible $\mathfrak{gl}(n)$-submodules of $E_1^{p,1}$

weight λ	$\mathbf{E_1^{0,1}} = \oplus_\lambda \mathbf{V}_\lambda$ $\mathbf{V}_\lambda$	$\mathbf{E_1^{1,1}} = \oplus_\lambda \mathbf{V}_\lambda \otimes \mathbf{V_0}$ $\mathbf{V}_\lambda \otimes \mathbf{V_0}$	$\mathbf{E_1^{2,1}} = \oplus_\lambda \mathbf{V}_\lambda \otimes \mathbf{S^2V_0}$ $\mathbf{V}_\lambda \otimes \mathbf{S^2V_0}$
$-2\varepsilon_n$	$-2\varepsilon_n$	$\varepsilon_1 - 2\varepsilon_n$ $-\varepsilon_n$	$2\varepsilon_1 - 2\varepsilon_n$ $\varepsilon_1 - \varepsilon_n$ 0
$2\varepsilon_1 - \varepsilon_n$	$2\varepsilon_1 - \varepsilon_n$	$3\varepsilon_1 - \varepsilon_n$ $2\varepsilon_1$ $2\varepsilon_1 + \varepsilon_2 - \varepsilon_n$	$4\varepsilon_1 - \varepsilon_n$ $3\varepsilon_1$ $3\varepsilon_1 + \varepsilon_2 - \varepsilon_n$ $2\varepsilon_1 + 2\varepsilon_2 - \varepsilon_n$ $2\varepsilon_1 + \varepsilon_2$
$\varepsilon_1 - 2\varepsilon_n$	$\varepsilon_1 - 2\varepsilon_n$	$2\varepsilon_1 - 2\varepsilon_n$ $\varepsilon_1 + \varepsilon_2 - 2\varepsilon_n$ $\varepsilon_1 - \varepsilon_n$	$3\varepsilon_1 - 2\varepsilon_n$ $2\varepsilon_1 - \varepsilon_n$ ε_1 $2\varepsilon_1 + \varepsilon_2 - 2\varepsilon_n$ $\varepsilon_1 + \varepsilon_2 - \varepsilon_n$
$-\varepsilon_n$	$-\varepsilon_n$	$\varepsilon_1 - \varepsilon_n$ 0	$2\varepsilon_1 - \varepsilon_n$ ε_1

Table 5. Irreducible $\mathfrak{gl}(n)$-submodules of $E_1^{p,2}$

weight λ	$\mathbf{E_1^{0,2}} = \oplus_\lambda \mathbf{V}_\lambda$ $\mathbf{V}_\lambda$	$\mathbf{E_1^{1,2}} = \oplus_\lambda \mathbf{V}_\lambda \otimes \mathbf{V_0}$ $\mathbf{V}_\lambda \otimes \mathbf{V_0}$
$-2\varepsilon_{n-1} - 2\varepsilon_n$	$-2\varepsilon_{n-1} - 2\varepsilon_n$	$\varepsilon_1 - 2\varepsilon_{n-1} - 2\varepsilon_n$ $-\varepsilon_{n-1} - 2\varepsilon_n$
$2\varepsilon_1 - \varepsilon_{n-1} - \varepsilon_n$	$2\varepsilon_1 - \varepsilon_{n-1} - \varepsilon_n$	$3\varepsilon_1 - \varepsilon_{n-1} - \varepsilon_n$ $2\varepsilon_1 - \varepsilon_n$
$\varepsilon_1 - \varepsilon_{n-1} - 2\varepsilon_n$	$\varepsilon_1 - \varepsilon_{n-1} - 2\varepsilon_n$	$2\varepsilon_1 - \varepsilon_{n-1} - 2\varepsilon_n$ $\varepsilon_1 - 2\varepsilon_n$ $\varepsilon_1 - \varepsilon_{n-1} - \varepsilon_n$
$-\varepsilon_{n-1} - \varepsilon_n$	$-\varepsilon_{n-1} - \varepsilon_n$	$\varepsilon_1 - \varepsilon_{n-1} - \varepsilon_n$ $-\varepsilon_n$

Table 6. Irreducible $\mathfrak{gl}(n)$-submodules of $E_0^{p,0}$

$\mathbf{U}$	$\mathbf{E_0^{1,0} = \oplus_U U \otimes V_0}$ $\mathbf{U \otimes V_0}$	$\mathbf{E_0^{2,0} = \oplus_U U \otimes S^2 V_0}$ $\mathbf{U \otimes S^2 V_0}$	$\mathbf{E_0^{3,0} = \oplus_U U \otimes S^3 V_0}$ $\mathbf{U \otimes S^3 V_0}$
V_0(mult2)	$2\varepsilon_1$ $\varepsilon_1 + \varepsilon_2$	$3\varepsilon_1$ $2\varepsilon_1 + \varepsilon_2$	$4\varepsilon_1$ $3\varepsilon_1 + \varepsilon_2$
V_0^*(mult2)	$\varepsilon_1 - \varepsilon_n$ 0	$2\varepsilon_1 - \varepsilon_n$ ε_1	$3\varepsilon_1 - \varepsilon_n$ $2\varepsilon_1$
$S^2 V_0$	$3\varepsilon_1$ $2\varepsilon_1 + \varepsilon_2$	$4\varepsilon_1$ $3\varepsilon_1 + \varepsilon_2$ $2\varepsilon_1 + 2\varepsilon_2$	$5\varepsilon_1$ $4\varepsilon_1 + \varepsilon_2$ $3\varepsilon_1 + 2\varepsilon_2$
$E^2 V_0^*$	$\varepsilon_1 - \varepsilon_{n-1} - \varepsilon_n$ $-\varepsilon_n$	$2\varepsilon_1 - \varepsilon_{n-1} - \varepsilon_n$ $\varepsilon_1 - \varepsilon_n$	$3\varepsilon_1 - \varepsilon_{n-1} - \varepsilon_n$ $2\varepsilon_1 - \varepsilon_n$
$\mathfrak{sl}(n)$	$2\varepsilon_1 - \varepsilon_n$ ε_1 $\varepsilon_1 + \varepsilon_2 - \varepsilon_n$	$3\varepsilon_1 - \varepsilon_n$ $2\varepsilon_1$ $2\varepsilon_1 + \varepsilon_2 - \varepsilon_n$ $\varepsilon_1 + \varepsilon_2$	$4\varepsilon_1 - \varepsilon_n$ $3\varepsilon_1 + \varepsilon_2 - \varepsilon_n$ $3\varepsilon_1$ $2\varepsilon_1 + \varepsilon_2$
$\mathbb{C}$(mult3)	ε_1	$2\varepsilon_1$	$3\varepsilon_1$

Table 7. Irreducible $\mathfrak{gl}(n)$-submodules of $E^3 V$

Space	**Highest vectors**	**Highest weights**
$E^3 V_0$	$e_1 \wedge e_2 \wedge e_3$	0
$(E^2 V_0)(V_0^*)$	$(e_1 \wedge e_2) f_3$ $\sum_{i=1}^3 (e_1 \wedge e_i) f_i$	$-2\varepsilon_3$ ε_1
$V_0(S^2 V_0^*)$	$e_1 f_3^2$ $\sum_{i=1}^3 e_i f_i f_3$	$\varepsilon_1 - 2\varepsilon_3$ $-\varepsilon_3$
$S^3 V_0^*$	f_3^3	$-3\varepsilon_3$

Table 8. Irreducible $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-submodules of $\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^*$

$\mathfrak{gl}(\mathbf{m}) \oplus \mathfrak{gl}(\mathbf{n})$**-module**	**Highest weight**	**Highest vector**
$(V \otimes V^*)/\mathbb{C} \otimes (U^* \otimes V)$	$2\varepsilon_1 - \varepsilon_m - \delta_n$ $\varepsilon_1 - \delta_n$ $\varepsilon_1 + \varepsilon_2 - \varepsilon_m - \delta_n$ (if $m \geq 3$)	$(e_1 \otimes \tilde{e}_m) \otimes (\tilde{f}_n \otimes e_1)$ $v_\lambda^1 = \sum_{i=1}^m (e_i \otimes \tilde{e}_i) \otimes (\tilde{f}_n \otimes e_1) -$ $-m \sum_{i=1}^m (e_1 \otimes \tilde{e}_i) \otimes (\tilde{f}_n \otimes e_i)$ $(e_1 \otimes \tilde{e}_m) \otimes (\tilde{f}_n \otimes e_2) -$ $-(e_2 \otimes \tilde{e}_m) \otimes (\tilde{f}_n \otimes e_1)$
$(U \otimes U^*)/\mathbb{C} \otimes (U^* \otimes V)$	$\varepsilon_1 + \delta_1 - 2\delta_n$ $\varepsilon_1 - \delta_n$ $\varepsilon_1 + \delta_1 - \delta_{n-1} - \delta_n$ (if $n \geq 3$)	$(f_1 \otimes \tilde{f}_n) \otimes (\tilde{f}_n \otimes e_1)$ $v_\lambda^2 = \sum_{i=1}^n (f_i \otimes \tilde{f}_i) \otimes (\tilde{f}_n \otimes e_1) -$ $-n \sum_{i=1}^n (f_i \otimes \tilde{f}_n) \otimes (\tilde{f}_i \otimes e_1)$ $(f_1 \otimes \tilde{f}_{n-1}) \otimes (\tilde{f}_n \otimes e_1) -$ $-(f_1 \otimes \tilde{f}_n) \otimes (\tilde{f}_{n-1} \otimes e_1)$
$\mathbb{C} \otimes (U^* \otimes V)$	$\varepsilon_1 - \delta_n$	$v_\lambda^3 = (n \sum_{i=1}^m e_i \otimes \tilde{e}_i +$ $+m \sum_{i=1}^n f_i \otimes \tilde{f}_i) \otimes (\tilde{f}_n \otimes e_1)$

Table 9. Irreducible $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-submodules of $(U^* \otimes V) \otimes (U^* \otimes V)$

$\mathfrak{gl}(\mathbf{m}) \oplus \mathfrak{gl}(\mathbf{n})$-module	Highest weight	Highest vector
$\Lambda^2 U^* \otimes S^2 V$	$2\varepsilon_1 - 2\delta_n$	$(\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_n \otimes e_1)$
$S^2 U^* \otimes \Lambda^2 V$	$\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n$	$(\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_{n-1} \otimes e_2) -$ $-(\tilde{f}_n \otimes e_2) \otimes (\tilde{f}_{n-1} \otimes e_1) -$ $-(\tilde{f}_{n-1} \otimes e_1) \otimes (\tilde{f}_n \otimes e_2) +$ $+(\tilde{f}_{n-1} \otimes e_2) \otimes (\tilde{f}_n \otimes e_1)$
$\Lambda^2 U^* \otimes \Lambda^2 V$	$\varepsilon_1 + \varepsilon_2 - 2\delta_n$	$(\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_n \otimes e_2) -$ $-(\tilde{f}_n \otimes e_2) \otimes (\tilde{f}_n \otimes e_1)$
$S^2 U^* \otimes S^2 V$	$2\varepsilon_1 - \delta_{n-1} - \delta_n$	$(\tilde{f}_{n-1} \otimes e_1) \otimes (\tilde{f}_n \otimes e_1) -$ $-(\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_{n-1} \otimes e_1)$

Table 10. Irreducible $\mathfrak{gl}(n)$-submodules of $C^{k,2}_{\mathfrak{gl}(n)}$

k	Highest weight	Highest vector
2	$\lambda_1 = \delta_1 - 3\delta_n$	$v_{\lambda_1} = (f_1 \otimes \tilde{f}_n) \otimes \tilde{f}_n^2$
	$\lambda_2 = -2\delta_n$	$v_{\lambda_2} = \sum_{j=1}^n (f_j \otimes \tilde{f}_n) \otimes \tilde{f}_j \tilde{f}_n$
	$\lambda_3 = -2\delta_n$	$v_{\lambda_3} = \sum_{j=1}^n (f_j \otimes \tilde{f}_j) \otimes \tilde{f}_n^2$
	$\lambda_4 = \delta_1 - \delta_{n-1} - 2\delta_n (\text{ if } n \geq 3)$	$v_{\lambda_4} = (f_1 \otimes \tilde{f}_{n-1}) \otimes \tilde{f}_n^2 - (f_1 \otimes \tilde{f}_n) \otimes \tilde{f}_{n-1} \tilde{f}_n$
	$\lambda_5 = -\delta_{n-1} - \delta_n$	$v_{\lambda_5} = \sum_{j=1}^n ((f_j \otimes \tilde{f}_{n-1}) \otimes \tilde{f}_j \tilde{f}_n - (f_j \otimes \tilde{f}_n) \otimes \tilde{f}_j \tilde{f}_{n-1})$
$3 \leq k \leq n+1$	$\lambda_1 = \delta_1 - \delta_{n-k+2} - \ldots - \delta_{n-1} - 3\delta_n$	$v_{\lambda_1} = (f_1 \otimes \tilde{f}_{n-k+2} \wedge \ldots \wedge \tilde{f}_n) \otimes \tilde{f}_n^2$
	$\lambda_2 = -\delta_{n-k+2} - \ldots - \delta_{n-1} - 2\delta_n$	$v_{\lambda_2} = \sum_{j=1}^n (f_j \otimes \tilde{f}_{n-k+2} \wedge \ldots \wedge \tilde{f}_n) \otimes \tilde{f}_j \tilde{f}_n$
$3 \leq k \leq n$	$\lambda_3 = -\delta_{n-k+3} - \ldots - \delta_n - 2\delta_n$	$v_{\lambda_3} = \sum_{j=1}^n (f_j \otimes \tilde{f}_j \wedge \tilde{f}_{n-k+3} \wedge \ldots \wedge \tilde{f}_n) \otimes \tilde{f}_n^2$
	$\lambda_4 = \delta_1 - \delta_{n-k+1} - \ldots - \delta_{n-1} - 2\delta_n$	$v_{\lambda_4} = \sum_{j=0}^{k-1} (-1)^{(k-1)j} (f_1 \otimes \tilde{f}_{s^j(n-k+1)} \wedge \ldots$ $\wedge \tilde{f}_{s^j(n-1)}) \otimes \tilde{f}_{s^j(n)} \tilde{f}_n$
	$\lambda_5 = -\delta_{n-k+1} - \delta_{n-k+2} - \ldots - \delta_n$	$v_{\lambda_5} = \sum_{j=0}^{k-1} (-1)^{(k-1)j} \sum_{i=1}^n (f_i \otimes \tilde{f}_{s^j(n-k+1)} \wedge \ldots$ $\wedge \tilde{f}_{s^j(n-1)}) \otimes \tilde{f}_i \tilde{f}_{s^j(n)}$
$3 \leq k \leq n-1$	$\lambda_6 = -\delta_{n-k+2} - \ldots - \delta_{n-1} - 2\delta_n$	$v_{\lambda_6} = \sum_{j=0}^{k-2} (-1)^{(k-2)j} \sum_{i=1}^n (f_i \otimes \tilde{f}_i \wedge \tilde{f}_{t^j(n-k+2)} \wedge \ldots$ $\wedge \tilde{f}_{t^j(n-1)}) \otimes \tilde{f}_{t^j(n)} \tilde{f}_n$

Table 11. Irreducible $\mathfrak{gl}(n)$-submodules of $C^{k+1,1}_{\mathfrak{gl}(n)}$

k	Highest weight	Highest vector
$2 \leq k \leq n$	$\beta_1 = \delta_1 - \delta_{n-k+1} - \ldots - \delta_{n-1} - 2\delta_n$	$v_{\beta_1} = (f_1 \otimes \tilde{f}_{n-k+1} \wedge \ldots \wedge \tilde{f}_{n-1} \wedge \tilde{f}_n) \otimes \tilde{f}_n$
	$\beta_2 = -\delta_{n-k+1} - \ldots - \delta_{n-1} - \delta_n$	$v_{\beta_2} = \sum_{j=1}^n f_j \otimes \tilde{f}_{n-k+1} \wedge \ldots \wedge \tilde{f}_{n-1} \wedge \tilde{f}_n) \otimes \tilde{f}_j$
$2 \leq k \leq n-1$	$\beta_3 = -\delta_{n-k+2} - \ldots - \delta_{n-1} - 2\delta_n$	$v_{\beta_3} = \sum_{j=6}^n (f_j \otimes \tilde{f}_j \wedge \tilde{f}_{n-k+2} \wedge \ldots \wedge \tilde{f}_{n-1} \wedge \tilde{f}_n) \otimes \tilde{f}_n$
	$\beta_4 = -\delta_{n-k+1} - \ldots - \delta_{n-1} - \delta_n$	$v_{\beta_4} = \sum_{j=0}^{k-1} (-1)^{(k-3)j} \sum_{i=1}^n (f_i \otimes \tilde{f}_i \wedge \tilde{f}_{s^j(n-k+1)} \wedge \ldots$ $\wedge f_{s^j(n-1)}) \otimes \tilde{f}_{s^j(n)}$
$2 \leq k \leq n-2$	$\beta_5 = \delta_1 - \delta_{n-k} - \ldots - \delta_{n-1} - \delta_n$	$v_{\beta_5} = \sum_{j=0}^{k} (-1)^{kj} (f_1 \otimes \tilde{f}_{r^j(n-k)} \wedge \ldots$ $\wedge \tilde{f}_{r^j(n-1)}) \otimes \tilde{f}_{r^j(n)}$

Table 12. Irreducible $\mathfrak{gl}(n)$-submodules of $\mathfrak{g}_k$

k	Highest weight	Highest vector
$2 \leq k \leq n-1$	$\gamma_1 = \delta_1 - \delta_{n-k} - \delta_{n-k+1} - \ldots - \delta_n$	$v_{\gamma_1} = f_1 \otimes \tilde{f}_{n-k} \wedge \ldots \wedge \tilde{f}_n$
$2 \leq k \leq n-2$	$\gamma_2 = -\delta_{n-k+1} - \delta_{n-k+2} - \ldots - \delta_n$	$v_{\gamma_2} = \sum_{j=1}^n f_j \otimes \tilde{f}_j \wedge \tilde{f}_{n-k+1} \wedge \ldots \wedge \tilde{f}_n$

Table 13. Structure functions of $\mathfrak{sl}(m|n)$ endowed with the standard $\mathbb{Z}$-grading

m	**n**	$H^{1,2}_{\mathfrak{g}_0}$	$H^{2,2}_{\mathfrak{g}_0}$
3	2	$2\varepsilon_1-\varepsilon_3+\delta_1-2\delta_2$	——
2	3	$2\varepsilon_1-\varepsilon_2+\delta_1-2\delta_3$	——
≥ 4	2	$2\varepsilon_1-\varepsilon_m+\delta_1-2\delta_2$	$\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_m-\delta_1-\delta_2$
2	≥ 4	$2\varepsilon_1-\varepsilon_2+\delta_1-2\delta_n$	$\varepsilon_1+\varepsilon_2+\delta_1-\delta_{n-2}-\delta_{n-1}-\delta_n$
≥ 3	≥ 3	$2\varepsilon_1-\varepsilon_m+\delta_1-2\delta_n$ $\varepsilon_1+\varepsilon_2-\varepsilon_m+\delta_1-\delta_{n-1}-\delta_n$ $\varepsilon_1-\delta_n$ (if $m=n$)	——
2	2	$2\varepsilon_1-\varepsilon_2+\delta_1-2\delta_2$ $\varepsilon_1-\delta_2$	——

Table 14. Irreducible $\mathfrak{gl}(m)\oplus\mathfrak{gl}(n)$submodules of $\mathfrak{g}_0\otimes\Lambda^2\mathfrak{g}_{-1}^*$

$\mathfrak{gl}(\mathbf{m})\oplus\mathfrak{gl}(\mathbf{n})$-module	**Highest weight**	**Highest vector**
$V\otimes V^*/\mathbb{C}\otimes\Lambda^2U^*\otimes S^2V$	$3\varepsilon_1-\varepsilon_m-2\delta_n$	$(e_1\otimes\tilde{e}_m)\otimes(\tilde{f}_n\otimes e_1)\wedge(\tilde{f}_n\otimes e_1)$
	$2\varepsilon_1-2\delta_n$	$v^1_\lambda=\sum_{i=1}^m(e_i\otimes\tilde{e}_i)\otimes(\tilde{f}_n\otimes e_1)\wedge(\tilde{f}_n\otimes e_1)-$ $m\sum_{i=1}^m(e_1\otimes\tilde{e}_i)\otimes(\tilde{f}_n\otimes e_i)\wedge(\tilde{f}_n\otimes e_1)$
	$\varepsilon_1+\varepsilon_2-2\delta_n$	$v^1_\lambda=\sum_{i=1}^m((e_1\otimes\tilde{e}_i)\otimes(\tilde{f}_n\otimes e_i)\wedge(\tilde{f}_n\otimes e_2)-$ $(e_2\otimes\tilde{e}_i)\otimes(\tilde{f}_n\otimes e_i)\wedge(\tilde{f}_n\otimes e_1))$
	$2\varepsilon_1+\varepsilon_2-\varepsilon_m-2\delta_n$ $(m\geq 3)$	$(e_1\otimes\tilde{e}_m)\otimes(\tilde{f}_n\otimes e_1)\wedge(\tilde{f}_n\otimes e_2)-$ $(e_2\otimes\tilde{e}_m)\otimes(\tilde{f}_n\otimes e_1)\wedge(\tilde{f}_n\otimes e_1)$
$V\otimes V^*/\mathbb{C}\otimes S^2U^*\otimes\Lambda^2V$	$2\varepsilon_1+\varepsilon_2-\varepsilon_m-\delta_{n-1}-\delta_n$	$(e_1\otimes\tilde{e}_m)\otimes(\tilde{f}_{n-1}\otimes e_1)\wedge(\tilde{f}_n\otimes e_2)-$ $(e_1\otimes\tilde{e}_m)\otimes(\tilde{f}_n\otimes e_1)\wedge(\tilde{f}_{n-1}\otimes e_2)$
	$2\varepsilon_1-\delta_{n-1}-\delta_n$ $(m\geq 3)$	$v^1_\lambda=\sum_{i=1}^m(e_1\otimes\tilde{e}_i)\otimes(\tilde{f}_{n-1}\otimes e_i)\wedge(\tilde{f}_n\otimes e_1)-$ $(e_1\otimes\tilde{e}_i)\otimes(\tilde{f}_n\otimes e_i)\wedge(\tilde{f}_{n-1}\otimes e_1)$
	$\varepsilon_1+\varepsilon_2-\delta_{n-1}-\delta_n$ $(m\geq 3)$	$v^1_\lambda=\sum_{i=1}^m((e_1\otimes\tilde{e}_i)\otimes(\tilde{f}_{n-1}\otimes e_i)\wedge(\tilde{f}_n\otimes e_2)-$ $(e_1\otimes\tilde{e}_i)\otimes(\tilde{f}_n\otimes e_i)\wedge(\tilde{f}_{n-1}\otimes e_2)-$ $(e_2\otimes\tilde{e}_i)\otimes(\tilde{f}_{n-1}\otimes e_i)\wedge(\tilde{f}_n\otimes e_1)+$ $(e_2\otimes\tilde{e}_i)\otimes(\tilde{f}_n\otimes e_i)\wedge(\tilde{f}_{n-1}\otimes e_1))$
	$\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_m-\delta_{n-1}-\delta_n$ $(m\geq 4)$	$\sum_{j=0}^2((e_{s^j(1)}\otimes\tilde{e}_m)\otimes(\tilde{f}_{n-1}\otimes e_{s^j(2)})\wedge(\tilde{f}_n\otimes e_{s^j(3)})-$ $(e_{s^j(1)}\otimes\tilde{e}_m)\otimes(\tilde{f}_n\otimes e_{s^j(2)})\wedge(\tilde{f}_{n-1}\otimes e_{s^j(3)}))$
$U\otimes U^*/\mathbb{C}\otimes\Lambda^2U^*\otimes S^2V$	$2\varepsilon_1+\delta_1-3\delta_n$	$(f_1\otimes\tilde{f}_n)\otimes(\tilde{f}_n\otimes e_1)\wedge(\tilde{f}_n\otimes e_1)$
	$2\varepsilon_1-2\delta_n$	$v^2_\lambda=\sum_{i=1}^n(f_i\otimes\tilde{f}_i)\otimes(\tilde{f}_n\otimes e_1)\wedge(\tilde{f}_n\otimes e_1)-$ $n\sum_{i=1}^n(f_i\otimes\tilde{f}_n)\otimes(\tilde{f}_i\otimes e_1)\wedge(\tilde{f}_n\otimes e_1)$
	$2\varepsilon_1-\delta_{n-1}-\delta_n$	$v^2_\lambda=\sum_{i=1}^n((f_i\otimes\tilde{f}_{n-1})\otimes(\tilde{f}_i\otimes e_1)\wedge(\tilde{f}_n\otimes e_1)-$ $(f_i\otimes\tilde{f}_n)\otimes(\tilde{f}_i\otimes e_1)\wedge(\tilde{f}_{n-1}\otimes e_1))$
	$2\varepsilon_1+\delta_1-\delta_{n-1}-2\delta_n$ $(n\geq 3)$	$(f_1\otimes\tilde{f}_n)\otimes(\tilde{f}_{n-1}\otimes e_1)\wedge(\tilde{f}_n\otimes e_1)-$ $(f_1\otimes\tilde{f}_{n-1})\otimes(\tilde{f}_n\otimes e_1)\wedge(\tilde{f}_n\otimes e_1)$
$U\otimes U^*/\mathbb{C}\otimes S^2U^*\otimes\Lambda^2V$	$\varepsilon_1+\varepsilon_2+\delta_1-\delta_{n-1}-2\delta_n$	$(f_1\otimes\tilde{f}_n)\otimes(\tilde{f}_{n-1}\otimes e_1)\wedge(\tilde{f}_n\otimes e_2)-$ $(f_1\otimes\tilde{f}_n)\otimes(\tilde{f}_n\otimes e_1)\wedge(\tilde{f}_{n-1}\otimes e_2)$
	$\varepsilon_1+\varepsilon_2-2\delta_n$ $(n\geq 3)$	$v^2_\lambda=\sum_{i=1}^n((f_i\otimes\tilde{f}_n)\otimes(\tilde{f}_i\otimes e_1)\wedge(\tilde{f}_n\otimes e_2)-$ $(f_i\otimes\tilde{f}_n)\otimes(\tilde{f}_n\otimes e_1)\wedge(\tilde{f}_i\otimes e_2))$
	$\varepsilon_1+\varepsilon_2-\delta_{n-1}-\delta_n$ $(n\geq 3)$	$v^2_\lambda=\sum_{i=1}^n((f_i\otimes\tilde{f}_n)\otimes(\tilde{f}_i\otimes e_1)\wedge(\tilde{f}_{n-1}\otimes e_2)-$ $(f_i\otimes\tilde{f}_n)\otimes(\tilde{f}_i\otimes e_2)\wedge(\tilde{f}_{n-1}\otimes e_1)-$ $(f_i\otimes\tilde{f}_{n-1})\otimes(\tilde{f}_i\otimes e_1)\wedge(\tilde{f}_n\otimes e_2)+$ $(f_i\otimes\tilde{f}_{n-1})\otimes(\tilde{f}_i\otimes e_2)\wedge(\tilde{f}_n\otimes e_1))$
	$\varepsilon_1+\varepsilon_2+\delta_1-\delta_{n-2}-\delta_{n-1}-\delta_n$ $(n\geq 4)$	$\sum_{j=0}^2((f_1\otimes\tilde{f}_{t^j(n-2)})\otimes(\tilde{f}_{t^j(n-1)}\otimes e_1)\wedge(\tilde{f}_{t^j(n)}\otimes e_2)-$ $(f_1\otimes\tilde{f}_{t^j(n-2)})\otimes(\tilde{f}_{t^j(n-1)}\otimes e_2)\wedge(\tilde{f}_{t^j(n)}\otimes e_1))$
$\mathbb{C}\otimes\Lambda^2U^*\otimes S^2V$	$2\varepsilon_1-2\delta_n$	$v^3_\lambda=(n\sum_{i=1}^m e_i\otimes\tilde{e}_i+$ $+m\sum_{i=1}^n f_i\otimes\tilde{f}_i)\otimes(\tilde{f}_n\otimes e_1)\wedge(\tilde{f}_n\otimes e_1)$
$\mathbb{C}\otimes S^2U^*\otimes\Lambda^2V$	$\varepsilon_1+\varepsilon_2-\delta_{n-1}-\delta_n$	$v^3_\lambda=(n\sum_{i=1}^m e_i\otimes\tilde{e}_i+$ $+m\sum_{i=1}^n f_i\otimes\tilde{f}_i)\otimes((\tilde{f}_{n-1}\otimes e_1)\wedge(\tilde{f}_n\otimes e_2)-$ $(\tilde{f}_n\otimes e_1)\wedge(\tilde{f}_{n-1}\otimes e_2))$

Table 15. Irreducible $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$-submodules of $\mathfrak{g}_1 \otimes \Lambda^2\mathfrak{g}_{-1}^*$

$\mathfrak{gl}(\mathbf{m}) \oplus \mathfrak{gl}(\mathbf{n})$-module	**Highest weight**	**Highest vector**
$(\Lambda^2 U^* \otimes U^*) \otimes (S^2 V \otimes V)$	$3\varepsilon_1 - 3\delta_n$	$(\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_n \otimes e_1)$
	$2\varepsilon_1 + \varepsilon_2 - 3\delta_n$	$(\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_n \otimes e_2) \wedge (\tilde{f}_n \otimes e_1) -$ $(\tilde{f}_n \otimes e_2) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_n \otimes e_1)$
	$3\varepsilon_1 - \delta_{n-1} - 2\delta_n$	$(\tilde{f}_{n-1} \otimes e_1) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_n \otimes e_1) -$ $(\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_{n-1} \otimes e_1) \wedge (\tilde{f}_n \otimes e_1)$
	$2\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - 2\delta_n$	$v_\lambda^1 = (\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_{n-1} \otimes e_1) \wedge (\tilde{f}_n \otimes e_2) -$ $(\tilde{f}_n \otimes e_2) \otimes (\tilde{f}_{n-1} \otimes e_1) \wedge (\tilde{f}_n \otimes e_1) -$ $(\tilde{f}_{n-1} \otimes e_1) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_n \otimes e_2) +$ $(\tilde{f}_{n-1} \otimes e_2) \otimes (\tilde{f}_n \otimes e_1) \wedge (\tilde{f}_n \otimes e_1)$
$(S^2 U^* \otimes U^*) \otimes (\Lambda^2 V \otimes V)$	$2\varepsilon_2 + \varepsilon_2 - \delta_{n-1} - 2\delta_n$	$v_\lambda^2 = (\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_{n-1} \otimes e_1) \wedge (\tilde{f}_n \otimes e_2) -$ $(\tilde{f}_n \otimes e_1) \otimes (\tilde{f}_{n-1} \otimes e_2) \wedge (\tilde{f}_n \otimes e_1)$
	$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \delta_{n-1} - 2\delta_n$ $(m \geq 3)$	$\sum_{j=0}^2((\tilde{f}_n \otimes e_{s^j(1)}) \otimes (\tilde{f}_{n-1} \otimes e_{s^j(2)}) \wedge (\tilde{f}_n \otimes e_{s^j(3)}) -$ $(\tilde{f}_n \otimes e_{s^j(1)}) \otimes (\tilde{f}_n \otimes e_{s^j(2)}) \wedge (\tilde{f}_{n-1} \otimes e_{s^j(3)})$
	$2\varepsilon_1 + \varepsilon_2 - \delta_{n-2} - \delta_{n-1} - \delta_n$ $(n \geq 3)$	$\sum_{j=0}^2((\tilde{f}_{t^j(n-2)} \otimes e_1) \otimes (\tilde{f}_{t^j(n-1)} \otimes e_2) \wedge (\tilde{f}_{t^j(n)} \otimes e_1) -$ $(\tilde{f}_{t^j(n-2)} \otimes e_1) \otimes (\tilde{f}_{t^j(n-1)} \otimes e_1) \wedge (\tilde{f}_{t^j(n)} \otimes e_2))$
	$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \delta_{n-2} - \delta_{n-1} - \delta_n$ $(m,\ n \geq 3)$	$\sum_{i=0}^2 \sum_{j=0}^2((\tilde{f}_{t^j(n-2)} \otimes e_{s^i(1)}) \otimes (\tilde{f}_{t^j(n-1)} \otimes e_{s^i(2)}) \wedge$ $(\tilde{f}_{t^j(n)} \otimes e_{s^i(3)}) - (\tilde{f}_{t^j(n-2)} \otimes e_{s^i(1)}) \otimes$ $(\tilde{f}_{t^j(n)} \otimes e_{s^i(2)}) \wedge (\tilde{f}_{t^j(n-1)} \otimes e_{s^i(3)}))$

Table 16. Spencer cohomology of $\mathfrak{sl}(\mathbf{m}|\mathbf{n})$ endowed with a $\mathbb{Z}$-grading, where $\mathfrak{g}_0 = \mathfrak{c}(\mathfrak{sl}(m|q) \oplus \mathfrak{sl}(n-q))$

$\mathbf{m}$	$\mathbf{q}$	$\mathbf{n-q}$	$\mathbf{H}_{\mathfrak{g}_0}^{1,2}$	$\mathbf{H}_{\mathfrak{g}_0}^{2,2}$
≥ 2	≥ 1	≥ 3	$m \neq q \pm 1$ $2\varepsilon_1 - \varepsilon_{m+q} + \delta_1 - 2\delta_{n-q}$ $\varepsilon_1 + \varepsilon_2 - \varepsilon_{m+q} + \delta_1 - \delta_{n-q-1} - \delta_{n-q}$ $\varepsilon_1 - \delta_{n-q}$ (if $m = n$)	—
≥ 3	≥ 1	2	$m \neq q - 1$ $2\varepsilon_1 - \varepsilon_{m+q} + \delta_1 - 2\delta_2$ $\varepsilon_1 - \delta_2$ (if $m = n$)	$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_{m+q} - \delta_1 - \delta_2$
2	≥ 2	2	$q \neq 3$ $2\varepsilon_1 - \varepsilon_{q+2} + \delta_1 - 2\delta_2$	$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_{q+2} - \delta_1 - \delta_2$
2	1	2	$2\varepsilon_1 - \varepsilon_3 + \delta_1 - 2\delta_2$	$\varepsilon_1 + \varepsilon_3 - \delta_1 - \delta_2$
1	≥ 1	≥ 3	$q \neq 2$ $2\varepsilon_1 - \varepsilon_{q+1} + \delta_1 - 2\delta_{n-q}$ $\varepsilon_1 + \varepsilon_2 - \varepsilon_{q+1} + \delta_1 - \delta_{n-q-1} - \delta_{n-q}$	—
1	≥ 1	2	$q \neq 2$ $2\varepsilon_1 - \varepsilon_{q+1} + \delta_1 - 2\delta_2$	$\varepsilon_1 + 2\varepsilon_2 - \varepsilon_{q+1} - \delta_1 - \delta_2$ $(q \neq 1)$ $2\varepsilon_2 - \delta_1 - \delta_2$ $(q = 1)$
0	2	2	—	$3\varepsilon_1 - \varepsilon_2 - \delta_1 - \delta_2$ $\varepsilon_1 + \varepsilon_2 + \delta_1 - 3\delta_2$
0	2	≥ 3	$2\varepsilon_1 - \varepsilon_2 + \delta_1 - \delta_{n-3} - \delta_{n-2}$	$\varepsilon_1 + \varepsilon_2 + \delta_1 - 3\delta_{n-2}$
0	≥ 3	2	$\varepsilon_1 + \varepsilon_2 - \varepsilon_q + \delta_1 - 2\delta_2$	$3\varepsilon_1 - \varepsilon_q - \delta_1 - \delta_2$
0	≥ 3	≥ 3	$2\varepsilon_1 - \varepsilon_q + \delta_1 - \delta_{n-q-1} - \delta_{n-q}$ $\varepsilon_1 + \varepsilon_2 - \varepsilon_q + \delta_1 - 2\delta_{n-q}$	—

Table 17. Spencer cohomology of $\mathfrak{sl}(m|n)$ endowed with a $\mathbb{Z}$-grading, where $\mathfrak{g}_0 = \mathfrak{c}(\mathfrak{sl}(m-p|q) \oplus \mathfrak{sl}(p|n-q))$

$\mathbf{m-p}$	$\mathbf{q}$	$\mathbf{p}$	$\mathbf{n-q}$	$\mathbf{H^{1,2}_{\mathfrak{g}_0}}$	$\mathbf{H^{2,2}_{\mathfrak{g}_0}}$
0	2	2	0	$2\varepsilon_1-\varepsilon_2+\delta_1-2\delta_2$ $\varepsilon_1-\delta_2$	——
0	2	3	0	$2\varepsilon_1-\varepsilon_2+\delta_1-2\delta_3$	——
0	3	2	0	$2\varepsilon_1-\varepsilon_3+\delta_1-2\delta_2$	——
0	2	≥ 4	0	$2\varepsilon_1-\varepsilon_2+\delta_1-2\delta_p$	$\varepsilon_1+\varepsilon_2+\delta_1-\delta_{p-2}-\delta_{p-1}-\delta_p$
0	≥ 4	2	0	$2\varepsilon_1-\varepsilon_q+\delta_1-2\delta_2$	$\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_q-\delta_1-\delta_2$
0	≥ 3	≥ 3	0	$2\varepsilon_1-\varepsilon_q+\delta_1-2\delta_p$ $\varepsilon_1+\varepsilon_2-\varepsilon_q+\delta_1-\delta_{p-1}-\delta_p$ $\varepsilon_1-\delta_p$ $(m=n)$	——
0	2	≥ 1	≥ 1	$n \neq p+q+1$ $2\varepsilon_1-\varepsilon_2+\delta_1-\delta_{p+n-3}-\delta_{p+n-2}$ $\varepsilon_1-\delta_{p+n-2}$ $(m=n)$	$\varepsilon_1+\varepsilon_2+\delta_1-3\delta_{p+n-2}$
≥ 1	≥ 1	2	0	$m \neq p+q+1$ $\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-2\delta_2$ $\varepsilon_1-\delta_2$ $(m=n)$	$3\varepsilon_1-\varepsilon_{m-p+q}-\delta_1-\delta_2$
0	≥ 3	≥ 1	≥ 1	$n \neq p+q\pm 1$ $\varepsilon_1+\varepsilon_2-\varepsilon_q+\delta_1-2\delta_{p+n-q}$ $2\varepsilon_1-\varepsilon_q+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}$ $\varepsilon_1-\delta_{p+n-q}$ $(m=n)$	——
≥ 1	≥ 1	≥ 3	0	$m \neq p+q\pm 1$ $2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-\delta_{p-1}-\delta_p$ $\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-2\delta_p$ $\varepsilon_1-\delta_p$ $(m=n)$	——
≥ 1	≥ 1	≥ 1	≥ 1	$m,n \neq p+q\pm 1$ $2\varepsilon_1-\varepsilon_{m-p+q}+\delta_1-2\delta_{p+n-q}$ $\varepsilon_1+\varepsilon_2-\varepsilon_{m-p+q}+\delta_1-\delta_{p+n-q-1}-\delta_{p+n-q}$ $\varepsilon_1-\delta_{p+n-q}$ $(m=n\geq 3)$	——

Table 18. Spencer cohomology of $\mathfrak{osp}(m|2n)$

$\mathbf{r}$	$\mathbf{n}$	$\mathbf{H^{2,2}_{\mathfrak{g}_0}}$	$\mathbf{S^2(\mathfrak{g}_{-1})}$
0	1	——	0 (if $m=2$) $0,\ \delta_1$ (if $m=3$)
0	≥ 2	$2\delta_1+2\delta_2$	$\delta_1+\delta_2,\ 0$
1	1	$\varepsilon_1+\delta_1$	$2\varepsilon_1,\ 0$
1	≥ 2	$2\varepsilon_1+\delta_1+\delta_2$	$2\varepsilon_1,\ 0$
≥ 2	≥ 1	$2\varepsilon_1+2\varepsilon_2$	$2\varepsilon_1,\ 0$