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GEOGRAPHY OF IRREDUCIBLE PLANE SEXTICS

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Abstract. We complete the equisingular deformation classification of irreducible singular plane sextic curves. As a by-product, we also compute the fundamental groups of the complement of all but a few maximizing sextics.

1. Introduction

Throughout the paper, all varieties are over the field $\mathbb{C}$ of complex numbers.

Our principal result is the completion of the classification of irreducible plane sextics (curves of degree 6) up to equisingular deformation. We confine ourselves to *simple* sextics only, *i.e.*, those with **A**–**D**–**E** singularities (see §2.2). The non-simple ones require completely different techniques and are well known; surprisingly, their study is much easier: the statements were announced by the second author long ago, and formal proofs can be found in [13]. Note also that degree 6 is the first nontrivial case (see [13] for the statements on quintics; quartics were known to Klein) and, probably, the last case that can be completely understood, thanks to the close relation between plane sextics and $K3$-surfaces.

The systematic study of simple sextics based on the theory of $K3$-surfaces was initiated by U. Persson [28], who proved that the total Milnor number μ of such a curve does not exceed 19. Based on this approach, T. Urabe [30] listed the possible sets of singularities with $\mu \leqslant 16$, and this result was extended to a complete list of the sets of singularities realized by simple sextics by J. G. Yang [31]. Later, using the arithmetical reduction [8], I. Shimada [29] gave a complete description of the moduli spaces of the *maximizing* ($\mu = 19$) sextics. In the meanwhile, a number of independent (not explicitly related to the $K3$-surfaces) attempts to attack the classification problem has also been made, see, *e.g.*, [2, 3] (defining equations of a number of maximizing sextics), [25, 26] (sets of singularities and explicit equations of sextics of torus type), [9, 10, 11] (sextics admitting stable projective symmetries), [13] (sextics with a triple point), *etc.*

At some point it was clearly understood, partially in conjunction with Oka's conjecture [18] and partially due to the arithmetical reduction of the problem [8], that irreducible sextics D should be subdivided into classes according to the maximal generalized dihedral quotient Q_D that the fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus D)$ admits. If this quotient is large, $|Q_D| > 6$, the curves are relatively few in number and can easily be listed manually (see [7] and §2.5), using Nikulin's sufficient uniqueness conditions [24]. The present paper fills the gap and covers the two remaining cases: non-special sextics ($Q_D = 0$, see Theorem 2.4) and 1-torus sextics ($Q_D = \mathbb{D}_6$, see Theorem 2.9). On the arithmetical side, our computation is based on the stronger (non-)uniqueness criteria due to Miranda–Morrison [21, 22, 23]. For an even further

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illustration of the power of [23], we solve a few more subtle geometric problems, namely, we compute the monodromy representation of the fundamental groups of the equisingular strata (in other words, we classify sextics with marked singular points, see §4.7 and Theorem 4.10), we discuss whether the strata are real and whether they contain real curves (the interesting discovery here is Proposition 2.6), and we give a complete description of the adjacencies of the strata (see §6.5 and Propositions 6.5, 6.7, 6.8).

There are three sets of singularities that deserve special attention: to the best of our knowledge, phenomena of this kind have not been observed before. It is quite common that the (discrete) moduli spaces of maximizing sextics are disconnected, see [29]. For about a dozen of the sets of singularities with $\mu = 18$, the moduli space (of dimension 1) consists of two complex conjugate components (see Table 4; the first such example, *viz.* $\mathbf{E}_6 \oplus \mathbf{A}_{11} \oplus \mathbf{A}_1$, was found in [1]). We discover a set of singularities, *viz.* $\mathbf{E}_6 \oplus 2\mathbf{A}_5 \oplus \mathbf{A}_1$, with $\mu = 17$ and disconnected moduli space (two conjugate components of dimension 2), and another one, $2\mathbf{A}_9$, with $\mu = 18$ and the moduli space consisting of two disjoint *real* components (see Proposition 2.5). Finally, the moduli space corresponding to the set of singularities $\mathbf{A}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5$, $\mu = 18$, consists of a single component, which is hence real, but it contains no real curves (see Proposition 2.6).

As another important by-product of Theorems 2.4 and 2.9, we obtain Corollaries 2.8 and 2.11, computing the fundamental groups of the complements of all but a few maximizing irreducible sextics. In fact, no computation is found in this paper: we merely use the classification, the degeneration principle, and previously known groups. Most statements on the fundamental groups were known conjecturally; more precisely, the groups of *some* sextics with certain sets of singularities were known, and our principal contribution is the connectedness of the moduli spaces.

1.1. **Contents of the paper.** The principal results of the paper are stated in §2, after the necessary terminology and notation have been introduced. For the reader's convenience, we also discuss the other irreducible simple sextics (see §2.5) and list the known fundamental groups. In §3, we recall the fundamentals of Nikulin's theory of discriminant forms and lattice extensions, give a brief introduction to Miranda–Morrison's theory [23], and recast some of their results in a form more suitable for our computations. In §4, we recall the notion of (abstract) homological type and the arithmetical reduction [8] of the classification problem (see §4.1 and §4.2) and begin the proof of our principal results, classifying the plane sextics up to equisingular deformation *and* complex conjugation. As a digression, we classify also sextics with marked singular points, see §4.7. With the classification in hand, the computation of the fundamental groups is almost straightforward; it is outlined in §5. Finally, in §6, we discuss real strata and real curves, completing the deformation classification of simple sextics. As another digression, in §6.5 we describe the adjacencies of the non-real strata.

1.2. **Acknowledgements.** We are grateful to V. Nikulin, who drew our attention to Miranda–Morrison's works [21, 22]. To a large extent, this text was written during the second author's stay at *Max-Planck-Institut für Mathematik*, partially supported by the "Tropical Geometry and Topology" program. We would like to extend our gratitude to the institute and its friendly staff and to the organizers of the program.

2. Principal results

2.1. **Notation.** We use the notation $\mathbb{G}_n := \mathbb{Z}/n\mathbb{Z}$ (reserving $\mathbb{Z}_p$ and $\mathbb{Q}_p$ for p-adic numbers) and $\mathbb{D}_{2n}$ for the cyclic group of order n and dihedral group of order $2n$, respectively. As usual, $SL(n, \Bbbk)$ is the group of $(n \times n)$-matrices M over a field $\Bbbk$ such that $\det M = 1$.

The notation $\mathbb{B}_n$ stands for the braid group on n stings. The *reduced braid group* (or the *modular group*) is the quotient $\Gamma = \mathbb{B}_3/(\sigma_1\sigma_2)^3$ of $\mathbb{B}_3$ by its center; one has $\Gamma = PSL(2, \mathbb{Z}) = \mathbb{G}_2 * \mathbb{G}_3$. The braid group is generated by the *Artin generators* σ_i, $i = 1, \ldots, n-1$, subject to the relations

$$[\sigma_i, \sigma_j] = 1 \quad \text{if } |i - j| > 1, \qquad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}.$$

Throughout the paper, all group actions are right, and we use the notation $(x, g) \mapsto x \uparrow g$. The standard action of $\mathbb{B}_n$ on the free group $\langle \alpha_1, \ldots, \alpha_n \rangle$ is as follows:

$$\sigma_i : \begin{cases} \alpha_i \mapsto \alpha_i\alpha_{i+1}\alpha_i^{-1}, \\ \alpha_{i+1} \mapsto \alpha_i, \\ \alpha_j \mapsto \alpha_j, & \text{if } j \neq i, i+1 \end{cases}$$

The element $\rho_n := \alpha_1 \ldots \alpha_n \in \langle \alpha_1, \ldots, \alpha_n \rangle$ is preserved by $\mathbb{B}_n$. Given a pair α_1, α_2, we use the notation $\{\alpha_1, \alpha_2\}_n := \alpha_2^{-1}(\alpha_2 \uparrow \sigma_1^n) \in \langle \alpha_1, \alpha_2 \rangle$ for $n \in \mathbb{Z}$.

We denote by $\mathbb{P} = \{2, 3, \ldots\}$ the set of all primes.

The group of units of a commutative ring R is denoted by $R^\times$. We recall that $\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2 = \{\pm 1\}$ for $p \in \mathbb{P}$ odd, and $\mathbb{Z}_2^\times/(\mathbb{Z}_2^\times)^2 = (\mathbb{Z}/8)^\times \cong \{\pm 1\} \times \{\pm 1\}$ is generated by $7 \bmod 8$ and $5 \bmod 8$. If $m \in \mathbb{Z}$ is prime to p, its class in $\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$ is the Legendre symbol $(\frac{m}{p}) \in \{\pm 1\}$ if p is odd or $m \bmod 8 \in (\mathbb{Z}/8)^\times$ if $p = 2$.

2.2. **Simple sextics.** A *sextic* is a plane curve $D \subset \mathbb{P}^2$ of degree six. A sextic is *simple* if all its singular points are simple, *i.e.*, those of type $\mathbf{A}$–$\mathbf{D}$–$\mathbf{E}$, see [16]. If this is the case, the minimal resolution of singularities X of the double covering of $\mathbb{P}^2$ ramified at D is a $K3$-surface. The intersection index form $H_2(X) \cong 2\mathbf{E}_8 \oplus 3\mathbf{U}$ is (the only) even unimodular lattice of signature $(\sigma_+, \sigma_-) = (3, 19)$ (see §3.4; here, $\mathbf{U}$ is the hyperbolic plane). We fix the notation $\mathbf{L} := 2\mathbf{E}_8 \oplus 3\mathbf{U}$.

For each simple singular point P of D, the components of the exceptional divisor $E \subset X$ over P span a root lattice in $\mathbf{L}$ (see §3.3). The (obviously orthogonal) sum of these sublattices is denoted by $\mathbf{S}(D)$ and is referred to as the *set of singularities* of D. (Recall that the types of the individual singular points are uniquely recovered from $\mathbf{S}(D)$, see §3.3.) The rank $\operatorname{rk}\mathbf{S}(D)$ equals the total Milnor number $\mu(D)$. Since $\mathbf{S}(D) \subset \mathbf{L}$ is negative definite, one has $\mu(D) \leqslant 19$, see [28]. If $\mu(D) = 19$, the sextic D is called *maximizing*. We emphasize that both the inequality and the term apply to simple sextics only.

An irreducible sextic $D \subset \mathbb{P}^2$ is called *special* (more precisely, $\mathbb{D}_{2n}$*-special*) if its fundamental group $\pi_1 := \pi_1(\mathbb{P}^2 \smallsetminus D)$ factors to a dihedral group $\mathbb{D}_{2n}$, $n \geqslant 3$.

A sextic D is said to be of *torus type* if its defining polynomial f can be written in the form $f = f_2^3 + f_3^2$, where f_2 and f_3 are homogenous polynomials of degree 2 and 3, respectively. A representation $f = f_2^3 + f_3^2$ as above, up to the obvious equivalence, is called a *torus structure* on D. According to [7], an irreducible sextic D may have one, four, or twelve distinct torus structures, and we call D a 1-, 4-, or 12-torus sextic, respectively. An irreducible sextic is of torus type if and only if it is $\mathbb{D}_6$-special, see [7]. In this case, the group $\pi_1(\mathbb{P}^2 \smallsetminus D)$ factors to Γ, see [32].

The points of the intersection $f_2 = f_3 = 0$ are singular for D; they are called the *inner* singularities of D (with respect to the given torus structure), whereas the other singular points are called *outer*. When listing the set of singularities of a 1-torus sextic (or describing a particular torus structure), it is common to enclose the inner singularities in parentheses, *cf.* Table 3. Conversely, the presence of a pair of parentheses in the notation indicates that the sextic is of torus type.

Denote by $\mathcal{M} \cong \mathbb{P}^{27}$ the space of all plane sextics. This space is subdivided into equisingular strata $\mathcal{M}(\mathbf{S})$; we consider only those with $\mathbf{S}$ simple. The space of all simple sextics and each of its strata $\mathcal{M}(\mathbf{S})$ are further subdivided into families $\mathcal{M}_*$, $\mathcal{M}_*(\mathbf{S})$, where the subscript $*$ refers to the sequence of invariant factors of a certain finite group, see §4.1 for the precise definition. Our primary concern are the spaces

- $\mathcal{M}_1(\mathbf{S})$: non-special irreducible sextics, see Theorem 4.7, and
- $\mathcal{M}_3(\mathbf{S})$: irreducible 1-torus sextics, see Theorem 4.8.

In this notation, irreducible 4- and 12-torus sextics constitute $\mathcal{M}_{3,3}$ and $\mathcal{M}_{3,3,3}$, respectively, whereas irreducible $\mathbb{D}_{2n}$-special sextics, $n = 5, 7$, constitute $\mathcal{M}_n$. For each subscript $*$, we denote by $\bar{\mathcal{M}}_*(\mathbf{S})$ and $\partial\mathcal{M}_*(\mathbf{S}) := \bar{\mathcal{M}}_*(\mathbf{S}) \smallsetminus \mathcal{M}_*(\mathbf{S})$ the closure and boundary of $\mathcal{M}_*(\mathbf{S})$ *in* $\mathcal{M}_*$.

If $\mathbf{S}$ is a simple set of singularities, the dimension of the *equisingular moduli space* $\mathcal{M}(\mathbf{S})/PGL(3, \mathbb{C})$ equals $19 - \mu(\mathbf{S})$, as follows from the theory of $K3$-surfaces.

The coordinatewise conjugation $(z_0 : z_1 : z_2) \mapsto (\bar{z}_0 : \bar{z}_1 : \bar{z}_2)$ in $\mathbb{P}^2$ induces a real structure (*i.e.*, anti-holomorphic involution) $\operatorname{conj}\colon \mathcal{M} \to \mathcal{M}$, which takes a sextic to its conjugate. A sextic $D \in \mathcal{M}$ is *real* if $\operatorname{conj}(D) = D$. A connected component $\mathcal{C} \subset \mathcal{M}_*(\mathbf{S})$ is *real* if it is preserved by conj as a set; this property of $\mathcal{C}$ is independent of the choice of coordinates in $\mathbb{P}^2$. Clearly, any connected component containing a real curve is real. The converse is not true; however, in the realm of irreducible sextics, the only exception is $\mathcal{M}_1(\mathbf{A}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5)$, see Proposition 2.6.

Most results of the paper are stated in terms of degenerations/perturbations of sets of singularities and/or sextics (or, equivalently, in terms of adjacencies of the equisingular strata of $\mathcal{M}$). As shown in [20], the deformation classes of perturbations of a simple singular point P of type $\mathbf{S}$ are in a one-to-one correspondence with the isomorphism classes of primitive extensions $\mathbf{S}' \rightarrowtail \mathbf{S}$ of root lattices, see §3.3 and §3.4. Thus, by a *degeneration* of sets of singularities we merely mean a class of primitive extensions $\mathbf{S}' \rightarrowtail \mathbf{S}$ of root lattices. Recall (see [17]) that $\mathbf{S}'$ admits a degeneration to $\mathbf{S}$ if and only if the Dynkin graph of $\mathbf{S}'$ is an induced subgraph of that of $\mathbf{S}$. A degeneration $D' \rightarrowtail D$ of simple sextics gives rise to a degeneration $\mathbf{S}(D') \rightarrowtail \mathbf{S}(D)$. According to [10], the converse also holds: given a simple sextic D and a root lattice $\mathbf{S}'$, any degeneration $\mathbf{S}' \rightarrowtail \mathbf{S}(D)$ is realized by a degeneration $D' \rightarrowtail D$ of simple sextics, so that $\mathbf{S}(D') = \mathbf{S}'$.

2.3. **Lists and fundamental groups.** A complete list of the sets of singularities realized by simple plane sextics is found in [31], and the deformation classification of all maximizing simple sextics is obtained in [29] (see also [13] for an alternative approach to sextics with a triple singular point). The relevant part of these results is collected in Tables 1, 2 (irreducible maximizing non-special sextics) and Table 3 (irreducible maximizing 1-torus sextics). In the tables, the column (r, c) refers to the numbers of real (r) and pairs of complex conjugate (c) curves realizing the given set of singularities; thus, the total number of connected components of the stratum $\mathcal{M}_1(\mathbf{S})$ (or $\mathcal{M}_3(\mathbf{S})$ for Table 3) is $n := r + 2c$. Some sets of singularities are prefixed

TABLE 1. The spaces $\mathcal{M}_1(\mathbf{S})$, $\mu(\mathbf{S}) = 19$, with a triple point in $\mathbf{S}$

Singularities	(r, c)
$2\mathbf{E}_8 \oplus \mathbf{A}_3$	$(1,0)$
$2\mathbf{E}_8 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1,0)$
$\mathbf{E}_8 \oplus \mathbf{E}_7 \oplus \mathbf{A}_4$	$(0,1)$
$\mathbf{E}_8 \oplus \mathbf{E}_7 \oplus 2\mathbf{A}_2$	$(1,0)$
$\mathbf{E}_8 \oplus \mathbf{E}_6 \oplus \mathbf{D}_5$	$(1,0)$
$\mathbf{E}_8 \oplus \mathbf{E}_6 \oplus \mathbf{A}_5$	$(0,1)$
$\mathbf{E}_8 \oplus \mathbf{E}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$(1,0)$
$\mathbf{E}_8 \oplus \mathbf{E}_6 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	$(1,0)$
$\mathbf{E}_8 \oplus \mathbf{D}_{11}$	$(1,0)$
$\mathbf{E}_8 \oplus \mathbf{D}_9 \oplus \mathbf{A}_2$	$(1,0)$
$\mathbf{E}_8 \oplus \mathbf{D}_7 \oplus \mathbf{A}_4$	$(1,0)$
$\mathbf{E}_8 \oplus \mathbf{D}_5 \oplus \mathbf{A}_6$	$(0,1)$
$\mathbf{E}_8 \oplus \mathbf{D}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(1,0)$
$\mathbf{E}_8 \oplus \mathbf{A}_{11}$	$(0,1)$
$\mathbf{E}_8 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_1$	$(1,1)$
$\mathbf{E}_8 \oplus \mathbf{A}_9 \oplus \mathbf{A}_2$	$(1,0)$
$\mathbf{E}_8 \oplus \mathbf{A}_8 \oplus \mathbf{A}_3$	$(1,0)$
$\mathbf{E}_8 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1,1)$
$\mathbf{E}_8 \oplus \mathbf{A}_7 \oplus \mathbf{A}_4$	$(0,1)$
$\mathbf{E}_8 \oplus \mathbf{A}_7 \oplus 2\mathbf{A}_2$	$(1,0)$
$\mathbf{E}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5$	$(0,1)$
$\mathbf{E}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$(1,1)$
$\mathbf{E}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	$(1,0)$
$\mathbf{E}_8 \oplus \mathbf{A}_6 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$	$(1,0)$
$\mathbf{E}_8 \oplus \mathbf{A}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(2,0)$
[1] $\mathbf{E}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2^*$	$(1,0)$
$\mathbf{E}_7 \oplus 2\mathbf{E}_6^*$	$(1,0)$
$\mathbf{E}_7 \oplus \mathbf{E}_6 \oplus \mathbf{A}_6$	$(0,1)$
$\mathbf{E}_7 \oplus \mathbf{E}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(2,0)$
$\mathbf{E}_7 \oplus \mathbf{A}_{12}$	$(0,1)$
$\mathbf{E}_7 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_2$	$(2,0)$
$\mathbf{E}_7 \oplus \mathbf{A}_8 \oplus \mathbf{A}_4$	$(0,1)$
$\mathbf{E}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(2,0)$
$\mathbf{E}_7 \oplus 2\mathbf{A}_6$	$(0,1)$
[2] $\mathbf{E}_7 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2^*$	$(1,0)$
$2\mathbf{E}_6^* \oplus \mathbf{A}_7$	$(1,0)$
$2\mathbf{E}_6^* \oplus \mathbf{A}_6 \oplus \mathbf{A}_1$	$(1,0)$
[3] $2\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$	$(1,0)$
$\mathbf{E}_6 \oplus \mathbf{D}_{13}$	$(1,0)$
$\mathbf{E}_6 \oplus \mathbf{D}_{11} \oplus \mathbf{A}_2$	$(1,0)$
$\mathbf{E}_6 \oplus \mathbf{D}_9 \oplus \mathbf{A}_4$	$(1,0)$
$\mathbf{E}_6 \oplus \mathbf{D}_7 \oplus \mathbf{A}_6$	$(1,0)$
$\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_8$	$(1,1)$
$\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_6 \oplus \mathbf{A}_2$	$(2,0)$
$\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus 2\mathbf{A}_4$	$(1,0)$
$\mathbf{E}_6 \oplus \mathbf{A}_{13}$	$(0,1)$
$\mathbf{E}_6 \oplus \mathbf{A}_{12} \oplus \mathbf{A}_1$	$(0,1)$
$\mathbf{E}_6 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_3$	$(2,0)$
$\mathbf{E}_6 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1,1)$
$\mathbf{E}_6 \oplus \mathbf{A}_9 \oplus \mathbf{A}_4$	$(1,1)$
$\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$(1,1)$
$\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_6$	$(0,1)$
$\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(2,0)$
$\mathbf{E}_6 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$	$(1,0)$
$\mathbf{E}_6 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1,1)$
$\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_4$	$(2,0)$
$\mathbf{D}_{19}$	$(1,0)$
$\mathbf{D}_{17} \oplus \mathbf{A}_2$	$(1,0)$
$\mathbf{D}_{15} \oplus \mathbf{A}_4$	$(1,0)$
$\mathbf{D}_{13} \oplus \mathbf{A}_6$	$(0,1)$
$\mathbf{D}_{13} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(1,0)$
$\mathbf{D}_{11} \oplus \mathbf{A}_8$	$(1,0)$
$\mathbf{D}_{11} \oplus \mathbf{A}_6 \oplus \mathbf{A}_2$	$(1,0)$
$\mathbf{D}_{11} \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_2^*$	$(1,0)$
$\mathbf{D}_9 \oplus \mathbf{A}_{10}$	$(1,0)$
$\mathbf{D}_9 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4$	$(1,0)$
$\mathbf{D}_9 \oplus 2\mathbf{A}_4^* \oplus \mathbf{A}_2$	$(1,0)$
$\mathbf{D}_7 \oplus \mathbf{A}_{12}$	$(1,1)$
$\mathbf{D}_7 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_2$	$(0,1)$
$\mathbf{D}_7 \oplus \mathbf{A}_8 \oplus \mathbf{A}_4$	$(2,0)$
$\mathbf{D}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(1,0)$
$\mathbf{D}_7 \oplus 2\mathbf{A}_6$	$(0,1)$
$\mathbf{D}_5 \oplus \mathbf{A}_{14}$	$(0,1)$
$\mathbf{D}_5 \oplus \mathbf{A}_{12} \oplus \mathbf{A}_2$	$(1,0)$
$\mathbf{D}_5 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_4$	$(1,1)$
$\mathbf{D}_5 \oplus \mathbf{A}_{10} \oplus 2\mathbf{A}_2^*$	$(1,0)$
$\mathbf{D}_5 \oplus \mathbf{A}_8 \oplus \mathbf{A}_6$	$(0,1)$
$\mathbf{D}_5 \oplus \mathbf{A}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(1,1)$
$\mathbf{D}_5 \oplus \mathbf{A}_6 \oplus 2\mathbf{A}_4$	$(2,0)$
$\mathbf{D}_5 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_2^*$	$(1,0)$

with a link of the form $^{[n]}$: this link refers to the listings of the fundamental groups found below. Some pairs of singular points are marked with a *. This marking is related to the real structures; it is explained in §6.2.

TABLE 2. The spaces $\mathcal{M}_1(\mathbf{S})$, $\mu(\mathbf{S}) = 19$, with double points only

Singularities	(r, c)
$\mathbf{A}_{19}$	$(2, 0)$
$\mathbf{A}_{18} \oplus \mathbf{A}_1$	$(1, 1)$
$\mathbf{A}_{16} \oplus \mathbf{A}_3$	$(2, 0)$
$\mathbf{A}_{16} \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1, 1)$
$\mathbf{A}_{15} \oplus \mathbf{A}_4$	$(0, 1)$
[6] $\mathbf{A}_{14} \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$(0, 3)$
[6] $\mathbf{A}_{13} \oplus \mathbf{A}_6$	$(0, 2)$
$\mathbf{A}_{13} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(2, 0)$
[6] $\mathbf{A}_{12} \oplus \mathbf{A}_7$	$(0, 1)$
[4] $\mathbf{A}_{12} \oplus \mathbf{A}_6 \oplus \mathbf{A}_1$	$(1, 1)$
$\mathbf{A}_{12} \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$	$(1, 0)$
[4] $\mathbf{A}_{12} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1, 1)$
[5] $\mathbf{A}_{11} \oplus 2\mathbf{A}_4^*$	$(2, 0)$
$\mathbf{A}_{10} \oplus \mathbf{A}_9$	$(2, 0)$
[4] $\mathbf{A}_{10} \oplus \mathbf{A}_8 \oplus \mathbf{A}_1$	$(1, 1)$
$\mathbf{A}_{10} \oplus \mathbf{A}_7 \oplus \mathbf{A}_2$	$(2, 0)$
[6] $\mathbf{A}_{10} \oplus \mathbf{A}_6 \oplus \mathbf{A}_3$	$(0, 1)$
[4] $\mathbf{A}_{10} \oplus \mathbf{A}_6 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1, 1)$
$\mathbf{A}_{10} \oplus \mathbf{A}_5 \oplus \mathbf{A}_4$	$(2, 0)$
[6] $\mathbf{A}_{10} \oplus 2\mathbf{A}_4^* \oplus \mathbf{A}_1$	$(1, 1)$
$\mathbf{A}_{10} \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	$(1, 0)$
$\mathbf{A}_{10} \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$	$(2, 0)$
[4] $\mathbf{A}_9 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4$	$(1, 1)$
[6] $\mathbf{A}_8 \oplus \mathbf{A}_7 \oplus \mathbf{A}_4$	$(0, 1)$
[4] $\mathbf{A}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$(1, 1)$
[6] $\mathbf{A}_7 \oplus 2\mathbf{A}_6$	$(0, 1)$
$\mathbf{A}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(2, 0)$
$\mathbf{A}_7 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2^*$	$(1, 0)$
$2\mathbf{A}_6^* \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(2, 0)$
$\mathbf{A}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_4^*$	$(2, 0)$

The fundamental groups of most irreducible maximizing sextics are computed in [13, 15]; the latest computations, using S. Orevkov's recent equations [27], are contained in [12]. (Due to [27], the defining equations of *all* maximizing irreducible sextics with double points only are known now.) Quite a few sporadic computations of the fundamental groups are also found in [2, 3, 6, 10, 11, 18, 19, 26, 33] and a number of other papers, see [13] for more detailed references.

The known fundamental groups $\pi_1 := \pi_1(\mathbb{P}^2 \smallsetminus D)$ of the maximizing non-special irreducible sextics D are as follows (depending on the set of singularities):

(1) for $\mathbf{E}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$, the group is the central product
$$\pi_1 = SL(2, \mathbb{F}_5) \odot \mathbb{G}_{12} := \bigl(SL(2, \mathbb{F}_5) \times \mathbb{G}_{12}\bigr)/(-\mathrm{id} = 6),$$
where $-\mathrm{id}$ is the generator of the center $\mathbb{G}_2 \subset SL(2, \mathbb{F}_5)$;

(2) for $\mathbf{E}_7 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2$, the group is $\pi_1 = SL(2, \mathbb{F}_{19}) \times \mathbb{G}_6$;

(3) for $2\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$, the group is $\pi_1 = SL(2, \mathbb{F}_5) \rtimes \mathbb{G}_6$, the generator of $\mathbb{G}_6$ acting on $SL(2, \mathbb{F}_5)$ by (any) order 2 outer automorphism;

(4) for the six sets of singularities marked with [4] in Table 2, one has $(r, c) = (1, 1)$, and *only for the real curve* the group $\pi_1 = \mathbb{G}_6$ is known;

(5) for $\mathbf{A}_{11} \oplus 2\mathbf{A}_4$, only for one of the two curves the group $\pi_1 = \mathbb{G}_6$ is known;

(6) for the seven sets of singularities marked with [6] in Table 2, the fundamental group is still unknown.

In all other cases, the fundamental group is abelian: $\pi_1 = \mathbb{G}_6$.

The fundamental groups of sextics of torus type are large and more difficult to describe. To simplify the description, we introduce a few *ad hoc* groups:

$$G(\bar{s}) := \bigl\langle \alpha_1, \alpha_2, \alpha_3 \bigm| \rho_3^4 = (\alpha_1\alpha_2)^3,\ \{\alpha_2 \uparrow \sigma_1^i, \alpha_3\}_{s_i} = 1,\ i = 0, \ldots, 5 \bigr\rangle, \tag{2.1}$$

where $\bar{s} = (s_0, \ldots, s_5) \in \mathbb{Z}^6$ is an integral vector,

$$\begin{aligned} L_{p,q,r} := \bigl\langle \alpha_1, \alpha_2 \bigm| (\alpha_1\alpha_2\alpha_1)^3 = \alpha_2\alpha_1\alpha_2,\ &\{\alpha_2, (\alpha_1\alpha_2)\alpha_1(\alpha_1\alpha_2)^{-1}\}_p \\ &= \{\alpha_1, \alpha_2\alpha_1\alpha_2^{-1}\}_q = \{\alpha_2, (\alpha_1\alpha_2^2)\alpha_1(\alpha_1\alpha_2^2)^{-1}\}_r = 1 \bigr\rangle, \end{aligned} \tag{2.2}$$

TABLE 3. The spaces $\mathcal{M}_3(\mathbf{S})$, $\mu(\mathbf{S}) = 19$

Singularities	(r,c)
[1] $(3\mathbf{E}_6) \oplus \mathbf{A}_1$	$(1,0)$
[2] $(2\mathbf{E}_6 \oplus \mathbf{A}_5) \oplus \mathbf{A}_2$	$(2,0)$
[3] $(2\mathbf{E}_6 \oplus 2\mathbf{A}_2^*) \oplus \mathbf{A}_3$	$(1,0)$
$(\mathbf{E}_6 \oplus \mathbf{A}_{11}) \oplus \mathbf{A}_2$	$(1,0)$
$(\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2) \oplus \mathbf{A}_3$	$(1,0)$
$(\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1,1)$
[4] $(\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2^*) \oplus \mathbf{A}_4$	$(2,0)$
$\mathbf{D}_5 \oplus (\mathbf{A}_8 \oplus 3\mathbf{A}_2^*)$	$(1,0)$

Singularities	(r,c)
$(\mathbf{A}_{17} \oplus \mathbf{A}_2)$	$(1,0)$
$(\mathbf{A}_{14} \oplus \mathbf{A}_2) \oplus \mathbf{A}_3$	$(1,0)$
$(\mathbf{A}_{14} \oplus \mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1,0)$
$(\mathbf{A}_{11} \oplus 2\mathbf{A}_2^*) \oplus \mathbf{A}_4$	$(1,0)$
$(2\mathbf{A}_8) \oplus \mathbf{A}_3$	$(1,0)$
[6] $(\mathbf{A}_8 \oplus \mathbf{A}_5 \oplus \mathbf{A}_2) \oplus \mathbf{A}_4$	$(0,1)$
[5] $(\mathbf{A}_8 \oplus 3\mathbf{A}_2^*) \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$(1,0)$

where $p, q, r \in \mathbb{Z}$, and

$$E_{p,q} := \bigl\langle \alpha_1, \alpha_2, \alpha_3 \bigm| \rho_3\alpha_2\rho_3^{-1} = \alpha_2^{-1}\alpha_1\alpha_2 = \rho_3^{-1}\alpha_3\rho_3, \quad \rho_3^4 = (\alpha_1\alpha_2)^3,\ \{\alpha_2, \alpha_3\}_p = \{\alpha_1, \alpha_3\}_q = 1 \bigr\rangle, \tag{2.3}$$

where $p, q \in \mathbb{Z}$. Then, the fundamental groups of the maximizing irreducible 1-torus sextics are as follows:

(1) for $(3\mathbf{E}_6) \oplus \mathbf{A}_1$, the group is $\pi_1 = \mathbb{B}_4/\sigma_2\sigma_1^2\sigma_2\sigma_3^3$;
(2) for $(2\mathbf{E}_6 \oplus \mathbf{A}_5) \oplus \mathbf{A}_2$, the groups are $E_{3,6}$, see (2.3), and $L_{3,6,0}$, see (2.2);
(3) for $(2\mathbf{E}_6 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_3$, the group is $E_{4,3}$, see (2.3);
(4) for $(\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_4$, the groups are $L_{5,6,3}$ and $G(6,5,3,3,5,6)$, see (2.2) and (2.1), respectively;
(5) for $(\mathbf{A}_8 \oplus 3\mathbf{A}_2) \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$, the group is

$$\pi_1 = \bigl\langle \alpha_1, \alpha_2, \alpha_3 \bigm| [\alpha_2, \alpha_3] = \{\alpha_1, \alpha_2\}_3 = \{\alpha_1, \alpha_3\}_9 = 1, \quad \alpha_3\alpha_1\alpha_2^{-1}\alpha_3\alpha_1\alpha_3(\alpha_3\alpha_1)^{-2}\alpha_2 = (\alpha_1\alpha_3)^2\alpha_2^{-1}\alpha_1\alpha_3\alpha_2\alpha_1 \bigr\rangle;$$

(6) for the set of singularities $(\mathbf{A}_8 \oplus \mathbf{A}_5 \oplus \mathbf{A}_2) \oplus \mathbf{A}_4$, the group is unknown.

In all other cases, the fundamental group is $\pi_1 = \Gamma$. In each of items 2 and 4, it is not known whether the two groups are isomorphic. The groups corresponding to distinct sets of singularities (listed above) are distinct, except that it is not known whether the group in item 5 is isomorphic to Γ.

2.4. **Statements.** There are 110 maximizing sets of simple singularities realized by non-special irreducible sextics. We found that 2996 sets of simple singularities are realized by non-maximizing non-special irreducible sextics. (This statement is almost contained in [31], although no distinction between special and non-special curves is made there, nor a description of *non-maximizing* irreducible sextics.) The corresponding counts for irreducible 1-torus sextics are 15 and 105, respectively, see [25]. Our principal results (the deformation classification and a few consequences on the fundamental group) are stated in the rest of this section, with references to the proofs given in the headers.

Theorem 2.4 (see §4.3 and §6.1)**.** *The space $\mathcal{M}_1(\mathbf{S})$ is nonempty if and only if either $\mathbf{S}$ is in one of the following two exceptional degeneration chains*

$$2\mathbf{D}_8 \rightarrowtail \mathbf{D}_9 \oplus \mathbf{D}_8 \rightarrowtail 2\mathbf{D}_9, \qquad 2\mathbf{D}_4 \oplus 4\mathbf{A}_2 \rightarrowtail \mathbf{D}_7 \oplus \mathbf{D}_4 \oplus 3\mathbf{A}_2 \rightarrowtail 2\mathbf{D}_7 \oplus 2\mathbf{A}_2$$

TABLE 4. Disconnected spaces $\mathcal{M}_1(\mathbf{S})$, $\mu(\mathbf{S}) < 19$

Singularities	(r,c)	Singularities	(r,c)
$\mathbf{E}_8 \oplus 2\mathbf{A}_5$	$(0,1)$	$\mathbf{D}_6 \oplus 2\mathbf{A}_6$	$(0,1)$
$\mathbf{E}_7 \oplus \mathbf{E}_6 \oplus \mathbf{A}_5$	$(0,1)$	$\mathbf{D}_5 \oplus 2\mathbf{A}_6 \oplus \mathbf{A}_1$	$(0,1)$
$\mathbf{E}_7 \oplus \mathbf{A}_7 \oplus \mathbf{A}_4$	$(0,1)$	$2\mathbf{A}_9$	$(2,0)$
$\mathbf{E}_6 \oplus \mathbf{A}_{11} \oplus \mathbf{A}_1$	$(0,1)$	$\mathbf{A}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5$	$(1,0)$
$\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_5$	$(0,1)$	$3\mathbf{A}_6$	$(0,1)$
$\mathbf{E}_6 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5 \oplus \mathbf{A}_1$	$(0,1)$	$2\mathbf{A}_6 \oplus 2\mathbf{A}_3$	$(0,1)$
$\mathbf{E}_6 \oplus 2\mathbf{A}_5 \oplus \mathbf{A}_1$	$(0,1)$	$2\mathbf{A}_7 \oplus \mathbf{A}_4$	$(0,1)$

or $\mathbf{S}$ *degenerates to one of the maximizing sets of singularities listed in Tables 1, 2. The numbers* (r,c) *of connected components of* $\mathcal{M}_1(\mathbf{S})$ *are as shown in Tables 1, 2, and 4; in all other cases,* $\mathcal{M}_1(\mathbf{S})$ *is connected and contains real curves.*

Two lines in Table 4 deserve separate statements: to our knowledge, phenomena of this kind have not been observed before.

Proposition 2.5 (see §4.5)**.** *Let* $\mathbf{S}_0 := 2\mathbf{A}_9$*,* $\mathbf{S}_1 := \mathbf{A}_{19}$*, and* $\mathbf{S}_2 := \mathbf{A}_{10} \oplus \mathbf{A}_9$*. The space* $\mathcal{M}_1(\mathbf{S}_i)$*,* $i = 0, 1, 2$*, consists of two connected components* $\mathcal{M}_1^{\pm}(\mathbf{S}_i)$*, each containing real curves, so that* $\partial\mathcal{M}_1^{\epsilon}(\mathbf{S}_0) = \mathcal{M}_1^{\epsilon}(\mathbf{S}_1) \cup \mathcal{M}_1^{\epsilon}(\mathbf{S}_2)$ *for each* $\epsilon = \pm$*.*

Proposition 2.6 (see §6.3)**.** *The space* $\mathcal{M}_1(\mathbf{A}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5) = \mathcal{M}(\mathbf{A}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5)$ *is connected* (*hence, its only component is real*)*, but it contains no real curves.*

In the other cases in Table 4, the space $\mathcal{M}_1(\mathbf{S})$ consists of two complex conjugate components. The first such example, *viz.* $\mathbf{S} = \mathbf{E}_6 \oplus \mathbf{A}_{11} \oplus \mathbf{A}_1$, was discovered in [1]. The adjacencies of these non-real components are studied in §6.5. Note that one set of singularities, *viz.* $\mathbf{E}_6 \oplus 2\mathbf{A}_5 \oplus \mathbf{A}_1$, has Milnor number 17; it gives rise to an interesting adjacency phenomenon, see Proposition 6.7.

Corollary 2.7 (see §4.4)**.** *With the same six exceptions as in Theorem 2.4, any non-special irreducible simple sextic degenerates to a maximizing sextic with these properties, see Tables 1 and 2.*

Corollary 2.8 (see §5.2)**.** *Let* $D \subset \mathbb{P}^2$ *be a non-special irreducible simple plane sextic. If* $\mu(D) = 19$*, the fundamental group* $\pi_1 := \pi_1(\mathbb{P}^2 \smallsetminus D)$ *is as shown in Tables 1 and 2. Otherwise, one has*

- $\pi_1 = SL(2, \mathbb{F}_3) \times \mathbb{G}_2$ *for* $2\mathbf{D}_7 \oplus 2\mathbf{A}_2$*,* $\mathbf{D}_7 \oplus \mathbf{D}_4 \oplus 3\mathbf{A}_2$*, and* $2\mathbf{D}_4 \oplus 4\mathbf{A}_2$*,*
- $\pi_1 = SL(2, \mathbb{F}_5) \odot \mathbb{G}_{12}$*, see §2.3, for* $2\mathbf{A}_4 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2$*,*

and $\pi_1 = \mathbb{G}_6$ *in all other cases.*

The remaining statements deal with sextics of torus type, and we introduce the notion of weight. The *weight* $w(P)$ of a simple singular point P is defined *via* $w(\mathbf{A}_{3p-1}) = p$, $w(\mathbf{E}_6) = 2$, and $w(P) = 0$ otherwise. The weight of a set of singularities $\mathbf{S}$ (or a simple sextic D) is the total weight of its singular points. Recall (see [7]) that, if D is a 1-torus sextic, then $6 \leqslant w(D) \leqslant 7$. Conversely, if D is an irreducible sextic and either $w(D) = 7$ or $w(D) = 6$ and D has at least one singular point $P \neq \mathbf{A}_1$ of weight 0, then D is a 1-torus sextic.

Theorem 2.9 (see §4.6 and §6.4)**.** *A set of singularities* $\mathbf{S}$ *with* $w(\mathbf{S}) \geqslant 6$ *is realized by an irreducible simple* 1*-torus sextic* D *if and only if* $\mathbf{S}$ *degenerates to one of the* *maximizing sets listed in Table 3. Furthermore, if $\mu(\mathbf{S}) \leqslant 18$, a sextic D as above is unique up to equisingular deformation and the space $\mathcal{M}_3(\mathbf{S})$ contains real curves.*

Corollary 2.10 (see §4.6)**.** *Any irreducible simple 1-torus sextic degenerates to a maximizing sextic with these properties, see Table 3.*

There are 51 sets of singularities $\mathbf{S}$ (all of weight 6) realized by both 1-torus and non-special irreducible sextics. Formally, these sets of singularities can be extracted from Theorems 2.4 and 2.9; an explicit list is found in [1]. The corresponding sextics constitute the so-called *classical Zariski pairs*.

Corollary 2.11 (see §5.3)**.** *Let $D \subset \mathbb{P}^2$ be an irreducible simple 1-torus sextic. If $\mu(D) = 19$, the fundamental group $\pi_1 := \pi_1(\mathbb{P}^2 \smallsetminus D)$ is as shown in Table 3. Otherwise, one has $\pi_1 = \mathbb{B}_4/\sigma_2\sigma_1^2\sigma_2\sigma_3^3$ for the sets of singularities*

$$(2\mathbf{E}_6 \oplus 2\mathbf{A}_2) \oplus 2\mathbf{A}_1, \quad (\mathbf{E}_6 \oplus 4\mathbf{A}_2) \oplus 3\mathbf{A}_1, \quad (\mathbf{E}_6 \oplus 4\mathbf{A}_2) \oplus \mathbf{A}_3 \oplus \mathbf{A}_1,$$
$$(6\mathbf{A}_2) \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_1, \quad (6\mathbf{A}_2) \oplus 4\mathbf{A}_1,$$

and $\pi_1 = \Gamma$ in all other cases.

Remark 2.12. In Corollary 2.11, the non-maximizing 1-torus sextics with the group $\pi_1 = \mathbb{B}_4/\sigma_2\sigma_1^2\sigma_2\sigma_3^3$ can be characterized as the degenerations of $(6\mathbf{A}_2) \oplus 4\mathbf{A}_1$.

2.5. **Other irreducible sextics.** For the reader's convenience and completeness of the exposition, we recall the classification of the other irreducible simple sextics, *viz.* the $\mathbb{D}_{10}$- and $\mathbb{D}_{14}$-special sextics and the 4- and 12-torus ones. The fundamental groups are computed in several papers, see [13] for detailed references.

Theorem 2.13 (see [7])**.** *The space $\mathcal{M}_5$ consists of eight connected components, one component $\mathcal{M}_5(\mathbf{S})$ for each of the following sets of singularities* $\mathbf{S}$:

$$2\mathbf{A}_9, \quad \mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2, \quad \mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_1, \quad \mathbf{A}_9 \oplus 2\mathbf{A}_4,$$
$$4\mathbf{A}_4 \oplus \mathbf{A}_2, \quad 4\mathbf{A}_4 \oplus 2\mathbf{A}_1, \quad 4\mathbf{A}_4 \oplus \mathbf{A}_1, \quad 4\mathbf{A}_4.$$

All components are real and contain real curves. ▷

The fundamental group $\pi_1 := \pi_1(\mathbb{P}^2 \smallsetminus D)$ of a *simple* sextic $D \in \mathcal{M}_5(\mathbf{S})$ can be described as follows. Denoting temporarily by G' the derived subgroup $[G, G]$, one always has $\pi_1/\pi_1'' = \mathbb{D}_{10} \times \mathbb{G}_3$. Besides,

(1) if $\mathbf{S} = \mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2$, then π_1'' is the only perfect group of order 120;
(2) if $\mathbf{S} = 4\mathbf{A}_4 \oplus 2\mathbf{A}_1$, then $\pi_1''/\pi_1''' = \mathbb{G}_2^4$ and $\pi_1''' = \mathbb{G}_2$, so that $\operatorname{ord} \pi_1 = 960$;
(3) in all other cases, $\pi_1 = \mathbb{D}_{10} \times \mathbb{G}_3$.

The precise presentations in (1) and (2) are rather lengthy, and we refer to [11].

Theorem 2.14 (see [7])**.** *The space $\mathcal{M}_7$ consists of two connected components, one component $\mathcal{M}_7(\mathbf{S})$ for each of the following sets of singularities* $\mathbf{S}$:

$$3\mathbf{A}_6 \oplus \mathbf{A}_1, \quad 3\mathbf{A}_6.$$

Both components are real and contain real curves. ▷

The fundamental groups of all $\mathbb{D}_{14}$-special sextics are $\mathbb{D}_{14} \times \mathbb{G}_3$.

Remark 2.15. The sets of singularities $2\mathbf{A}_9$, $\mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_1$, $\mathbf{A}_9 \oplus 2\mathbf{A}_4$, $4\mathbf{A}_4 \oplus \mathbf{A}_1$, $4\mathbf{A}_4$ (*cf.* Theorem 2.13) and $3\mathbf{A}_6$ (*cf.* Theorem 2.14) are also realized by non-special irreducible sextics, each by a single connected deformation family.

Theorem 2.16 (see [7]). *The union* $\mathcal{M}_{3.3} \cup \mathcal{M}_{3,3,3}$ *consists of eight connected components, one component for each of the following sets of singularities* $\mathbf{S}$:

- $\mathcal{M}_{3,3}$ (4-*torus sextics,* idem *weight* $w = 8$): $\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 4\mathbf{A}_2$, $\mathbf{E}_6 \oplus 6\mathbf{A}_2$, $2\mathbf{A}_5 \oplus 4\mathbf{A}_2$, $\mathbf{A}_5 \oplus 6\mathbf{A}_2 \oplus \mathbf{A}_1$, $\mathbf{A}_5 \oplus 6\mathbf{A}_2$, $8\mathbf{A}_2 \oplus \mathbf{A}_1$, $8\mathbf{A}_2$;
- $\mathcal{M}_{3,3,3}$ (12-*torus sextics,* idem $w = 9$): $9\mathbf{A}_2$.

All components are real and contain real curves. ▷

All sets of singularities of weight 8 degenerate to $\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 4\mathbf{A}_2$ and can be characterized as perturbations of the latter preserving all four torus structures. Note that $9\mathbf{A}_2$ does *not* degenerate to a maximizing sextic, irreducible or not!

Introduce the group

$$H := \langle \alpha, \bar{\alpha}, \beta, \gamma, \bar{\gamma} \mid \{\alpha, \beta\}_3 = \{\bar{\alpha}, \beta\}_3 = \{\gamma, \beta\}_3 = \{\bar{\gamma}, \beta\}_3 = \beta\gamma\alpha\beta\bar{\gamma}\bar{\alpha} = 1 \rangle.$$

In this notation (see also (2.1)), the fundamental group $\pi_1 := \pi_1(\mathbb{P}^2 \smallsetminus D)$ of a sextic D with a set of singularities $\mathbf{S}$ of weight 8 or 9 is as follows:

(1) if $\mathbf{S} = 9\mathbf{A}_2$ ($w = 9$), then

$$\pi_1 = H_3 := H/\langle \{\bar{\gamma}, \alpha\} = \{\gamma, \bar{\alpha}\} = \{\gamma, \bar{\gamma}\} = [\beta, \alpha^{-1}\gamma^{-1}\bar{\alpha}\bar{\gamma}] = 1,\ \bar{\gamma}^{-1}\alpha\bar{\gamma} = \gamma^{-1}\bar{\alpha}\gamma \rangle;$$

(2) if $\mathbf{S} = \mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 4\mathbf{A}_2$, then

$$\pi_1 = H_2 := H/\langle \bar{\alpha}\gamma\alpha = \alpha\bar{\gamma}\bar{\alpha} = \gamma\alpha\bar{\gamma} = \bar{\gamma}\bar{\alpha}\gamma \rangle \cong G(3,6,3,3,6,3);$$

(3) if $\mathbf{S} = \mathbf{A}_5 \oplus 6\mathbf{A}_2 \oplus \mathbf{A}_1$, then

$$\pi_1 = H_1 := H/\langle \{\alpha, \gamma\}_3 = \{\bar{\alpha}, \bar{\gamma}\}_3 = [\gamma, \bar{\gamma}] = 1,\ \gamma\alpha\bar{\gamma} = \bar{\gamma}\bar{\alpha}\gamma \rangle;$$

(4) for all other sextics of weight 8,

$$\pi_1 = H_0 := H/\langle \alpha = \bar{\alpha}, \gamma = \bar{\gamma}, \{\alpha, \gamma\}_3 = 1 \rangle \cong G(3,3,3,3,3,3).$$

All perturbation epimorphisms $H_3 \twoheadrightarrow H_0$ and $H_2 \twoheadrightarrow H_1 \twoheadrightarrow H_0$, *cf.* Theorem 5.1, lift to the identity $H \to H$. We do not know whether the emimorphism $H_2 \twoheadrightarrow H_1$ is proper; the others are.

3. Integral lattices

3.1. **Finite quadratic forms** (see [24]). A *finite quadratic form* is a finite abelian group $\mathcal{N}$ equipped with a symmetric bilinear form $b\colon \mathcal{N} \otimes \mathcal{N} \to \mathbb{Q}/\mathbb{Z}$ and a *quadratic extension* of b, *i.e.*, a map $q\colon \mathcal{N} \to \mathbb{Q}/2\mathbb{Z}$ such that $q(x+y) - q(x) - q(y) = 2b(x,y)$ for all $x, y \in \mathcal{N}$ (where 2 is the isomorphism $\times 2\colon \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/2\mathbb{Z}$); clearly, b is determined by q. If q is understood, we abbreviate $b(x,y) = x \cdot y$ and $q(x) = x^2$. In what follows, we consider *nondegenerate* forms only, *i.e.*, such that the associated homomorphism $\mathcal{N} \to \operatorname{Hom}(\mathcal{N}, \mathbb{Q}/\mathbb{Z})$, $x \mapsto (y \mapsto x \cdot y)$ is an isomorphism.

Each finite quadratic form $\mathcal{N}$ splits into orthogonal sum $\mathcal{N} = \bigoplus_{p \in \mathbb{P}} \mathcal{N}_p$ of its p-primary components $\mathcal{N}_p := \mathcal{N} \otimes \mathbb{Z}_p$. The *length* $\ell(\mathcal{N})$ of $\mathcal{N}$ is the minimal number of generators of $\mathcal{N}$. Obviously, $\ell(\mathcal{N}) = \max_{p \in \mathbb{P}} \ell_p(\mathcal{N})$, where $\ell_p(\mathcal{N}) := \ell(\mathcal{N}_p)$. The notation $-\mathcal{N}$ stands for the group $\mathcal{N}$ with the form $x \mapsto -x^2$.

We describe nondegenerate finite quadratic forms by expressions of the form $\langle q_1 \rangle \oplus \ldots \oplus \langle q_r \rangle$, where $q_i := \frac{m_i}{n_i} \in \mathbb{Q}$, g.c.d.$(m_i, n_i) = 1$, $m_i n_i = 0 \bmod 2$; the group is generated by pairwise orthogonal elements $\alpha_1, \ldots, \alpha_r$ (numbered in the order of appearance), so that $\alpha_i^2 = q_i \bmod 2\mathbb{Z}$ and the order of α_i is n_i. (In the 2-torsion, there also may be indecomposable summands of length 2, but we do not need them.) Describing an automorphism σ of such a group, we only list the images of the generators α_i that are moved by σ.

A finite quadratic form is called *even* if $x^2 = 0 \bmod \mathbb{Z}$ for each element $x \in \mathcal{N}$ of order two; otherwise, the form is called *odd*. In other words, $\mathcal{N}$ is odd if and only it contains $\langle \pm\frac{1}{2}\rangle$ as an orthogonal summand.

Given a prime $p \in \mathbb{P}$, the *determinant* $\det_p \mathcal{N}$ is defined as the determinant of the 'matrix' of the quadratic form on $\mathcal{N}_p$ in an appropriate basis (see [24] and [23] for details and alternative definitions). The determinant is well defined modulo squares; if $\mathcal{N}_p$ is nondegenerate, one has $\det_p \mathcal{N} = u/|\mathcal{N}_p|$ for some $u \in \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$. If $p = 2$, the determinant $\det_2 \mathcal{N}$ is well defined only if $\mathcal{N}_2$ is even. By definition, one always has $|\mathcal{N}| \det_p \mathcal{N} \in \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$.

The group of q-autoisometries of $\mathcal{N}$ is denoted by $\operatorname{Aut}\mathcal{N}$; obviously, one has $\operatorname{Aut}\mathcal{N} = \prod_{p\in\mathbb{P}} \operatorname{Aut}\mathcal{N}_p$. An element $\xi \in \mathcal{N}_p$ is called a *mirror* if, for some integer k, one has $p^k\xi = 0$ and $\xi^2 = 2u/p^k \bmod 2\mathbb{Z}$, g.c.d.$(u, p) = 1$. If this is the case, the map $x \mapsto 2(x\cdot\xi)/\xi^2 \bmod p^k$ is a well defined functional $\mathcal{N}_p \to \mathbb{Z}/p^k$; hence, one has a *reflection* $t_\xi \in \operatorname{Aut}\mathcal{N}_p$,

$$t_\xi\colon x \mapsto x - \frac{2(x\cdot\xi)}{\xi^2}\xi.$$

Note that $t_\xi = \mathrm{id}$ whenever $2\xi = 0$ and $\xi^2 = \frac{1}{2} \bmod \mathbb{Z}$.

3.2. **Lattices and discriminant forms** (see [24]). An (*integral*) *lattice* N is a finitely generated free abelian group equipped with a symmetric bilinear form $b\colon N \otimes N \to \mathbb{Z}$. If b is understood, we abbreviate $b(x, y) = x\cdot y$ and $b(x, x) = x^2$. A lattice N is called *even* if $x^2 = 0 \bmod 2$ for all $x \in N$; it is called *odd* otherwise. The determinant $\det N$ of a lattice N is the determinant of the Gram matrix of b. As the transition matrix from one integral basis to another has determinant ±1, the determinant $\det N \in \mathbb{Z}$ is well-defined. The lattice N is called *non-degenerate* if $\det N \neq 0$ and *unimodular* if $\det N = \pm1$. The signature (σ_+N, σ_-N) of a non-degenerate lattice N is the pair of the inertia indices of the bilinear form b.

For a lattice N, the bilinear form extends to a $\mathbb{Q}$-valued bilinear form on $N \otimes \mathbb{Q}$. If N is non-degenerate, the dual group $N^\sharp := \operatorname{Hom}(N, \mathbb{Z})$ can be identified with the subgroup $\{x \in N \otimes \mathbb{Q} \mid x\cdot y \in \mathbb{Z} \text{ for all } y \in N\}$. The lattice N is a finite index subgroup of $N^\sharp$. The quotient $\operatorname{discr} N := N^\sharp/N$ is called the *discriminant group* of N; it is often denoted by $\mathcal{N}$, and we use the shortcut $\operatorname{discr}_p N = \mathcal{N}_p$ for the p-primary components. One has $\det N = (-1)^{\sigma_- N}|\mathcal{N}|$. The group $\mathcal{N}$ inherits from $N \otimes \mathbb{Q}$ a symmetric bilinear form $b\colon \mathcal{N} \otimes \mathcal{N} \to \mathbb{Q}/\mathbb{Z}$, called the *discriminant form*, and, if N is even, a quadratic extension of b.

Convention 3.1. Unless specified otherwise, all lattices considered below are non-degenerate and even. The discriminant group of such a lattice is always regarded as a finite quadratic form.

The *genus* $g(N)$ of a nondegenerate even lattice N can be defined as the set of isomorphism classes of all even lattices L such that $\operatorname{discr} L \cong \mathcal{N}$ and $\sigma_\pm L = \sigma_\pm N$. If N is indefinite and $\operatorname{rk} N \geqslant 3$, then $g(N)$ is a finite abelian group.

An *isometry* of lattices is a homomorphism of abelian groups preserving the forms. (Note that we do not assume the surjectivity.) The group of auto-isometries of a lattice N is denoted by $O(N)$. There is an obvious natural homomorphism $\mathrm{d}\colon O(N) \to \operatorname{Aut}\mathcal{N}$, and we denote by $\mathrm{d}_p\colon O(N) \to \operatorname{Aut}\mathcal{N}_p$ its restrictions to the p-primary components. For an element $u \in N$ such that $2u/u^2 \in N^\sharp$, the *reflection*

$t_u\colon x \mapsto 2u(x\cdot u)/u^2$ is an involutive isometry of N. Each image $\mathrm{d}_p(t_u)$, $p \in \mathbb{P}$, is also a reflection. If $u^2 = \pm 1$ or ± 2, then $\mathrm{d}(t_u) = \mathrm{id}$.

3.3. **Root lattices** (see [4]). In this paper, a *root lattice* is a negative definite lattice generated by vectors of square (-2) (*roots*). Any root lattice has a unique decomposition into orthogonal sum of indecomposable ones, which are of types $\mathbf{A}_p$, $p \geqslant 1$, $\mathbf{D}_q$, $q \geqslant 4$, $\mathbf{E}_6$, $\mathbf{E}_7$, or $\mathbf{E}_8$.

Given a root lattice S, the vertices of the Dynkin graph $\mathfrak{G} := \mathfrak{G}_S$ can be identified with the elements of a basis for S constituting a single Weyl chamber. This identification defines a homomorphism $\operatorname{Sym}\mathfrak{G} \to O(S)$, $s \mapsto s_*$, where $\operatorname{Sym}\mathfrak{G}$ is the group of symmetries of $\mathfrak{G}$. The image consists of the isometries preserving the distinguished Weyl chamber. For indecomposable root lattices, the groups $\operatorname{Sym}\mathfrak{G}$ are as follows:

- $\operatorname{Sym}\mathfrak{G} = 1$ if S is $\mathbf{A}_1$, $\mathbf{E}_7$, or $\mathbf{E}_8$,
- $\operatorname{Sym}\mathfrak{G} \cong \mathbb{S}_3 \cong \mathbb{D}_6$ if S is $\mathbf{D}_4$, and
- $\operatorname{Sym}\mathfrak{G} = \mathbb{G}_2$ in all other cases.

In the latter case, unless $S = \mathbf{D}_{\mathrm{even}}$, the generator of $\operatorname{Sym}\mathfrak{G}$ induces $-\mathrm{id}$ on $\mathcal{S}$. If $S = \mathbf{E}_8$, then $\mathcal{S} = 0$. For $S = \mathbf{A}_1$, $\mathbf{A}_7$, or $\mathbf{D}_{\mathrm{even}}$, the groups $\mathcal{S}$ are $\mathbb{F}_2$-modules and $-\mathrm{id} = \mathrm{id}$ in $\operatorname{Aut}\mathcal{S}$.

A choice of a Weyl chamber gives rise to a decomposition $O(S) = R(S) \rtimes \operatorname{Sym}\mathfrak{G}$, where $R(S) \subset O(S)$ is the subgroup generated by reflections t_u, $u \in S$, $u^2 = -2$. Furthermore,

$$\operatorname{Ker}[\mathrm{d}\colon O(S) \to \operatorname{Aut}\mathcal{S}] = R(S) \rtimes \operatorname{Sym}_0\mathfrak{G},$$

where $\operatorname{Sym}_0\mathfrak{G}$ is the group of permutations of the $\mathbf{E}_8$-type components of $\mathfrak{G}$. Thus, denoting by $\operatorname{Sym}'\mathfrak{G} \subset \operatorname{Sym}\mathfrak{G}$ the group of symmetries acting identically on the union of the $\mathbf{E}_8$-type components, we obtain an isomorphism $\operatorname{Sym}'\mathfrak{G} = \operatorname{Im}\mathrm{d}$. For future references, we combine these statements in a separate lemma.

Lemma 3.2. *Let S be a root lattice. Then, the epimorphism $\mathrm{d}\colon O(S) \twoheadrightarrow \operatorname{Im}\mathrm{d}$ has a splitting $\operatorname{Im}\mathrm{d} = \operatorname{Sym}'\mathfrak{G}_S \hookrightarrow O(S)$, and one always has $-\mathrm{id} \in \operatorname{Im}\mathrm{d}$.* ◁

3.4. **Lattice extensions** (see [24]). An *extension* of a lattice S is an isometry $S \to L$. Two extensions $S \to L_1, L_2$ are (*strictly*) *isomorphic* if there is a bijective isometry $L_1 \to L_2$ identical on S. More generally, given a subgroup $O' \subset O(S)$, two extensions are *O'-isomorphic* if they are related by a bijective isometry whose restriction to S is an element of O'.

We use the notation $S \hookrightarrow L$ for finite index extension ($[L:S] < \infty$). There is a unique embedding $L \subset S \otimes \mathbb{Q}$ and, hence, inclusions $S \subset L \subset L^\sharp \subset S^\sharp$. The *kernel* of a finite index extension $S \hookrightarrow L$ is the subgroup $\mathcal{K} := L/S \subset S^\sharp/S = \mathcal{S}$. Since L is an even integral lattice, the kernel $\mathcal{K}$ is isotropic, *i.e.*, the restriction to $\mathcal{K}$ of the quadratic form $q : \mathcal{S} \to \mathbb{Q}/2\mathbb{Z}$ is identically zero. Conversely, given an isotropic subgroup $\mathcal{K} \subset \mathcal{S}$, the subgroup $L = \{x \in S^\sharp \mid (x \bmod S) \in \mathcal{K}\} \subset S^\sharp$ is an extension of S. Thus, we have the following theorem.

Theorem 3.3 (Nikulin [24]). *The map $L \mapsto \mathcal{K} = L/S \subset \mathcal{S}$ establishes a one-to-one correspondence between the set of strict isomorphism classes of finite index extension $S \hookrightarrow L$ and that of isotropic subgroup $\mathcal{K} \subset \mathcal{S}$. One has $\mathcal{L} = \mathcal{K}^\perp/\mathcal{K}$.* ▷

An isometry $a \in O(S)$ extends to a finite index extension L if and only if $\mathrm{d}(a)$ preserves the kernel $\mathcal{K}$ (as a set). Hence, O'-isomorphism classes of finite index extensions of S correspond to the $\mathrm{d}(O')$-orbits of isotropic subgroups $\mathcal{K} \subset \mathcal{S}$.

Another extreme case is that of a *primitive* extension $S \to L$, *i.e.*, such that the group L/S is torsion free; we use the notation $S \rightarrowtail L$. If L is unimodular, one has $\operatorname{discr} S^\perp \cong -\mathcal{S}$: the graph of this anti-isometry is the kernel of the finite index extension $S \oplus S^\perp \hookrightarrow L$. Hence, the genus $g(S^\perp)$ is determined by those of S and L. If L is also indefinite, it is unique in its genus. Then, for each representative $N \in g(S^\perp)$, an extension $S \rightarrowtail L$ with $S^\perp \cong N$ is determined by a bijective anti-isometry $\varphi\colon \mathcal{S} \to \mathcal{N}$ (L is the finite index extension of $S \oplus N$ whose kernel is the graph of φ), and the latter induces a homomorphism $\mathrm{d}^\varphi\colon O(S) \to \operatorname{Aut}\mathcal{N}$. If φ is not fixed, this map is well defined up to an inner automorphism of $\operatorname{Aut}\mathcal{N}$.

Theorem 3.4 (Nikulin [24])**.** *Let L be an indefinite unimodular even lattice, $S \subset L$ a nondegenerate primitive sublattice, and $O' \subset O(S)$ a subgroup. Then, the O'-isomorphism classes of primitive extensions $S \rightarrowtail L$ are enumerated by the pairs (N, c_N), where $N \in g(S^\perp)$ and $c_N \in \mathrm{d}^\varphi(O')\backslash \operatorname{Aut}\mathcal{N}/\operatorname{Im}\mathrm{d}$ is a double coset (for given N and some anti-isometry $\varphi\colon \mathcal{S} \to \mathcal{N}$).* ▷

Theorem 3.5 (Nikulin [24])**.** *Let $S \rightarrowtail L$ be a lattice extension as in Theorem 3.4, $N = S^\perp$, and $\varphi\colon \mathcal{S} \to \mathcal{N}$ the corresponding anti-isometry. Then, a pair of isometries $a_S \in O(S)$, $a_N \in O(N)$ extends to L if and only if $\mathrm{d}^\varphi(a_S) = \mathrm{d}(a_N)$.* ▷

Fix the notation $\mathbf{L} := 2\mathbf{E}_8 \oplus 3\mathbf{U}$, where $\mathbf{U}$ is the *hyperbolic plane*, $\mathbf{U} = \mathbb{Z}u + \mathbb{Z}v$, $u^2 = v^2 = 0$, $u\cdot v = 1$, and $\mathbf{E}_8$ is the root lattice, see §3.3. For the ease of references, we recast Nikulin's existence theorem from [24] to the particular case of primitive extensions $S \rightarrowtail \mathbf{L}$. Note that we do not need the restriction on the Brown invariant: by the additivity, it would hold automatically.

Theorem 3.6 (Nikulin [24])**.** *Given a nondegenerate even lattice S, a primitive extension $S \rightarrowtail \mathbf{L}$ exists if and only if the following conditions hold: $\sigma_+ S \leqslant 3$, $\sigma_- S \leqslant 19$, $\ell(\mathcal{S}) \leqslant \delta := 22 - \operatorname{rk} S$, and*

- *for all odd $p \in \mathbb{P}$, either $\ell_p(\mathcal{S}) < \delta$ or $|\mathcal{S}| \det_p \mathcal{S} = (-1)^{\sigma_+ S - 1} \bmod (\mathbb{Z}_p^\times)^2$;*
- *either $\ell_2(\mathcal{S}) < \delta$, or $\mathcal{S}_2$ is odd, or $|\mathcal{S}| \det_2 \mathcal{S} = \pm 1 \bmod (\mathbb{Z}_2^\times)^2$.* ▷

3.5. **Miranda–Morrison results** (see [21, 22, 23])**.** Classically, the uniqueness of a lattice N in its genus and the surjectivity of the map $\mathrm{d}\colon\ O(N) \to \operatorname{Aut}\mathcal{N}$ are established using the sufficient conditions found in [24]. Unfortunately, these results do not cover our needs, and we use the stronger criteria developed in [21, 22, 23]. Throughout the rest of this section, we assume that

($*$) N is a nondegenerate indefinite even lattice, $\operatorname{rk} N \geqslant 3$.

Warning 3.7. The convention used in this paper (following Nikulin [24] and, eventually, Gauss) differs slightly from that of Miranda–Morrison, where quadratic and bilinear forms are related *via* $q(x + y) - q(x) - q(y) = b(x, y)$. Roughly, the values of all quadratic (but not bilinear) forms in [21, 22, 23], both on lattices and finite groups, should be multiplied by 2. In particular, all lattices in [21, 22, 23] are even by definition. Note though that this multiplication by 2 is partially incorporated in [21, 22, 23]: for example, the isomorphism class of a finite quadratic form generated by an element α with $q(\alpha) = (u/p^k) \bmod \mathbb{Z}$, which is $(2u/p^k) \bmod 2\mathbb{Z}$ in our notation, is designated by the class of $2u$ in $(\mathbb{Z}_p^\times)/(\mathbb{Z}_p^\times)^2$.

Given a lattice N and a prime $p \in \mathbb{P}$, we define the number $e_p := e_p(N) \in \mathbb{N}$ and the subgroup $\tilde\Sigma_p := \tilde\Sigma_p(N) \subset \Gamma_0 := \{\pm1\} \times \{\pm1\}$ as in (3.11). Algorithms computing $e_p(N)$ and $\tilde\Sigma_p(N)$ are given explicitly in [22]. Computations are in terms of $\operatorname{rk} N$, $\det N$, and $\mathcal{N}$ only, which means that the genus $g(N)$ determines $e_p(N)$, $\tilde{\Sigma}_p(N)$ and, moreover, $\operatorname{Coker}\mathrm{d}$. One has $e_p = 1$ and $\tilde{\Sigma}_p = \Gamma_0$ for almost all $p \in \mathbb{P}$.

Theorem 3.8 (Miranda–Morrison [21, 22])**.** *For N as in ($*$), there is an $\mathbb{F}_2$-module $\mathrm{E}(N)$ and an exact sequence*

$$O(N) \xrightarrow{\mathrm{d}} \operatorname{Aut}\mathcal{N} \xrightarrow{\mathrm{e}} \mathrm{E}(N) \to g(N) \to 1,$$

where $g(N)$ is the genus group of N. One has $|\mathrm{E}(N)| = e(N)/[\Gamma_0 : \tilde{\Sigma}(N)]$, where $e(N) := \prod_{p\in\mathbb{P}} e_p(N)$ and $\tilde{\Sigma}(N) := \bigcap_{p\in\mathbb{P}} \tilde{\Sigma}_p(N)$. ▷

The group $\mathrm{E}(N)$ and homomorphism $\mathrm{e}\colon \operatorname{Aut}\mathcal{N} \to \mathrm{E}(N)$ given by Theorem 3.8 will be called, respectively, the *Miranda–Morrison group* and *Miranda–Morisson homomorphism* of N. The next statement follows from Theorem 3.4, Theorem 3.8, and the fact that a unimodular even indefinite lattice is unique in its genus.

Corollary 3.9 (Miranda–Morrison [21, 22])**.** *Let L be a unimodular even lattice and $S \subset L$ a primitive sublattice such that $N := S^\perp$ is as in ($*$). Then the strict isomorphism classes of primitive extensions $S \rightarrowtail L$ are in a canonical one-to-one correspondence with the Miranda–Morrison group $\mathrm{E}(N)$.* ▷

Generalizing, fix an anti-isometry $\varphi\colon \mathcal{S} \to \mathcal{N}$ and consider the induced map $\mathrm{d}^\varphi\colon O(S) \to \operatorname{Aut} N$, see §3.4. Since $\operatorname{Im}\mathrm{d} \subset \operatorname{Aut}\mathcal{N}$ is a normal subgroup with abelian quotient, this map factors to a homomorphism $\mathrm{d}^\perp\colon O(S) \to \operatorname{Aut}\mathcal{N} \to \mathrm{E}(N)$ independent of φ. Then, the following statement is an immediate consequence of Theorems 3.4 and 3.8.

Corollary 3.10. *Let $S \subset L$ be as in Corollary 3.9, and let $O' \subset O(S)$ be a subgroup. Then the O'-isomorphism classes of primitive extensions $S \rightarrowtail L$ are in a one-to-one correspondence with the $\mathbb{F}_2$-module $\mathrm{E}(N)/\mathrm{d}^\perp(O')$.* ◁

Theorem 3.8 and Corollary 3.9 cover most of our needs. However, in a few special cases, we need the more advanced treatment of [23]. Introduce the groups

$$\Gamma_{p,0} := \{\pm1\} \times \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2 \subset \Gamma_p := \{\pm1\} \times \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2, \quad p \in \mathbb{P},$$

and

$$\Gamma_{\mathbb{A},0} := \prod_p \Gamma_{p,0} \subset \Gamma_{\mathbb{A}} := \Gamma_{\mathbb{A},0} \cdot \sum_p \Gamma_p \subset \Gamma := \prod_p \Gamma_p.$$

(Since the groups involved are multiplicative, although abelian, we follow [23] and use $\cdot$ to denote the sum of subgroups. However, we retain the notation $\sum$ and $\prod$ to distinguished between direct sums and products. Thus, the adelic version $\Gamma_{\mathbb{A}}$ is the set of sequences $\{(s_p, t_p)\} \in \Gamma$ such that $(s_p, t_p) \in \Gamma_{p,0}$ for almost all p.) Let also $\Gamma_{\mathbb{Q}} := \{\pm1\} \times \mathbb{Q}^\times/(\mathbb{Q}^\times)^2 \subset \Gamma_{\mathbb{A}}$. Then $\Gamma_{\mathbb{A},0} \cdot \Gamma_{\mathbb{Q}} = \Gamma_{\mathbb{A}}$ and the intersection $\Gamma_{\mathbb{A},0} \cap \Gamma_{\mathbb{Q}}$ is the group $\Gamma_0 = \{\pm1\} \times \{\pm1\}$ introduced above. We recall that $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ is the $\mathbb{F}_2$-module on the basis $\{-1\} \cup \mathbb{P}$, *i.e.*, it is the set of all square free integers.

On various occasions we will also consider the following subgroups:

- $\Gamma_p^{++} := \{1\} \times \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2 \subset \Gamma_{p,0}$;
- $\Gamma_{2,2} \subset \Gamma_2^{++}$ is the subgroup generated by $(1, 5)$;
- $\Gamma_{\mathbb{Q}}^{--} \subset \Gamma_{\mathbb{Q}}$ is the subgroup generated by $(-1, -1)$ and $(1, p)$, $p \in \mathbb{P}$;
- $\Gamma_0^{--} := \Gamma_{\mathbb{Q}}^{--} \cap \Gamma_0 \subset \Gamma_0$ is the subgroup generated by $(-1, -1)$.

We denote by $\iota_p\colon \Gamma_p \hookrightarrow \Gamma_{\mathbb{A}}$, $p \in \mathbb{P}$, and $\iota_{\mathbb{Q}}\colon \Gamma_{\mathbb{Q}} \hookrightarrow \Gamma_{\mathbb{A}}$ the inclusions. The images $\iota_{\mathbb{Q}}(1,q)$ and $\iota_q(1,q)$, $q \in \mathbb{P}$, differ by an element of $\prod_p \Gamma_p^{++}$, *viz.*, by the sequence $\{(1,s_p)\}$, where $s_q = 1$ and s_p is the class of q in $\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$ for $p \neq q$.

Defined and computed in [23] are certain $\mathbb{F}_2$-modules

$$\Sigma_p^\sharp(N) := \Sigma^\sharp(N \otimes \mathbb{Z}_p) \subset \Sigma_p(N) := \Sigma(N \otimes \mathbb{Z}_p),$$

which depend on the genus of N only. One has $\Sigma_p^\sharp \subset \Gamma_{p,0}$, $\Sigma_p \subset \Gamma_p$, and $\Sigma_p \subset \Gamma_{p,0}$ for almost all p. (In fact, for almost all $p \in \mathbb{P}$ one has $\Sigma_p^\sharp = \Sigma_p = \Gamma_{p,0}$.) Hence,

$$\Sigma^\sharp(N) := \prod_p \Sigma_p^\sharp(N) \subset \Gamma_{\mathbb{A},0}, \qquad \Sigma(N) := \prod_p \Sigma_p(N) \subset \Gamma_{\mathbb{A}}.$$

In these notations, the invariants used in Theorem 3.8 are

$$(3.11) \qquad e_p(N) = [\Gamma_{p,0} : \Sigma_p^\sharp(N)], \qquad \tilde{\Sigma}_p(N) = \Sigma_0^\sharp(N \otimes \mathbb{Z}_p) := \varphi_p^{-1}(\Sigma_p^\sharp(N)),$$

where $\varphi_p\colon \Gamma_0 \to \Gamma_{p,0}$ is the projection, and $\mathrm{E}(N)$ is the quotient $\Gamma_{\mathbb{A},0}/\Sigma^\sharp(N)\cdot\Gamma_0$. (Clearly, $\tilde{\Sigma}(N) = \Sigma^\sharp(N) \cap \Gamma_0$.) Unfortunately, the map $\prod_p \operatorname{Aut}\mathcal{N}_p \to \mathrm{E}(N)$ given by Theorem 3.8 does not respect the product structures. The following statement refines Theorem 3.8, separating the genus group and the p-primary components.

Theorem 3.12 (Miranda–Morrison [23])**.** *Let N be as in* ($*$)*. Then*:

1. *there is an isomorphism $g(N) = \Gamma_{\mathbb{A}}/\Sigma(N)\cdot\Gamma_{\mathbb{Q}}$ (hence, N is unique in its genus if and only if $\Gamma_{\mathbb{A}} = \Sigma(N)\cdot\Gamma_{\mathbb{Q}}$);*
2. *there is a commutative diagram*

$$\begin{array}{ccc}
\operatorname{Aut}\mathcal{N} = \prod_p \operatorname{Aut}\mathcal{N}_p & \xrightarrow{\ \gamma\ } & \prod_p \Sigma_p(N)/\Sigma_p^\sharp(N) \\
\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\
\operatorname{Coker} \mathrm{d} & \xrightarrow{\ \cong\ } & \Sigma(N)/\Sigma^\sharp(N)\cdot(\Sigma(N)\cap\Gamma_{\mathbb{Q}}),
\end{array}$$

where all maps are epimorphisms, γ is the product of certain epimorphisms $\gamma_p\colon \operatorname{Aut}\mathcal{N}_p \twoheadrightarrow \Sigma_p(N)/\Sigma_p^\sharp(N)$, $p \in \mathbb{P}$, and β is the quotient projection. ▷

3.6. **A few simple consequences.** The homomorphism γ in Theorem 3.12(2) is easily computed on reflections: for a mirror $\xi \in \mathcal{N}_r$, $r \in \mathbb{P}$, modulo $\Sigma_r^\sharp(N)$ one has

$$\gamma_r(t_\xi) = (-1, mr^k), \quad \text{where} \quad \xi^2 = \frac{2m}{r^k} \bmod 2\mathbb{Z},\ \text{g.c.d.}(m,r) = 1,\ k \in \mathbb{N}.$$

If $r = 2$ and $\xi^2 = 0 \bmod \mathbb{Z}$, this value is only well defined $\bmod\, \Gamma_2^{++}$; if $r = 2$ and $\xi^2 = \frac{1}{2} \bmod \mathbb{Z}$, it is well defined $\bmod\, \Gamma_{2,2}$. In these two cases, the disambiguation of $\gamma_r(t_\xi)$ needs more information about ξ and N: one needs to represent ξ in the form $\frac{1}{2}x$ for some $x \in N \otimes \mathbb{Z}_2$. Given another prime p, consider the homomorphism $\chi_p\colon \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2 \twoheadrightarrow \{\pm 1\}$,

$$\chi_p(m) := \left(\frac{m}{p}\right) \quad \text{if } p \neq 2, \qquad \chi_2(m) := m \bmod 4,$$

and define the *p-norm* $|\xi|_p \in \{\pm 1\}$ and the 'Kronecker symbol' $\delta_p(\xi) \in \{\pm 1\}$ *via*

$$|\xi|_p := \begin{cases} \chi_p(q^k), & \text{if } r \neq p, \\ \chi_p(m), & \text{if } r = p, \end{cases} \qquad \delta_p(\xi) = (-1)^{\delta_{p,r}},$$

where $\delta_{p,r}$ is the conventional Kronecker symbol. (If $p = 2$ and $\xi^2 = 0 \bmod \mathbb{Z}$, then $|\xi|_2$ is undefined.) Finally, introduce a few *ad hoc* notations for a lattice N:

- the group $\mathrm{E}_p(N) = \{\pm 1\}$ if $p = 1 \bmod 4$ and $e_p(N) \cdot |\tilde{\Sigma}_p(N)| = 8$; in all other cases, $\mathrm{E}_p(N) = 1$;
- the map $\bar{\gamma}_p$ sending a mirror ξ to $|\xi|_p \in \mathrm{E}_p(N)$, with the convention that $\bar{\gamma}_p(\xi) = 1$ whenever $\mathrm{E}_p(N) = 1$;
- the map $\bar{\beta}_p$ sending a mirror ξ to an element of Γ_0: if $p = 1 \bmod 4$, then $\bar{\beta}_p(\xi) = (\delta_p(\xi) \cdot |\xi|_p, 1)$; otherwise, $\bar{\beta}_p(\xi) = \delta_p(\xi) \times |\xi|_p$.

Following [23], we say that a lattice N is *p-regular*, $p \in \mathbb{P}$, if $\Sigma_p^\sharp(N) = \Gamma_{p,0}$, *i.e.*, if $e_p(N) = 1$. We will also say that the prime p is *regular* with respect to N; otherwise, p is *irregular*. In several statements below, we make a technical assumption that $\Sigma_2^\sharp(N) \supset \Gamma_{2,2}$; this inclusion does hold for the transcendental lattices of all primitive homological types (see §4.1) except $\mathbf{S} = \mathbf{A}_{15} \oplus \mathbf{A}_3$, see [23].

Lemma 3.13. *Let N be a lattice as in $(*)$, $\Sigma_2^\sharp(N) \supset \Gamma_{2,2}$, and assume that N has one irregular prime p. Then $\mathrm{E}(N) = \mathrm{E}_p(N)$ and $\mathrm{e}(t_\xi) = \bar{\gamma}_p(\xi)$ for a mirror ξ.*

Lemma 3.14. *Let N be a lattice as in $(*)$, $\Sigma_2^\sharp(N) \supset \Gamma_{2,2}$, and assume that N has two irregular primes p, q. Then*

$$\mathrm{E}(N) = \mathrm{E}_p(N) \times \mathrm{E}_q(N) \times (\Gamma_0 / \tilde{\Sigma}_p(N) \cdot \tilde{\Sigma}_q(N))$$

and one has $\mathrm{e}(t_\xi) = \bar{\gamma}_p(\xi) \times \bar{\gamma}_q(\xi) \times (\bar{\beta}_p(\xi) \cdot \bar{\beta}_q(\xi))$ for a mirror $\xi \in \mathcal{N}$, provided that $\xi^2 \neq 0 \bmod \mathbb{Z}$ if $p = 2$ or $q = 2$.

Corollary 3.15. *Under the hypotheses of Lemma 3.14, assume, in addition, that $|\mathrm{E}(N)| = |\mathrm{E}_p(N)| = 2$. Then $\mathrm{E}(N) = \mathrm{E}_p(N)$ and $\mathrm{e}(t_\xi) = |\xi|_p$ for a mirror ξ.* ◁

Proof of Lemmas 3.13 and 3.14. Let $\Gamma'_{p,0} := \Gamma_{p,0}$ for $p \neq 2$ and $\Gamma'_{2,0} := \Gamma_{2,0}/\Gamma_{2,2}$, so that we can identify $\Gamma'_{p,0} \cong \{\pm 1\} \times \{\pm 1\}$ for all $p \in \mathbb{P}$. If $p \neq 1 \bmod 4$, the map $\varphi_p \colon \Gamma_0 \to \Gamma'_{p,0}$ is an epimorphism; if $p = 1 \bmod 4$, one has $\varphi_p(\Gamma_0) = \{\pm 1\} \times \{1\}$. Modulo $\Gamma_{\mathbb{Q}}^{--}$, the image $\gamma(t_\xi)$ equals $\bar{\gamma}(t_\xi) := \{(\delta_s(\xi), |\xi|_s)\} \in \prod \Gamma'_{s,0}$.

Now, the first statement of each lemma is a computation of the group $\mathrm{E}(N) = \Gamma_{\mathbb{A},0}/\Sigma^\sharp(N) \cdot \Gamma_0$, which can be done in $\Gamma'_{p,0}$ or $\Gamma'_{p,0} \times \Gamma'_{q,0}$; our group $\mathrm{E}_p(N)$ is the quotient $\Gamma_{p,0}/\Sigma_p^\sharp(N) \cdot \operatorname{Im} \varphi_p$. The second statement is the computation of the image of $\bar{\gamma}(\xi)$ in $\mathrm{E}(N)$: the maps $\bar{\gamma}_p$ and $\bar{\beta}_p$ are the projections $\Gamma_{p,0} \to \mathrm{E}_p(N)$ and $\Gamma_{p,0} \to \operatorname{Im} \varphi_p$, respectively. For the latter, we use the following fact, see [23]: if a prime $p = 1 \bmod 4$ is irregular for N and $\Sigma_p^\sharp(N) \not\subset \operatorname{Im} \varphi_p$, then $\Sigma_p^\sharp(N)$ is generated by $(-1, -1)$. □

3.7. **The positive sign structure.** A *positive sign structure* on a lattice N is a choice of an orientation of a maximal positive definite subspace of $N \otimes \mathbb{R}$. (Recall that the orthogonal projection of one such subspace to another is an isomorphism and, hence, all these spaces admit a coherent orientation.) We will use the map $\det_+ \colon O(N) \to \{\pm 1\}$ sending an auto-isometry to $+1$ or -1 if it preserves or, respectively, reverses a positive sign structure. Thus, $O_+(N) := \operatorname{Ker} \det_+$ is the subgroup of auto-isometries preserving positive sign structures. (In the notation of [23], one has $\det_+ = \det \cdot \operatorname{spin}$ and $O_+ = O^{--}$.) The following statement is essentially contained in [23].

Proposition 3.16 (Miranda–Morrison [23])**.** *Let N be a lattice as in $(*)$. Then one has $\tilde{\Sigma}(N) \subset \Gamma_0^{--}$ if and only if $\det_+ a = 1$ for all $a \in \operatorname{Ker}[\mathrm{d} \colon O(N) \to \operatorname{Aut} \mathcal{N}]$.* ▷

Thus, if $\tilde{\Sigma}(N) \subset \Gamma_0^{--}$, there is a well defined descent $\det_+ \colon \operatorname{Im} \mathrm{d} \to \{\pm 1\}$. The next lemma computes the values of $\det_+$ on reflections.

Lemma 3.17. *Let N be a lattice as in (∗), $\Sigma_2^\sharp(N) \supset \Gamma_{2,2}$, and assume that there is a prime p such that $\tilde\Sigma_p(N) \subset \Gamma_0^{--}$. Then, for a mirror $\xi \in \mathcal{N}$ such that $t_\xi \in \operatorname{Im} \mathrm{d}$ and $\xi^2 \ne 0 \bmod \mathbb{Z}$ if $p = 2$, one has $\det_+ t_\xi = \delta_p(\xi) \cdot |\xi|_p$.*

Proof. The proof is similar to that of Lemmas 3.13 and 3.14: we *assume* that the element $\bar\gamma(t_\xi) \cdot \iota_{\mathbb{Q}}(\delta_p(\xi), \delta_p(\xi))$ representing t_ξ lies in $\Sigma^\sharp(N) \cdot \Gamma_0$ and compute its image in $\Sigma^\sharp(N) \cdot \Gamma_0/\Sigma^\sharp(N) \cdot \Gamma_0^{--} = \{\pm 1\}$. This can be done in $\Gamma_{p,0}$. □

Proposition 3.16 can be restated in a form closer to Theorem 3.8: introducing the group $\mathrm{E}_+(N) := \Gamma_{\mathbb{A},0}/\Sigma^\sharp(N) \cdot \Gamma_0^{--}$, one has an exact sequence

$$O_+(N) \xrightarrow{\mathrm{d}} \operatorname{Aut} \mathcal{N} \xrightarrow{\mathrm{e}_+} \mathrm{E}_+(N) \to g(N) \to 1. \tag{3.18}$$

The groups $\mathrm{E}_+(N)$, as well as a few other counterparts, are also computed in [22]: for the order $|\mathrm{E}_+(N)|$, one merely replaces $\tilde\Sigma(N)$ with $\tilde\Sigma(N) \cap \Gamma_0^{--}$ in Theorem 3.8. In the special case of at most two irregular primes, the computation is very similar to §3.6. For an irregular prime p, denote $\tilde\Sigma_p^+(N) := \tilde\Sigma_p(N) \cap \Gamma_0^{--} \subset \Gamma_0^{--}$ and introduce the groups $\mathrm{E}_p^+(N)$ and maps $\bar\gamma_p^+$, $\bar\beta_p^+$ defined on the set of mirrors and taking values in $\mathrm{E}_p^+(N)$ and $\Gamma_0^{--} = \{\pm 1\}$, respectively, as follows:

- if $p = 1 \bmod 4$, then $\mathrm{E}_p^+(N) = \mathrm{E}_p(N)$, $\bar\gamma_p^+ = \bar\gamma_p$, and $\bar\beta_p^+(\xi) = \delta_p(\xi) \cdot |\xi|_p$;
- if $p \ne 1 \bmod 4$, then $\mathrm{E}_p^+(N) = \Gamma_0/\tilde\Sigma_p(N) \cdot \Gamma_0^{--}$ (if $p \ne 2$ or $\Sigma_2^\sharp(N) \supset \Gamma_{2,2}$, one has $\mathrm{E}_p^+(N) = \{\pm 1\}$ if $e_p(N) \cdot |\tilde\Sigma_p^+(N)| = 4$ and $\mathrm{E}_p^+(N) = 1$ otherwise);
- if $p \ne 1 \bmod 4$ and $\mathrm{E}_p^+(N) \ne 1$, then $\bar\gamma_p^+(\xi) = \delta_p(\xi) \cdot |\xi|_p$ and $\bar\beta_p^+(\xi) = |\xi|_p$;
- if $p \ne 1 \bmod 4$ and $\mathrm{E}_p^+(N) = 1$, then $\bar\gamma_p^+(\xi) = 1$ and $\bar\beta_p^+(\xi)$ is the image of $\bar\beta(\xi) = \delta_p(\xi) \times |\xi|_p$, see §3.6, under the projection $\Gamma_0 \to \Gamma_0/\tilde\Sigma_p(N) = \Gamma_0^{--}$.

(In the last case, one has $\bar\beta_p^+(\xi) = |\xi|_p$ unless $p = 2$.) The proof of the next two statements repeats literally that of Lemmas 3.13 and 3.14.

Lemma 3.19. *Let N be a lattice as in (∗), $\Sigma_2^\sharp(N) \supset \Gamma_{2,2}$, and assume that N has a single irregular prime p. Then one has $\mathrm{E}_+(N) = \mathrm{E}_p^+(N)$ and $\mathrm{e}_+(t_\xi) = \bar\gamma_p^+(\xi)$ for a mirror $\xi \in \mathcal{N}$ such that $\xi^2 \ne 0 \bmod \mathbb{Z}$ if $p = 2$.* ◁

Lemma 3.20. *Let N be a lattice as in (∗), $\Sigma_2^\sharp(N) \supset \Gamma_{2,2}$, and assume that N has two irregular primes p, q. Then*

$$\mathrm{E}_+(N) = \mathrm{E}_p^+(N) \times \mathrm{E}_q^+(N) \times (\Gamma_0^{--}/\tilde\Sigma_p^+(N) \cdot \tilde\Sigma_q^+(N))$$

and one has $\mathrm{e}_+(t_\xi) = \bar\gamma_p^+(\xi) \times \bar\gamma_q^+(\xi) \times (\bar\beta_p^+(\xi) \cdot \bar\beta_q^+(\xi))$ for a mirror $\xi \in \mathcal{N}$ such that $\xi^2 \ne 0 \bmod \mathbb{Z}$ if $p = 2$ or $q = 2$. ◁

Corollary 3.21. *Under the hypotheses of Lemma 3.20, assume, in addition, that $|\mathrm{E}_+(N)| = |\mathrm{E}_p^+(N)| = 2$. Then $\mathrm{E}_+(N) = \mathrm{E}_p^+(N)$ and $\mathrm{e}(t_\xi) = \bar\gamma_p^+(\xi)$ for a mirror $\xi \in \mathcal{N}$ such that $\xi^2 \ne 0 \bmod \mathbb{Z}$ if $p = 2$.* ◁

4. The deformation classification

4.1. **The homological type.** Consider a simple sextic $D \subset \mathbb{P}^2$. Recall (see §2.2) that we denote by $X \to \mathbb{P}^2$ the minimal resolution of singularities of the double covering of $\mathbb{P}^2$ ramified at D, and that the set of singularities of D can be identified with the sublattice $\mathbf{S} \subset \mathbf{L}$ spanned by the classes of the exceptional divisors. Let $\tau\colon X \to X$ be the deck translation of the covering.

Lemma 4.1. *The induced action of τ on the Dynkin graph $\mathfrak{G} := \mathfrak{G}_{\mathbf{S}}$ preserves the components of $\mathfrak{G}$; it acts by the only nontrivial symmetry on the components of type $\mathbf{A}_{p\geqslant 2}$, $\mathbf{D}_{\mathrm{odd}}$, or $\mathbf{E}_6$, and by the identity otherwise.* ◁

Remark 4.2. In other words, $\tau\colon \mathfrak{G} \to \mathfrak{G}$ can be characterized as the 'simplest' symmetry of $\mathfrak{G}$ inducing $-\mathrm{id}$ on $\operatorname{discr}\mathbf{S}$.

In addition to $\mathbf{S}$, we have the class $h \in \mathbf{L}$ of the pull-back of a generic line in $\mathbb{P}^2$. Obviously, h is orthogonal to $\mathbf{S}$ and $h^2 = 2$. Let $\mathbf{S}_h := \mathbf{S} \oplus \mathbb{Z}h$. The triple $\mathcal{H} := (\mathbf{S}, h, \mathbf{L})$, *i.e.*, the lattice extension $\mathbf{S}_h \hookrightarrow \mathbf{L}$ regarded up to isometries of $\mathbf{L}$ preserving $\mathbf{S}$ (as a set) and h, is called the *homological type* of D. This extension is subject to certain restrictions, which are summarized in the following definitions.

Definition 4.3. Let $\mathbf{S}$ be a root lattice. A *homological type* (extending $\mathbf{S}$) is an extension $\mathbf{S}_h := \mathbf{S} \oplus \mathbb{Z}h \hookrightarrow \mathbf{L}$ satisfying the following conditions:

(1) any vector $v \in (\mathbf{S} \otimes \mathbb{Q}) \cap \mathbf{L}$ with $v^2 = -2$ is in $\mathbf{S}$;
(2) there is no vector $v \in \tilde{\mathbf{S}}_h := (\mathbf{S}_h \otimes \mathbb{Q}) \cap \mathbf{L}$ with $v^2 = 0$ and $v \cdot h = 1$.

Note that condition (2) in this definition can be restated as follows: if a is a generator of an orthogonal summand $\mathbf{A}_1 \subset \mathbf{S}$, the vector $a + h$ is primitive in $\mathbf{L}$.

Given a homological type $\mathcal{H} = (\mathbf{S}, h, \mathbf{L})$, we let

- $\tilde{\mathbf{S}} := (\mathbf{S} \otimes \mathbb{Q}) \cap \mathbf{L}$ be the primitive hull of $\mathbf{S}$,
- $\tilde{\mathbf{S}}_h := (\mathbf{S}_h \otimes \mathbb{Q}) \cap \mathbf{L}$ be the primitive hull of $\mathbf{S}_h$, and
- $\mathbf{T} := \mathbf{S}_h^\perp$ with $\mathcal{T} = \operatorname{discr}\mathbf{T}$ be the *transcendental lattice.*

Since $\sigma_+\mathbf{T} = 2$, all positive definite 2-spaces in $\mathbf{T} \otimes \mathbb{R}$ can be oriented in a coherent way. A choice $\mathfrak{o}$ of one of these coherent orientations, *i.e.*, a positive sign structure on $\mathbf{T}$, see §3.7, is called an *orientation* of $\mathcal{H}$. The homological type of a plane sextic D has a canonical orientation, *viz.* the one given by the real and imaginary parts of the class of a holomorphic form ω on X.

An *automorphism* of a homological type $\mathcal{H} = (\mathbf{S}, h, \mathbf{L})$ is an autoisometry of $\mathbf{L}$ preserving $\mathbf{S}$ (as a set) and h. The group of automorphisms of $\mathcal{H}$ is denoted by $\operatorname{Aut}\mathcal{H}$. Let $\operatorname{Aut}_+\mathcal{H} \subset \operatorname{Aut}\mathcal{H}$ be the subgroup of the automorphisms inducing id on $\mathbf{T}$. On the other hand, we have the group $\operatorname{Aut}_h \tilde{\mathbf{S}}_h \subset O(\tilde{\mathbf{S}}_h)$ of the isometries of $\tilde{\mathbf{S}}_h$ preserving h. There are obvious homomorphisms

$$\operatorname{Aut}_+\mathcal{H} \hookrightarrow \operatorname{Aut}\mathcal{H} \to \operatorname{Aut}_h \tilde{\mathbf{S}}_h \hookrightarrow O(\mathbf{S}), \tag{4.4}$$

where the latter inclusion is due to item 1 in Definition 4.3, as $\mathbf{S} \subset \tilde{\mathbf{S}}_h$ is recovered as the sublattice generated by the roots orthogonal to h. If the primitive extension $\tilde{\mathbf{S}}_h \rightarrowtail \mathbf{L}$ is defined by an anti-isometry $\varphi\colon \operatorname{discr}\tilde{\mathbf{S}}_h \to \mathcal{T}$ (see §3.4), so that we have a homomorphism $\mathrm{d}^\varphi\colon \operatorname{Aut}_h \tilde{\mathbf{S}}_h \to \operatorname{Aut}\mathcal{T}$, then, for $\epsilon = +$ or empty,

$$\operatorname{Im}[\operatorname{Aut}_\epsilon \mathcal{H} \to \operatorname{Aut}_h \tilde{\mathbf{S}}_h] = (\mathrm{d}^\varphi)^{-1}\mathrm{d}(O_\epsilon(\mathbf{T})). \tag{4.5}$$

The deformation classification of sextics is based on the following statement.

Theorem 4.6 (see [8])**.** *The map sending a plane sextic $D \subset \mathbb{P}^2$ to its oriented homological type establishes a bijection between the set of equisingular deformation classes of simple sextics and the set of isomorphism classes of oriented homological types. Complex conjugate sextics have isomorphic homological types that differ by the orientations.* ▷

A homological type is called *symmetric* if it admits an orientation reversing automorphism. According to Theorem 4.6, symmetric are the homological types corresponding to *real*, *i.e.*, conjugation invariant components of $\mathcal{M}(\mathbf{S})$.

Recall that, in §2.2, the equisingular strata $\mathcal{M}(\mathbf{S})$ were subdivided into families $\mathcal{M}_*(\mathbf{S})$. The precise definition is as follows: the subscript $*$ is the sequence of invariant factors of the kernel $\mathcal{K}$ of the finite index extension $\mathbf{S}_h \hookrightarrow \tilde{\mathbf{S}}_h$. (Obviously, $\mathcal{K}$ is invariant under equisingular deformations.) Theorems 4.7 and 4.8 below single out the families $\mathcal{M}_1$ and $\mathcal{M}_3$, which are of our primary interest; they correspond to $\mathcal{K} = 0$ and $\mathcal{K} = \mathbb{G}_3$, respectively.

A homological type $\mathcal{H} = (\mathbf{S}, h, \mathbf{L})$ is called *primitive* if $\mathbf{S}_h \subset \mathbf{L}$ is a primitive sublattice, *i.e.*, if $\mathcal{K} = 0$. In this case, one has $\operatorname{discr} \tilde{\mathbf{S}}_h = \mathcal{S} \oplus \langle \frac{1}{2} \rangle$ and the inclusion $\operatorname{Aut}_h \tilde{\mathbf{S}}_h \hookrightarrow O(\mathbf{S})$, see (4.4), is an isomorphism.

Theorem 4.7 (see [7])**.** *A simple plane sextic D is irreducible and non-special if and only if its homological type is primitive.* ▷

The fact that primitive homological types give rise to irreducible sextics was also observed in [31], where the primitivity is stated as a sufficient condition.

Theorem 4.8 (see [7])**.** *A simple plane sextic D is irreducible and p-torus, $p = 1$, 4, or 12, if and only if the kernel $\mathcal{K}$ of the extension $\mathbf{S}_h \hookrightarrow \tilde{\mathbf{S}}_h$ is, respectively, $\mathbb{G}_3$, $\mathbb{G}_3 \oplus \mathbb{G}_3$, or $\mathbb{G}_3 \oplus \mathbb{G}_3 \oplus \mathbb{G}_3$.* ▷

There is a similar characterization of other special sextics: a sextic is irreducible and $\mathbb{D}_{2n}$-special, $n > 3$, if and only if the kernel $\mathcal{K}$ is $\mathbb{G}_n$; one necessarily has $n = 5$ or 7. Note that these statements cover all possibilities for the kernel $\mathcal{K}$ free of 2-torsion, and $\mathcal{K}$ has 2-torsion if and only if the sextic is reducible, see, *e.g.*, [13].

4.2. **Extending a fixed set of singularities S to a sextic.** By Theorem 4.6, given a simple set of singularities $\mathbf{S}$, the connected components of the space $\mathcal{M}(\mathbf{S})$ modulo the complex conjugation $\operatorname{conj}\colon \mathbb{P}^2 \to \mathbb{P}^2$ are enumerated by the isomorphism classes of the homological types extending $\mathbf{S}$. If a subscript $*$ is specified, the set $\pi_0(\mathcal{M}_*(\mathbf{S})/\operatorname{conj})$ is enumerated by the extensions with the kernel $\mathcal{K}$ of the finite index extension $\mathbf{S}_h \hookrightarrow \tilde{\mathbf{S}}_h$ in the given isomorphism class.

We are interested in the sets of singularities $\mathbf{S}$ with $\mu(\mathbf{S}) \leqslant 18$. In this case, $\mathbf{T}$ is indefinite and $\operatorname{rk} \mathbf{T} \geqslant 3$; hence, Miranda–Morrison's results apply and, with $\mathcal{K}$ and, hence, $\tilde{\mathbf{S}}_h$ fixed, the further extensions $\tilde{\mathbf{S}}_h \rightarrowtail \mathbf{L}$ are enumerated by the cokernel of the well-defined homomorphism $\mathrm{d}^\perp\colon \operatorname{Aut}_h \tilde{\mathbf{S}}_h \to \mathrm{E}(\mathbf{T})$, see §3.5. In the special case $\mathcal{K} = 0$, due to the isomorphism $\operatorname{Aut}_h \tilde{\mathbf{S}}_h = O(\mathbf{S})$, we have a canonical bijection

$$\pi_0(\mathcal{M}_1(\mathbf{S})/\operatorname{conj}) = \operatorname{Coker}[\mathrm{d}^\perp\colon O(\mathbf{S}) \to \mathrm{E}(\mathbf{T})], \tag{4.9}$$

assuming that $\mathbf{S}_h$ does admit a primitive extension to $\mathbf{L}$ and taking for $\mathbf{T}$ any representative of the genus $\mathbf{S}_h^\perp$.

4.3. **Proof of Theorem 2.4.** By Theorems 4.6 and 4.7, for the first part of the statement it suffices to list (using Theorem 3.6) all sets of singularities extending to a primitive homological type; the resulting list is compared against the list of all perturbations of the maximizing sets obtained. Since the homological type is primitive, we have $\operatorname{discr} \tilde{\mathbf{S}}_h = \mathcal{S} \oplus \langle \frac{1}{2} \rangle$.

For the second part, let $\mathbf{S}$ be one of the sets of singularities found, $\mu(\mathbf{S}) \leqslant 18$, and let $\mathbf{T}$ be a representative of the genus $g(\mathbf{S}_h^\perp)$. In most cases, Theorem 3.8 gives

Table 5. Exceptional sets of singularities (see §4.3)

[1] $\mathbf{E}_6 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2$	[3] $\mathbf{E}_7 \oplus \mathbf{A}_7 \oplus 2\mathbf{A}_2$	[3] $\mathbf{A}_7 \oplus \mathbf{A}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$
[1] $\mathbf{A}_5 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$	[3] $\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_5$	[4] $2\mathbf{A}_6 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$
[2] $3\mathbf{A}_4 \oplus 3\mathbf{A}_2$	[3] $2\mathbf{A}_7 \oplus 2\mathbf{A}_2$	[5] $2\mathbf{A}_9$

us $\mathrm{E}(\mathbf{T}) = 0$ and, due to Corollary 3.9, a primitive homological type extending $\mathbf{S}$ is unique up to strict isomorphism. In the remaining cases, it suffices to show that the map $\mathrm{d}^\perp\colon O(\mathbf{S}) \to \mathrm{E}(\mathbf{T})$ is onto, see (4.9).

There are 32 sets of singularities containing a point of type $\mathbf{A}_4$ and satisfying the hypotheses of Lemma 3.13 or Corollary 3.15 (with $p = 5$); in these cases, a nontrivial symmetry of any type $\mathbf{A}_4$ points maps to the generator $-1 \in \mathrm{E}(\mathbf{T})$. The remaining nine sets of singularities are collected in Table 5, with references to the list below, where we indicate the Miranda–Morrison homomorphism $\mathrm{e}\colon \operatorname{Aut}\mathcal{T} \to \mathrm{E}(\mathbf{T})$ (given by Lemma 3.14) and automorphism(s) of $\mathbf{S}$ generating $\mathrm{E}(\mathbf{T})$.

(1) $\mathrm{e}\colon t_\xi \mapsto \delta_3(\xi)\cdot\delta_5(\xi)\cdot|\xi|_5 \in \{\pm1\}$; a transposition $\mathbf{A}_4 \leftrightarrow \mathbf{A}_4$;
(2) $\mathrm{e}\colon t_\xi \mapsto (\delta_3(\xi)\cdot\delta_5(\xi)\cdot|\xi|_5, |\xi|_5) \in \{\pm1\}\times\{\pm1\}$; a symmetry of $\mathbf{A}_4$ *and* a transposition $\mathbf{A}_4 \leftrightarrow \mathbf{A}_4$ (two generators);
(3) $\mathrm{e}\colon t_\xi \mapsto \delta_2(\xi)\cdot\delta_3(\xi)\cdot|\xi|_2\cdot|\xi|_3 \in \{\pm1\}$; a transposition $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$ or a symmetry of $\mathbf{A}_4$, $\mathbf{A}_5$, or $\mathbf{E}_6$;
(4) $\mathrm{e}\colon t_\xi \mapsto \delta_3(\xi)\cdot\delta_7(\xi)\cdot|\xi|_3\cdot|\xi|_7 \in \{\pm1\}$; a transposition $\mathbf{A}_1 \leftrightarrow \mathbf{A}_1$;
(5) $\mathrm{e}\colon t_\xi \mapsto |\xi|_5 \in \{\pm1\}$; none.

The last case $\mathbf{S} = 2\mathbf{A}_9$ is special: the map $\mathrm{d}^\perp\colon O(\mathbf{S}) \to \mathrm{E}(\mathbf{T})$ is not surjective and there are two deformation families, as stated.

To complete the proof, we need to analyze whether the space $\mathcal{M}_1(\mathbf{S})$ contains a real curve and, if it does not, whether the homological type $\mathcal{H}$ extending $\mathbf{S}$ is symmetric. This is done in §6.2 below. □

4.4. **Proof of Corollary 2.7.** Unless $\mathbf{S} = 2\mathbf{A}_9$, the statement follows immediately from Theorem 2.4. Indeed, there is a degeneration $\mathbf{S} \rightarrowtail \mathbf{S}'$ to a maximizing set of singularities $\mathbf{S}'$. Due to [10, Proposition 5.1.1], there is a degeneration $D \rightarrowtail D'$ of *some* sextics $D \in \mathcal{M}_1(\mathbf{S})$ and $D' \in \mathcal{M}_1(\mathbf{S}')$. Since $\mathcal{M}_1(\mathbf{S})/\operatorname{conj}$ is connected, a degeneration exists for *any* sextic $D \in \mathcal{M}_1(\mathbf{S})$. The exceptional case $\mathbf{S} = 2\mathbf{A}_9$ with disconnected moduli space is given by Proposition 2.5, see §4.5 below. □

4.5. **Proof of Proposition 2.5.** For $\mathbf{S}_0 = 2\mathbf{A}_9$, one has $\mathbf{T} \cong \mathbb{Z}u \oplus \mathbb{Z}v \oplus \mathbb{Z}w$, with $u^2 = v^2 = 10$, $w^2 = -2$. The group $\mathcal{T}$ is $\langle\frac{2}{5}\rangle \oplus \langle\frac{2}{5}\rangle \oplus \langle\frac{1}{2}\rangle \oplus \langle\frac{1}{2}\rangle \oplus \langle\frac{3}{2}\rangle$, and $\operatorname{Aut}\mathcal{T}$ is generated by

$$\sigma_{1,2}\colon \alpha_{1,2} \mapsto -\alpha_{1,2}, \quad \sigma_3\colon \alpha_1 \leftrightarrow \alpha_2, \quad \sigma_4\colon \alpha_3 \leftrightarrow \alpha_4.$$

Let $\mathcal{S}_h := \operatorname{discr}\tilde{\mathbf{S}}_h = \mathcal{S}_0 \oplus \langle\frac{1}{2}\rangle$. According to §3.3, the image of $\mathrm{d}\colon O(\mathbf{S}_0) \to \operatorname{Aut}\mathcal{S}_h$ is generated by $-\mathrm{id}$ on each of the two copies of $\operatorname{discr}\mathbf{A}_9$ and by the transposition of the two copies. Since $|\mathrm{E}(\mathbf{T})| = 2$, the image $\operatorname{Im}[\mathrm{d}\colon O(\mathbf{T}) \to \operatorname{Aut}\mathcal{T}]$ is generated by the images σ_1, σ_2, $\sigma_3\sigma_4$ of the auto-isometries $u \mapsto -u$, $v \mapsto -v$, $u \leftrightarrow v$, respectively. It is straightforward that $\operatorname{Im}\mathrm{d}^\perp = 0 \subset \mathrm{E}(\mathbf{T})$; hence, by Corollary 3.10, $2\mathbf{A}_9 \oplus \mathbb{Z}h$ extends to $\mathbf{L}$ in two ways. The proof of the fact that both homological types are represented by real curves is postponed till §6.1 below.

The two homological types can be distinguished as follows. In $\mathcal{T}$, there are two non-characteristic elements of square $\frac{1}{2}$ and two pairs of opposite elements of square

$\frac{2}{5}$, and the map $\frac{1}{2}u \mapsto \frac{1}{5}u$, $\frac{1}{2}v \mapsto \pm\frac{1}{5}v$ establishes a bijection between these two-element sets. A similar bijection in the other group $\mathcal{S}_h$ is due to the decomposition $\mathcal{S}_h = 2\operatorname{discr}\mathbf{A}_9 \oplus \langle\frac{1}{2}\rangle$. The two homological types extending $2\mathbf{A}_9$ differ by whether the anti-isometry $\mathcal{S}_h \to \mathcal{T}$ does or does not respect these bijections.

Now, a simple computation shows that each of the two sublattices $\mathbf{S}_0 \oplus \mathbb{Z}h \subset \mathbf{L}$ extends to both $\mathbf{S}_i \oplus \mathbb{Z}h \subset \mathbf{L}$, $i = 1, 2$ (where $\mathbf{S}_1 = \mathbf{A}_{19}$ and $\mathbf{S}_2 = \mathbf{A}_{10} \oplus \mathbf{A}_9$ are as in the statement), and these are all possible degenerations of $\mathbf{S}_0$. On the other hand, each $\mathbf{S}_i$, $i = 1, 2$, extends to two distinct real homological types, see [29], and each of the resulting families admits a unique, up to deformation, perturbation to $2\mathbf{A}_9$, *cf.* [10, Proposition 5.1.1]. These observations complete the proof. ☐

4.6. **Proof of Theorem 2.9 and Corollary 2.10.** Let $\mathbf{S}$ be a set of singularities of weight 6 or 7. As shown in [7], up to automorphism of $\mathbf{S}$, there is at most one isotropic order 3 element $\beta \in \mathcal{S}$ satisfying condition (1) in Definition 4.3. Such an element does exist if and only if $w(\mathbf{S}) = 6$ or $w(\mathbf{S}) = 7$ and $\mathbf{S}$ contains $\mathbf{A}_2$ as a direct summand. (In the latter case, the extra $\mathbf{A}_2$ point becomes an outer singularity; all other singular points of positive weight are inner.) This element β has the form $\sum_i(\pm\alpha_i)$, where α_i are the only (up to sign) order 3 elements in the discriminants of the inner singular points. Important for Theorems 3.6 and 3.8 is the relation between $\mathcal{S}$ and $\tilde{\mathcal{S}} := \operatorname{discr}\tilde{\mathbf{S}}$. One has:

- $\ell_p(\tilde{\mathcal{S}}) = \ell_p(\mathcal{S})$ and $\det_p\tilde{\mathcal{S}} = \det_p\mathcal{S}$ for all primes $p \neq 3$;
- $|\tilde{\mathcal{S}}| = \frac{1}{9}|\mathcal{S}|$ and $\det_3\tilde{\mathcal{S}} = -9\det_3\mathcal{S}$;
- $\ell_3(\tilde{\mathcal{S}}) = \ell_3(\mathcal{S}) - \delta$, where $\delta = 1$ if $\mathbf{S}$ contains (as a direct summand) $\mathbf{A}_{17}$ or $2\mathbf{A}_8$ and $\delta = 2$ otherwise.

Now, as in §4.3, we compare two lists: the sets of singularities extending to a homological types with kernel $\mathbb{G}_3$ (using Theorem 3.6) and those obtained by perturbations from the maximizing sets, see Table 3. These lists coincide. For each set of singularities $\mathbf{S}$ found, Theorem 3.8 gives us $\mathrm{E}(\mathbf{T}) = 0$; hence, there is a unique homological type and the space $\mathcal{M}_3(\mathbf{S})/\operatorname{conj}$ is connected. In view of the first part, this fact implies Corollary 2.10, and it remains to analyze the real structures. This is done in §6.4 below. ☐

4.7. **Digression: permutations of the singular points.** Consider a sextic D with the set of singularities $\mathbf{S}$, and let $\mathcal{M}(D)$ be the *connected* equisingular stratum containing D. Denoting by $\mathbb{S}(\mathbf{S})$ the group of the type-preserving permutations of the singular points constituting $\mathbf{S}$, we have the so-called *monodromy representation* $\pi_1(\mathcal{M}(D)) \to \mathbb{S}(\mathbf{S})$. In this section, we are interested in the image $\mathbb{S}_+(D)$ of this homomorphism. In other words, we can consider the covering $\tilde{\mathcal{M}}(D) \to \mathcal{M}(D)$ whose points are sextics with marked singular points; then, $[\mathbb{S}(\mathbf{S}) : \mathbb{S}_+(D)]$ is the number of the connected components of $\tilde{\mathcal{M}}(D)$.

Theorem 4.10. *The permutation group $\mathbb{S}_+ := \mathbb{S}_+(D)$ of a non-special irreducible simple sextic D with the set of singularities $\mathbf{S}$ is as follows*:

- *if $\mu(D) = 19$, then $\mathbb{S}_+$ is the group of permutations of the $\mathbf{E}_8$ points of $\mathbf{S}$*;
- *if $\mathbf{S}$ is one of the sets of singularities listed in Table 6, then $\mathbb{S}_+$ is as shown in the table (see the explanation after the statement).*

In all other cases, one has $\mathbb{S}_+ = \mathbb{S}(\mathbf{S})$.

The groups $\mathbb{S}_+(D)$ are encoded in Table 6 by means of one or several subsets $\mathbf{S}_1, \mathbf{S}_2, \ldots$ enclosed in brackets: a permutation $\sigma \in \mathbb{S}(\mathbf{S})$ belongs to $\mathbb{S}_+(D)$ if and

Table 6. Permutation groups (see Theorem 4.10)

$[3\mathbf{E}_6]$	$[2\mathbf{A}_2] \oplus \mathbf{E}_7 \oplus \mathbf{A}_7$	$[2\mathbf{A}_4 \oplus 2\mathbf{A}_2] \oplus \mathbf{D}_6$
$[2\mathbf{E}_6] \oplus \mathbf{D}_6$	$[2\mathbf{A}_7 \oplus 2\mathbf{A}_2]$	$[2\mathbf{A}_4 \oplus 2\mathbf{A}_2] \oplus 2\mathbf{A}_3$
$[2\mathbf{E}_6] \oplus \mathbf{A}_6$	$[2\mathbf{A}_2 \oplus 2\mathbf{A}_1] \oplus 2\mathbf{A}_6$	$[2\mathbf{A}_4 \oplus 3\mathbf{A}_2] \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$
$[2\mathbf{E}_6] \oplus \mathbf{A}_5 \oplus \mathbf{A}_1$	$[2\mathbf{A}_4] \oplus \mathbf{E}_6 \oplus 2\mathbf{A}_2$	$[3\mathbf{A}_4] \oplus [3\mathbf{A}_2]$
$[2\mathbf{E}_6] \oplus \mathbf{A}_5$	$[2\mathbf{A}_4] \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$	

only if the restriction of σ to each subset $\mathbf{S}_i$ is even. Note that, in many cases, this condition actually implies that $\mathbb{S}_+(D)$ is the trivial group.

Proof. If $\mu(D) = 19$, then $\mathbb{S}_+(D)$ is the group of projective symmetries of D; these groups are described in [9].

In general, let $(\mathcal{H}, \mathfrak{o})$ be the oriented homological type of D. From the description of the equisingular moduli spaces of sextics, see, *e.g.*, [8], it is immediate that the monodromy representation can be factored as

$$\pi_1(\mathcal{M}(D)) \twoheadrightarrow \operatorname{Aut}_+ \mathcal{H} \to O(\mathbf{S}) \twoheadrightarrow \mathbb{S}(\mathbf{S}),$$

where the arrow in the middle is the homomorphism (4.4). If $\mathcal{H}$ is primitive and $\mu(\mathbf{S}) \leqslant 18$, we have a well-defined homomorphism $\mathrm{d}^\perp \colon O(\mathbf{S}) \to \mathrm{E}_+(\mathbf{T})$, *cf.* §3.5, where $\mathbf{T}$ is the transcendental lattice; this homomorphism factors through $\mathrm{d}' \colon \operatorname{Sym}' \mathfrak{G}_{\mathbf{S}} \to \mathrm{E}_+(\mathbf{T})$, see Lemma 3.2. Hence, combining the above observation with (3.18) and (4.5), we conclude that $\mathbb{S}_+ \subset \mathbb{S}(\mathbf{S})$ is the image of $\operatorname{Ker} \mathrm{d}'$.

The groups $\mathrm{E}_+(\mathbf{T})$ are computed using Lemmas 3.19 and 3.20. For most curves, one has $\mathrm{E}_+(\mathbf{T}) = 1$ and hence $\mathbb{S}_+ = \mathbb{S}(\mathbf{S})$.

There are 171 sets of singularities $\mathbf{S}$ containing a point of type $\mathbf{A}_2$ and satisfying the hypotheses of Lemma 3.19 or Corollary 3.21 with $p = 3$. For such curves, a non-trivial symmetry of $\mathbf{A}_2$ maps to the generator $-1 \in \mathrm{E}_+(\mathbf{T})$; hence, $\mathbb{S}_+ = \mathbb{S}(\mathbf{S})$.

Similarly, there are 28 sets of singularities $\mathbf{S}$ containing a point of type $\mathbf{A}_4$ and satisfying the hypotheses of Lemma 3.19 or Corollary 3.21 with $p = 5$: a non-trivial symmetry of $\mathbf{A}_4$ maps to the generator $-1 \in \mathrm{E}_+(\mathbf{T})$.

In the very few remaining cases, the group $\operatorname{Sym}' \mathfrak{G}_{\mathbf{S}}$, identified with its image in $\operatorname{Aut} \mathcal{S}$, see Lemma 3.2, is generated by reflections, and the map d' is computed explicitly using Lemmas 3.19 and 3.20. Details are left to the reader. □

5. The fundamental group

5.1. **The degeneration principle.** Our computation of the fundamental groups is indirect; it is based on a few previously known results and the following statement, often referred to as the *degeneration principle*.

Theorem 5.1 (Zariski [32])**.** *If a plane curve D' degenerates to a reduced plane curve D, there is an epimorphism $\pi_1(\mathbb{P}^2 \smallsetminus D) \twoheadrightarrow \pi_1(\mathbb{P}^2 \smallsetminus D')$.* ▷

Corollary 5.2. *If a plane sextic D' degenerates to D and $\pi_1(\mathbb{P}^2 \smallsetminus D) = \mathbb{G}_6$, then also $\pi_1(\mathbb{P}^2 \smallsetminus D') = \mathbb{G}_6$.* ◁

Corollary 5.3. *If a sextic D' of torus type degenerates to D and $\pi_1(\mathbb{P}^2 \smallsetminus D) = \Gamma$, then also $\pi_1(\mathbb{P}^2 \smallsetminus D') = \Gamma$.*

Proof. Since any sextic D' of torus type is a degeneration of Zariski's six-cuspidal sextic, there is an epimorphism $\pi_1(\mathbb{P}^2 \smallsetminus D') \twoheadrightarrow \Gamma$, see [32] and Theorem 5.1. Since Γ is a Hopfian group, the statement follows from Theorem 5.1. □

5.2. **Proof of Corollary 2.8.** We need a slightly stronger statement, which is proved in the same way as Corollary 2.7, see §4.4, by comparing two independent lists: with few exceptions listed below, any non-special irreducible plane sextic degenerates to one with known *abelian* fundamental group.

The exceptions are the six sets of singularities listed in Theorem 2.4 and

$$2\mathbf{A}_4 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2 \rightarrowtail \mathbf{E}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2,$$
$$3\mathbf{A}_4 \oplus 3\mathbf{A}_2,\ 2\mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 3\mathbf{A}_2 \oplus \mathbf{A}_1 \rightarrowtail \mathbf{E}_7 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2.$$

The fundamental groups of the curves listed in Theorem 2.4 are computed in [13], using the degenerations

$$2\mathbf{D}_9 \rightarrowtail \mathbf{D}_{10} \oplus \mathbf{D}_9, \quad 2\mathbf{D}_7 \oplus 2\mathbf{A}_2 \rightarrowtail \mathbf{D}_{10} \oplus \mathbf{D}_7 \oplus \mathbf{A}_2$$

to reducible maximizing sextics. The groups of *some* curves realizing the three other sets of singularities are computed together with those of the corresponding maximizing sextics, by analyzing the perturbations (see [13] for references). In view of the uniqueness given by Theorem 2.4, the results hold for *all* curves. □

5.3. **Proof of Corollary 2.11.** With one exception, *viz.* the set of singularities $(\mathbf{A}_8 \oplus \mathbf{A}_5 \oplus \mathbf{A}_2) \oplus \mathbf{A}_4$, the fundamental groups of all maximizing irreducible sextics of torus type are known, see [13, 14] for references. Comparing the two lists, one can easily see that all but 14 non-maximizing deformation families degenerate to maximizing sextics D with $\pi_1(\mathbb{P}^2 \smallsetminus D) = \Gamma$ known; for these curves, the fundamental group is Γ due to Corollary 5.3. All sextics with at least one type $\mathbf{E}_6$ type point are treated in [13]. The remaining exceptions are

$$(6\mathbf{A}_2) \oplus 4\mathbf{A}_1 \rightarrowtail (6\mathbf{A}_2) \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_1,$$

studied in [6] as perturbations of $(3\mathbf{E}_6) \oplus \mathbf{A}_1$, and

$$(6\mathbf{A}_2) \oplus \mathbf{A}_4 \oplus \mathbf{A}_1 \rightarrowtail (\mathbf{A}_5 \oplus 4\mathbf{A}_2) \oplus \mathbf{A}_4 \oplus \mathbf{A}_1,$$

studied in [14] as perturbations of $(\mathbf{A}_8 \oplus 3\mathbf{A}_2) \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$. □

6. Real structures

6.1. **Real sextics.** A *real structure* on a complex analytic variety X is an anti-holomorphic involution $c\colon X \to X$. A *real variety* is a pair (X, c), where X is a complex variety and c is a real structure. The fixed point set $X_{\mathbb{R}} := \operatorname{Fix} c$ is called the *real part* of X. (We routinely omit c in the notation when it is understood.)

Let (X, c) be a real surface. A curve $D \subset X$ is said to be *real* if $c(D) = D$. If $\bar{X} \to X$ is a double covering branched over a (nonempty) real curve, the real structure c lifts to two distinct real structures on $\bar{X}$; the two lifts differ by the deck translation of the covering, and all three involutions commute.

Any real structure on $\mathbb{P}^2$ is equivalent to the standard complex conjugation; in appropriate homogeneous coordinates, it is given by $(z_0 : z_1 : z_2) \mapsto (\bar{z}_0 : \bar{z}_1 : \bar{z}_2)$. In these coordinates, real curves are those defined by real polynomials.

Theorem 6.1. *A homological type $\mathcal{H}$ is realized by a real sextic if and only if $\mathcal{H}$ admits an* involutive *orientation reversing automorphism.*

Proof. The necessity is obvious: the real structure on $\mathbb{P}^2$ lifts to a real structure on the covering $K3$-surface X, which induces an involutive automorphism of the homological type.

For the converse, let $a \in \operatorname{Aut}\mathcal{H}$ be an automorphism as in the statement. Due to Lemma 3.2, the restriction $a|_{\mathbf{S}}$ has the form $r \circ (-s_*)$, where $r \in \operatorname{Ker} \mathrm{d}$ and s_* is induced by an *involutive* symmetry $s \in \operatorname{Sym}' \mathfrak{G}_{\mathbf{S}}$. Since $\operatorname{Ker} \mathrm{d} \subset \operatorname{Aut}\mathcal{H}$ (in the obvious way: automorphisms extend to $\mathbf{S}^\perp$ by the identity, see Theorem 3.5), the involution $r^{-1} \circ a$ is also in $\operatorname{Aut}\mathcal{H}$. Let $c := r^{-1} \circ a \circ t_h \in O(\mathbf{L})$; it is still an involution and $c|_{\mathbf{T}} = a|_{\mathbf{T}}$.

Let $T_\pm$ be the (± 1)-eigenspaces of the action of c on $\mathbf{T} \otimes \mathbb{R}$. Since c reverses the orientation, one has $\sigma_+ T_\pm = 1$. Hence, one can choose generic (*i.e.*, maximally irrational) vectors $\omega_\pm \in T_\pm$ such that $\omega_+^2 = \omega_-^2 > 0$ and take $\omega := \omega_+ + i\omega_-$ for the class of a holomorphic form. Let, further, S_- be the (-1)-eigenspace of the action of c on $\tilde{\mathbf{S}}_h \otimes \mathbb{R}$. Since $h \in S_-$, one has $\sigma_+ S_- = 1$. By the construction, $-c$ preserves a Weyl chamber of $\mathbf{S}$; hence, condition (1) in Definition 4.3 implies that S_- is *not* orthogonal to a vector $v \in \tilde{\mathbf{S}}_h$ of square (-2) and one can find a generic vector $\rho \in S_-$, $\rho^2 > 0$, and take it for the class of a Kähler form. These choices define a 2-polarized $K3$-surface X with $\operatorname{Pic} X = \tilde{\mathbf{S}}_h$ and, by an equivariant version of the global Torelli theorem, c is induced by a real structure on X commuting with the deck translation τ of the ramified covering $X \to \mathbb{P}^2$ defined by h. This real structure descends to $\mathbb{P}^2$ and makes the sextic corresponding to X (*i.e.*, the branch curve) real. □

Let D be a real sextic with the set of singularities $\mathbf{S}$. The real structure c lifts to two real structures on the covering $K3$-surface; they take exceptional divisors to exceptional divisors and, hence, induce two involutive symmetries $c_\pm \colon \mathfrak{G} \to \mathfrak{G}$ of the Dynkin graph $\mathfrak{G} := \mathfrak{G}_{\mathbf{S}}$. Define another symmetry $c_0 \colon \mathfrak{G} \to \mathfrak{G}$ as follows: on each connected component $\mathfrak{G}_i$ of $\mathfrak{G}$ *fixed* by $c_\pm$ and of type other than $\mathbf{D}_{\text{even}}$ let $c_0 = \mathrm{id}$; on all other components, let $c_0 = c_\pm$. In other words, since $c_- = c_+ \circ \tau$, we just let $v \uparrow c_0 = v$ for each vertex v such that $v \uparrow c_+ \neq v \uparrow c_-$, see Lemma 4.1.

Corollary 6.2. *If a homological type $\mathcal{H}$ is realized by a real sextic (D, c), then any c_0-invariant perturbation $\mathcal{H}'$ of $\mathcal{H}$ is also realized by a real sextic D'.*

Note that we do *not* assert that D' degenerates to D in the class of real sextics. A real perturbation can be found if $\mathcal{H}'$ is invariant under one of $c_\pm$.

Proof of Corollary 6.2. Let $c_* \colon \mathbf{L} \to \mathbf{L}$ be the automorphism of $\mathcal{H}$ induced by one of the two lifts of c. Composing c_* with $-\tau_*$ on *some* of the indecomposable summands of $\mathbf{S}$, we can change it to another involutive automorphism c' of $\mathcal{H}$ (see Lemma 3.2 and Theorem 3.5) inducing c_0 on $\mathfrak{G}$. Then c' preserves $\mathbf{S}'$; hence, $c' \circ t_h$ can be regarded as an involutive orientation reversing automorphism of $\mathcal{H}'$, and Theorem 6.1 applies. □

6.2. **End of the proof of Theorem 2.4.** It is easily confirmed that most sets of singularities $\mathbf{S}$ with $\mu(\mathbf{S}) \leqslant 18$ are *symmetric* perturbations of maximizing sets of singularities realized by real sextics, see Tables 1 and 2. (In the tables, marked with a * are pairs of isomorphic singular points permuted by the complex conjugation. These pairs should be taken into account when analyzing symmetric perturbations. Note that singular points of type $\mathbf{D}_{\text{even}}$ do not appear in irreducible maximizing sextics.) Due to Corollary 6.2, these sets of singularities are realized by real curves.

The remaining 25 sets of singularities are listed in Table 7. Each of these sets $\mathbf{S}$ extends to a unique (up to isomorphism) primitive homological type $\mathcal{H}$, and we

Table 7. Exceptional sets of singularities

$[3\mathbf{A}_6]_7$	$[\mathbf{E}_6 \oplus \mathbf{A}_5]_3 \oplus \mathbf{A}_6 \oplus \mathbf{A}_1$	$\star\, 2\mathbf{E}_7 \oplus \mathbf{A}_4$
$[2\mathbf{A}_6]_7 \oplus \mathbf{D}_6$	$[\mathbf{E}_6 \oplus 2\mathbf{A}_5]_3 \oplus \mathbf{A}_1$	$\star\, \mathbf{E}_7 \oplus \mathbf{D}_5 \oplus \mathbf{A}_6$
$[2\mathbf{A}_6]_7 \oplus \mathbf{D}_5 \oplus \mathbf{A}_1$	$[\mathbf{E}_7 \oplus \mathbf{A}_7]_2 \oplus \mathbf{A}_4$	$\star\, \mathbf{E}_7 \oplus \mathbf{A}_{11}$
$[2\mathbf{A}_6]_7 \oplus 2\mathbf{A}_3$	$[2\mathbf{A}_7]_2 \oplus \mathbf{A}_4$	$\star\, \mathbf{E}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5$
$[2\mathbf{A}_5]_3 \oplus \mathbf{E}_8$	$\mathbf{A}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5$	$\star\, 2\mathbf{D}_9$
$[\mathbf{E}_6 \oplus \mathbf{A}_{11}]_3 \oplus \mathbf{A}_1$	$2\mathbf{D}_7 \oplus 2\mathbf{A}_2$	$\star\, \mathbf{D}_9 \oplus \mathbf{D}_8$
$[\mathbf{E}_6 \oplus \mathbf{A}_5]_3 \oplus \mathbf{E}_7$	$\mathbf{D}_7 \oplus \mathbf{D}_4 \oplus 3\mathbf{A}_2$	$\star\, 2\mathbf{D}_8$
$[\mathbf{E}_6 \oplus \mathbf{A}_5]_3 \oplus \mathbf{A}_7$	$2\mathbf{D}_4 \oplus 4\mathbf{A}_2$	$\star\, \mathbf{D}_5 \oplus \mathbf{A}_7 \oplus \mathbf{A}_6$
		$\mathbf{E}_7 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_3$

denote by $\mathbf{T}$ the corresponding transcendental lattice. In each case, the natural homomorphism $\mathrm{d}\colon O(\mathbf{T}) \to \operatorname{Aut}\mathcal{T}$ is surjective.

By Theorem 3.5, the homological type $\mathcal{H}$ is symmetric if and only if there is an isometry $a \in O(\mathbf{T})$ with $\det_+ a = -1$ and such that $\mathrm{d}(a) \in \mathrm{d}^\varphi(O(\mathbf{S}))$, where d^φ is induced by any anti-isometry $\varphi\colon \mathcal{S} \oplus \langle \frac{1}{2} \rangle \to \mathcal{T}$. If (and only if) a as above can be chosen *involutive*, then so is $\mathrm{d}(a)$ and, due to Lemma 3.2, a extends to $\mathbf{L}$ by an *involutive* isometry of $\mathbf{S}$; hence, $\mathcal{M}_1(\mathbf{S})$ contains real curves, see Theorem 6.1.

Lemma 6.3. *The first twelve sets of singularities in Table 7 (those with a $[\,\cdot\,]_p$ pattern) extend to* asymmetric *primitive homological types.*

Proof. Let $\mathbf{S}$ be one of the sets of singularities in question. Then $\tilde{\Sigma}(\mathbf{T}) \subset \Gamma_0^{--}$, see §3.7, and there is a well defined map $\det_+\colon \operatorname{Aut}\mathcal{T} \to \{\pm 1\}$. We use Lemma 3.17 (with the 'test prime' p indicated in the table) to show that $\det_+$ takes value $+1$ on the image of $O(\mathbf{S})$. If $p = 7$ (the first four lines), the latter image is generated by reflections t_ξ such that either

- $\xi^2 = \frac{6}{7}$ (a symmetry of the Dynkin graph of $\mathbf{A}_6$), or
- $\xi^2 = \frac{12}{7}$ (interchanging of two copies of $\mathbf{A}_6$), or
- $\xi \in \mathcal{T}_2$ (isometries involving the other singular points);

on the other hand, one has $(\frac{-3}{7}) = (\frac{-6}{7}) = (\frac{2}{7}) = 1$. If $p = 3$ (the next six sets of singularities), the image of $O(\mathbf{S})$ is generated by the following automorphisms a:

- t_ξ with $\xi^2 = \frac{4}{3}$ (a symmetry of the Dynkin graph of $\mathbf{E}_6$ or $\mathbf{A}_5$),
- $t_\xi t_\eta$ with $\xi^2 = \frac{2}{3}$, $\eta^2 = 1$ (interchanging of two copies of $\mathbf{A}_5$),
- $t_\xi t_\eta$ with $\xi^2 = \frac{2}{3}$, $\eta^2 = \frac{1}{4}$ (a symmetry of the Dynkin graph of $\mathbf{A}_{11}$),
- t_ξ with $\xi^2 = \frac{7}{8}$ or $\xi^2 = \frac{6}{7}$ (a symmetry of the Dynkin graph of $\mathbf{A}_7$ or $\mathbf{A}_6$).

In each case, Lemma 3.17 (with $p = 3$) implies that $\det_+ a = 1$. Finally, if $p = 2$ (the last two sets of singularities), we have reflections t_ξ such that either

- $\xi^2 = \frac{7}{8}$ (a symmetry of the Dynkin graph of $\mathbf{A}_7$), or
- $\xi^2 = \frac{7}{4}$ (interchanging of two copies of $\mathbf{A}_7$), or
- $\xi^2 = \frac{4}{5}$ (a symmetry of the Dynkin graph of $\mathbf{A}_4$).

Lemma 3.17 (with $p = 2$) implies that $\det_+ t_\xi = 1$. ☐

Listed in the last column in Table 7 are the sets of singularities $\mathbf{S}$ extending to symmetric homological types due to Proposition 3.16. However, since we want to represent these types by real sextics, we will attempt to find *involutive* orientation reversing automorphisms, see Theorem 6.1. A simplest automorphism with this property would be a reflection t_a, $a \in \mathbf{T}$, $a^2 = 2$.

Lemma 6.4. *If* $\mathbf{S}$ *is one of the sets of singularities marked with a* $\star$ *in Table 7, the lattice* $\mathbf{T}$ *contains a vector* a *with* $a^2 = 2$.

Proof. It suffices to find an embedding $\mathbf{S}_h \oplus \mathbb{Z}a \hookrightarrow \mathbf{L}$, $a^2 = 2$, with the image of $\mathbf{S}_h$ primitive. In each case, there is an element $\alpha \in \operatorname{discr} \mathbf{S}_h$ with $\alpha^2 = -\frac{1}{2} \bmod 2\mathbb{Z}$. Let $\beta \in \operatorname{discr}(\mathbb{Z}a) = \langle \frac{1}{2} \rangle$ be the generator, and let $\mathbf{S}'_h$ be the finite index extension of $\mathbf{S}_h$ with the kernel generated by $\alpha + \beta$. On a case-by-case basis one confirms that Theorem 3.6 implies the existence of a primitive embedding $\mathbf{S}'_h \hookrightarrow \mathbf{L}$. (In the last case, the set of singularities $\mathbf{D}_5 \oplus \mathbf{A}_7 \oplus \mathbf{A}_6$, the element α above should be chosen carefully, *viz.* $\alpha = 2\alpha_1 + 4\alpha_2 + \alpha_4$ in $\operatorname{discr} \mathbf{S}_h = \langle \frac{3}{4} \rangle \oplus \langle \frac{9}{8} \rangle \oplus \langle \frac{8}{7} \rangle \oplus \langle \frac{1}{2} \rangle$.) □

The set of singularities $\mathbf{A}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5$ is considered in Proposition 2.6, see §6.3 below, and the remaining four deformation families are real and contain real curves; for proof, we construct explicit reflections in $O(\mathbf{T})$.

If $\mathbf{S} = 2\mathbf{D}_7 \oplus 2\mathbf{A}_2$, then $\mathbf{T} = \mathbb{Z}u \oplus \mathbb{Z}v \oplus \mathbb{Z}w$ with $u^2 = 4$, $v^2 = -12$, $w^2 = 6$, and the reflection t_u extends to an involutive automorphism of $\mathcal{H}$ (*via* $-\mathrm{id}$ on one of the $\mathbf{D}_7$ components). Hence, $\mathcal{M}_1(\mathbf{S})$ contains a real curve; by Corollary 6.2, so do $\mathcal{M}_1(\mathbf{D}_7 \oplus \mathbf{D}_4 \oplus 3\mathbf{A}_2)$ and $\mathcal{M}_1(2\mathbf{D}_4 \oplus 4\mathbf{A}_2)$.

Finally, if $\mathbf{S} = \mathbf{E}_7 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_3$, then $\mathbf{T} = \mathbb{Z}u \oplus \mathbb{Z}v \oplus \mathbb{Z}w$ with $u^2 = v^2 = 10$, $w^2 = -4$. Since $\mathrm{d}\colon O(2\mathbf{A}_4) \to \operatorname{discr} 2\mathbf{A}_4$ is obviously onto, the reflection t_u extends to an involutive automorphism of $\mathcal{H}$. □

6.3. **Proof of Proposition 2.6.** One has $\mathcal{T} = \langle \frac{7}{8} \rangle \oplus \langle \frac{6}{7} \rangle \oplus \langle \frac{4}{3} \rangle \oplus \langle \frac{3}{2} \rangle \oplus \langle \frac{3}{2} \rangle$, and the image of $O(\mathbf{S})$ in $\operatorname{Aut} \mathcal{T}$ is generated by the reflections t_{α_i}, $i = 1, 2, 3$. Furthermore, one has $\tilde{\Sigma}_2(\mathbf{T}) = \Gamma_0^{--}$ and the map $\det_+ \colon \operatorname{Aut} \mathcal{T} \to \{\pm 1\}$ is well defined. Applying Lemma 3.17 with $p = 2$, one finds that $\det_+ t_{\alpha_1} = 1$ and $\det_+ t_{\alpha_2} = \det_+ t_{\alpha_3} = -1$. In particular, it follows that the homological type is symmetric, *i.e.*, $\mathcal{M}_1(\mathbf{S})$ consists of a single real component.

Up to sign, any involutive isometry $a \in O(\mathbf{T})$ with $\det_+ a = -1$ is a reflection, $a = \pm t_x$ for some $x \in \mathbf{T}$, $x^2 > 0$: one can take for x a primitive vector generating the (-1)-eigenlattice of $\pm a$, whichever has rank one. As explained above, t_x must induce $-\mathrm{id}$ in one *and only one* of the components $\mathcal{T}_3$, $\mathcal{T}_7$. Hence, $x^2 = 2^k q$, where $k = 1, 3$ and $q = 3, 7$. (Recall that $x \in (\frac{1}{2}x^2)\mathbf{T}^\sharp$; if $k = 2$, then $\xi := \frac{1}{2}x \in \mathcal{T}_2$ has square $0 \bmod \mathbb{Z}$ and t_ξ is not in the image of $O(\mathbf{S})$.) Obviously, $\eta := \frac{1}{q}x$ is a generator of $\mathcal{T}_q$; on the other hand, one can see that $\eta^2/\alpha^2 \notin (\mathbb{Z}_q^\times)^2$, where $\alpha = \alpha_2$ or α_3 for $q = 7$ or 3, respectively. This is a contradiction. □

6.4. **End of the proof of Theorem 2.9.** As in §6.2, one can easily see that each set of singularities $\mathbf{S}$ can be obtained by a symmetric perturbation from a maximizing real one, see Table 3. Furthermore, the perturbation can be chosen of *torus type*, *i.e.*, each inner singular point of weight w is perturbed to a collection of points of total weight w. Such perturbations are known to preserve the torus structure. Hence, by Corollary 6.2, the space $\mathcal{M}_3(\mathbf{S})$ contains a real curve. □

6.5. **Adjacencies of the strata.** Recall that, with the exception of the set of singularities $\mathbf{S} = 2\mathbf{A}_9$, the spaces $\mathcal{M}_1(\mathbf{S})/\operatorname{conj}$ are connected for all non-maximizing sextics (see Theorem 2.4). Together with [10, Proposition 5.1.1] and [20], this fact gives us a clear picture of the adjacencies of the *real* strata; the only doubtful case of the two components of $\mathcal{M}(2\mathbf{A}_9)$ is treated in Proposition 2.5.

Consider the adjacency graph $\mathfrak{C}$ of the strata $\mathcal{M}_1(\mathbf{S}) \subset \mathcal{M}_1$ containing non-real components, and let $\tilde{\mathfrak{C}}$ be the adjacency graph of these non-real components. One

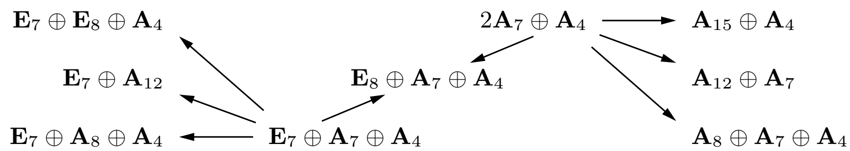

FIGURE 1. The graph $\mathfrak{C}_2$

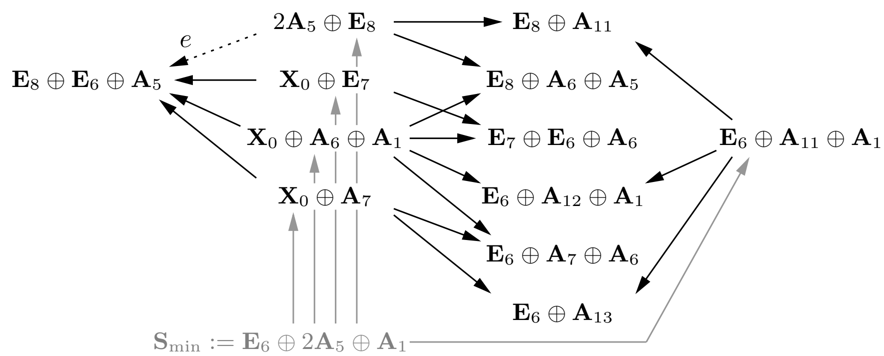

FIGURE 2. The graph $\mathfrak{C}_3$ (where $\mathbf{X}_0 := \mathbf{E}_6 \oplus \mathbf{A}_5$)

can interpret the vertices and edges of $\mathfrak{C}$ as, respectively, asymmetric primitive homological types and isomorphism classes of their degenerations, whereas those of $\tilde{\mathfrak{C}}$ are oriented homological types and their orientation preserving degenerations. With two exceptions, *viz.* $\mathbf{A}_{14} \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$ and $\mathbf{A}_{13} \oplus \mathbf{A}_6$, see Table 2, a vertex of $\mathfrak{C}$ is determined by the corresponding set of singularities. Most degenerations are of corank one, in which case a degeneration $\mathbf{S}' \rightarrowtail \mathbf{S}$ is uniquely determined by the pair $(\mathbf{S}', \mathbf{S})$, see, *e.g.*, [17]. The forgetful projection $\tilde{\mathfrak{C}} \to \mathfrak{C}$ is a double covering, and we are interested in the structure of this map, in particular, in the connected components of $\tilde{\mathfrak{C}}$.

The graph $\mathfrak{C}$ has several isolated vertices, *viz.* $\mathbf{D}_7 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_2$, $\mathbf{D}_5 \oplus \mathbf{A}_{14}$, three vertices representing $\mathbf{A}_{14} \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$, and all maximizing sets of singularities that are also represented by real curves. The rest splits into three larger components, which we denote by $\mathfrak{C}_p$, $p = 2, 3, 7$, and call *clusters*. For a fixed p, the vertices of $\mathfrak{C}_p$ are all sets of singularities in Table 7 containing a $[\,\cdot\,]_p$ pattern and all their *asymmetric* degenerations, see Figures 1–3. Denote by $\tilde{\mathfrak{C}}_p \subset \tilde{\mathfrak{C}}$ the pull-back of $\mathfrak{C}_p$, $p = 2, 3, 7$. Each double covering $\tilde{\mathfrak{C}}_p \to \mathfrak{C}_p$ is described by its characteristic class, which we denote by $\omega_p \in H^1(\mathfrak{C}_p; \mathbb{F}_2)$.

Let $\mathcal{C}_p := \bigcup \mathcal{M}_1(\mathbf{S})$, the union running over all $\mathbf{S} \in \mathfrak{C}_p$, $p = 2, 3, 7$. These subspaces of $\mathcal{M}$ are also called *clusters*; their connected components are in a one-to-one correspondence with those of $\tilde{\mathfrak{C}}_p$.

The graph $\mathfrak{C}_2$ is shown in Figure 1. Since it is simply connected, we have the following immediate statement.

Proposition 6.5. *The double covering $\tilde{\mathfrak{C}}_2 \to \mathfrak{C}_2$ is trivial. Hence, the cluster $\mathcal{C}_2$ consists of two complex conjugate components.* ◁

The graph $\mathfrak{C}_3$ is depicted in Figure 2, where *only corank one degenerations are shown.* This graph has a minimal vertex $\mathbf{S}_{\min} := \mathbf{E}_6 \oplus 2\mathbf{A}_5 \oplus \mathbf{A}_1$, shown in grey.

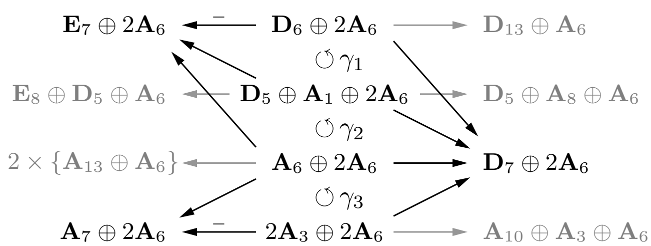

FIGURE 3. The graph $\mathfrak{C}_7$

The closure of $\mathcal{C}_3$ contains four real strata

$$2\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus \mathbf{A}_1 \rightarrowtail \mathbf{E}_7 \oplus 2\mathbf{E}_6,\ 2\mathbf{E}_6 \oplus \mathbf{A}_7,\ 2\mathbf{E}_6 \oplus \mathbf{A}_6 \oplus \mathbf{A}_1.$$

In all four, the real structure interchanges the two $\mathbf{E}_6$ points; for the non-maximizing set of singularities $2\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus \mathbf{A}_1$, this fact can be proved similar to Lemma 6.3.

Regarded as a diagram, $\mathfrak{C}_3$ is not quite commutative. There are *two* isomorphism classes of degenerations $\mathbf{S}_{\min} \rightarrowtail \mathbf{E}_8 \oplus \mathbf{E}_6 \oplus \mathbf{A}_5$; in the self-explanatory notation, they are

$$[\mathbf{E}_6 \oplus \mathbf{A}_1] \oplus [\mathbf{A}_5] \oplus [\mathbf{A}_5],\ [\mathbf{A}_5 \oplus \mathbf{A}_1] \oplus [\mathbf{E}_6] \oplus [\mathbf{A}_5] \rightarrowtail \mathbf{E}_8 \oplus \mathbf{E}_6 \oplus \mathbf{A}_5. \tag{6.6}$$

The former factors through the edge $e\colon 2\mathbf{A}_5 \oplus \mathbf{E}_8 \rightarrowtail \mathbf{E}_8 \oplus \mathbf{E}_6 \oplus \mathbf{A}_5$ represented by a dotted arrow in Figure 2, and the latter factors through the three other edges ending at $\mathbf{E}_8 \oplus \mathbf{E}_6 \oplus \mathbf{A}_5$. Denote by $e^\sharp \in H^1(\mathfrak{C}_3; \mathbb{F}_2)$ the class sending a cycle α, regarded as a sequence of undirected edges, to the multiplicity of e in α. Formally, $e^\sharp$ is the image of the generator of the group $H^1(e, \partial e; \mathbb{F}_2) = \mathbb{F}_2$ under the relativization homomorphism $H^1(e, \partial e; \mathbb{F}_2) = H^1(\mathfrak{C}_3, \mathfrak{C}_3 \smallsetminus e; \mathbb{F}_2) \to H^1(\mathfrak{C}_3; \mathbb{F}_2)$.

Proposition 6.7. *The characteristic class ω_3 of the double covering $\tilde{\mathfrak{C}}_3 \to \mathfrak{C}_3$ is $\omega_3 = e^\sharp \neq 0$. In particular, the cluster $\mathcal{C}_3$ is connected.*

Proof. Let $\mathfrak{C}_3'$ be the graph obtained from $\mathfrak{C}_3$ by removing the (open) edge e, and let $\tilde{\mathfrak{C}}_3' \subset \tilde{\mathfrak{C}}_3$ be the pull-back of $\mathfrak{C}_3'$. As explained above, $\mathfrak{C}_3'$ is a commutative diagram. Hence, the restricted covering $\tilde{\mathfrak{C}}_3' \to \mathfrak{C}_3'$ is trivial: an orientation of the homological type extending $\mathbf{S}_{\min}$ induces an orientation of all other homological types. On the other hand, both degenerations (6.6) factor through $2\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus \mathbf{A}_1$ and differ by a transposition of the two $\mathbf{E}_6$ type points, which extends to an orientation reversing automorphism of the homological type. Hence, the double covering $\tilde{\mathfrak{C}}_3 \to \mathfrak{C}_3$ is not trivial and the obstruction is $e^\sharp$. □

The graph $\mathfrak{C}_7$ is depicted in Figure 3, where shown in black are the vertices and edges constituting undirected cycles. (There are two vertices corresponding to the set of singularities $\mathbf{A}_{13} \oplus \mathbf{A}_6$, see Table 2, each connected by an edge to $3\mathbf{A}_6$.) The group $H_1(\mathfrak{C}_7; \mathbb{F}_2) \cong \mathbb{F}_2^3$ is generated by the three four-edge cycles γ_1, γ_2, γ_3, and the characteristic class ω_7 is determined by its values on these cycles.

Proposition 6.8. *The characteristic class ω_7 of the double covering $\tilde{\mathfrak{C}}_7 \to \mathfrak{C}_7$ is $\gamma_1, \gamma_3 \mapsto 1$, $\gamma_2 \mapsto 0$. In particular, the cluster $\mathcal{C}_7$ is connected.*

Proof. Consider a quadratic $\mathbb{F}_7$-module $\mathcal{X}$. Recall that the group $\operatorname{Aut}\mathcal{X}$ is generated by reflections and there are well defined homomorphisms $\det, \operatorname{spin}\colon \operatorname{Aut}\mathcal{X} \to \{\pm 1\}$ sending a reflection t_ξ to (-1) and the class $14\xi^2 \bmod (\mathbb{Z}_7^\times)^2 \in \mathbb{Z}_7^\times/(\mathbb{Z}_7^\times)^2 = \{\pm 1\}$, respectively, see, *e.g.*, [5]. Assuming that $|\mathcal{X}|\cdot\det_7\mathcal{X}=1 \bmod (\mathbb{Z}_7^\times)^2$, define a *spin-orientation* of $\mathcal{X}$ as a class of orthogonal bases $\alpha:=\{\alpha_1,\ldots,\alpha_\ell\}$, $\alpha_i=\frac{2}{7} \bmod 2\mathbb{Z}$, two bases α', α'' being equivalent if the isometry $\sigma\colon \alpha_i'\mapsto\alpha_i''$, $i=1,\ldots,\ell$, has $\operatorname{spin}\sigma=1$. Note that the order or the signs of the basis vectors are not important: isometries reversing the spin-orientation are more subtle. In particular, the group $\operatorname{discr}_7\mathbf{S}$ for any $\mathbf{S}\in\mathfrak{C}_7$ has a *canonical* spin-orientation.

Let $\mathrm{ds}:=\det\cdot\operatorname{spin}$. In a similar way, using bases with $\alpha_i^2=-\frac{2}{7} \bmod 2\mathbb{Z}$, we can define the notion of ds-*orientation* for a $\mathbb{F}_7$-module $\mathcal{Y}$ satisfying $|\mathcal{Y}|\cdot\det_7\mathcal{Y}=(-1)^\ell \bmod (\mathbb{Z}_7^\times)^2$, where $\ell:=\ell(\mathcal{Y})$. An anti-isometry $\mathcal{X}\to\mathcal{Y}$ takes spin-orientations to ds-orientations. There is a *unique* ds-orientation on $\langle-\frac{2}{7}\rangle$; hence, a ds-orientation on $\mathcal{Y}$ induces a ds-orientation on any *codimension one* submodule $\mathcal{Z}\subset\mathcal{Y}$ satisfying $|\mathcal{Z}|\cdot\det_7\mathcal{Z}=(-1)^{\ell-1} \bmod (\mathbb{Z}_7^\times)^2$. A similar statement holds for spin-orientations.

The essence of the proof of Lemma 6.3 is the fact that, for any vertex $\mathbf{S}\in\mathfrak{C}_7$, one has $\operatorname{Im}[\mathrm{d}_7\colon O_+(\mathbf{T})\to\operatorname{Aut}\mathcal{T}_7]\subset\operatorname{Ker}\mathrm{ds}$. (If $\mu(\mathbf{S})=19$, this follows from [29].) Hence, there is a bijection $\operatorname{conv}\colon\mathfrak{o}\mapsto\mathfrak{s}$ between positive sign structures on $\mathbf{T}$ and ds-orientations on $\mathcal{T}_7$. (The particular choice of conv is not important; it can be fixed separately for each isomorphism class.) Thus, an oriented homological type $(\mathcal{H},\mathfrak{o})$ can be declared *positive* or *negative* according to whether the anti-isometry $\mathcal{S}\to\mathcal{T}$ does or does not take the canonical spin-orientation of $\mathcal{S}_7$ to $\operatorname{conv}(\mathfrak{o})$.

Given a lattice extension $\iota\colon\mathbf{S}\hookrightarrow\mathbf{S}'$, the homomorphisms $\iota\otimes\mathbb{Q}$ and $\iota^\sharp$ induce additive relations $\iota_*\colon\mathcal{S}_7 \dashrightarrow \mathcal{S}_7'$ and $\iota^\sharp\colon\mathcal{S}_7'\dashrightarrow\mathcal{S}_7$. If ι is one of the *black* arrows in Figure 3, both ι_* and $\iota^\sharp$ are true homomorphisms; they give rise, in a canonical way, to either an isomorphism $\mathcal{S}_7=\mathcal{S}_7'$ or a splitting $\mathcal{S}_7=\mathcal{S}_7'\oplus\langle\frac{2}{7}\rangle$ (if $\mathbf{S}=3\mathbf{A}_6$), which respect the canonical spin-orientation. Passing to the transcendental lattices, we conclude that, in either case, a ds-orientation on $\mathcal{T}_7$ induces one on $\mathcal{T}_7'$. On the other hand, $\mathbf{T}'\subset\mathbf{T}$ is a maximal positive definite sublattice and $\mathbf{T}$ and $\mathbf{T}'$ have a *common* positive sign structure $\mathfrak{o}=\mathfrak{o}'$. Hence, we can assign to ι a sign $\epsilon=\pm1$ so that the ds-orientation on $\mathcal{T}_7'$ induced by $\operatorname{conv}(\mathfrak{o})$ equals $\epsilon\operatorname{conv}(\mathfrak{o}')$. This sign depends on the conventions conv, but the product $\epsilon:=\epsilon_1\epsilon_2\epsilon_3\epsilon_4$ over a four-edge cycle $c:=(\iota_1,\iota_2,\iota_3,\iota_4)$ does not, as each convention is used twice. It is immediate from the definitions that $\epsilon=(-1)^{\omega_7(c)}$. Now, the statement of the proposition is proved by a routine computation of the signs, *cf.* Example 6.9 below. ☐

Example 6.9. We illustrate the computation of the signs in the previous proof. All rank two lattices involved are of the form $\mathbb{Z}u\oplus\mathbb{Z}v$, $u^2=2^{r+1}\cdot7$, $v^2=2^{s+1}\cdot7$, $r,s\geqslant0$, and *for such lattices*, we define conv to take the positive basis $\{u,v\}$ to a basis $\{\alpha_1,\alpha_2\}$ with $\alpha_1:=\frac{1}{7}(2^{r+2}u+2^{s+1}v)$. (For a module of length two, one vector of square $-\frac{2}{7} \bmod 2\mathbb{Z}$ is enough to define a ds-orientation. For the comparison purposes, it is convenient to consider the basis $\beta_1:=\frac{1}{7}\cdot2^ru$, $\beta_2:=\frac{1}{7}\cdot2^sv$ with $\beta_1^2=\beta_2^2=\frac{2}{7} \bmod 2\mathbb{Z}$, so that $\alpha_1=4\beta_1+2\beta_2$. In terms of the β-basis, the transposition of the two vectors or changing the sign of one of them reverses the orientation.) To avoid choices for rank three lattices, we consider a pair of arrows $\mathbf{S}'\hookleftarrow\mathbf{S}\rightarrowtail\mathbf{S}''$. Let $\mathbf{S}=\mathbf{D}_6\oplus2\mathbf{A}_6$ (the upper pair in Figure 3). Then $\mathbf{T}=\mathbb{Z}u\oplus\mathbb{Z}v\oplus\mathbb{Z}w$, $u^2=v^2=14$, $w^2=-2$, and $\mathbf{T}',\mathbf{T}''\subset\mathbf{T}$ are spanned, respectively, by $u'=u$, $v'=v$ and $u''=3u+7w$, $v''=v$. (Since we know that the sign is well defined, it suffices to consider a particular pair of sublattices.) The orientations of the two bases are coherent, and the coefficient $3\notin(\mathbb{Z}_7^\times)^2$ in the expression for u'' tells us that the product of the signs associated with this pair of arrows is (-1): one has $\beta_1''=-\beta_1'$ and $\beta_2''=\beta_2'$. A similar computation, slightly more involved if $\mathbf{S} = 3\mathbf{A}_6$, shows that the sign convention for rank three lattices can be chosen so that only two arrows have associated sign (-1), see Figure 3.

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