Contact Dehn surgery, symplectic fillings, and Property P for knots

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Mathematische Arbeitstagung June 10-16, 2005
Max-Planck-Institut für Mathematik, Bonn, Germany

1 Property P for knots

According to a fundamental theorem of Lickorish and Wallace from the 1960s, every closed, orientable 3–manifold can be obtained by performing Dehn surgery on a link in the 3–sphere. Previous to the recent work of Perelman, which is expected to close the coffin on the Poincaré conjecture, it was a natural question for geometric topologists whether one might be able to produce a counterexample to that conjecture by a single Dehn surgery. This led to the definition of the following property, whose name is generally regarded as a little unfortunate.

Definition. A knot K in S^3 has **Property P** if every nontrivial surgery along K yields a non-simply-connected 3–manifold.

Our knots are always understood to be smooth, or at least tame, i.e. equivalent to a smooth one.

Let me briefly recall the notion of Dehn surgery along a knot K in the 3–sphere S^3. Write $\nu K \cong S^1 \times D^2$ for a (closed) tubular neighbourhood of K. On the boundary $\partial(\nu K) \cong T^2$ of this tubular neighbourhood there are two distinguished curves (which we implicitly identify with the classes they represent in the homology group $H_1(T^2)$):

1. The meridian μ, defined as a simple closed curve that generates the kernel of the homomorphism on H_1 induced by the inclusion $T^2 \to \nu K$.

2. The preferred longitude λ, defined as a simple closed curve that generates the kernel of the homomorphism on H_1 induced by the inclusion $T^2 \to C := \overline{S^3 \setminus \nu K}$.

This preferred longitude can also be characterised by the property that it has linking number zero with K. The knot K bounds an embedded surface in S^3 (called a **Seifert surface** for K), and λ can be obtained by pushing K along that surface. For that reason, the trivialisation of the normal bundle of K defined by λ is called the **surface framing** of K.

Given an orientation of S^3, orientations of μ and λ are chosen such that (μ, λ) is a positive basis for $H_1(T^2)$, with T^2 oriented as the boundary of νK. In the contact geometric setting below, the orientation of S^3 will be the one induced from the contact structure.

Let p, q be coprime integers. The manifold $K_{p/q}$ obtained from S^3 by **Dehn surgery** along K with **surgery coefficient** $p/q \in \mathbb{Q} \cup \{\infty\}$ is defined as

$$K_{p/q} := \overline{S^3 \setminus \nu K} \cup_g S^1 \times D^2,$$

where the gluing map g sends the meridian $* \times \partial D^2$ to $p\mu + q\lambda$. The resulting manifold is completely determined by the knot and the surgery coefficient.

A simple Mayer-Vietoris argument shows that $H_1(K_{p/q}) \cong \mathbb{Z}_{|p|}$. Therefore, saying that a knot K has Property P is equivalent to

$$\pi_1(K_{1/q}) = 1 \quad \text{only for} \quad q = 0.$$

(Observe that $p/q = \infty$ corresponds to a trivial surgery.)

Example. The unknot does *not* have Property P. Indeed, every $(1/q)$–surgery on the unknot yields S^3, which is seen as follows. If K is the unknot, then the closure C of $S^3 \setminus \nu K$ is also a solid torus. Write μ_C and λ_C for meridian and preferred longitude on ∂C. We may assume $\mu = \lambda_C$ and $\lambda = -\mu_C$. When performing $(1/q)$–surgery on K, a solid torus is glued to C by sending its meridian μ_0 to $\mu + q\lambda = \lambda_C - q\mu_C$. Now, there clearly is a diffeomorphism of C that sends μ_C to itself and λ_C to $\lambda_C - q\mu_C$. It follows that the described surgery is equivalent to the one where we send μ_0 to $\lambda_C = \mu$, which is a trivial ∞–surgery.

In the early 1970s, Bing and Martin, as well as González-Acuña, conjectured that every nontrivial knot has Property P. By work of Kronheimer and Mrowka [9], this is now a theorem.

Theorem 1 (Kronheimer-Mrowka). *Every nontrivial knot in S^3 has Property P.*

Before describing the role that contact geometry has played in the proof of this theorem, I want to indicate the importance of this theorem beyond the negative statement that counterexamples to the Poincaré conjecture cannot result from a single surgery.

Proposition 2. *If two knots K, K' in S^3 have homeomorphic complements and one of the knots has property P, then the knots are equivalent, i.e. there is a homeomorphism of S^3 mapping K to K'.*

Proof. According to a result of Edwards [3], two compact 3–manifolds with boundary are homeomorphic if and only if their interiors are homeomorphic. Thus, if $S^3 \setminus K$ is homeomorphic to $S^3 \setminus K'$, then there is a homeomorphism $\varphi\colon\ C \to C'$, where $C := \overline{S^3 \setminus \nu K}$ and $C' := \overline{S^3 \setminus \nu K'}$.

Suppose K has Property P. This implies that there is a unique way of attaching a solid torus $S^1 \times D^2$ to C such that the resulting manifold is the 3–sphere. Hence φ extends to a homeomorphism $S^3 \to S^3$, i.e. the knots K and K' are equivalent. $\square$

Observe that in this proof we only used the weaker property that nontrivial surgery along K does not yield the standard 3–sphere. This had been proved earlier (for K different from the unknot) by Gordon and Luecke [8]. Since the unknot is characterised by its complement being a solid torus, the result of Kronheimer and Mrowka (or the weaker one by Gordon and Luecke) yields the following corollary.

Corollary 3. *If two knots in S^3 have homeomorphic complements, then the knots are equivalent.* □

Of course, together with a positive answer to the Poincaré conjecture, the result of Gordon-Luecke implies that of Kronheimer-Mrowka.

2 Contact Dehn surgery

This section gives a brief report on joint work with Fan Ding [1]. Recall that a (coorientable) **contact structure** ξ on a differential 3–manifold is a tangent 2–plane field defined as the kernel of a global differential 1–form α that satisfies the nonintegrability condition $\alpha \wedge d\alpha \neq 0$ (meaning that $\alpha \wedge d\alpha$ vanishes nowhere). An example is the standard contact structure

$$\xi_{st} = \ker(x\,dy - y\,dx + z\,dt - t\,dz)$$

on $S^3 \subset \mathbb{R}^4$. This can also be characterised as the complex line in the tangent bundle of S^3 with respect to complex multiplication induced from the inclusion $S^3 \subset \mathbb{C}^2$.

I shall have to use a few notions from contact geometry without time for much explanation (tight and overtwisted contact structures, convex surfaces in contact 3–manifolds). For more details see the introductory lectures by Etnyre [5] or the *Handbook* chapter by the present author [7].

A (smooth) knot K in a contact 3–manifold (M, ξ) is called **Legendrian** if it is everywhere tangent to ξ. The normal bundle of such a knot has a canonical trivialisation, determined by a vector field along K that is everywhere transverse to ξ. This will be referred to as the **contact framing**. We now consider Dehn surgery along K with coefficient p/q as before, but we define the surgery coefficient with respect to the contact framing.

It turns out that for $p \neq 0$, one can always extend the contact structure $\xi|_{M \setminus \nu K}$ to one on the surgered manifold in such a way that the extended contact structure is tight on the glued-in solid torus $S^1 \times D^2$. Moreover, subject to this tightness condition there are but finitely many choices for such an extension, and for $p/q = 1/k$ with $k \in \mathbb{Z}$ the extension is in fact unique. These observations hinge on the fact that $\partial(\nu K)$ is a convex surface, i.e. a surface admitting a transverse flow preserving the contact structure. On solid tori with convex boundary condition, tight contact structures have been classified by Giroux and Honda.

We can therefore speak sensibly of **contact** $(1/k)$**–surgery**. The following theorem is proved in [1].

Theorem 4. *Let (M, ξ) be a closed, connected contact 3–manifold. Then (M, ξ) can be obtained from (S^3, ξ_{st}) by contact (± 1)–surgery along a Legendrian link.*

Remarks. (1) There is a related theorem, due to Lutz-Martinet in the early 1970s, cf. [7], saying that every (closed, orientable) 3–manifold admits a contact structure in each homotopy class of tangent 2–plane fields. The original proof is based on surgery along a link in S^3 *transverse* to ξ_{st}. For an alternative proof using Legendrian surgery see [2].

(2) From the topological point of view, surgeries with integer surgery coefficient are best, since they correspond to attaching 2–handles to the boundary of a 4–manifold. Thus, contact (± 1)–surgeries are best from both the topological and contact geometric viewpoint.

(3) Contact $(-1/k)$–surgery is the inverse of contact $(1/k)$–surgery (along appropriately related knots).

(4) Contact (-1)–surgery is symplectic handlebody surgery in the sense of Eliashberg and Weinstein, cf. [2], and preserves the property of being strongly symplectically fillable (see below).

3 Symplectic fillings

Contact geometry enters the proof of Theorem 1 via the notion of symplectic fillings. Observe that a contact 3–manifold (M, ξ) is naturally oriented — the sign of the volume form $\alpha \wedge d\alpha$ does not depend on the choice of 1–form α defining a given ξ; similarly, a symplectic 4–manifold (W, ω), i.e. with ω a closed 2–form satisfying $\omega^2 \neq 0$, is naturally oriented by the volume form ω^2.

Definition. (a) The symplectic 4–manifold (W, ω) is called a **weak (symplectic) filling** of the contact manifold (M, ξ) if $\partial W = M$ as oriented manifolds (outward normal followed by orientation of M gives orientation of W) and $\omega|_\xi \neq 0$.

(b) The symplectic 4–manifold (W, ω) is called a **strong (symplectic) filling** of the contact manifold (M, ξ) if $\partial W = M$ and there is a Liouville vector field X defined near ∂W, pointing outwards along ∂W, and satisfying $\xi = \ker(i_X \omega|_{TM})$. Here **Liouville vector field** means that the Lie derivative $\mathcal{L}_X \omega$, which is the same as $d(i_X \omega)$ because of $d\omega = 0$ and Cartan's formula, is required to be equal to ω.

For instance, (S^3, ξ_{st}) is strongly filled by the standard symplectic 4–disc D^4 with $\omega_{st} = dx \wedge dy + dz \wedge dt$. The Liouville vector field here is the radial vector field $X = r\partial_r/2$.

It is clear that every strong filling is also a weak filling. The converse is false: There are contact structures that are weakly but not strongly fillable; such examples are due to Eliashberg and Ding-Geiges.

The contact geometric result that allowed Kronheimer and Mrowka to conclude their proof of Property P was first proved by Eliashberg [4].

Theorem 5 (Eliashberg). *Any weak symplectic filling of a contact 3–manifold embeds symplectically into a closed symplectic 4–manifold.*

An alternative proof was given by Etnyre [6]. Both proofs rely on open book decompositions adapted to contact structures. Theorem 5 being a cobordism theoretic result, it is arguably more natural to give a surgical proof. Özbağcı and Stipsicz [10] were the first to observe that such a proof, based on Theorem 4, can indeed be devised. In the remainder of this section, I shall sketch this surgical argument.

Theorem 5 is proved by showing that any contact 3–manifold can be capped off symplectically, or has what is called a **concave** filling that can be glued to the given (convex) filling. (For instance, a strong concave filling corresponds to a Liouville vector field pointing inwards along the boundary.) Such a cap, attached to the (convex) symplectic filling of the contact manifold, gives the desired closed symplectic manifold.

(i) Strong fillings can be capped off: Let (W, ω) be a strong filling of (M, ξ). By Theorem 4, there is a Legendrian link $\mathbb{L} = \mathbb{L}^- \sqcup \mathbb{L}^+$ in (S^3, ξ_{st}) such that contact (-1)–surgery along the components of $\mathbb{L}^-$ and contact $(+1)$–surgery along those of $\mathbb{L}^+$ produces (M, ξ). By Remarks (3) and (4) we can attach symplectic 1–handles to the boundary (M, ξ) of (W, ω) corresponding to contact (-1)–surgeries that undo the contact $(+1)$–surgeries along $\mathbb{L}^+$. The result will be a symplectic manifold (W', ω') strongly filling a contact manifold (M', ξ'), and the latter can be obtained from $(S^3, \xi_{st}) = \partial(D^4, \omega_{st})$ by performing contact (-1)–surgeries (along $\mathbb{L}^-$) only.

A handlebody obtained from (D^4, ω_{st}) by attaching symplectic handles in this way is in fact a Stein filling of its boundary contact manifold, and for those a symplectic cap had been found earlier by Akbulut-Özbağcı and Lisca-Matić. The cap that fits on the Stein filling also fits on the strong filling (W', ω'), since strongly convex and strongly concave fillings of a given contact manifold can always be glued together, using the Liouville flow to define collar neighbourhoods of the boundary.

(ii) Reduce the problem to the consideration of homology spheres only: Let (W, ω) be a weak filling of (M, ξ). We want to attach a (weak) symplectic cobordism from (M, ξ) to some integral homology sphere Σ^3 with contact structure ξ', so as to get a weak filling of (Σ^3, ξ') containing (M, ξ) as a separating hypersurface.

We start from a contact surgery presentation of (M, ξ) as in (i). For each component L_i of $\mathbb{L}$ we choose a Legendrian knot K_i in (S^3, ξ_{st}) only linked with that component, with linking number 1. These K_i can be chosen in such a way that surgery with framing -1 relative to the contact framing is the same as surgery with coefficient 0 relative to the surface framing. (In case you know the term: The Thurston-Bennequin invariant of K_i can be chosen to be equal to 1). Performing these surgeries has the effect of killing all integral homology.

Since ω is exact in the neighbourhood $S^1 \times D^2 \times (-\varepsilon, 0]$ of a Legendrian knot in the boundary (M, ξ) of (W, ω), these surgeries can be performed by attaching symplectic handles as in the case of a strong filling. The collection of these handles gives the desired (weak) symplectic cobordism.

(iii) Pass from a weak filling of a homology sphere to a strong filling: We begin with the symplectic manifold (W', ω') with boundary (Σ^3, ξ') constructed in (ii). We want to modify ω' in a collar neighbourhood $\Sigma^3 \times [0,1]$ of the boundary $\Sigma^3 \equiv \Sigma^3 \times \{1\}$ such that the resulting symplectic manifold is a strong filling of the new induced contact structure ξ'' on the boundary. By (i) this can then be capped off.

Since $H^2(\Sigma^3) = 0$, we can write $\omega = d\eta$ with some 1–form η in a collar neighbourhood as described. (We see that it would be enough to have Σ^3 a rational homology sphere.) Choose a 1-form α on Σ^3 with $\xi' = \ker \alpha$ and $\alpha \wedge \omega|_{T\Sigma^3} > 0$, which is possible for a weak filling. Then set

$$\tilde{\omega} = d(f\eta) + d(g\alpha)$$

on $\Sigma^3 \times [0,1]$, where the smooth functions $f(t)$ and $g(t)$, $t \in [0,1]$, are chosen as follows: Fix a small $\varepsilon > 0$. Choose $f\colon [0,1] \to [0,1]$ identically 1 on $[0,\varepsilon]$ and identically 0 near 1. Choose $g\colon [0,1] \to \mathbb{R}_0^+$ identically 0 near 0 and with $g'(t) > 0$ for $t > \varepsilon/2$.

We compute

$$\tilde{\omega} = f'\, dt \wedge \eta + f\omega + g'\, dt \wedge \alpha + g\, d\alpha,$$

whence

$$\begin{aligned} \tilde{\omega}^2 \;\; = \;\; & ff'\, dt \wedge \eta \wedge \omega + f'g\, dt \wedge \eta \wedge d\alpha + f^2\omega^2 \\ & + fg'\, \omega \wedge dt \wedge \alpha + fg\, \omega \wedge d\alpha + gg'\, dt \wedge \alpha \wedge d\alpha. \end{aligned}$$

The terms appearing with the factors f^2, fg' and gg' are positive volume forms. By choosing g small on $[0,\varepsilon]$ and g' large compared with $|f'|$, one can ensure that these positive terms dominate the three terms we cannot control. Then $\tilde{\omega}$ is a symplectic form on the collar, and in terms of the coordinate $s = \log g(t)$, the symplectic form looks like $d(e^s\alpha)$ near the boundary, with Liouville vector field ∂_s.

4 Proof of Property P for nontrivial knots

Here is a very rough sketch of the proof by Kronheimer and Mrowka. It relies heavily on pretty much everything known under the sun about gauge theory.

Let K be a nontrivial knot. It had been proved earlier by Culler-Gordon-Luecke-Shalen that $\pi_1(K_{1/q})$ is nontrivial for $q \notin \{0, \pm 1\}$. It therefore suffices to find a nontrivial homomorphism $\pi_1(K_1) \to \mathrm{SO}(3)$.

Arguing by contradiction, we assume that no such homomorphism exists. This implies the vanishing of the instanton Floer homology group $HF(K_1)$. By the Floer exact triangle one finds that the group $HF(K_0)$ vanishes likewise, and so does the Fukaya-Floer homology group.

For K nontrivial, results of Gabai say that K_0 is different from $S^1 \times S^2$ and admits a taut 2–dimensional foliation. Eliashberg and Thurston, in their theory of confoliations, deduce from this the existence of a symplectic structure on $K_0 \times [-1,1]$ weakly filling contact structures on the boundary components. According to Theorem 5, by capping off these boundaries we find a symplectic manifold V containing K_0 as a separating hypersurface (and satisfying some mild cohomological conditions).

Now, on the one hand, the Donaldson invariants of V can be expressed as a pairing on the Fukaya-Floer homology group of K_0 and therefore have to vanish.

On the other hand, results of Taubes say that the Seiberg-Witten invariants of V are nontrivial. By a conjecture of Witten, proved in the relevant case by Feehan-Leness, the Donaldson invariants are likewise nontrivial. This contradiction proves Theorem 1.

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