QUADRATIC ALGEBRA OF SQUARE GROUPS

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ABSTRACT. Square groups are quadratic analogues of abelian groups. Many properties of abelian groups are shown to hold for square groups. In particular, there is a symmetric monoidal tensor product of square groups generalizing the classical tensor product.

There is a long-standing problem of algebra to extend the symmetric monoidal structure of abelian groups, given by the tensor product, to a non abelian setting, see for example [10]. In this paper we show the somewhat surprising fact that such an extension is possible. Morover our non abelian tensor product remains even right exact and balanced. We describe the new non-abelian tensor product in the context of quadratic algebra which extends linear algebra.

"Linear algebra" is the algebra of rings and modules. A ring is a monoid in the symmetric monoidal category of abelian groups

$$(\mathsf{Ab}, \otimes, \mathbb{Z}).$$

The monoidal structure is given by the tensor product of abelian groups, with the group of integers $\mathbb{Z}$ as the unit object. Moreover a module is an object in Ab together with an action of such a monoid.

In "quadratic algebra" abelian groups are replaced by square groups. In fact, if one considers endofunctors of the category of groups which preserve filtered colimits and reflexive coequalizers, then abelian groups can be identified with linear endofunctors and square groups can be identified with quadratic endofunctors, [7]. The abelian group $\mathbb{Z}$ corresponds to the linear endofunctor which carries a group G to its abelization $G^{\mathrm{ab}} = G/[G,G]$. The square group $\mathbb{Z}_{\mathrm{nil}}$ corresponds to the quadratic endofunctor which carries G to the class two nilpotent group $G^{\mathrm{nil}} = G/[G,[G,G]]$. The category SG of square groups contains the category Ab of abelian groups as a full subcategory since a linear endofunctor is also quadratic. Therefore the question arises whether the symmetric monoidal structure of Ab extends to a symmetric monoidal structure of SG. The main purpose of this paper is the proof that this is, in fact, the case.

Let G and H be (additively written) groups. One can consider the group with generators $g * h$ for $g \in G$ and $h \in H$, subject to the following relations:

$$g * (h_1 + h_2) = g * h_1 + g * h_2,$$
$$(g_1 + g_2) * h = g_1 * h + g_2 * h.$$

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It is well known and easy to prove that this group is isomorphic to $G^{\mathrm{ab}} \otimes H^{\mathrm{ab}}$. Thus the naive definition of the tensor product of nonabelian groups does not really provide a new object. A more sophisticated tensor product was constructed by Brown and Loday [10]. However, their tensor product does not define a symmetric monoidal structure.

The definition of the tensor product of square groups relies on the structure (H, P) of a square group M. In $M\underline{\circledcirc}N$ one has among others the relations

$$\begin{aligned}
&x\underline{\circledcirc}(y_1 + y_2) = x\underline{\circledcirc}y_1 + x\underline{\circledcirc}y_2,\\
&(x_1 + x_2)\underline{\circledcirc}y = x_1\underline{\circledcirc}y + x_2\underline{\circledcirc}y + (x_2 \mid x_1)_H \bar{\otimes} H(y)
\end{aligned}$$

which replace the naive relations above. It is a somewhat surprising fact that these relations for $M\underline{\circledcirc}N$ lead to a symmetric monoidal structure of the category of square groups extending the tensor product of abelian groups. We show:

Theorem. *There is a tensor product of square groups M, N denoted by $M\underline{\circledcirc}N$ such that*

$$(\mathsf{SG}, \underline{\circledcirc}, \mathbb{Z}_{\mathrm{nil}})$$

is a symmetric monoidal category. Moreover, if M and N are abelian groups then

$$M\underline{\circledcirc}N = M \otimes N.$$

A monoid R in $(\mathsf{SG}, \underline{\circledcirc}, \mathbb{Z}_{\mathrm{nil}})$ is termed a quadratic ring. An R-quadratic module is a square group with an action of the monoid R. This leads to the wide area of quadratic algebra generalizing classical linear algebra. For example we describe in this paper the Tor-exact sequence for square groups. Also we study various special classes of square groups, like abelian square groups, quadratic $\mathbb{Z}$-modules and free square groups.

The category SG has another (very non-symmetric) monoidal category structure $\square$ induced by composition of endofunctors (see [7] or Section 15.3). In Theorem 51 we describe, by means of abelian groups with cosymmetry, a subcategory of SG on which the products $\square$ and $\underline{\circledcirc}$ coincide.

As we will see in sequel publications quadratic rings and related quadratic pair algebras play an important rôle in secondary homotopy theory as well as in the theory of Mac Lane cohomology of rings (see [14], [21], [9] and Chapter 13 of [18]). Namely, quadratic pair algebras are natural objects representing classes in the third dimensional Mac Lane cohomology. Moreover the secondary homotopy groups of each ring spectrum form a quadratic pair algebra [6]. In particular, the sphere spectrum yields a quadratic pair algebra encoding all its secondary homotopy structure like triple Toda brackets. Also in order to study these examples it is necessary to develop the quadratic algebra of square groups.

Concerning (symmetric) monoidal categories and (symmetric) monoidal functors we use the terminology following [16]. In particular, a lax monoidal functor is a functor F together with coherent morphisms $\phi_{A,B} : F(A) \otimes F(B) \to F(A \otimes B)$. Moreover F is a monoidal functor if $\phi_{A,B}$ are isomorphisms for all A and B.

1. The monoidal category of square groups

In this section we recall the notion of square group (see also Section 15.3 below) and we give an explicit construction of the tensor product of square groups. We formulate our main results concerning symmetric monoidal structure and right exactness of this tensor product.

Let G be an additively written group and A be an abelian group. We call a map $f : G \to A$ *quadratic* if for any $x, y \in G$ the *cross-effect*

$$(x \mid y)_f := f(x+y) - f(x) - f(y)$$

is linear in x and y, that is $(- \mid y)_f$ and $(x \mid -)_f$ are homomorphisms $G \to A$.

Definition 1. A *square group* is a diagram

$$M = (\ M_{\mathrm{e}} \xrightarrow{H} M_{\mathrm{ee}} \xrightarrow{P} M_{\mathrm{e}}\ )$$

where the ee-level M_{ee} is an abelian group and the e-level M_{e} is a group. Both groups are written additively. Moreover P is a homomorphism and H is a quadratic map. In addition the following identities

$$\begin{aligned}(Pa \mid y)_H = 0 = (x \mid Pb)_H\,,\\ P\,(x \mid y)_H = -x - y + x + y,\\ PHP(a) = 2P(a)\end{aligned}$$

are satisfied for all $x, y \in M_{\mathrm{e}}$ and $a, b \in M_{\mathrm{ee}}$.

As an example we have the square group

$$\mathbb{Z}_{\mathrm{nil}} = (\ \mathbb{Z} \xrightarrow{H} \mathbb{Z} \xrightarrow{P} \mathbb{Z}\ )$$

with $H(n) = \binom{n}{2} = \frac{n(n-1)}{2}$ and $P = 0$. Let SG be the category of square groups.

In any square group M the image of P is a normal subgroup containing the commutator subgroup (see [7], or Section 5.1), thus $\mathsf{Coker}(P)$ is a well-defined abelian group. Let $\bar{x} \in \mathsf{Coker}(P)$ be the element represented by $x \in M_{\mathrm{e}}$. The cross effect of H induces a homomorphism (see [7] or Corollary 15 below)

$$(- \mid -)_H : \mathsf{Coker}(P) \otimes \mathsf{Coker}(P) \to M_{\mathrm{ee}}.$$

Moreover, there is a well-defined homomorphism

$$\Delta : \mathsf{Coker}(P) \to M_{\mathrm{ee}}$$

given by $\Delta(\bar{x}) = HPH(x) + H(x+x) - 4H(x)$ (see [7], or Corollary 15). Furthermore, the map

$$T = HP - \mathsf{Id} : M_{\mathrm{ee}} \to M_{\mathrm{ee}}$$

is an endomorphism of the abelian group M_{ee} with $T^2 = \mathsf{Id}$ (see [7], or Proposition 14). Sometimes we write P^M, H^M, Δ^M, T^M in order to make clear the rôle of M.

The aim of this work is to introduce a symmetric monoidal category structure $\underline{\circledcirc}$ on the category SG of square groups.

Definition 2. For two square groups M, N we introduce the tensor product $M\underline{\circledcirc}N$ which is a square group defined as follows. The group $(M\underline{\circledcirc}N)_{\mathrm{e}}$ is given by generators of the form $x\underline{\circledcirc}y$ for $x \in M_{\mathrm{e}}$, $y \in N_{\mathrm{e}}$, and $a\bar{\otimes}b$ for $a \in M_{\mathrm{ee}}$, $b \in N_{\mathrm{ee}}$, subject to the relations

(1) the symbol $a\bar{\otimes}b$ is bilinear and central in $(M\underline{\circledcirc}N)_{\mathrm{e}}$;
(2) $x\underline{\circledcirc}(y_1 + y_2) = x\underline{\circledcirc}y_1 + x\underline{\circledcirc}y_2$;
(3) $(x_1 + x_2)\underline{\circledcirc}y = x_1\underline{\circledcirc}y + x_2\underline{\circledcirc}y + (x_2 \mid x_1)_H \bar{\otimes}H(y)$;
(4) $x\underline{\circledcirc}P(b) = (x \mid x)_H \bar{\otimes}b$;
(5) $P(a)\underline{\circledcirc}y = a\bar{\otimes}\Delta(y)$;
(6) $T(a)\bar{\otimes}T(b) = -a\bar{\otimes}b$.

Next the abelian group $(M\underline{\circledcirc}N)_{\mathrm{ee}}$ is defined to be $M_{\mathrm{ee}} \otimes N_{\mathrm{ee}}$. The homomorphism

$$P : (M\underline{\circledcirc}N)_{\mathrm{ee}} \to (M\underline{\circledcirc}N)_{\mathrm{e}}$$

is given by

$$P(a \otimes b) = a\bar{\otimes}b$$

In order to define H, we first observe that

$$\mathsf{Coker}(P^{M\underline{\circledcirc}N}) = \mathsf{Coker}(P^M) \otimes \mathsf{Coker}(P^N).$$

Therefore we have the following homomorphism

$$\begin{aligned}\rho : \mathsf{Coker}(P^{M\underline{\circledcirc}N}) &\otimes \mathsf{Coker}(P^{M\underline{\circledcirc}N}) \\ &= \mathsf{Coker}(P^M) \otimes \mathsf{Coker}(P^N) \otimes \mathsf{Coker}(P^M) \otimes \mathsf{Coker}(P^N) \to M_{\mathrm{ee}} \otimes N_{\mathrm{ee}}\end{aligned}$$

with $\rho(\bar{a} \otimes \bar{b} \otimes \bar{a}' \otimes \bar{b}') = (a \mid a')_H \otimes (b \mid b')_H$. Now

$$H : (M\underline{\circledcirc}N)_{\mathrm{e}} \to (M\underline{\circledcirc}N)_{\mathrm{ee}}$$

is the unique quadratic map with the map ρ as its cross-effect satisfying

$$H(x\underline{\circledcirc}y) = (x \mid x)_H \otimes H(y) + H(x) \otimes \Delta(y)$$

and

$$H(a\bar{\otimes}b) = a \otimes b - T(a) \otimes T(b).$$

The following is the main result of the paper:

Theorem 3. *The tensor product of square groups gives rise to a well-defined bifunctor*

$$-\underline{\circledcirc}- : \mathsf{SG} \times \mathsf{SG} \to \mathsf{SG}$$

which equips the category SG *with a symmetric monoidal structure, with the unit object* $\mathbb{Z}_{\mathrm{nil}}$. *The associativity and commutativity isomorphisms on the* ee*-level are the usual isomorphisms for the tensor product of abelian groups, while on the* e*-level the isomorphism* $((M\underline{\circledcirc}N)\underline{\circledcirc}K)_{\mathrm{e}} \cong (M\underline{\circledcirc}(N\underline{\circledcirc}K))_{\mathrm{e}}$ *is given by*

$$\begin{aligned}(x\underline{\circledcirc}y)\underline{\circledcirc}z &\mapsto x\underline{\circledcirc}(y\underline{\circledcirc}z)\\ (a\bar{\otimes}b)\underline{\circledcirc}z &\mapsto a\bar{\otimes}(b \otimes \Delta(z))\\ (a \otimes b)\bar{\otimes}c &\mapsto a\bar{\otimes}(b \otimes c)\end{aligned}$$

and the isomorphism $(M \underline{\circledcirc} N)_{\mathrm{e}} \cong (N \underline{\circledcirc} M)_{\mathrm{e}}$ *is given by*

$$\begin{aligned} x \underline{\circledcirc} y &\mapsto y \underline{\circledcirc} x - H(y) \bar{\otimes} TH(x) \\ a \bar{\otimes} b &\mapsto b \bar{\otimes} a. \end{aligned}$$

Under the associativity isomorphism the element $x \underline{\circledcirc} (b \bar{\otimes} c)$ *corresponds to* $(x|x)_H \bar{\otimes} (b \otimes c)$.

Proof of this result occupies Section 6.

The category of square groups with $M_{\mathrm{ee}} = 0$ is equivalent to the category of abelian groups. Thus we identify the category Ab with this subcategory of SG. Then the restriction of $\underline{\circledcirc}$ to Ab coincides with the usual tensor product of abelian groups. The following generalizes some well-known properties of the tensor product of abelian groups.

Proposition 4. *For any square group A the tensor product $A \underline{\circledcirc} - : \mathsf{SG} \to \mathsf{SG}$ preserves filtered colimits, reflexive coequalizers and finite products. It is right exact and balanced, that is for any short exact sequence of square groups*

$$0 \to B_1 \xrightarrow{\mu} B \xrightarrow{\sigma} B_2 \to 0$$

the induced sequence

$$A \underline{\circledcirc} B_1 \to A \underline{\circledcirc} B \to A \underline{\circledcirc} B_2 \to 0$$

is exact and the first arrow $A \underline{\circledcirc} \mu$ is a monomorphism provided A is a projective object in the category SG.

The proof of this result is given in Section 9.

The functor $A \underline{\circledcirc} - : \mathsf{SG} \to \mathsf{SG}$ does not preserves coproducts. However the following result is true. For an abelian group A we define the square group $A^{\otimes}$ by

$$(A^{\otimes})_{\mathrm{e}} = A, \quad (A^{\otimes})_{\mathrm{ee}} = A \oplus A,$$

where $P(a,b) = a + b$ and $H(a) = (a,a)$.

Proposition 5. *Let A, B, M be square groups. Then one has the short exact sequence of square groups*

$$0 \to (M_{\mathrm{ee}} \otimes \mathsf{Coker}(P^A) \otimes \mathsf{Coker}(P^B))^{\otimes} \to M \underline{\circledcirc} (A \vee B) \to (M \underline{\circledcirc} A) \times (M \underline{\circledcirc} B) \to 0.$$

Here $\vee$ denotes the coproduct in the category of square groups.

The proof of this result is given at the end of Section 11.

2. Symmetric definition of the tensor product

The tensor product $M \underline{\circledcirc} N$ in Definition 2 is the *right linear version.* There is also a *left linear version* $M \circledcirc N$ defined below. Moreover we introduce a *symmetric version* $M \odot N$ and we show that there are natural isomorphisms

$$M \circledcirc N \cong M \odot N \cong M \underline{\circledcirc} N.$$

The different versions of the tensor product are identified in this way. The symmetry of the tensor product is most apparent in $M \odot N$ where, however, redundant generators are needed. The non-symmetric versions $M \underline{\circledcirc} N$ and $M \circledcirc N$ have the advantage of a smaller set of generators. Most calculations in the paper are using the right linear version $M \underline{\circledcirc} N$.

Definition 6. For square groups M, N let $M \circledcirc N$ be the square group with $(M \circledcirc N)_{\mathrm{ee}} = M_{\mathrm{ee}} \otimes N_{\mathrm{ee}}$ and $(M \circledcirc N)_{\mathrm{e}}$ given by generators $x \circledcirc y$ for $x \in M_{\mathrm{e}}$, $y \in N_{\mathrm{e}}$, and $a \bar{\otimes} b$ for $a \in M_{\mathrm{ee}}$, $b \in N_{\mathrm{ee}}$, subject to the relations

(1) the symbol $a \bar{\otimes} b$ is bilinear and central in $(M \circledcirc N)_{\mathrm{e}}$;
(2) $x \circledcirc (y_1 + y_2) = x \circledcirc y_1 + x \circledcirc y_2 + H(x) \bar{\otimes} (y_2 \mid y_1)_H$;
(3) $(x_1 + x_2) \circledcirc y = x_1 \circledcirc y + x_2 \circledcirc y$;
(4) $x \circledcirc P(b) = \Delta(x) \bar{\otimes} b$;
(5) $P(a) \circledcirc y = a \bar{\otimes} (y \mid y)_H$;
(6) $T(a) \bar{\otimes} T(b) = -a \bar{\otimes} b$.

Here the homomorphism P is given as in Definition 2 and H is the quadratic map with the cross effect ρ as in Definition 2 and

$$H(x \circledcirc y) = \Delta(x) \otimes H(y) + H(x) \otimes (y \mid y)_H\,,$$
$$H(a \bar{\otimes} b) = a \otimes b - T(a) \otimes T(b).$$

Next we introduce the symmetric version of the tensor product.

Definition 7. For two square groups A, B we define their tensor product $A \odot B$ which is again a square group defined as follows. The group $(A \odot B)_{\mathrm{e}}$ is defined by generators of the form $x \underline{\circledcirc} y$, $x \circledcirc y$ for $x \in A_{\mathrm{e}}$, $y \in B_{\mathrm{e}}$ and $a \bar{\otimes} b$ for $a \in A_{\mathrm{ee}}$, $b \in B_{\mathrm{ee}}$, subject to the relations

(1) the symbol $a \bar{\otimes} b$ is bilinear and central in $(A \underline{\circledcirc} B)_{\mathrm{e}}$;
(2) $x \underline{\circledcirc} (y_1 + y_2) = x \underline{\circledcirc} y_1 + x \underline{\circledcirc} y_2$;
(3) $(x_1 + x_2) \circledcirc y = x_1 \circledcirc y + x_2 \circledcirc y$;
(4) $P(a) \underline{\circledcirc} y = a \bar{\otimes} \Delta(y)$;
(5) $x \circledcirc P(b) = \Delta(x) \bar{\otimes} b$;
(6) $T(a) \bar{\otimes} T(b) = -a \bar{\otimes} b$;
(7) $x \underline{\circledcirc} y - x \circledcirc y = H(x) \bar{\otimes} TH(y)$.

Next the abelian group $(A \odot B)_{\mathrm{ee}}$ is defined to be $A_{\mathrm{ee}} \otimes B_{\mathrm{ee}}$. The homomorphism

$$P : (A \odot B)_{\mathrm{ee}} \to (A \odot B)_{\mathrm{e}}$$

is given by

$$P(a \otimes b) = a \bar{\otimes} b$$

and the map

$$H : (A \odot B)_{\mathrm{e}} \to (A \odot B)_{\mathrm{ee}}$$

is the unique quadratic map with

$$H(x \underline{\circledcirc} y) = (x|x)_H \otimes H(y) + H(x) \otimes \Delta(y),$$
$$H(x \circledcirc y) = \Delta(x) \otimes H(y) + H(x) \otimes (y|y)_H$$

and

$$H(a \bar{\otimes} b) = a \otimes b - T(a) \otimes T(b)$$

such that its cross-effect coincides with the bilinear map ρ in Definition 2.

Proposition 8. *For any square groups A, B the above data define square groups $A \odot B$ and $A \circledcirc B$ and both of them are isomorphic to $A\underline{\circledcirc}B$.*

The proof of this fact is given in Section 13.

Corollary 9. *The symmetry isomorphism*

$$\tau(A,B) : A \odot B \to B \odot A,$$

corresponding to the symmetry for $A\underline{\circledcirc}B$ under the above isomorphism, is given by

$$x\underline{\circledcirc}y \mapsto y \circledcirc x,$$
$$x \circledcirc y \mapsto y\underline{\circledcirc}x.$$
$$a\bar{\otimes}b \mapsto b\bar{\otimes}a.$$

Remark 10. The notation above is chosen to be compatible with the notation for *exterior cup products* $f\#g$ and $f\underline{\#}g$ in topology, see [1], [12]. Here $f\#g$, being left linear, corresponds to $x \circledcirc y$ and $f\underline{\#}g$, being right linear, corresponds to $x\underline{\circledcirc}y$, see [6]. In fact, the construction of $A \odot B$ above originates from properties of the exterior cup products. Compare also the tensor product of "quadratic modules" in [2].

3. Preliminaries on Nil_2-groups

Groups will be written additively. In particular, for elements $a, b \in G$ of a group G their commutator will be denoted by $[a,b] = -a - b + a + b$. For any group G we denote by $\mathscr{Z}(G)$ the center of G.

A group G is of *nilpotence class two*, or is a *nil$_2$-group*, if all triple commutators of G vanish, $[[G,G],G] = 0$. The category of all such groups and their homomorphisms will be denoted by Nil.

For any $G \in \mathsf{Nil}$ there is a well-defined homomorphism $\Lambda^2(G^{\mathrm{ab}}) \to G$ given by $\hat{a} \wedge \hat{b} \mapsto [a,b]$. Here and elsewhere $\hat{x}$ denotes the class of $x \in G$ in $G^{\mathrm{ab}} = G/[G,G]$. Moreover, one has the inclusion $[G,G] \subseteq \mathscr{Z}(G)$ and for any $a, b \in G$ and any $n \in \mathbb{Z}$ one has

$$na + nb = n(a+b) + \binom{n}{2}[a,b]. \tag{1}$$

The category Nil has all limits and colimits. For G_1 and G_2 in Nil let $G_1 \vee G_2$ denote their coproduct in Nil. Then one has the following central extension

$$0 \to G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}} \xrightarrow{i} G_1 \vee G_2 \to G_1 \times G_2 \to 0. \tag{2}$$

Here the homomorphism i is given by $\hat{x} \otimes \hat{y} \mapsto [i_1(x), i_2(y)]$ for $x \in G_1, y \in G_2$, where $i_t : G_t \to G_1 \vee G_2$, $t = 1, 2$ is the canonical inclusion.

The inclusion functor $\mathsf{Nil} \subset \mathsf{Groups}$ has a left adjoint, given by

$$G \mapsto G^{\mathrm{nil}} := G/[[G,G],G].$$

The forgetful functor $\mathsf{Nil} \to \mathsf{Sets}$ has a left adjoint, whose value on a set S is known as *the free nilpotent group of class two generated by* S and is denoted by $\langle S \rangle^{\mathrm{nil}}$. One has $\langle S \rangle^{\mathrm{nil}} = (F_S)^{\mathrm{nil}}$, where F_S is the free group on S.

The following is an easy consequence of the theorem of Witt on the lower central series of a free group:

Lemma 11. *For a free nil$_2$-group G one has the central extension*

$$0 \to \Lambda^2(G^{\mathrm{ab}}) \to G \to G^{\mathrm{ab}} \to 0.$$

It follows that there is a normal form of elements in $\langle S\rangle^{\mathrm{nil}}$ for each linear ordering of the set S. Namely, all elements of $\langle S\rangle^{\mathrm{nil}}$ can be written in a unique way in the form

$$n_1x_1 + ... + n_px_p + m_1[y_1, z_1] + ... + m_q[y_q, z_q] \tag{3}$$

with $x_i, y_j, z_j \in S$ and $n_i, m_j \in \mathbb{Z}\setminus\{0\}$ for all i, j and moreover $x_1 < ... < x_p$, $y_1 < z_1$, ..., $y_q < z_q$ with respect to the given ordering of S and $(y_1, z_1) < ... < (y_q, z_q)$ with respect to the induced (left) lexicographic ordering of $S \times S$.

4. Preliminaries on quadratic maps

Let G and G' be additively written groups of nilpotence class two. Recall that a map $f : G \to G'$ is quadratic if for any $a, b \in G$ the cross-effect

$$(a \mid b)_f := -f(b) - f(a) + f(a + b)$$

is a central element in G' and is linear in a and b. Then $f(0) = 0$ and the cross-effect yields a well-defined homomorphism $(- \mid -)_f : G^{\mathrm{ab}} \otimes G^{\mathrm{ab}} \to \mathscr{Z}(G')$. Moreover, the following holds (see [15])

$$f(na) = nf(a) + \binom{n}{2}(a \mid a)_f\,, \quad n \in \mathbb{Z}, \tag{4}$$

$$f([a, b]) = (a \mid b)_f - (b \mid a)_f\,. \tag{5}$$

Lemma 12. *For any set S, any abelian group A, any map*

$$f_0 : S \to A$$

and any homomorphism

$$\Phi : \mathbb{Z}[S]^{\otimes 2} \to A$$

there exists a unique quadratic map

$$f : \langle S\rangle^{\mathrm{nil}} \to A$$

satisfying

$$f(s) = f_0(s)$$

for $s \in S \subset \langle S\rangle^{\mathrm{nil}}$ and

$$f(u + v) = f(u) + f(v) + \Phi(\hat{u} \otimes \hat{v})$$

for $u, v \in \langle S\rangle^{\mathrm{nil}}$. In particular there is a unique map

$$H : \langle S\rangle^{\mathrm{nil}} \to \mathbb{Z}[S] \otimes \mathbb{Z}[S]$$

satisfying

$$H(x) = 0$$

for $x \in S$ *and*

$$H(u+v) = H(u) + H(v) + \hat{v} \otimes \hat{u}$$

for $u, v \in \langle S\rangle^{\mathrm{nil}}$. *Thus* H *is a quadratic map with*

$$\begin{aligned} (u \mid v)_H &= \hat{v} \otimes \hat{u}, \\ H(-u) &= -H(u) + \hat{u} \otimes \hat{u}, \\ H([u,v]) &= \hat{v} \otimes \hat{u} - \hat{u} \otimes \hat{v}. \end{aligned}$$

Proof. Uniqueness is clear from the hypothesis. For the existence, using the above normal form (3), we can explicitly define

$$\begin{aligned} &f(n_1x_1 + ... + n_px_p + m_1[y_1,z_1] + ... + m_q[y_q,z_q]) \\ &= n_1H_0(x_1) + ... + n_pH_0(x_p) + \binom{n_1}{2}\Phi(x_1 \otimes x_1) + ... + \binom{n_p}{2}\Phi(x_p \otimes x_p) + \sum_{1 \leqslant i < j \leqslant p} n_in_j\Phi(x_i \otimes x_j) \\ &\qquad + m_1\Phi(y_1 \otimes z_1 - z_1 \otimes y_1) + ... + m_q\Phi(y_q \otimes z_q - z_q \otimes y_q). \end{aligned}$$

It is easy to see that the so defined f satisfies the required equalities. □

The following key lemma is useful for constructing quadratic maps on groups given in terms of generators and relations.

Lemma 13. *Suppose a nil*$_2$*-group* G *is given by a set* $S = \{x_i; i \in I\}$ *of generators subject to the relations* $\{r_j; j \in J\}$. *For any abelian group* A, *any* I*-tuple* $(a_i)_{i \in I}$ *of elements in* A *and any homomorphism*

$$\Phi : G^{\mathrm{ab}} \otimes G^{\mathrm{ab}} \to A$$

there exists a unique quadratic map

$$f : \langle S\rangle^{\mathrm{nil}} \to A$$

satisfying

$$f(x_i) = a_i, \quad (x_i|x_{i'})_f = \Phi(x_i, x_{i'}), \quad i, i' \in I.$$

Moreover this map factors through the quotient map $q : \langle S\rangle^{\mathrm{nil}} \twoheadrightarrow G$ *to yield a quadratic map* $G \to A$ *if and only if it satisfies*

$$f(r_j) = 0, \quad j \in J.$$

Proof. Existence of $f : \langle S\rangle^{\mathrm{nil}} \to A$ is a direct consequence of Lemma 12. It is clear that if f factors trough G then $f(r_j) = 0$ for all $j \in J$. Conversely, assume this condition holds. We have to show that $f(x) = f(y)$ provided $y = r + x$, where r lies in the smallest normal subgroup B of $\langle S\rangle$ containing all r_j, $j \in J$. Observe that $\Phi(r, -) = 0$ for any $r \in B$, so that restriction of f to B is a homomorphism and moreover $f(y) = f(r) + f(x)$, so it remains to prove that f vanishes identically on B. For this, it suffices to show that for any $r \in B$ with $f(r) = 0$ one also has $f(-z + r + z) = 0$ for any $z \in \langle S \rangle$. Indeed

$$\begin{aligned} f(-z+r+z) &= f(-z) + f(r+z) + \varPhi(-z, r+z) \\ &= f(-z) + f(z) + \varPhi(-z, z) \\ &= f(-z+z) \\ &= 0. \end{aligned}$$

□

5. Square groups and pre-square groups

In this section we list main properties of square groups and introduce various special cases of square groups. Also quotients, coproducts, central extensions in the category of square groups are described. We then outline the simplicial theory of square groups. Finally we discuss pre-square groups needed in the proof of our main results.

5.1. **Properties of square groups.** Let

$$M = (\ M_{\mathrm{e}} \xrightarrow{H} M_{\mathrm{ee}} \xrightarrow{P} M_{\mathrm{e}}\ )$$

be a square group as in Section 1. Then $P\,(x \mid y)_H = -x - y + x + y$ implies that $[M_{\mathrm{e}}, M_{\mathrm{e}}] \subset \mathsf{Im}(P)$, while $(Pa \mid y)_H = 0 = (x \mid Pb)_H$ shows that $\mathsf{Im}(P) \subset \mathscr{Z}(M_{\mathrm{e}})$. In particular M_{e} is a nil$_2$-group and $\mathsf{Coker}(P)$ is a well-defined abelian group.

Proposition 14.

i) *One has*

$$H(-x-y+x+y) = (x \mid y)_H - (y \mid x)_H\,.$$

Moreover, the function

$$T = HP - \mathsf{Id}$$

is an involutive automorphism of M_{ee}, i. e. $T^2 = \mathsf{Id}_{M_{\mathrm{ee}}}$. Furthermore one has

$$PT = P, \quad T\,(x \mid y)_H + (y \mid x)_H = 0.$$

ii) *The function $\Delta : M_{\mathrm{e}} \to M_{\mathrm{ee}}$ is linear, where*

$$\begin{aligned} \Delta(x) &= HPH(x) - 2H(x) + (x \mid x)_H \\ &= HPH(x) + H(2x) - 4H(x) \\ &= \{x \mid x\}_H - H(x) + TH(x) \\ &= H(-x) + TH(x) \end{aligned}$$

and furthermore one has

$$\Delta P = 0, \quad P\Delta = 0, \quad \Delta + T\Delta = 0.$$

iii) *For any integer* n*, the map* $n^*: M_{\mathrm{e}} \to M_{\mathrm{e}}$ *defined by* $n^*(x) = nx + \binom{n}{2}PH(x)$ *is a homomorphism. Moreover one has*

$$(nm)^* = n^* m^*$$

and

$$P(n^2 a) = n^*(P(a)), \quad (n^* x \mid n^* y)_H = n^2 (x \mid y)_H.$$

Proof. i) Since H is quadratic and M_{ee} is abelian we can use identity (5) to get the first identity. We have

$$(a \mid b)_T = (a \mid b)_{HP} = (Pa \mid Pb)_H = 0.$$

Thus T is a homomorphism. Furthermore, one has

$$T^2 = (HP - \mathsf{Id})(HP - \mathsf{Id}) = HPHP - 2HP + \mathsf{Id} = H(P + P) - 2HP + \mathsf{Id} = \mathsf{Id}.$$

Similarly

$$PT = P(HP - \mathsf{Id}) = PHP - P = 2P - P = P$$

and

$$T\,(x \mid y)_H = HP\,(x \mid y)_H - (x \mid y)_H = H([x, y]) - (x \mid y)_H = -\,(y \mid x)_H\,.$$

ii) Since H takes values in an abelian group, we have

$$\begin{aligned}\Delta(x + y) &= HPH(x + y) - 2H(x + y) + (x + y \mid x + y)_H \\ &= HPH(x) + HPH(y) + H([x, y]) - 2H(x) - 2H(y) - 2\,(x \mid y)_H + (x + y \mid x + y)_H \\ &= \Delta(x) + \Delta(y) + H([x, y]) - (x \mid y)_H + (y \mid x)_H = \Delta(x) + \Delta(y).\end{aligned}$$

Hence Δ is additive. To get the other expressions for Δ observe that $H(2x) = 2H(x) + (x \mid x)_H$ as well as $TH = HPH - H$ and $H(-x) = -H(x) + (x \mid x)_H$. Moreover, we have $\Delta P = HPHP - 2HP = 0$ and $P\Delta(x) = PHPH(x) - 2PH(x) + [x, x] = 0$. Similarly

$$\begin{aligned}T\Delta(x) &= THPH(x) - 2TH(x) + T((x \mid x)_H) = \\ &= HPHPH(x) - HPH(x) - 2HPH(x) + 2H(x) - (x \mid x)_H \\ &= -HPH + 2H(x) - (x \mid x)_H = -\Delta(x).\end{aligned}$$

iii) We have

$$\begin{aligned}n^*(x + y) &= n(x + y) + \binom{n}{2} PH(x + y) \\ &= nx + ny + \binom{n}{2}(-[x, y] + PH(x) + PH(y) + P(x \mid y)_H) \\ &= nx + ny + \binom{n}{2}(PH(x) + PH(y)) = n^*(x) + n^*(y).\end{aligned}$$

Thus n^* is indeed a homomorphism. We also have

$$n^*(m^*(x)) = n(mx + \binom{m}{2} PH(x)) + \binom{n}{2} PH(mx + \binom{m}{2} PH(x)).$$

Since $PH(x)$ is a central element, we obtain

$$n^*(m^*(x)) = nmx + (n\binom{m}{2} + \binom{n}{2}m + 2\binom{n}{2}\binom{m}{2})PH(x)$$
$$= nmx + \binom{nm}{2}PH(x) = (mn)^*(x).$$

Furthermore, we have

$$n^*(Pa) = nP(a) + \binom{n}{2}PHP(a) = nP(a) + (n^2-n)P(a) = n^2P(a)$$

and

$$(n^*(x) \mid n^*(y))_H = (nx + P(\binom{n}{2}H(x)) \mid ny + P(\binom{m}{2}H(y)))_H$$
$$= (nx \mid ny)_H = n^2(x \mid y)_H.$$

□

Corollary 15. *The cross-effect and* Δ *yield homomorphisms*

$$(-,-)_H : \mathsf{Coker}(P^M) \otimes \mathsf{Coker}(P^M) \to M_{\mathrm{ee}}$$

and

$$\Delta : \mathsf{Coker}(P^M) \to \mathsf{Ker}(P^M) \subset M_{\mathrm{ee}}.$$

Moreover Δ *yields the natural homomorphism*

$$k^M : \mathsf{Coker}(P^M) \to \mathsf{Ker}(\frac{M_{\mathrm{ee}}}{\mathsf{Id}-T} \xrightarrow{P^M} M_{\mathrm{e}})$$

One also has

$$k^M(\bar{x}) \equiv (x|x)_H$$

in $\mathsf{Ker}(\frac{M_{\mathrm{ee}}}{\mathsf{Id}-T} \xrightarrow{P^M} M_{\mathrm{e}})$ *for any* $x \in M_{\mathrm{e}}$.

5.2. **Abelian square groups and quadratic $\mathbb{Z}$-modules.** A square group M is called *abelian* if H is a homomorphism, that is $(x \mid y)_H = 0$ for all $x, y \in M_{\mathrm{e}}$, equivalently abelian square group consists of two abelian groups M_{e} and M_{ee} together two homomorphisms $P : M_{\mathrm{ee}} \to M_{\mathrm{e}}$, $H : M_{\mathrm{e}} \to M_{\mathrm{ee}}$ such that $PHP = 2P$. Abelian square groups correspond to quadratic functors $\mathsf{Groups} \to \mathsf{Ab}$ preserving filtered colimits and reflexive coequalizers. The category of abelian square groups is denoted by $\mathsf{Ab}(\mathsf{SG})$.

Let us recall that a quadratic $\mathbb{Z}$-module (see [3], [7]) is a square group M for which $(- \mid -)_H = 0$ and $\Delta = 0$. Equivalently a quadratic $\mathbb{Z}$-module is given by two abelian groups M_{e} and M_{ee} together with two homomorphisms $P : M_{\mathrm{ee}} \to M_e$ $H : M_{\mathrm{e}} \to M_{\mathrm{ee}}$ such that $PHP = 2P$ and $HPH = 2H$. Quadratic $\mathbb{Z}$-modules correspond to quadratic functors $\mathsf{Ab} \to \mathsf{Ab}$ preserving filtered colimits and reflexive coequalizers. The category of quadratic $\mathbb{Z}$-modules is denoted by QZ.

Thus we have the following full embeddings:

$$\mathsf{Ab} \subset \mathsf{QZ} \subset \mathsf{Ab}(\mathsf{SG}) \subset \mathsf{SG}.$$

Here abelian groups corresponds to square groups M with $M_{\mathrm{ee}} = 0$.

For any abelian group A, let $A^{\otimes}$ be the quadratic $\mathbb{Z}$-module defined as in Proposition 5.

Lemma 16. *For any square group M and for any abelian group A one has the isomorphism*

$$\mathsf{Hom}_{\mathsf{SG}}(A^{\otimes}, M) \cong \mathsf{Hom}(A, M_{\mathrm{ee}}).$$

Proof. Take a homomorphism $g : A \to M_{\mathrm{ee}}$. We define $f = (f_{\mathrm{e}}, f_{\mathrm{ee}}) : A^{\otimes} \to M$ by $f_{\mathrm{e}}(a) = Pg(a)$ and $f_{\mathrm{ee}}(a, b) = g(a) + Tg(b)$. Then f is a morphism of square groups and one easily sees that in this way one gets all such maps. Indeed, one takes $g(a) = f_{\mathrm{ee}}(a, 0)$. $\square$

Of special interest is the quadratic $\mathbb{Z}$-module $\mathbb{Z}^{\otimes}$ since by Lemma 16

$$\mathsf{Hom}_{\mathsf{SQ}}(\mathbb{Z}^{\otimes}, M) \cong M_{\mathrm{ee}}.$$

We will need also the following construction. Let L be an abelian group and let τ be an involution on L. Then $E(L, \tau)$ is the quadratic $\mathbb{Z}$-module with

$$E(L,\tau)_{\mathrm{e}} = \mathsf{Coker}(L \xrightarrow{\mathsf{Id}+\tau} L)$$

$$E(L,\tau)_{\mathrm{e}} = L$$

where P is the natural projection onto quotient, while H is induced by the homomorphism $\mathsf{Id} - \tau$.

5.3. **Sets versus square groups.** There is a functor

$$\mathbb{Z}_{\mathrm{nil}}[-] : \mathsf{Sets} \to \mathsf{SG}$$

which is constructed as follows. For a set S one puts

$$\mathbb{Z}_{\mathrm{nil}}[S]_{\mathrm{ee}} = \mathbb{Z}[S] \otimes \mathbb{Z}[S],$$

where $\mathbb{Z}[S]$ is the free abelian group generated by S. We take $\mathbb{Z}_{\mathrm{nil}}[S]_{\mathrm{e}}$ to be $\langle S \rangle^{\mathrm{nil}}$, the free nil_2-group generated by S. The homomorphism P is given by $P(s \otimes t) = [t, s]$, $s, t \in S$, while the quadratic map H is uniquely defined by

$$H(s) = 0, \quad (s \mid t)_H = t \otimes s \quad s, t \in S.$$

If S is a singleton, we obtain $\mathbb{Z}_{\mathrm{nil}} = \mathbb{Z}_{\mathrm{nil}}[S]$, see Definition 1.

For general S one has

$$\mathsf{Coker}(P^{\mathbb{Z}_{\mathrm{nil}}[S]}) = \mathbb{Z}[S]$$

and the homomorphism

$$\Delta : \mathbb{Z}[S] = \mathsf{Coker}(P^{\mathbb{Z}_{\mathrm{nil}}[S]}) \to \mathbb{Z}[S] \otimes \mathbb{Z}[S] = \mathbb{Z}_{\mathrm{nil}}[S]_{\mathrm{ee}}$$

is given by $\Delta(s) = (s, s)$. Moreover, the homomorphism

$$T : \mathbb{Z}[S] \otimes \mathbb{Z}[S] \to \mathbb{Z}[S] \otimes \mathbb{Z}[S]$$

is given by $T(s \otimes t) = -t \otimes s$.

It turns out that the functor $\mathbb{Z}_{\mathrm{nil}}[-]:\mathsf{Sets}\to\mathsf{SG}$ is a left adjoint. Let M be a square group. An element $x\in M_{\mathrm{e}}$ is called *linear* if $H(x)=0$. Let $\mathbb{L}(M)$ be the subset of linear elements in M_{e} so that we obtain a functor

$$\mathbb{L}:\mathsf{SG}\to\mathsf{Sets}.$$

Proposition 17. *The functor* $\mathbb{Z}_{\mathrm{nil}}[-]:\mathsf{Sets}\to\mathsf{SG}$ *is left adjoint to the functor* $\mathbb{L}$.

Proof. Let S be a set and M be a square group. Given a morphism $f:\mathbb{Z}_{\mathrm{nil}}[S]\to M$ of square groups, the composite $S\subset\mathbb{Z}_{\mathrm{nil}}[S]\to M$ is a map f_0; moreover its image is contained in $\mathbb{L}(M)$ since $H(s)=0$ for $s\in S$ and f is compatible with H.

Conversely we must show that any map $f_0:S\to\mathbb{L}(M)\subset M_{\mathrm{e}}$ extends uniquely to a square group morphism $f:\mathbb{Z}_{\mathrm{nil}}[S]\to M$. First, there is clearly a unique group homomorphism $f_{\mathrm{e}}:\mathbb{Z}_{\mathrm{nil}}[S]_{\mathrm{e}}\to M_{\mathrm{e}}$ extending f_0, as $\mathbb{Z}_{\mathrm{nil}}[S]_{\mathrm{e}}$ is the free nil$_2$-group on S and M_{e} is a nil$_2$-group. Moreover, by compatibility of a morphism of square groups with H we necessarily have

$$f_{\mathrm{ee}}(s\otimes s')=f_{\mathrm{ee}}((s'|s)_H)=(f_0(s')|f_0(s))_H$$

for any $s,s'\in S$. Hence we also have a unique choice for $f_{\mathrm{ee}}:\mathbb{Z}[M]\otimes\mathbb{Z}[M]\to M_{\mathrm{ee}}$ and one has

$$f_{\mathrm{ee}}(\bar{x}\otimes\bar{y})=(f_{\mathrm{e}}(y)|f_{\mathrm{e}}(x))_H$$

for any $x,y\in\mathbb{Z}_{\mathrm{nil}}[M]_{\mathrm{e}}$.

It remains to show that the pair $(f_{\mathrm{e}},f_{\mathrm{ee}})$ yields a morphism of square groups

$$(f_{\mathrm{e}},f_{\mathrm{ee}}):\mathbb{Z}_{\mathrm{nil}}[S]\to M.$$

Indeed, compatibility with P is clear since

$$f_{\mathrm{e}}P(s\otimes s')=f_{\mathrm{e}}P((s'|s)_H)=f_{\mathrm{e}}[s',s]=[f_0(s'),f_0(s)]=P((f_0(s')|f_0(s))_H)=Pf_{\mathrm{ee}}(s\otimes s')$$

as f_{e}, f_{ee} and P are group homomorphisms. Since image of f_0 is in $\mathbb{L}(M)$, compatibility with H holds on elements of S; moreover if it holds on x and y, one has

$$\begin{aligned}f_{\mathrm{ee}}H(x+y)=f_{\mathrm{ee}}(H(x)+H(y)+\bar{y}\otimes\bar{x})&=Hf_{\mathrm{e}}(x)+Hf_{\mathrm{e}}(y)+(f_{\mathrm{e}}(x)|f_{\mathrm{e}}(y))_H\\&=H(f_{\mathrm{e}}(x)+f_{\mathrm{e}}(y))=Hf_{\mathrm{e}}(x+y).\end{aligned}$$

This finishes the proof. □

5.4. **Normal subobjects and quotients of square groups.** Obviously for any morphism $f:M\to N$ of square groups kernels of f_{e} and f_{ee} determine a sub-square group $\mathsf{Ker}(f)$ of M. Sub-square groups of this form can be characterized as those $K\rightarrowtail M$ for which K_{e} is normal in M_{e} and moreover one has

$$(M_{\mathrm{e}}|K_{\mathrm{e}})_H,(K_{\mathrm{e}}\mid M_{\mathrm{e}})_H\subset K_{\mathrm{ee}},$$

i. e. for any $x\in M_{\mathrm{e}}$, $k\in K_{\mathrm{e}}$ one has $(x|k)_H\in K_{\mathrm{ee}}$ and $(k|x)_H\in K_{\mathrm{ee}}$. Such sub-square groups will be called *normal*. For any normal sub-square group $K\triangleleft M$ the quotient M/K is defined, with $(M/K)_{\mathrm{e}}=M_{\mathrm{e}}/K_{\mathrm{e}}$, $(M/K)_{\mathrm{ee}}=M_{\mathrm{ee}}/K_{\mathrm{ee}}$; here $P:M_{\mathrm{ee}}/K_{\mathrm{ee}}\to M_{\mathrm{e}}/K_{\mathrm{e}}$ is the uniquely determined homomorphism whereas $H : M_{\mathrm{e}}/K_{\mathrm{e}} \to M_{\mathrm{ee}}/K_{\mathrm{ee}}$ is the uniquely determined map by virtue of

$$H(x+k) - H(x) = H(k) + (x|k)_H \in K_{\mathrm{ee}}$$

for any $x \in M_{\mathrm{e}}$, $k \in K_{\mathrm{e}}$.

Cokernels of morphisms in SG are defined as follows. For a morphism $f : M \to N$ of square groups, let $\mathsf{Coker}(f)$ be the quotient of N by the smallest normal sub-square group generated by $\mathsf{Im}(f)$. Thus one has

$$\mathsf{Coker}(f)_{\mathrm{e}} = \mathsf{Coker}(f_{\mathrm{e}}),$$

i. e. $\mathsf{Coker}(f)_{\mathrm{e}}$ is the quotient of N_{e} by the normal subgroup generated by the image of f_{e}, and

$$\mathsf{Coker}(f)_{\mathrm{ee}} = \mathsf{Coker}(f_{\mathrm{ee}})/(\mathsf{Im}(f_{\mathrm{e}})|N)_H,$$

that is, $\mathsf{Coker}(f)_{\mathrm{ee}}$ is the quotient of N_{ee} by elements of the form $f_{\mathrm{ee}}(a)$ for $a \in M_{\mathrm{ee}}$ and $(f_{\mathrm{e}}(x)|y)_H$ for $x \in M_{\mathrm{e}}$, $y \in N_{\mathrm{e}}$.

Lemma 18. *Let $f : M \to N$ be a morphism in SG. Then the following conditions are equivalent:*

(1) *f_{e} and f_{ee} are surjective;*
(2) *for any morphism $h : N \to N'$ in SG with $hf = 0$ one has $h = 0$;*
(3) *$\mathsf{Coker}(f) = 0$.*

Proof. We show that (3) $\Longrightarrow$ (1). The rest is trivial. Assume (3) holds. It follows that $\mathsf{Coker}(f_{\mathrm{e}}) = 0$. Hence f_{e} is surjective (see for example Exercise 5, Section 5, Chapter 1 in [19]). Moreover (3) implies

$$N_{\mathrm{ee}} = f_{\mathrm{ee}}(M_{\mathrm{ee}}) + (f_{\mathrm{e}}(M_{\mathrm{e}}) \mid N_{\mathrm{e}})_H + (N_{\mathrm{e}} \mid f_{\mathrm{e}}(M_{\mathrm{e}}))_H.$$

Since $N_{\mathrm{e}} = f_{\mathrm{e}}(M_{\mathrm{e}})$, we see that

$$N_{\mathrm{ee}} = f_{\mathrm{ee}}(M_{\mathrm{ee}}) + \big(f_{\mathrm{e}}(M_{\mathrm{e}}) \mid f_{\mathrm{e}}(M_{\mathrm{e}})\big)_H = f_{\mathrm{ee}}(M_{\mathrm{ee}}) + f_{\mathrm{ee}}\big((M_{\mathrm{e}} \mid M_{\mathrm{e}})_H\big)$$

and hence f_{ee} is surjective. □

A morphism $f : M \to N$ in the category SG is called an *epimorphism* provided it satisfies the conditions of Lemma 18. One easily deduces from Lemma 18 that the class of effective epimorphisms (Section 4, Chapter 2, [23]) in the category SG coincides with the class of epimorphisms.

5.5. **Central extensions of square groups.** A sequence of square groups

$$0 \to A \to B \to C \to 0$$

is called *short exact* if it is exact on the e-level and the ee-level. We will say that it is a *central extension* if A_{e} is a central in B_{e} and $(x \mid y)_H = 0$ provided $x \in A_{\mathrm{e}}$ and $y \in B_{\mathrm{e}}$, or $x \in B_{\mathrm{e}}$ and $y \in A_{\mathrm{e}}$. In particular A is a normal sub-square group of B and H is linear on A_{e}.

Lemma 19. *Let*

$$0 \to A \to B \to C \to 0$$

be a short exact sequence in SG*. Then there is a well-defined square group* A' *defined by*

$$A'_{\mathrm{e}} := \{x \in A_{\mathrm{e}} \mid x + y = y + x \ \& \ (x \mid y)_H = 0 = (y \mid x)_H \,,\ y \in B_{\mathrm{e}}\},$$
$$A'_{\mathrm{ee}} := \{a \in A_{\mathrm{ee}} \mid P(a) \in A'_{\mathrm{e}}\},$$

and H *and* P *being the restriction of* P^A *and* H^A*. Moreover the columns and the bottom row of the commutative diagram*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & A' \arrow[r, "\mathsf{Id}"] \arrow[d] & A' \arrow[d] & & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "\mathsf{Id}"] & 0 \\
0 \arrow[r] & A/A' \arrow[r] \arrow[d] & B/A' \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

are central extensions of square groups.

Proof. Take any element x from A_{e}. Then $PH(x) \in A'_{\mathrm{e}}$. In particular $H(A'_{\mathrm{e}}) \subset A'_{\mathrm{ee}}$ and therefore A' is well-defined. It is obvious that the columns are central extensions of square groups. Take now any elements $x \in A_{\mathrm{e}}$ and $y \in B_{\mathrm{e}}$. Then the commutator $[x, y]$ projects to zero in C and therefore $[x, y] \in A_{\mathrm{e}}$. Since $N_{\mathrm{e}} \in \mathsf{Nil}$ and the cross-effect of H vanishes on commutators it follows that $[x, y] \in A'_{\mathrm{e}}$. Thus $(A/A')_{\mathrm{e}}$ is a central subgroup of $(B/A')_{\mathrm{e}}$. It remains to show that $H(x, y) \in A'_{\mathrm{ee}}$. But this follows immediately from the facts that the image of $H(x, y)$ in C_{ee} vanishes (thus $H(x, y) \in A_{\mathrm{ee}}$) and $PH(x, y) \in A'_{\mathrm{e}}$. $\square$

5.6. **Coproduct of square groups.** Let M and N be square groups. Then their coproduct $M \vee N$ in the category of square groups has the following form

$$M \vee N = ((M \vee N)_{\mathrm{e}} \xrightarrow{H} (M \vee N)_{\mathrm{ee}} \xrightarrow{P} (M \vee N)_{\mathrm{e}})$$

where

$$(M \vee N)_{\mathrm{ee}} = M_{\mathrm{ee}} \oplus N_{\mathrm{ee}} \oplus \mathsf{Coker}(P^M) \otimes \mathsf{Coker}(P^N) \oplus \mathsf{Coker}(P^N) \otimes \mathsf{Coker}(P^M)$$

while $(M \vee N)_{\mathrm{e}}$ is the quotient of the coproduct $M_{\mathrm{e}} \vee N_{\mathrm{e}}$ in the category Nil by the following relations

$$\begin{aligned}
P^M(a) + y &= y + P^M(a), && a \in M_{\mathrm{ee}}, y \in N_{\mathrm{e}},\\
P^N(b) + x &= x + P^N(b), && b \in N_{\mathrm{ee}}, x \in M_{\mathrm{e}}.
\end{aligned}$$

Moreover, P and H of $M \vee N$ are given by

$$P(a+b+x_1\otimes y_1+y_2\otimes x_2)=P^M(a)+P^N(b)+[x_1,y_1]+[y_2,x_2],$$
$$H(x+y+[x_1,y_1])=H^M(x)+H^N(y)+x\otimes y+x_1\otimes y_1-y_1\otimes x_1$$

where $x,x_1,x_2\in M_{\mathrm{e}}$, $y,y_1,y_2\in N_{\mathrm{e}}$, $a\in M_{\mathrm{ee}}$, $b\in N_{\mathrm{ee}}$.

Let us also observe that one has a central extension of the form

$$0\to \mathsf{Coker}(P^M)\otimes \mathsf{Coker}(P^N)\to (M\vee N)_{\mathrm{e}}\to M_{\mathrm{e}}\times N_{\mathrm{e}}\to 0,$$

which implies the short exact sequence of square groups

$$0\to (\mathsf{Coker}(P^M)\otimes \mathsf{Coker}(P^N))^{\otimes}\xrightarrow{j} M\vee N\to M\times N\to 0. \tag{6}$$

Here $j_{\mathrm{e}}(x\otimes y)=[x,y]$ and $j_{\mathrm{ee}}(x_1\otimes y_1,x_2\otimes y_2)=x_1\otimes y_1+y_2\otimes x_2$.

Since the map P^M is surjective for $M=A^{\otimes}$, we obtain that for any abelian group A and any square group M one has

$$A^{\otimes}\vee M\cong A^{\otimes}\times M.$$

In particular for abelian groups A and B one gets

$$A^{\otimes}\vee B^{\otimes}\cong (A\oplus B)^{\otimes}.$$

5.7. **Free and projective square groups.** We need the following square group $\mathbb{Z}^Q$ defined by

$$(\mathbb{Z}^Q)_{\mathrm{e}}=\mathbb{Z}\oplus\mathbb{Z},\quad (\mathbb{Z}^Q)_{\mathrm{ee}}=\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z},$$

with the maps P and H given by $P(a,b,c)=(0,a+2b)$ and $H(m,n)=(m,n,\binom{m}{2})$. One easily shows that for a square group M one has the natural isomorphism

$$\mathsf{Hom}_{\mathsf{SG}}(\mathbb{Z}^Q,M)\cong M_{\mathrm{e}}.$$

For any set S we put

$$V(S):=\bigvee_{s\in S}\mathbb{Z}^Q.$$

It follows that

$$\mathsf{Hom}_{\mathsf{SG}}(V(S),M)\cong \mathsf{Hom}_{\mathsf{Sets}}(S,M_{\mathrm{e}}).$$

Thus the functor $V:\mathsf{Sets}\to\mathsf{SG}$ is left adjoint to the functor

$$\mathsf{SG}\to\mathsf{Sets},\quad M\mapsto M_{\mathrm{e}}.$$

Now we give the following explicate construction of $V(S)$. We consider three further copies of S, which are denoted respectively by HS, PHS and $HPHS$. For an element $s\in S$ the elements Hs, PHs, $HPHs$ correspond to s in these copies. Then we take

$$V(S)_{\mathrm{e}}=\langle S\rangle^{\mathrm{nil}}\times\mathbb{Z}[PHS],$$
$$V(S)_{\mathrm{ee}}=\mathbb{Z}[HS]\oplus\mathbb{Z}[S\times S]\oplus\mathbb{Z}[HPHS].$$

Moreover, H is the unique quadratic map with

$$\begin{aligned}
H(s) &= Hs,\\
H(PHs) &= HPHs,\\
(s \mid t)_H &= (s,t),\\
(s \mid PHt)_H &= 0 = (PHs \mid t)_H = (PHs \mid PHt)_H\,.
\end{aligned}$$

Here $s, t \in S$. The homomorphism P is given by

$$P(Hs) = PHs, \quad P(s,t) = [s,t], \quad P(HPHs) = 2PHs.$$

The fact that this is really isomorphic to $V(S)$ can be deduced either from Section 5.6 or directly by the universal property.

Properties of the functor $A \mapsto A^{\otimes}$ imply that the functor $\mathsf{Sets} \times \mathsf{Sets} \to \mathsf{SG}$ given by

$$(S,T) \mapsto V(S) \times (\mathbb{Z}[T])^{\otimes} \cong V(S) \vee (\mathbb{Z}[T])^{\otimes}$$

is left adjoint to the forgetful functor

$$\mathsf{SG} \to \mathsf{Sets} \times \mathsf{Sets}, \quad M \mapsto (M_{\mathrm{e}}, M_{\mathrm{ee}}).$$

A square group is called *free* if it is isomorphic to $V(S) \times (\mathbb{Z}[T])^{\otimes}$. A square group is called *projective* provided it has the familiar lifting property with respect to epimorphisms of square groups. Any free square group is projective and any projective square group is a retract of a free square group.

Lemma 20. *For any square group M there exists an epimorphism $F \twoheadrightarrow M$, where F is free.*

Proof. One can take $F = V(M_{\mathrm{e}}) \times (\mathbb{Z}[M_{\mathrm{ee}}])^{\otimes}$ with the morphism adjoint to $(\mathsf{Id}_{M_{\mathrm{e}}}, \mathsf{Id}_{M_{\mathrm{ee}}})$. $\square$

5.8. **Simplicial objects in the category SG.** Let $\mathsf{Simpl}(\mathsf{SG})$ be the category of simplicial objects in the category SG of square groups. Any such simplicial object X defines two simplicial groups X_{e} and X_{ee} as well as a morphism $P : X_{\mathrm{ee}} \to X_{\mathrm{e}}$ of simplicial groups. The map H yields a morphism of simplicial sets $H : X_{\mathrm{e}} \to X_{\mathrm{ee}}$. If one passes to homotopy groups, then one obtains groups $\pi_i(X_{\mathrm{e}})$ and $\pi_i(X_{\mathrm{ee}})$ together with induced homomorphisms $P : \pi_i(X_{\mathrm{ee}}) \to \pi_i(X_{\mathrm{e}})$, $i \geqslant 0$. The map H yields a quadratic map $H : \pi_0(X_{\mathrm{e}}) \to \pi_0(X_{\mathrm{ee}})$ and homomorphisms $H : \pi_i(X_{\mathrm{e}}) \to \pi_i(X_{\mathrm{ee}})$ for $i \geqslant 1$. It is clear that the equation $PHP = 2P$ still holds for induced maps. It follows that for each $i \geqslant 1$ one obtains a well-defined abelian square group

$$\pi_i X \in \mathsf{Ab}(\mathsf{SG}), i \geqslant 1$$

with

$$(\pi_i X)_{\mathrm{e}} = \pi_i(X_{\mathrm{e}}), \quad (\pi_i X)_{\mathrm{ee}} = \pi_i(X_{\mathrm{ee}}).$$

We also have a well-defined square group

$$\pi_0(X) \in \mathsf{SG},$$

since all equations defining a square group hold in X_0, the zero component of X, and therefore they remain true in the quotient $\pi_0(X)$.

By 5.7 the category SG satisfies all conditions of Theorem 4 Section 4, Chapter 2 [23]. Hence the category $\mathsf{Simpl}(\mathsf{SG})$ of simplicial objects of SG possesses a closed model category structure, where a morphism f is a weak equivalence (resp. fibration) provided f_{e} and f_{ee} are weak equivalences (resp. fibrations) of underlying simplicial sets. According to [23] cofibrations are retracts of free maps as they are defined in [23]. Equivalently, a map $f : X \to Y$ is weak equivalence if and only if the induced map of square groups $f_i : \pi_i(X) \to \pi_i(Y)$ is an isomorphism for all $i \geqslant 0$.

5.9. **Pre-square groups.**

Definition 21. A *pre-square group* consists of a diagram

$$M = \left(M_{\mathrm{e}} \times M_{\mathrm{e}} \xrightarrow{\{-,-\}} M_{\mathrm{ee}} \xrightarrow{T} M_{\mathrm{ee}} \xrightarrow{P} M_{\mathrm{e}}\right).$$

Here M_{ee} is an abelian group and T is a homomorphism with $T^2 = \mathsf{Id}$ and M_{e} is a group written additively, P is a homomorphism and $\{-,-\}$ is a bilinear map, that is $\{x+y,z\} = \{x,z\}+\{y,z\}$ and $\{x,y+z\} = \{x,y\}+\{x,z\}$, for all $x,y,z \in M_{\mathrm{e}}$. Moreover one requires the following identities:

(a) $PT = P$,
(b) $T\{x,y\} + \{y,x\} = 0$, $x,y \in M_{\mathrm{e}}$,
(c) $P\{x,y\} = -x-y+x+y$, $x,y \in M_{\mathrm{e}}$,
(d) $\{x,Pa\} = 0$, $x \in M_{\mathrm{e}}$, $a \in M_{\mathrm{ee}}$.

It follows from (b) that one has $\{Pa,x\} = 0$. It follows from (c) and (d) that Pa lies in the center of M_{e}. Thus $\mathsf{Coker}(P)$ is well-defined and by (c) it is an abelian group. It follows that M_{e} is a group of nilpotence class 2. Bilinearity of the bracket together with (d) shows that there is a well-defined homomorphism

$$\{-,-\} : \mathsf{Coker}(P) \otimes \mathsf{Coker}(P) \to M_{\mathrm{ee}}.$$

We let PSG denote the category of pre-square groups. It is clear that the full subcategory of PSG consisting of pre-square groups with trivial $M_{\mathrm{ee}} = 0$ is equivalent to the category of abelian groups. In what follows we identify abelian groups with such pre-square groups.

Thanks to Proposition 14, for any square group the following object

$$\wp(M) = (M_{\mathrm{e}}, M_{\mathrm{ee}}, T = HP - \mathsf{Id}, (-,-)_H, P)$$

is a pre-square group. Thus we obtain the forgetful functor

$$\wp : \mathsf{SG} \to \mathsf{PSG}.$$

Comparing the definitions we immediately obtain the following easy, but useful result.

Lemma 22. *Let M be a pre-square group and let $H : M_{\mathrm{e}} \to M_{\mathrm{ee}}$ be a map. Then $(M_{\mathrm{e}}, M_{\mathrm{ee}}, P, H)$ is a square group with $\wp(M_{\mathrm{e}}, M_{\mathrm{ee}}, P, H) = M$ iff $\{x,y\} = (x \mid y)_H$ and $\mathsf{Id} + T = HP$.*

The following Lemma is an immediate consequence of Proposition 14.

Lemma 23. *For any square group M and any integer n, there is a morphism of pre-square groups*

$$n^* = (n^*)_M : \wp(M) \to \wp(M)$$

which on ee*-level is the multiplication by n^2, while on* e*-level it is given by*

$$x \mapsto nx + \binom{n}{2} PH(x).$$

6. The tensor product as a monoidal structure of SG

In this section we prove Theorem 3.

6.1. **The tensor product of a pre-square group and a square group.** Before considering the tensor product of square groups as a square group for technical reasons we construct first the bifunctor

$$\underline{\circledcirc} : \mathsf{PSG} \times \mathsf{SG} \to \mathsf{PSG}.$$

Definition 24. Let M be a pre-square group and N be a square group, then $M \underline{\circledcirc} N$ is the pre-square group, which on the ee-level is given by

$$(M \underline{\circledcirc} N)_{\mathrm{ee}} = M_{\mathrm{ee}} \otimes N_{\mathrm{ee}}.$$

Moreover $(M \underline{\circledcirc} N)_{\mathrm{e}}$ is generated by elements of the form $x \underline{\circledcirc} y$ for $x \in M_{\mathrm{e}}$, $y \in N_{\mathrm{e}}$ and $a \bar{\otimes} b$ for $a \in M_{\mathrm{ee}}$, $b \in N_{\mathrm{ee}}$, subject to the relations

1. the symbol $a \bar{\otimes} b$ is bilinear and central in $(M \underline{\circledcirc} N)_{\mathrm{e}}$,
2. $x \underline{\circledcirc} (y_1 + y_2) = x \underline{\circledcirc} y_1 + x \underline{\circledcirc} y_2$,
3. $(x_1 + x_2) \underline{\circledcirc} y = x_1 \underline{\circledcirc} y + x_2 \underline{\circledcirc} y + \{x_2, x_1\} \bar{\otimes} H(y)$,
4. $x \underline{\circledcirc} P(b) = \{x, x\} \bar{\otimes} b$,
5. $P(a) \underline{\circledcirc} y = a \bar{\otimes} \Delta(y)$,
6. $T(a) \bar{\otimes} T(b) = -a \bar{\otimes} b$.

The homomorphism

$$P : (M \underline{\circledcirc} N)_{\mathrm{ee}} \to (M \underline{\circledcirc} N)_{\mathrm{e}}$$

is given by

$$P(a \otimes b) = a \bar{\otimes} b.$$

The involution on $(M \underline{\circledcirc} N)_{\mathrm{ee}}$ is given by

$$T(a \otimes b) = -T(a) \otimes T(b),$$

while the bracket is given by

$$\{x \underline{\circledcirc} y, x' \underline{\circledcirc} y'\} = \{x, x'\} \otimes (y \mid y')_H,$$

$$\{u, a \bar{\otimes} b\} = 0 = \{a \bar{\otimes} b, u\},$$

where $u \in (M \underline{\circledcirc} N)_{\mathrm{e}}$.

We now show that $M\underline{\circledcirc}N$ is a well-defined pre-square group satisfying the identities in Definition 21. Identity a) follows from the identity 6) of the definition of $M\underline{\circledcirc}N$, while d) is a direct consequence of the description of P. Identity b) can be verified as follows

$$\begin{aligned} T\{x\underline{\circledcirc}y, x'\underline{\circledcirc}y'\} =& T(\{x,x'\}\otimes(y\mid y')_H) \\ =& -T(\{x,x'\})\otimes T((y\mid y')_H) \\ =& -\{x',x\}\otimes(y'\mid y)_H \\ =& -\{x'\otimes y', x\otimes y\}. \end{aligned}$$

It remains to check identity c). Since $P\{x\underline{\circledcirc}y, x'\underline{\circledcirc}y'\} = \{x,x'\}\bar{\otimes}(y\mid y')_H$, this must be equal to $-x\underline{\circledcirc}y - x'\underline{\circledcirc}y' + x\underline{\circledcirc}y + x'\underline{\circledcirc}y'$. Consider

$$\begin{aligned} (x'+x)\underline{\circledcirc}(y+y') =& (x'+x)\underline{\circledcirc}y + (x'+x)\underline{\circledcirc}y' \\ =& x'\underline{\circledcirc}y + x\underline{\circledcirc}y + \{x,x'\}\bar{\otimes}H(y) + x'\underline{\circledcirc}y' + x\underline{\circledcirc}y' + \{x,x'\}\bar{\otimes}H(y') \\ =& x'\underline{\circledcirc}y + x\underline{\circledcirc}y + x'\underline{\circledcirc}y' + x\underline{\circledcirc}y' + \{x,x'\}\bar{\otimes}(H(y)+H(y')) \\ =& x'\underline{\circledcirc}y + x\underline{\circledcirc}y + x'\underline{\circledcirc}y' + x\underline{\circledcirc}y' + \{x,x'\}\bar{\otimes}(H(y+y')) - \{x,x'\}\bar{\otimes}(y|y')_H. \end{aligned}$$

On the other hand the same expression expands to

$$\begin{aligned} (x'+x)\underline{\circledcirc}(y+y') &= x'\underline{\circledcirc}(y+y') + x\underline{\circledcirc}(y+y') + \{x,x'\}\bar{\otimes}H(y+y') \\ &= x'\underline{\circledcirc}y + x'\underline{\circledcirc}y' + x\underline{\circledcirc}y + x\underline{\circledcirc}y' + \{x,x'\}\bar{\otimes}H(y+y'). \end{aligned}$$

Comparing these expressions gives

$$-x\underline{\circledcirc}y - x'\underline{\circledcirc}y' + x\underline{\circledcirc}y + x'\underline{\circledcirc}y' = \{x,x'\}\bar{\otimes}(y|y')_H \tag{7}$$

which is the equality we need. Thus we have constructed a well-defined tensor product $\mathsf{PSG}\times\mathsf{SG}\to\mathsf{PSG}$.

Lemma 25. *Let M be a pre-square group and N be a square group. Then one has the following identities in $M\underline{\circledcirc}N$.*

(1) $\{x_2,x_1\}\bar{\otimes}b = \{x_1,x_2\}\bar{\otimes}Tb$,
(2) $(nx)\underline{\circledcirc}y = x\underline{\circledcirc}(ny + \binom{n}{2}PHy)$,
(3) $[x_1,x_2]\underline{\circledcirc}y = \{x_1,x_2\}\bar{\otimes}(HPHy - 2Hy)$.

Here $x,x_1,x_2\in M_{\mathrm{e}}$, $y\in N_{\mathrm{e}}$, $b\in M_{\mathrm{ee}}$ and $n\in\mathbb{Z}$.

Proof. We have

$$\{x_2,x_1\}\bar{\otimes}b = (-T\{x_1,x_2\})\bar{\otimes}TTb = \{x_1,x_2\}\bar{\otimes}Tb$$

and (1) is proved. For a given $y\in N_{\mathrm{e}}$ consider the map $f: M_{\mathrm{e}}\to M_{\mathrm{ee}}$ defined by $f(x) = x\underline{\circledcirc}y$. Then f is quadratic, with cross-effect given by

$$(x_1\mid x_2)_f = \{x_2,x_1\}\bar{\otimes}H(y).$$

Thus by identity (4) in Section 4 we have

$$(nx)\underline{\circledcirc}y = f(nx) = nf(x) + \binom{n}{2}(x\mid x)_f = n(x\underline{\circledcirc}y) + \binom{n}{2}(\{x,x\})\bar{\otimes}H(y).$$

Since $\underline{\circledcirc}$ is linear with respect to the second variable, (2) follows. Next we use identity (5) in section 4 to get

$$[x_1,x_2]\underline{\circledcirc}y = f([x_1,x_2]) = (x_1 \mid x_2)_f - (x_2 \mid x_1)_f = \{x_2,x_1\}\bar{\otimes}Hy - \{x_1,x_2\}\bar{\otimes}Hy.$$

By (1) in Lemma 25 we get

$$[x_1,x_2]\underline{\circledcirc}y = \{x_1,x_2\}\bar{\otimes}(THy - Hy).$$

But $TH = HPH - H$ and the result follows. □

6.2. **The tensor product of square groups.** Assume now that M and N are square groups. We have to show that $H_{M\underline{\circledcirc}N}$ is well-defined and $M\underline{\circledcirc}N$ is in fact a square group. To prove the first assertion we show that the conditions of Lemma 13 are indeed satisfied, namely, that the quadratic map H (when considered as a quadratic map from a free nil$_2$-group to $(M\otimes N)_{\mathrm{ee}}$) respects the relations from the definition. First, bilinearity of $a\bar{\otimes}b$ is respected since $H(a\bar{\otimes}b) = a\otimes b - T(a)\otimes T(b)$ is bilinear and the cross-effect of H vanishes on all elements of the form $a\bar{\otimes}b$. Centrality of $a\bar{\otimes}b$ is trivially respected as the values are taken in an abelian group. Next, the relation (2) in Definition 2 is respected since

$$\begin{aligned}
&H(x\underline{\circledcirc}y_1 + x\underline{\circledcirc}y_2 - x\underline{\circledcirc}(y_1+y_2))\\
&\quad=H(x\underline{\circledcirc}y_1 + x\underline{\circledcirc}y_2) + H(-x\underline{\circledcirc}(y_1+y_2)) + (x\underline{\circledcirc}y_1 + x\underline{\circledcirc}y_2 | - x\underline{\circledcirc}(y_1+y_2))_H\\
&\quad=H(x\underline{\circledcirc}y_1) + H(x\underline{\circledcirc}y_2) + (x\underline{\circledcirc}y_1|x\underline{\circledcirc}y_2)_H - H(x\underline{\circledcirc}(y_1+y_2))\\
&\qquad+ (x\underline{\circledcirc}(y_1+y_2)|x\underline{\circledcirc}(y_1+y_2))_H + (x\underline{\circledcirc}y_1 + x\underline{\circledcirc}y_2| - x\underline{\circledcirc}(y_1+y_2))_H\\
&\quad=(x|x)_H\otimes H(y_1) + H(x)\otimes\Delta(y_1) + (x|x)_H\otimes H(y_2) + H(x)\otimes\Delta(y_2)\\
&\qquad+ (x|x)_H\otimes(y_1|y_2)_H - (x|x)_H\otimes H(y_1+y_2)\\
&\qquad- H(x)\otimes\Delta(y_1+y_2) + (x|x)_H\otimes(y_1+y_2|y_1+y_2)_H\\
&\qquad- (x\underline{\circledcirc}y_1|x\underline{\circledcirc}(y_1+y_2))_H - (x\underline{\circledcirc}y_2|x\underline{\circledcirc}(y_1+y_2))_H\\
&\quad=(x|x)_H\otimes H(y_1) + (x|x)_H\otimes H(y_2) + (x|x)_H\otimes(y_1|y_2)_H\\
&\qquad- (x|x)_H\otimes(H(y_1)+H(y_2)+(y_1|y_2)_H)\\
&\qquad+ (x|x)_H\otimes(y_1|y_1)_H + (x|x)_H\otimes(y_1|y_2)_H\\
&\qquad+ (x|x)_H\otimes(y_2|y_1)_H + (x|x)_H\otimes(y_2|y_2)_H\\
&\qquad- (x|x)_H\otimes(y_1|y_1+y_2)_H - (x|x)_H\otimes(y_2|y_1+y_2)_H\\
&\qquad= 0.
\end{aligned}$$

For the relation (3) Definition 2 we have

$$\begin{aligned}
&H(x_1\underline{\circledcirc}y + x_2\underline{\circledcirc}y + (x_2|x_1)_H\bar{\otimes}H(y) - (x_1+x_2)\underline{\circledcirc}y)\\
&\quad=H(x_1\underline{\circledcirc}y + x_2\underline{\circledcirc}y + (x_2|x_1)_H\bar{\otimes}H(y)) + H(-(x_1+x_2)\underline{\circledcirc}y)\\
&\qquad+(x_1\underline{\circledcirc}y + x_2\underline{\circledcirc}y + (x_2|x_1)_H\bar{\otimes}H(y)| - (x_1+x_2)\underline{\circledcirc}y)_H\\
&\quad=H(x_1\underline{\circledcirc}y + x_2\underline{\circledcirc}y) + H((x_2|x_1)_H\bar{\otimes}H(y))\\
&\qquad- H((x_1+x_2)\underline{\circledcirc}y) + ((x_1+x_2)\underline{\circledcirc}y|(x_1+x_2)\underline{\circledcirc}y)_H\\
&\qquad- (x_1\underline{\circledcirc}y|(x_1+x_2)\underline{\circledcirc}y)_H - (x_2\underline{\circledcirc}y|(x_1+x_2)\underline{\circledcirc}y)_H\\
&\quad=H(x_1\underline{\circledcirc}y) + H(x_2\underline{\circledcirc}y) + (x_1\underline{\circledcirc}y|x_2\underline{\circledcirc}y)_H\\
&\qquad+ (x_2|x_1)_H\otimes H(y) - T((x_2|x_1)_H)\otimes TH(y)\\
&\qquad- (x_1+x_2|x_1+x_2)_H\otimes H(y) - H(x_1+x_2)\otimes\Delta(y)\\
&\qquad+ (x_1+x_2|x_1+x_2)_H\otimes(y|y)_H\\
&\qquad- (x_1|x_1+x_2)_H\otimes(y|y)_H - (x_2|x_1+x_2)_H\otimes(y|y)_H\\
&\quad=(x_1|x_1)_H\otimes H(y) + H(x_1)\otimes\Delta(y) + (x_2|x_2)_H\otimes H(y) + H(x_2)\otimes\Delta(y)\\
&\qquad+ (x_1|x_2)_H\otimes(y|y)_H + (x_2|x_1)_H\otimes H(y) + (x_1|x_2)_H\otimes TH(y)\\
&\qquad- (x_1+x_2|x_1+x_2)_H\otimes H(y) - (H(x_1)+H(x_2)+(x_1|x_2)_H)\otimes\Delta(y)\\
&\quad=(x_1|x_2)_H\otimes(y|y)_H + (x_1|x_2)_H\otimes TH(y)\\
&\qquad- (x_1|x_2)_H\otimes H(y) - (x_1|x_2)_H\otimes\Delta(y)\\
&\quad=0.
\end{aligned}$$

Next for the relation (4) Definition 2 we check

$$\begin{aligned}
H(x\underline{\circledcirc}P(b) - (x|x)_H\bar{\otimes}b) &= H(x\underline{\circledcirc}P(b)) - H((x|x)_H\bar{\otimes}b)\\
&= (x|x)_H\otimes HP(b) + H(x)\otimes\Delta P(b) - (x|x)_H\otimes b + T((x|x)_H)\otimes T(b)\\
&= (x|x)_H\otimes(HP(b) - b - T(b)) = 0
\end{aligned}$$

and for (5)

$$\begin{aligned}
H(P(a)\underline{\circledcirc}y - a\bar{\otimes}\Delta(y)) &= H(P(a)\underline{\circledcirc}y) - H(a\bar{\otimes}\Delta(y))\\
&= (P(a)|P(a))_H\otimes H(y) + HP(a)\otimes\Delta(y) - a\otimes\Delta(y) + T(a)\otimes T\Delta(y)\\
&= (HP(a) - a - T(a))\otimes\Delta(y) = 0.
\end{aligned}$$

Finally the relation (6) Definition 2 is respected since

$$\begin{aligned}
H(a\bar{\otimes}b + T(a)\bar{\otimes}T(b)) &= H(a\bar{\otimes}b) + H(T(a)\bar{\otimes}T(b))\\
&= a\otimes b - T(a)\otimes T(b) + T(a)\otimes T(b) - TT(a)\otimes TT(b) = 0.
\end{aligned}$$

Moreover we have to show that identities of square groups hold for $M\underline{\circledcirc}N$. But we have already proved that it is a pre-square group, thus by Lemma 22 we have only to check the identity $\mathsf{Id} + T = HP$, which holds because

$$HP(a \otimes b) = H(a \bar{\otimes} b) = a \otimes b - T(a) \otimes T(b)) = a \bar{\otimes} b + T(a \otimes b).$$

Thus we get the well-defined tensor product $-\underline{\circledcirc}- : \mathsf{SG} \times \mathsf{SG} \to \mathsf{SG}$.

Lemma 26. *Let M and N be square groups. Then one has the following identity in $M \underline{\circledcirc} N$.*

$$HPH(x) \bar{\otimes} b = H(x) \bar{\otimes} (b - Tb).$$

Here $x \in M_{\mathrm{e}}$ and $b \in M_{\mathrm{ee}}$.

Proof. We have

$$(HPHx) \bar{\otimes} b = (Hx + THx) \bar{\otimes} b = (Hx) \bar{\otimes} b + (THx) \bar{\otimes} b = (Hx) \bar{\otimes} b - (Hx) \bar{\otimes} Tb$$

and the result follows.

□

6.3. **Associativity.** Our next goal is to show that the tensor product in SG defines a symmetric monoidal structure. To construct associativity isomorphisms we introduce the triple tensor product $A \underline{\circledcirc} B \underline{\circledcirc} C$; we will then construct isomorphisms of this object to $(A \underline{\circledcirc} B) \underline{\circledcirc} C$ and $A \underline{\circledcirc} (B \underline{\circledcirc} C)$.

We define $(A \underline{\circledcirc} B \underline{\circledcirc} C)_{\mathrm{e}}$ by generators of the form $x \underline{\circledcirc} y \underline{\circledcirc} z$ for $x \in A_{\mathrm{e}}$, $y \in B_{\mathrm{e}}$, $z \in C_{\mathrm{e}}$ and $a \bar{\otimes} b \bar{\otimes} c$ for $a \in A$, $b \in B$, $c \in C$, subject to the relations

1. $a \bar{\otimes} b \bar{\otimes} c$ is central and trilinear;
2. $x \underline{\circledcirc} y \underline{\circledcirc} (z + z') = x \underline{\circledcirc} y \underline{\circledcirc} z + x \underline{\circledcirc} y \underline{\circledcirc} z'$;
3. $x \underline{\circledcirc} (y + y') \underline{\circledcirc} z = x \underline{\circledcirc} y \underline{\circledcirc} z + x \underline{\circledcirc} y' \underline{\circledcirc} z + (x|x)_H \bar{\otimes} (y'|y)_H \bar{\otimes} H(z)$;
4. $(x + x') \underline{\circledcirc} y \underline{\circledcirc} z = x \underline{\circledcirc} y \underline{\circledcirc} z + x' \underline{\circledcirc} y \underline{\circledcirc} z + (x'|x)_H \bar{\otimes} (y|y)_H \bar{\otimes} H(z) + (x'|x)_H \bar{\otimes} H(y) \bar{\otimes} \Delta(z)$;
5. $P(a) \underline{\circledcirc} y \underline{\circledcirc} z = a \bar{\otimes} \Delta(y) \bar{\otimes} \Delta(z)$;
6. $x \underline{\circledcirc} P(b) \underline{\circledcirc} z = (x|x)_H \bar{\otimes} b \bar{\otimes} \Delta(z)$;
7. $x \underline{\circledcirc} y \underline{\circledcirc} P(c) = (x|x)_H \bar{\otimes} (y|y)_H \bar{\otimes} c$;
8. $T(a) \bar{\otimes} T(b) \bar{\otimes} T(c) = a \bar{\otimes} b \bar{\otimes} c$.

Moreover we define $(A \underline{\circledcirc} B \underline{\circledcirc} C)_{\mathrm{ee}} = A \otimes B \otimes C$ and $P(a \otimes b \otimes c) = a \bar{\otimes} b \bar{\otimes} c$. Finally we let H be the unique quadratic map satisfying

$$H(x \underline{\circledcirc} y \underline{\circledcirc} z) = (x|x)_H \bar{\otimes} (y|y)_H \bar{\otimes} H(z) + (x|x)_H \bar{\otimes} H(y) \bar{\otimes} \Delta(z) + H(x) \bar{\otimes} \Delta(y) \Delta(z)$$

with cross-effect equal to

$$\rho : \mathsf{Coker}(P_{A \underline{\circledcirc} B \underline{\circledcirc} C}) \otimes \mathsf{Coker}(P_{A \underline{\circledcirc} B \underline{\circledcirc} C}) \xrightarrow{\rho} A_{\mathrm{ee}} \otimes B_{\mathrm{ee}} \otimes C_{\mathrm{ee}}$$

given by $\rho(\bar{a}_1 \otimes \bar{b}_1 \otimes \bar{c}_1 \otimes \bar{a}_2 \otimes \bar{b}_2 \otimes \bar{c}_2) = (a_1|a_2)_H \otimes (b_1|b_2)_H \otimes (c_1|c_2)_H$. Here we use the identification

$$\begin{aligned}&\mathsf{Coker}(P_{A \underline{\circledcirc} B \underline{\circledcirc} C}) \otimes \mathsf{Coker}(P_{A \underline{\circledcirc} B \underline{\circledcirc} C}) \\ &\qquad = \mathsf{Coker}(P_A) \otimes \mathsf{Coker}(P_B) \otimes \mathsf{Coker}(P_C) \otimes \mathsf{Coker}(P_A) \otimes \mathsf{Coker}(P_B) \otimes \mathsf{Coker}(P_C)\end{aligned}$$

Again by Lemma 13 such H exists and is unique and the same argument as in Section 6.2 shows that $A\underline{\circledcirc}B\underline{\circledcirc}C$ is a well-defined square group. By the same methods one shows that there is the unique morphism of square groups

$$\alpha = \alpha_{A,B,C} : ((A\underline{\circledcirc}B)\underline{\circledcirc}C) \to A\underline{\circledcirc}B\underline{\circledcirc}C$$

which is the canonical isomorphism $(A_{\mathrm{ee}} \otimes B_{\mathrm{ee}}) \otimes C_{\mathrm{ee}} \to A_{\mathrm{ee}} \otimes B_{\mathrm{ee}} \otimes C_{\mathrm{ee}}$ on the ee-level, while on the e-level it satisfies the identities

$$\begin{aligned} \alpha((x\underline{\circledcirc}y)\underline{\circledcirc}z) &= x\underline{\circledcirc}y\underline{\circledcirc}z, \\ \alpha((a\bar{\otimes}b)\underline{\circledcirc}z) &= a\bar{\otimes}b\bar{\otimes}\Delta(z), \\ \alpha((a \otimes b)\bar{\otimes}c) &= a\bar{\otimes}b\bar{\otimes}c. \end{aligned}$$

This is an isomorphism with inverse given by

$$\begin{aligned} \alpha^{-1}(x\underline{\circledcirc}y\underline{\circledcirc}z) &= (x\underline{\circledcirc}y)\underline{\circledcirc}z, \\ \alpha^{-1}(a\bar{\otimes}b\bar{\otimes}c) &= (a \otimes b)\bar{\otimes}c. \end{aligned}$$

Similarly there exists a unique morphism of square groups

$$\beta = \beta_{A,B,C} : A\underline{\circledcirc}(B\underline{\circledcirc}C) \to A\underline{\circledcirc}B\underline{\circledcirc}C$$

which is the canonical isomorphism $A_{\mathrm{ee}} \otimes B_{\mathrm{ee}} \otimes C_{\mathrm{ee}} \to A_{\mathrm{ee}} \otimes (B_{\mathrm{ee}} \otimes C_{\mathrm{ee}})$ on the ee-level and on the e-level satisfies

$$\begin{aligned} \beta(x\underline{\circledcirc}(y\underline{\circledcirc}z)) &= x\underline{\circledcirc}y\underline{\circledcirc}z, \\ \beta(x\underline{\circledcirc}(b\bar{\otimes}c)) &= (x|x)_H\bar{\otimes}b\bar{\otimes}c, \\ \beta(a\bar{\otimes}(b \otimes c)) &= a\bar{\otimes}b\bar{\otimes}c. \end{aligned}$$

This is an isomorphism with inverse given by

$$\begin{aligned} \beta^{-1}(x\underline{\circledcirc}y\underline{\circledcirc}z) &= x\underline{\circledcirc}(y\underline{\circledcirc}z), \\ \beta^{-1}(a\bar{\otimes}b\bar{\otimes}c) &= a\bar{\otimes}(b \otimes c). \end{aligned}$$

From these isomorphisms one obtains that $\underline{\circledcirc}$ is indeed associative with associativity isomorphisms given as in Theorem 3. The last statement of Theorem 3 follows from the fact that one has

$$x\underline{\circledcirc}(b\bar{\otimes}c) = x\underline{\circledcirc}P(b \otimes c) = (x|x)_H\bar{\otimes}(b \otimes c)$$

and this corresponds to $((x|x)_H \otimes b)\bar{\otimes}c$.

6.4. **The pentagon axiom.** The well-known pentagon axiom for monoidal categories claims that two natural ways from $((A\underline{\circledcirc}B)\underline{\circledcirc}C)\underline{\circledcirc}D$ to $A\underline{\circledcirc}(B\underline{\circledcirc}(C\underline{\circledcirc}D))$ are equal. That is the following diagram commutes.

$$\begin{tikzcd}[column sep=small]
 & & (A\underline{\circledcirc}B)\underline{\circledcirc}(C\underline{\circledcirc}D) \arrow[drr, "\alpha_{A,B,C\underline{\circledcirc}D}"] & & \\
((A\underline{\circledcirc}B)\underline{\circledcirc}C)\underline{\circledcirc}D \arrow[urr, "\alpha_{A\underline{\circledcirc}B,C,D}"] \arrow[dr, "\alpha_{A,B,C}\underline{\circledcirc}D"'] & & & & A\underline{\circledcirc}(B\underline{\circledcirc}(C\underline{\circledcirc}D)) \\
 & (A\underline{\circledcirc}(B\underline{\circledcirc}C))\underline{\circledcirc}D \arrow[rr, "\alpha_{A,B\underline{\circledcirc}C,D}"'] & & A\underline{\circledcirc}((B\underline{\circledcirc}C)\underline{\circledcirc}D) \arrow[ur, "A\underline{\circledcirc}\alpha_{B,C,D}"'] &
\end{tikzcd}$$

To show this statement in our circumstances let us observe that $(((A\underline{\circledcirc}B)\underline{\circledcirc}C)\underline{\circledcirc}D)_{\mathrm{e}}$ is generated by elements of the form $((x\underline{\circledcirc}y)\underline{\circledcirc}z)\underline{\circledcirc}w$, $((a\bar{\otimes}b)\underline{\circledcirc}z)\underline{\circledcirc}w$, $(((a\otimes b)\bar{\otimes}c)\underline{\circledcirc}w$ and $(((a\otimes b)\otimes c)\bar{\otimes}d$. Then the statement is clear for elements of the form $((x\underline{\circledcirc}y)\underline{\circledcirc}z)\underline{\circledcirc}w$ and $(((a\otimes b)\otimes c)\bar{\otimes}d$. It is also straightforward to check that both ways carry $(((a\otimes b)\bar{\otimes}c)\underline{\circledcirc}w$ to $a\bar{\otimes}(b\otimes(c\otimes\Delta(w)))$. It remains to consider $((a\bar{\otimes}b)\underline{\circledcirc}z)\underline{\circledcirc}w$. By one way it goes to $a\bar{\otimes}(b\otimes(\Delta(z)\otimes\Delta(w))$, while by the second way it goes to $a\bar{\otimes}(b\otimes\Delta(z\underline{\circledcirc}w))$. Thus the pentagon axiom follows from the following Lemma.

Lemma 27. *For the homomorphism*

$$\Delta_{M\underline{\circledcirc}N}:\mathsf{Coker}(P_{M\underline{\circledcirc}N})=\mathsf{Coker}(P_M)\otimes\mathsf{Coker}(P_N)\to(M\underline{\circledcirc}N)_{\mathrm{ee}}=M_{\mathrm{ee}}\otimes N_{\mathrm{ee}}$$

one has

$$\Delta_{M\underline{\circledcirc}N}=\Delta_M\otimes\Delta_N.$$

Proof. Since $a\bar{\otimes}b=P(a\otimes b)$ we have $\Delta(a\bar{\otimes}b)=0$. On the other hand

$$\begin{aligned}
\Delta(x\underline{\circledcirc}y)=&(x\underline{\circledcirc}y|x\underline{\circledcirc}y)_H-H(x\underline{\circledcirc}y)+TH(x\underline{\circledcirc}y)\\
=&(x|x)_H\otimes(y|y)_H-(x|x)_H\otimes H(y)-H(x)\otimes\Delta(y)\\
&+T((x|x)_H\otimes H(y)+H(x)\otimes\Delta(y))\\
=&(x|x)_H\otimes(y|y)_H-(x|x)_H\otimes H(y)-H(x)\otimes((y|y)_H-H(y)+TH(y))\\
&-T((x|x)_H)\otimes TH(y)-TH(x)\otimes T\Delta(y)\\
=&(x|x)_H\otimes(y|y)_H-(x|x)_H\otimes H(y)-H(x)\otimes(y|y)_H\\
&+H(x)\otimes H(y)-H(x)\otimes TH(y)+(x|x)_H\otimes TH(y)+TH(x)\otimes\Delta(y)\\
=&(x|x)_H\otimes(y|y)_H-(x|x)_H\otimes H(y)-H(x)\otimes(y|y)_H\\
&+H(x)\otimes H(y)-H(x)\otimes TH(y)\\
&+(x|x)_H\otimes TH(y)+TH(x)\otimes((y|y)_H-H(y)+TH(y))
\end{aligned}$$

and

$$\begin{aligned}
\Delta(x)\otimes\Delta(y)=&((x|x)_H-H(x)+TH(x))\otimes((y|y)_H-H(y)+TH(y))\\
=&(x|x)_H\otimes(y|y)_H-H(x)\otimes(y|y)_H+TH(x)\otimes(y|y)_H-(x|x)_H\otimes H(y)\\
&+H(x)\otimes H(y)-TH(x)\otimes H(y)+(x|x)_H\otimes TH(y)\\
&-H(x)\otimes TH(y)+TH(x)\otimes TH(y)
\end{aligned}$$

which coincides with $\Delta(x\underline{\circledcirc}y)$. $\square$

6.5. **Unit object for $\underline{\circledcirc}$.** We start to check that $\mathbb{Z}_{\mathrm{nil}}$ has the unit object property, where $(\mathbb{Z}_{\mathrm{nil}})_{\mathrm{e}} = (\mathbb{Z}_{\mathrm{nil}})_{\mathrm{ee}} = \mathbb{Z}$, $P = 0$ and $H(n) = \binom{n}{2}$. For any square group A we define

$$\iota(A)_{\mathrm{e}} : (\mathbb{Z}_{\mathrm{nil}}\underline{\circledcirc}A)_{\mathrm{e}} \to A_{\mathrm{e}}$$

by

$$\iota(A)_{\mathrm{e}}(n\underline{\circledcirc}x) = nx + \binom{n}{2}PH(x)$$

for $n \in \mathbb{Z}$, $x \in A_{\mathrm{e}}$ and

$$\iota(A)_{\mathrm{e}}(n\bar{\otimes}a) = nP(a).$$

Let us show that $\iota(A)_{\mathrm{e}}$ respects all relations of Definition 2. The relation (1) of Definition 2 is clear, because P is a homomorphism with values in the center of A_{e}. We have

$$\iota(A)_{\mathrm{e}}(n\underline{\circledcirc}(x_1 + x_2)) = n(x_1 + x_2) + \binom{n}{2}PH(x_1 + x_2).$$

Since $n(x_1 + x_2) = nx_1 + nx_2 - \binom{n}{2}[x_1, x_2]$ and $PH(x_1 + x_2) = PH(x_1) + PH(x_2) + [x_1, x_2]$, the relation (2) of Definition 2 follows. We also have

$$\begin{aligned}
\iota(A)_{\mathrm{e}}((n_1 + n_2)\underline{\circledcirc}x) &= (n_1 + n_2)x + \binom{n_1 + n_2}{2}PH(x)\\
&= n_1x + n_2x + \binom{n_1}{2}PH(x) + \binom{n_2}{2}PH(x) + n_1n_2PH(x)\\
&= \iota(A)_{\mathrm{e}}(n_1\underline{\circledcirc}x) + \iota(A)_{\mathrm{e}}(n_2\underline{\circledcirc}x) + \iota(A)_{\mathrm{e}}(n_1n_2\bar{\otimes}H(x))
\end{aligned}$$

and the relation (3) of Definition 2 follows, because for $H(n) = \binom{n}{2}$ one has $(n_1, n_2)_H = n_1n_2$. Similarly, we have

$$\begin{aligned}
\iota(A)_{\mathrm{e}}(n\underline{\circledcirc}P(a)) &= nP(a) + \binom{n}{2}PHP(a)\\
&= nP(a) + \binom{n}{2}2P(a)\\
&= n^2P(a)\\
&= \iota(A)_{\mathrm{e}}(n^2\bar{\otimes}a)
\end{aligned}$$

and the condition (4) of Definition 2 follows. Since $P = 0$ for $\mathbb{Z}_{\mathrm{nil}}$ we have

$$\iota(A)_{\mathrm{e}}(P(n)\underline{\circledcirc}x) = 0 = nP\Delta(x) = \iota(A)_{\mathrm{e}}(n\bar{\otimes}\Delta(x))$$

and the condition (5) of Definition 2 follows. Finally, one has

$$\iota(A)_{\mathrm{e}}(T(n)\bar{\otimes}T(a)) = -nPT(a) = -nP(a) = -\iota(A)_{\mathrm{e}}(n\bar{\otimes}a)$$

and the relation (6) of Definition 2 follows.

We now define

$$\iota(A)_{\mathrm{ee}} : (\mathbb{Z}_{\mathrm{nil}}\underline{\circledcirc}A)_{\mathrm{ee}} \to A_{\mathrm{ee}}$$

by

$$\iota(A)_{\mathrm{ee}}(n \otimes a) = na.$$

We claim that $\iota(A) = (\iota(A)_{\mathrm{e}}, \iota(A)_{\mathrm{ee}})$ defines the natural morphism of square groups $\iota(A) : \mathbb{Z}_{\mathrm{nil}} \underline{\circledcirc} A \to A$. Indeed, we have

$$\iota(A)_{\mathrm{e}}(P(a \otimes a)) = \iota(A)_{\mathrm{e}}(n \bar{\otimes} a) = nP(a) = P(na) = P(n \otimes a)$$

and compatibility with P follows. We have also

$$\begin{aligned}
\iota(A)_{\mathrm{ee}} H(n \underline{\circledcirc} x) &= \iota(A)_{\mathrm{ee}}(n^2 \otimes H(x) + \binom{n}{2} \otimes \Delta(x)) \\
&= n^2 H(x) + \binom{n}{2} \Delta(x) \\
&= n^2 H(x) + \binom{n}{2} HPH(x) - \binom{n}{2} 2Hx + \binom{n}{2} (x \mid x)_H \\
&= nH(x) + \binom{n}{2} (x \mid x)_H + \binom{n}{2} HPHx \\
&= H(nx + \binom{n}{2} PHx) \\
&= H\iota(A)_{\mathrm{e}}(n \underline{\circledcirc} x)
\end{aligned}$$

and the claim follows.

Next we show that $\iota(A) : \mathbb{Z}_{\mathrm{nil}} \underline{\circledcirc} A \to A$ is an isomorphism. The inverse is given by

$$\iota_{\mathrm{e}}^{-1}(A)(x) = 1 \underline{\circledcirc} x, \quad \iota_{\mathrm{ee}}^{-1}(A)(a) = 1 \otimes a.$$

Since

$$P\iota_{\mathrm{ee}}^{-1}(A)(a) = P(1 \otimes a) = 1 \bar{\otimes} a = 1 \underline{\circledcirc} Pa = \iota_{\mathrm{e}}^{-1}(A)(Pa)$$

and

$$H\iota_{\mathrm{e}}^{-1}(A)(x) = H(1 \underline{\circledcirc} x) = 1 \otimes Hx = \iota_{\mathrm{ee}}^{-1}(A)(Hx)$$

it follows that $\iota^{-1}(A) : A \to \mathbb{Z}_{\mathrm{nil}} \underline{\circledcirc} A$ is indeed a morphism of square groups. Since

$$\begin{aligned}
&\iota_{\mathrm{ee}}(A)\iota_{\mathrm{ee}}^{-1}(A)(a) = \iota_{\mathrm{ee}}(A)(1 \otimes a) = a, \\
&\iota_{\mathrm{e}}(A)\iota_{\mathrm{e}}^{-1}(A)(x) = \iota_{\mathrm{e}}(A)(1 \underline{\circledcirc} x) = x, \\
&\iota_{\mathrm{ee}}^{-1}(A)\iota_{\mathrm{ee}}(A)(n \otimes x) = 1 \otimes nx = n \otimes x
\end{aligned}$$

and

$$\iota_{\mathrm{e}}^{-1}(A)\iota_{\mathrm{e}}(A)(n \underline{\circledcirc} x) = 1 \underline{\circledcirc} nx + 1 \underline{\circledcirc} \binom{n}{2} Phx = n \underline{\circledcirc} x$$

we see that $\iota(A)$ is really an isomorphism. In a similar way we will see that

$$\varkappa(A) : A \underline{\circledcirc} \mathbb{Z}_{\mathrm{nil}} \to A$$

is an isomorphism of square groups, where A is an arbitrary square group and

$$\varkappa(A)_{\mathrm{e}}(x\underline{\circledcirc}n)=nx,\quad \varkappa(A)_{\mathrm{e}}(a\bar{\otimes}n)=nP(a),\quad \varkappa(A)_{\mathrm{ee}}(a\otimes n)=na.$$

Finally, we have to check that

$$\begin{tikzcd}
(A\underline{\circledcirc}\mathbb{Z}_{\mathrm{nil}})\underline{\circledcirc}B \arrow[r] \arrow[dr, "\varkappa\underline{\circledcirc}\mathsf{Id}"'] & A\underline{\circledcirc}(\mathbb{Z}_{\mathrm{nil}}\underline{\circledcirc}B) \arrow[d, "\mathsf{Id}\underline{\circledcirc}\iota"] \\
 & A\underline{\circledcirc}B
\end{tikzcd}$$

is a commutative diagram, where the top map is given by the associativity isomorphisms. Generators of $((A\underline{\circledcirc}\mathbb{Z}_{\mathrm{nil}})\underline{\circledcirc}B)_{\mathrm{e}}$ are of the form $(x\underline{\circledcirc}n)\underline{\circledcirc}y$, $(a\otimes n)\underline{\circledcirc}y$ and $(a\otimes n)\bar{\otimes}b$. Commutativity of the diagram is obvious for $(a\otimes n)\bar{\otimes}b$. On the other hand for $(a\otimes n)\underline{\circledcirc}y$ it means $nP(a)\underline{\circledcirc}y=a\bar{\otimes}n\Delta(a)$, which follows from (5) of Definition 2. Finally, commutativity of the diagram for element $(x\underline{\circledcirc}n)\underline{\circledcirc}y$ means $(nx)\underline{\circledcirc}y=x\underline{\circledcirc}(nx+\binom{n}{2}PHy)$, which can be checked as follows. By the identity (4) and (2) of Definition 2 we have

$$(nx)\underline{\circledcirc}y=n(x\underline{\circledcirc}y)+\binom{n}{2}(x|x)_H\bar{\otimes}H(y)$$

while $x\underline{\circledcirc}(nx+\binom{n}{2}PHy)=n(x\underline{\circledcirc}y)+\binom{n}{2}x\underline{\circledcirc}PHy=n(x\underline{\circledcirc}y)+\binom{n}{2}(x|x)_H\bar{\otimes}H(y)$. Here we used (2) and (4) of Definition 2. Now the proof that SG is a monoidal category is complete.

6.6. **Symmetry property of $\underline{\circledcirc}$.** Next we prove that the tensor product $\underline{\circledcirc}$ is symmetric monoidal. To this end we define

$$\tau(A,B)_{\mathrm{e}}(x\underline{\circledcirc}y)=y\underline{\circledcirc}x-H(y)\bar{\otimes}TH(x)$$

for $x\in A_{\mathrm{e}}$, $y\in B_{\mathrm{e}}$ and

$$\tau(A,B)_{\mathrm{ee}}(a\otimes b)=b\otimes a$$

for $a\in A_{\mathrm{ee}}$, $b\in B_{\mathrm{ee}}$, which then also necessarily determines

$$\tau(A,B)_{\mathrm{e}}(a\bar{\otimes}b)=b\bar{\otimes}a$$

and makes compatibility with P clear. Compatibility with H means

$$H(y\underline{\circledcirc}x-H(y)\bar{\otimes}TH(x))=H(y)\otimes(x|x)_H+\Delta(y)\otimes H(x).$$

Indeed we have

$$
\begin{aligned}
&H(y\underline{\circledcirc}x-H(y)\bar{\otimes}TH(x))\\
&\quad=H(y\underline{\circledcirc}x)-H(H(y)\bar{\otimes}TH(x))+(H(y)\bar{\otimes}TH(x)|H(y)\bar{\otimes}TH(x))_H\\
&\quad=(y|y)_H\otimes H(x)+H(y)\otimes\Delta(x)\\
&\qquad+HPH(y)\otimes HPTH(x)-HPH(y)\otimes TH(x)-H(y)\otimes HPTH(x)\\
&\qquad+(P(H(y)\otimes TH(x))|P(H(y)\otimes TH(x)))_H\\
&\quad=(\Delta(y)+H(y)-TH(y))\otimes H(x)+H(y)\otimes((x|x)_H-H(x)+TH(x))\\
&\qquad+HPH(y)\otimes HPH(x)-HPH(y)\otimes TH(x)-H(y)\otimes HPH(x)\\
&\quad=H(y)\otimes(x|x)_H+\Delta(y)\otimes H(x)\\
&\qquad-TH(y)\otimes H(x)+H(y)\otimes TH(x)\\
&\qquad+HPH(y)\otimes HPH(x)-HPH(y)\otimes TH(x)-H(y)\otimes HPH(x),
\end{aligned}
$$

so compatibility of τ with H amounts to showing that the sum

$$-TH(y)\otimes H(x)+H(y)\otimes TH(x)+HPH(y)\otimes HPH(x)-HPH(y)\otimes TH(x)-H(y)\otimes HPH(x)$$

is zero. Substituting here $HP=1+T$ gives

$$
\begin{aligned}
&-TH(y)\otimes H(x)+H(y)\otimes TH(x)+H(y)\otimes H(x)+H(y)\otimes TH(x)+TH(y)\otimes H(x)\\
&+TH(y)\otimes TH(x)-H(y)\otimes TH(x)-TH(y)\otimes TH(x)-H(y)\otimes H(x)-H(y)\otimes TH(x),
\end{aligned}
$$

which is indeed zero.

Naturality of τ is straightforward; to check $\tau(B,A)\tau(A,B)=\mathsf{Id}$, the only nontrivial part is to look at

$$
\begin{aligned}
\tau(B,A)_{\mathrm{e}}\tau(A,B)_{\mathrm{e}}(x\underline{\circledcirc}y)&=\tau(B,A)_{\mathrm{e}}(y\underline{\circledcirc}x-H(y)\bar{\otimes}TH(x))\\
&=x\underline{\circledcirc}y-H(x)\bar{\otimes}TH(y)-TH(x)\bar{\otimes}H(y).
\end{aligned}
$$

But

$$
\begin{aligned}
-H(x)\bar{\otimes}TH(y)-TH(x)\bar{\otimes}H(y)&=-P(H(x)\otimes TH(y))-P(TH(x)\otimes H(y))\\
&=-P(H(x)\otimes TH(y))-P(TH(x)\otimes TTH(y))\\
&=-P(H(x)\otimes TH(y))+PT(H(x)\otimes TH(y))\\
&=0.
\end{aligned}
$$

Finally we have to check that two hexagons commute. The first case amounts to checking that two ways from $(A\underline{\circledcirc}B)\underline{\circledcirc}C$ to $B\underline{\circledcirc}(C\underline{\circledcirc}A)$ are the same. That is the following diagram commutes.

$$\begin{tikzcd}
& (B\underline{\circledcirc}A)\underline{\circledcirc}C \arrow[r] & B\underline{\circledcirc}(A\underline{\circledcirc}C) \arrow[dr] & \\
(A\underline{\circledcirc}B)\underline{\circledcirc}C \arrow[ur] \arrow[dr] & & & B\underline{\circledcirc}(C\underline{\circledcirc}A) \\
& A\underline{\circledcirc}(B\underline{\circledcirc}C) \arrow[r] & (B\underline{\circledcirc}C)\underline{\circledcirc}A \arrow[ur] &
\end{tikzcd}$$

Again this is trivial for elements of the form $(a\otimes b)\bar{\otimes}c$. It is straightforward to check that both ways take the element $(a\bar{\otimes}b)\underline{\circledcirc}z$ to $b\bar{\otimes}(\Delta(z)\otimes a)$. For the element $(x\underline{\circledcirc}y)\underline{\circledcirc}z$ it amounts to showing that $y\underline{\circledcirc}(z\underline{\circledcirc}x)-(y|y)_H\bar{\otimes}(Hz\otimes THx)-Hy\bar{\otimes}(\Delta(z)\otimes THx) = y\underline{\circledcirc}(z\underline{\circledcirc}x)-y\underline{\circledcirc}(Hz\bar{\otimes}THx)-Hy\bar{\otimes}(\Delta(z)\otimes THx)$ which is obvious since of $y\underline{\circledcirc}(Hz\bar{\otimes}THx) = y\underline{\circledcirc}P(Hz\otimes THx) = (y|y)_H\bar{\otimes}(Hz\otimes THx)$. The second hexagon axiom amounts to showing that two ways from $A\underline{\circledcirc}(B\underline{\circledcirc}C)$ to $(C\underline{\circledcirc}A)\underline{\circledcirc}B$ are the same. That is the following diagram commutes.

$$\begin{tikzcd}
& A\underline{\circledcirc}(C\underline{\circledcirc}B) \arrow[r] & (A\underline{\circledcirc}C)\underline{\circledcirc}B \arrow[dr] & \\
A\underline{\circledcirc}(B\underline{\circledcirc}C) \arrow[ur] \arrow[dr] & & & (C\underline{\circledcirc}A)\underline{\circledcirc}B \\
& (A\underline{\circledcirc}B)\underline{\circledcirc}C \arrow[r] & C\underline{\circledcirc}(A\underline{\circledcirc}B) \arrow[ur] &
\end{tikzcd}$$

This is trivial for $a\bar{\otimes}(b\otimes c)$. One checks that both ways take the element $x\underline{\circledcirc}(b\bar{\otimes}c)$ to $(c\otimes(x|x)_H)\bar{\otimes}b$. Finally the element $x\underline{\circledcirc}(y\underline{\circledcirc}z)$ goes to

$$c\underline{\circledcirc}(a\underline{\circledcirc}b)-Hc\bar{\otimes}TH(a\underline{\circledcirc}b) = c\underline{\circledcirc}(a\underline{\circledcirc}b)-Hc\bar{\otimes}((a|a)_H\otimes TH(b))-Hc\bar{\otimes}(THa\otimes\Delta(b))$$

in $C\underline{\circledcirc}(A\underline{\circledcirc}B)$ and to

$$(c\underline{\circledcirc}a)\underline{\circledcirc}b-(Hc\otimes(a|a)_H)\bar{\otimes}TH(b)-(Hc\otimes THa)\bar{\otimes}\Delta(b))$$

in $(C\underline{\circledcirc}A)\underline{\circledcirc}B$. By the second way the element $x\underline{\circledcirc}(y\underline{\circledcirc}z)$ goes to

$$(c\underline{\circledcirc}a)\underline{\circledcirc}b-(Hc\bar{\otimes}THa)\underline{\circledcirc}b-(Hc\otimes(a|a)_H)\bar{\otimes}TH(b)$$

But these elements are the same, because

$$(Hc\bar{\otimes}THa)\underline{\circledcirc}b = P(Hc\otimes THa)\underline{\circledcirc}b = (Hc\otimes THa)\bar{\otimes}\Delta(b).$$

6.7. **The action of the monoidal category $(\mathsf{SG},\underline{\circledcirc})$ on the category of pre-square groups.** In Section 6.1 we defined the bifunctor

$$\underline{\circledcirc}:\mathsf{PSG}\times\mathsf{SG}\to\mathsf{PSG}$$

which yields a right action of the monoidal category $(\mathsf{SG}, \underline{\circledcirc})$ on the category PSG, meaning that for any $M \in \mathsf{PSG}$ and $N_1, N_2 \in \mathsf{SG}$ there are coherent isomorphisms

$$(M \underline{\circledcirc} N_1) \underline{\circledcirc} N_2 \cong M \underline{\circledcirc} (N_1 \underline{\circledcirc} N_2),$$
$$M \underline{\circledcirc} \mathbb{Z}_{\mathrm{nil}} \cong M.$$

The proof of this fact is quite similar to the proof of associativity and unit properties of $(\mathsf{SG}, \underline{\circledcirc})$, therefore we omit it.

Existence of the bifunctor $\underline{\circledcirc} : \mathsf{PSG} \times \mathsf{SG} \to \mathsf{PSG}$ is crucial in the following Lemma. First observe that if M, M' and N are square groups and $f : \wp(M) \to \wp(M')$ is a morphism of underlying pre-square groups, then $f \underline{\circledcirc} \mathsf{Id}$ defines a morphism of pre-square groups

$$\wp(M \underline{\circledcirc} N) \to \wp(M \underline{\circledcirc} N').$$

We can take $M' = M$ and $f = n^*$ (see Lemma 23). The following result shows that the action of integers respects the tensor product.

Lemma 28. *Let M and N be square groups. Then for any integer n one has*

$$(n^*)_M \underline{\circledcirc} \mathsf{Id}_N = (n^*)_{M \underline{\circledcirc} N}.$$

Proof. The result is obvious on the ee-level, while on the e-level it can be checked as follows

$$\begin{aligned}(n^*(x)) \underline{\circledcirc} y &= (nx + P(\binom{n}{2} Hx)) \underline{\circledcirc} y = (nx) \underline{\circledcirc} y + P(\binom{n}{2} Hx) \underline{\circledcirc} y \\ &= (nx) \underline{\circledcirc} y + (\binom{n}{2} Hx) \circledcirc \Delta(y) = n(x \underline{\circledcirc} y) + \binom{n}{2}((x|x)_H \circledcirc H(y) + H(x) \circledcirc \Delta(y)) \\ &= n(x \underline{\circledcirc} y) + \binom{n}{2} H(x \underline{\circledcirc} y)\end{aligned}$$

Here we used the fact that $(nx) \underline{\circledcirc} y = n(x \underline{\circledcirc} y) = \binom{n}{2}(x|x)_H \bar{\circledcirc} H(y)$, which follows from identity (4) in section 4. □

7. The tensor product of abelian square groups and quadratic $\mathbb{Z}$-modules

We now describe the tensor product of those square groups which are abelian or are quadratic $\mathbb{Z}$-modules. In this case the tensor product has a particularly simple form.

Proposition 29. *Let A be an abelian square group and B be a square group, then $A \underline{\circledcirc} B$ is an abelian square group. Moreover the abelian group $(A \underline{\circledcirc} B)_{\mathrm{e}}$ is given by the following pushout diagram of abelian groups.*

$$\begin{tikzcd} A_{\mathrm{ee}} \otimes \mathsf{Coker}(P^B) \arrow[r, "\mathsf{Id} \otimes \Delta"] \arrow[d, "P \otimes \mathsf{Id}"] & A_{\mathrm{ee}} \otimes B_{\mathrm{ee}} / (\mathsf{Id} + T \otimes T) \arrow[d] \\ A_{\mathrm{e}} \otimes \mathsf{Coker}(P^B) \arrow[r] & (A \underline{\circledcirc} B)_{\mathrm{e}} \end{tikzcd}$$

If additionally A is a quadratic $\mathbb{Z}$-module then $A \underline{\circledcirc} B$ is a quadratic $\mathbb{Z}$-module as well.

Proof. Since the cross-effect of H vanishes on A it follows that $\rho = 0$, where ρ is the same as in Definition 2. Hence $H^{A \underline{\circledcirc} B}$ is a homomorphism and therefore $A \underline{\circledcirc} B \in \mathsf{Ab}(\mathsf{SG})$. As a consequence $(A \underline{\circledcirc} B)_{\mathrm{e}}$ is an abelian group. Moreover for each $x \in A_{\mathrm{e}}$ one has $x \underline{\circledcirc} P(b) = 0$ for all $b \in B_{\mathrm{ee}}$, therefore the function $x \underline{\circledcirc} -$ factors through $\mathsf{Coker}(P_A)$. Finally, the relations (5) and (6) of Definition 2 show that $(A \underline{\circledcirc} B)_{\mathrm{e}}$ indeed fits into the above pushout diagram. If additionally $\Delta^A = 0$, then $\Delta^{A \underline{\circledcirc} B} = 0$ (see Lemma 27), hence $A \underline{\circledcirc} B$ is a quadratic $\mathbb{Z}$-module. $\square$

Proposition 30. *Let A be an abelian square group and B be a quadratic $\mathbb{Z}$-module. Then*

$$A \underline{\circledcirc} B \cong E(A_{\mathrm{ee}} \otimes B_{\mathrm{ee}}, T \otimes T) \oplus \mathsf{Coker}(P^A) \otimes \mathsf{Coker}(P^B)$$

is a quadratic $\mathbb{Z}$-module. Here the abelian group $\mathsf{Coker}(P^A) \otimes \mathsf{Coker}(P^B)$ is considered as a square group by putting 0 on the ee*-level and $E(-,-)$ is defined in Section 5.2.*

Proof. This is clear because $\Delta^B = 0$. $\square$

Lemma 31. *Let A be an abelian group and M be a square group. Then*

$$f : (A \otimes M_{\mathrm{ee}})^{\otimes} \to A^{\otimes} \underline{\circledcirc} M$$

is an isomorphism. Here

$$f_{\mathrm{e}}(a \otimes c) = (a, 0) \bar{\otimes} c$$

and

$$f_{\mathrm{ee}}(a \otimes c, b \otimes d) = (a, 0) \otimes c - (0, b) \otimes T(d),$$

with $a, b \in A$ and $c, d \in M_{\mathrm{ee}}$.

Proof. The statement is obvious on the ee-level. To see it on the e-level one observes that P is a split epimorphism on $A^{\otimes}$ and $\mathsf{Ker}(P)$ consists of elements $(a, -a) \in A^{\otimes}_{\mathrm{ee}}$, $a \in A$. By Proposition 29 one can check that $(A^{\otimes} \underline{\circledcirc} M)_{\mathrm{e}}$ is the quotient of $(A \oplus A) \otimes M_{\mathrm{ee}}$ by the relations

$$\begin{aligned} (a, -a) \otimes \Delta(x) &= 0, & a \in A,\ x \in M_{\mathrm{e}}, \\ (a, 0) \otimes c + (0, a) \otimes T(c) &= 0, & c \in M_{\mathrm{ee}}. \end{aligned}$$

Since $T\Delta + \Delta = 0$ we see that the first relation follows from the second one. Now the result is clear. $\square$

8. The tensor product $V(S) \underline{\circledcirc} M$

We consider the case when one of the factors in a tensor product is the free square group $V(S)$ (see Section 5.7). We take $S = \{1, 2, \cdots, n\}$. In this case we use the notation $V(n)$ instead of $V(S)$. If $n = 1$ then $V(1) \cong \mathbb{Z}^Q$ and we have the following explicit result.

Lemma 32. *For any square group A one has an isomorphism*

$$\mathbb{Z}^Q \underline{\circledcirc} A \cong (A_{\mathrm{ee}} \times A_{\mathrm{e}} \xrightarrow{H^{\mathbb{Z}^Q \underline{\circledcirc} A}} A_{\mathrm{ee}} \oplus A_{\mathrm{ee}} \oplus A_{\mathrm{ee}} \xrightarrow{P^{\mathbb{Z}^Q \underline{\circledcirc} A}} A_{\mathrm{ee}} \times A_{\mathrm{e}})$$

where

$$P^{\mathbb{Z}^Q \underline{\circledcirc} A}(a,b,c) = (a+b-Tb, Pa)$$

and

$$H^{\mathbb{Z}^Q \underline{\circledcirc} A}(a,x) = (a+T(a)+\Delta(x), -Ta, H(x)).$$

Proof. We just indicate the explicit isomorphism $\psi : A_{\mathrm{e}} \times A_{\mathrm{ee}} \to (\mathbb{Z}^Q \underline{\circledcirc} A)_c$ and its inverse:

$$\alpha_{\mathrm{e}}(x,a) = (1,0)\underline{\circledcirc}x + (1,0,0)\bar{\otimes}a,$$

$$\alpha_{\mathrm{e}}^{-1}((m,n)\underline{\circledcirc}x + (k_1,k_2,k_3)\bar{\otimes}a) = (mx + \binom{m}{2}PHx + k_1Pa, n\Delta(x) + k_1a + k_2a - k_2Ta).$$

□

For arbitrary n and a square group M let $G_n(M)$ be the group, which is $(M_{\mathrm{e}})^n \times (M_{\mathrm{ee}})^{\binom{n}{2}}$ as a set with (x_k, a_{ij}) as a generic element. Here $1 \leqslant k \leqslant n$ and $1 \leqslant i < j \leqslant n$. The group structure is given by

$$(x_k, a_{ij}) + (y_k, b_{ij}) = (x_k + y_k, a_{ij} + b_{ij} + (x_j \mid x_i)_H).$$

Proposition 33. *For any integer $n \geqslant 1$ and any square group M one has isomorphisms of groups*

$$(V(n)\underline{\circledcirc}M)_{\mathrm{ee}} \cong (M_{\mathrm{ee}})^{n^2+2n},$$
$$(V(n)\underline{\circledcirc}M)_{\mathrm{e}} \cong G_n(M) \times (M_{\mathrm{ee}})^n.$$

Proof. The isomorphism is obvious on the ee-level. We define

$$\alpha_{\mathrm{e}} : G_n(M) \times (M_{\mathrm{ee}})^n \to (V(n)\underline{\circledcirc}M)_{\mathrm{e}}$$

as follows. If a generic element of the group $G_n(M) \times (M_{\mathrm{ee}})^n$ is (x_k, a_{ij}, b_l), where $1 \leqslant k, l \leqslant n$, $1 \leqslant i < j \leqslant n$ and $x_k \in M_{\mathrm{e}}$, $a_{ij}, b_l \in M_{\mathrm{ee}}$, then we put

$$\alpha_{\mathrm{e}}(x_k, a_{ij}, b_l) = 1\underline{\circledcirc}x_1 + 2\underline{\circledcirc}x_2 + \cdots + n\underline{\circledcirc}x_n + \sum_{i<j}(j,i)\bar{\otimes}a_{ij} + \sum_l Hl\bar{\otimes}b_l.$$

One easily checks that α is a homomorphism. Actually it is an isomorphism with inverse α_{e}^{-1} which is uniquely determined by

$$\alpha_{\mathrm{e}}^{-1}(\sum_i PHi\underline{\circledcirc}x_i + \sum_l Hl\bar{\otimes}d_l + \sum_{ij}(i,j)\bar{\otimes}c_{ij} + \sum_k HPHk\bar{\otimes}b_k) =$$
$$(PHc_{kk}, c_{ji} + Tc_{ij}, d_l + b_l - Tb_l + \Delta(x_l)),$$

and

$$\alpha_{\mathrm{e}}^{-1}(m_1 1 + \cdots + m_n n + \sum_{i<j} m_{ij}[i,j])\underline{\circledcirc}x =$$
$$(m_k x + \binom{m_k}{2}PHx, n_i m_j Hx + m_{ij}(Hx - THx), 0).$$

□

9. Right exactness

We prove Proposition 4. First we check that for any square group A the tensor product functor $A\underline{\circledcirc}- : \mathsf{SG} \to \mathsf{SG}$ preserves reflexive coequalizers. Recall that a functor $R : \mathsf{SG} \to \mathsf{SG}$ preserves reflexive coequalizers if for any simplicial object B_* in SG the canonical morphism $\pi_0(R(B_*)) \to R(\pi_0(B_*))$ is an isomorphism. Observe that $\pi_0(B_*)$ is a coequalizer of two parallel arrows $d_1, d_0 : B_1 \to B_0$. We put $B = \pi_0(B_*)$. By universality property of coequalizers we have a canonical map $\pi_0(A\underline{\circledcirc}B_*) \to A\underline{\circledcirc}B$. This is an isomorphism, because it has an inverse, which is defined as follows. Take a generator $x\underline{\circledcirc}y$ with $x \in A_{\mathrm{e}}$ and $y \in B_{\mathrm{e}}$. We choose an element $z \in B_{0\mathrm{e}}$ in the class $y \in \pi_0(B_{0\mathrm{e}})$. The class of $x\underline{\circledcirc}\hat{y} \in A\underline{\circledcirc}B_0$ in the quotient $\pi_0(A\underline{\circledcirc}B_*)$ is independent of the choice. Indeed, if z' is also in the class y, then there exists $w \in B_{1\mathrm{e}}$ such that $d_0w = z$ and $d_1w = z'$ and therefore $x\underline{\circledcirc}z$ and $x\underline{\circledcirc}z'$ define the same element in $\pi_0(A\underline{\circledcirc}B_*)$. Based on this fact one easily checks that this assignment respects all relations for $\underline{\circledcirc}$ and indeed defines a morphism $A\underline{\circledcirc}B \to \pi_0(A\underline{\circledcirc}B_*)$ and hence $A\underline{\circledcirc}- : \mathsf{SG} \to \mathsf{SG}$ preserves reflexive coequalizers.

Next we prove that the tensor product $A\underline{\circledcirc}- : \mathsf{SG} \to \mathsf{SG}$ preserves finite products. Let $B, C \in \mathsf{SG}$. Then $B \times C$ in SG is constructed degreewise, i. e. $(B \times C)_{\mathrm{e}} = B_{\mathrm{e}} \times C_{\mathrm{e}}$, $(B\times C)_{\mathrm{ee}} = B_{\mathrm{ee}}\times C_{\mathrm{ee}}$, $P(b,c) = (Pb, Pc)$ and $H(y,z) = (Hy, Hz)$. Here $y \in B_{\mathrm{e}}, z \in C_{\mathrm{e}}$ and $b \in B_{\mathrm{ee}}, c \in C_{\mathrm{ee}}$. The projection $p_1 : B \times C \to B$ has the canonical section $i_1 : B \to B \times C$ given by $i_{1\mathrm{e}}(x) = (x, 0)$ and $i_{1\mathrm{ee}}(b) = (b, 0)$. This section yields a morphism of square groups $\mathsf{Id}_A\underline{\circledcirc}i_1 : A\underline{\circledcirc}B \to A\underline{\circledcirc}(B \times C)$. Similarly, one gets the morphism $\mathsf{Id}_A\underline{\circledcirc}i_2 : A\underline{\circledcirc}C \to A\underline{\circledcirc}(B \times C)$. The identity (7) in Section 6.1 shows that the images of $\mathsf{Id}_A\underline{\circledcirc}i_1$ and $\mathsf{Id}_A\underline{\circledcirc}i_2$ commute and therefore they yield the canonical morphism

$$i_* : (A\underline{\circledcirc}B) \times (A\underline{\circledcirc}C) \to A\underline{\circledcirc}(B \times C)$$

which obviously is left inverse to the canonical morphism $A\underline{\circledcirc}(B \times C) \to (A\underline{\circledcirc}B) \times (A\underline{\circledcirc}C)$ induced by projections. It is clear that both morphisms are isomorphisms on the ee-level. Thus we need only to show that the map i_* is surjective on the e-level. This follows immediately from the fact that $x\underline{\circledcirc}(y,z) = x\underline{\circledcirc}i_1(y) + x\underline{\circledcirc}i_2(z)$ and $a\underline{\circledcirc}(b,c) = a\underline{\circledcirc}i_1(b) + a\underline{\circledcirc}i_2(c)$.

We now prove that for any short exact sequence of square groups

$$0 \to B_1 \xrightarrow{\mu} B \xrightarrow{\sigma} B_2 \to 0$$

the induced sequence

$$A\underline{\circledcirc}B_1 \to A\underline{\circledcirc}B \to A\underline{\circledcirc}B_2 \to 0$$

is also exact. To this end one observes that by Lemma 19 it suffices to consider the case, when the extension is central. Consider the following diagram in SG

$$B_1 \times B \underset{g}{\overset{f}{\rightrightarrows}} B$$

where g is the projection on the second factor, while $f_{\mathrm{e}}(b_1, b) = \mu_{\mathrm{e}}(b_1) + b$ and $f_{\mathrm{ee}}(y_1, y) = \mu_{\mathrm{ee}}(y) + y$, $y \in B_{\mathrm{e}}, y_1 \in B_{1e}$, $b_1 \in B_{1ee}, b \in B_{\mathrm{ee}}$. Since B_1 is central in B, it follows that f

is a morphism in SG and in fact B_2 is isomorphic to the coequalizer of this diagram. Since $A\underline{\circledcirc}-$ preserves products and reflexive coequalizers we see that

$$A\underline{\circledcirc}B_1 \times A\underline{\circledcirc}B \overset{f_*}{\underset{g_*}{\rightrightarrows}} A\underline{\circledcirc}B \longrightarrow A\underline{\circledcirc}B_2$$

is a coequalizer. From this follows that the functor $A\underline{\circledcirc}(-)$ is right exact.

Finally, assume that A is a projective square group. We have to show that the functor $A\underline{\circledcirc}(-)$ is exact. We can assume that $A = V(S) \times (\mathbb{Z}[T])^{\otimes}$ for some $S \in \mathsf{Sets}$ and $T \in \mathsf{Sets}$, because any projective is a retract of a free one. Since the tensor product commutes with filtered colimits we can assume that S and T are finite sets. Since the tensor product commutes with finite products the result follows from Lemma 31 and Proposition 33.

10. $\mathbb{Z}_{\mathrm{nil}}[-]$ as a monoidal functor

The next result generalizes the well-known fact that the free abelian group functor

$$\mathbb{Z}[-] : (\mathsf{Sets}, \times) \to (\mathsf{Ab}, \otimes)$$

is a symmetric monoidal functor.

Proposition 34. *For any sets S and S', one has a natural isomorphism of square groups*

$$\delta : \mathbb{Z}_{\mathrm{nil}}[S \times S'] \cong \mathbb{Z}_{\mathrm{nil}}[S]\underline{\circledcirc}\mathbb{Z}_{\mathrm{nil}}[S']$$

which on the ee*-level is the canonical isomorphism $\mathbb{Z}[S \times S'] \cong \mathbb{Z}[S] \otimes \mathbb{Z}[S']$, given by $\delta_{\mathrm{ee}}((s, s') \otimes (t, t')) = s \otimes t \otimes s' \otimes t'$. On the* e*-level it is given by*

$$\delta_{\mathrm{e}}(s, s') = s\underline{\circledcirc}s', \quad s, s' \in S.$$

Thus the functor

$$\mathbb{Z}_{\mathrm{nil}}[-] : (\mathsf{Sets}, \times) \to (\mathsf{SG}, \underline{\circledcirc})$$

is symmetric monoidal.

Proof. Since $(\mathbb{Z}_{\mathrm{nil}}[S \times S'])_{\mathrm{e}}$ is a free nil$_2$-group on $S \times S'$, the homomorphism δ_{e} is well-defined. Let us first check that the pair $\delta = (\delta_{\mathrm{e}}, \delta_{\mathrm{ee}})$ defines a morphism of square groups. We have

$$P\delta_{\mathrm{ee}}((s, s') \otimes (t, t')) = P(s \otimes t \otimes s' \otimes t') = (s \otimes t)\bar{\otimes}(s' \otimes t').$$

Similarly we have

$$\begin{aligned}\delta_{\mathrm{e}}P((s, s') \otimes (t, t')) = \delta_{\mathrm{e}}([(t, t'), (s, s')]) = [\delta_{\mathrm{e}}(t, t'), \delta_{\mathrm{e}}(s, s')] = [t\underline{\circledcirc}t', s\underline{\circledcirc}s']\\ = (t|s)_H \otimes (t'|s')_H = s \otimes t \otimes s' \otimes t'.\end{aligned}$$

Comparing these expressions we see that δ is compatible with P. We have also

$$\delta_{\mathrm{ee}}((s, s')|(t, t'))_H = \delta\ ((t, t') \otimes (s, s')) = t \otimes s \otimes t' \otimes s'.$$

On the other hand we have

$$(\delta_{\mathrm{e}}(s, s') \mid \delta_{\mathrm{e}}(t, t'))_H = (s\underline{\circledcirc}s'|t\underline{\circledcirc}t')_H = (s|t)_H \otimes (s'|t')_H = t \otimes s \otimes t' \otimes s'.$$

Since

$$\delta_{\mathrm{ee}} H(s,s') = 0$$

and

$$H(\delta_{\mathrm{e}}((s,s')) = H(s\underline{\circledcirc}s') = (s|s')_H \otimes H(s') + H(s)\underline{\circledcirc}\Delta(s') = 0$$

we conclude that δ is compatible with H and therefore δ is, in fact, a morphism of square groups. Since δ_{ee} is an isomorphism, it suffices to show that δ_{e} is an isomorphism. We construct the inverse map

$$\eta : (\mathbb{Z}_{\mathrm{nil}}[S]\underline{\circledcirc}\mathbb{Z}_{\mathrm{nil}}[S'])_{\mathrm{e}} \to (\mathbb{Z}_{\mathrm{nil}}[S\times S'])_{\mathrm{e}}$$

as follows. Take $s\in S$. Since $(\mathbb{Z}_{\mathrm{nil}}[S'])_{\mathrm{e}}$ is a free nil$_2$-group on S', there is a unique homomorphism

$$f_s : (\mathbb{Z}_{\mathrm{nil}}[S'])_{\mathrm{e}} \to (\mathbb{Z}_{\mathrm{nil}}[S\times S'])_{\mathrm{e}}$$

such that $f_s(s') = (s,s')$, $s'\in S'$. In particular for any $y\in(\mathbb{Z}_{\mathrm{nil}}[S'])_{\mathrm{e}}$ the element $f_s(y)$ is well-defined. Since $(\mathbb{Z}_{\mathrm{nil}}[S])_{\mathrm{e}}$ is a free nil$_2$-group we can extend $s\mapsto f_s(y)$ to a map $x\mapsto f_x(y)\in(\mathbb{Z}_{\mathrm{nil}}[S\times S'])_{\mathrm{e}}$ in such a way that

$$f_{x_1+x_2}(y) = f_{x_1}(y) + f_{x_2}(y) + g(x_1\otimes x_2\otimes H(y)),$$

where $g:\mathbb{Z}[S]\otimes\mathbb{Z}[S]\otimes\mathbb{Z}[S']\otimes\mathbb{Z}[S']\to(\mathbb{Z}_{\mathrm{nil}}[S\times S'])_{\mathrm{e}}$ is given by $g(s\otimes t\otimes s'\otimes t') = [(t,t'),(s,s')]$. Thus we have

$$[f_{x_1}(y_1), f_{x_2}(y_2)] = -g(x_1,x_2,y_1,y_2).$$

We claim that for all $x\in(\mathbb{Z}_{\mathrm{nil}}[S])_{\mathrm{e}}$ and $y_1,y_2\in(\mathbb{Z}_{\mathrm{nil}}[S'])_{\mathrm{e}}$ one has

$$f_x(y_1+y_2) = f_x(y_1) + f_x(y_2).$$

By our construction the claim holds if $x\in S$. Therefore it suffices to show that the equation holds for $x = x_1+x_2$, provided it holds for x_1 and x_2. To this end, we proceed as follows: $f_{x_1+x_2}(y_1+y_2) = f_{x_1}(y_1)+f_{x_1}(y_2)+f_{x_2}(y_1)+f_{x_2}(y_2)+g(x_1\otimes x_2\otimes H(y_1+y_2)) = f_x(y_1)+f_x(y_2)+[f_{x_1}(y_1),f_{x_2}(y_2)]+g(x_1\otimes x_2\otimes y_2\otimes y_1)$ because $(y_1|y_1)_H = y_2\otimes y_1$ and last two summands cancel; hence the claim.

Now we are ready to define the map $\eta : (\mathbb{Z}_{\mathrm{nil}}[S]\underline{\circledcirc}\mathbb{Z}_{\mathrm{nil}}[S'])_{\mathrm{e}} \to (\mathbb{Z}_{\mathrm{nil}}[S\times S'])_{\mathrm{e}}$ by

$$\eta(x\underline{\circledcirc}y) = f_x(y),$$
$$\eta((s\otimes t)\bar{\otimes}(s'\otimes t')) = g(s\otimes t\otimes s'\otimes t').$$

We have to check that η respects the relations (1)-(6) of Definition 2. The relations (1) and (6) are clear, (3) holds by our construction and (2) we just checked. Let us check (4). Without loss of generality we can assume that $b = s'\otimes t'$ with $s',t'\in S'$. First we consider the case $x = s\in S$. Then one has $\eta(x\underline{\circledcirc}P(b)) = \eta(s\underline{\circledcirc}[t',s']) = [(s,t'),[(s,s')] = g(s\otimes s\otimes s'\otimes t') = \eta((s\otimes s)\bar{\otimes}s'\otimes t') = \eta((x|x)_H\bar{\bar{\otimes}}b)$. For general x it suffices to show that η respects the equality (4) for $x = x_1+x_2$ provided η respects it for x_1 and x_2. To check the last assertion one needs to show the equality for the cross-effects of both sides of the equality in question. But this is formal,

$$\eta((x_1|x_2)_H\bar{\otimes}HP(b)) = \eta((x_1|x_2)_H\bar{\otimes}(b+Tb)) = \eta((x_1|x_2)_H\bar{\otimes}b + (x_2|x_1)_H\bar{\otimes}b).$$

Here we used the fact that η respects the identity (6). Now we check that η respects the equation (5). Indeed, we can assume that $y = s' \in S'$ and $a = s \otimes t$, $s,t \in S$. Then we have

$$\eta(P(a)\underline{\circledcirc}y) = \eta([t,s]\underline{\circledcirc}s') = [(t,s'),(s,s')].$$

On the other hand we have

$$\eta(a\bar{\otimes}\Delta(b)) = \eta(s \otimes s \otimes s' \otimes s') = [(t,s'),(s,s')]$$

and the result follows. □

11. Torsion product of square groups

It follows from Proposition 4 that the functor $M\underline{\circledcirc}(-)$ respects weak equivalences of simplicial square groups provided M is projective. We will exploit this fact in this section.

Thanks to Section 5.8 for any square group M one can take a cofibrant replacement M^c of M. This means that there is given a weak equivalence $M^c \to M$ and M^c is cofibrant in the model category structure introduced in Section 5.8.

Lemma 35. *For any square groups M and N there is an isomorphism of square groups*

$$\pi_i(M^c\underline{\circledcirc}N) \cong \pi_i(M\underline{\circledcirc}N^c),\ i \geqslant 0$$

Proof. According to [23] the square groups $\pi_i(M^c\underline{\circledcirc}N)$ (as well as $\pi_i(M\underline{\circledcirc}N^c)$) do not depend on the cofibrant replacement. Moreover one can assume that each component of M^c and N^c is a free square group ([23]). One considers now the bisimplicial square group $M^c\underline{\circledcirc}N^c$. Both on the e- and on the ee-level one has the Quillen spectral sequences for double simplicial groups [22]. Since the tensor product with a free object respects weak equivalences, both spectral sequences degenerate yielding the isomorphism in question. □

We now put

$$\mathsf{Tor}^{\odot}{}_i(M,N) = \pi_i(M^c\underline{\circledcirc}N) \cong \pi_i(M\underline{\circledcirc}N^c).$$

Lemma 36. *For any square groups M and N one has natural isomorphisms $\mathsf{Tor}^{\odot}{}_0(M,N) \cong M\underline{\circledcirc}N$ and $\mathsf{Tor}^{\odot}{}_1(M,N) \in \mathsf{Ab}(\mathsf{SG})$ and for all $i \geqslant 2$ one has $\mathsf{Tor}^{\odot}{}_i(M,N) \in \mathsf{Ab}$.*

Proof. The statement on $\mathsf{Tor}^{\odot}{}_0$ follows from right exactness of $\underline{\circledcirc}$. As already mentioned, for any simplicial square group X one has $\pi_i(X) \in \mathsf{Ab}(\mathsf{SG})$ for all $i \geqslant 1$, hence the statement for $i = 1$. Finally let us observe that

$$(\mathsf{Tor}^{\odot}{}_i(M,N))_{\mathrm{ee}} = \mathsf{Tor}^{\mathbb{Z}}{}_i(M_{\mathrm{ee}},N_{\mathrm{ee}}),\quad i \geqslant 0,$$

hence $(\mathsf{Tor}^{\odot}{}_i(M,N))_{\mathrm{ee}} = 0$ for $i \geqslant 2$ and the result follows. □

Thus we have bifunctors

$$\mathsf{Tor}^{\odot}{}_1 : \mathsf{SG} \times \mathsf{SG} \to \mathsf{Ab}(\mathsf{SG})$$

and

$$\mathsf{Tor}^{\odot}{}_i : \mathsf{SG} \times \mathsf{SG} \to \mathsf{Ab}, \qquad i \geqslant 2.$$

Proposition 37. *If*

$$0 \to N_1 \to N \to N_2 \to 0$$

is a short exact sequence of square groups then for any square group M one has a long exact sequence of square groups

$$\cdots \to \mathsf{Tor}^{\odot}{}_2(M, N_2) \to \mathsf{Tor}^{\odot}{}_1(M, N_1) \to \mathsf{Tor}^{\odot}{}_1(M, N)$$
$$\to \mathsf{Tor}^{\odot}{}_1(M, N_2) \to M \underline{\circledcirc} N_1 \to M \underline{\circledcirc} N \to M \underline{\circledcirc} N_2 \to 0.$$

Furthermore $\mathsf{Tor}^{\odot}{}_i(M, N) = 0$ provided $i \geqslant 1$ and either M or N is projective.

Proof. If M is free, then one can take $M^c = M$. A projective object is a retracts of a free square group and hence the statement on projective objects is clear. For general M we choose an M^c with degreewise free square groups. According to Proposition 4 one has the short exact sequence of simplicial objects

$$0 \to M^c \underline{\circledcirc} N_1 \to M^c \underline{\circledcirc} N \to M^c \underline{\circledcirc} N_2 \to 0$$

yielding the long exact sequence for homotopy groups. Here we used the well-known fact that epimorphisms of simplicial groups are Kan fibrations. □

Corollary 38. *Let A be an abelian group and M be a square group. Then*

$$\mathsf{Tor}^{\odot}{}_1(A^{\otimes}, M) \cong (\mathsf{Tot}_1^{\mathbb{Z}}(A, M_{\mathrm{ee}}))^{\otimes}$$

and $\mathsf{Tor}^{\odot}{}_k(A^{\otimes}, M) = 0$ for $k \geq 2$.

Proof. Take a short exact sequence $0 \to F_1 \to F_0 \to A \to 0$ of abelian groups with free abelian group F_0. It yields a short exact sequence of square groups

$$0 \to F_1^{\otimes} \to F_0^{\otimes} \to A^{\otimes} \to 0.$$

Since free square groups $F_0^{\otimes}$ and $F_1^{\otimes}$ are free square groups the result follows from Lemma 31 and Proposition 37. □

Now we give the proof of Proposition 5.

Proof of Proposition 5. We use the short exact sequence (6) in Section 5.6. Proposition 37 shows that it yields the exact sequence of square groups

$$\mathsf{Tor}^{\odot}{}_1(M, A \vee B) \to \mathsf{Tor}^{\odot}{}_1(M, A) \times \mathsf{Tor}^{\odot}{}_1(M, B)$$
$$\to M \underline{\circledcirc} (\mathsf{Coker}(P^A) \otimes \mathsf{Coker}(P^B))^{\otimes} \xrightarrow{\mu} M \otimes (A \vee B) \to M \underline{\circledcirc} (A \times B) \to 0.$$

Let us observe that for any category $\mathscr{C}$ with coproducts and zero object and for any functor $F : \mathscr{C} \to \mathsf{Groups}$ with $F(0) = 0$ the natural homomorphism $r : F(X \vee Y) \to F(X) \times F(Y)$ is always an epimorphism. Indeed, if $s_1 \in F(X)$ and $s_2 \in F(Y)$ then $s = i_1(s_1) + i_2(s_2) \in F(X \vee Y)$ has the property that $r(s) = (s_1, s_2)$. Here $i_1 : X \to X \vee Y$ and $i_2 : Y \to X \vee Y$ are standard inclusions. Thus μ is a monomorphism and the result follows from Proposition 31 and the fact that $M \underline{\circledcirc}(-)$ preserves finite products. □

Corollary 39. *Let M be a square group. Then the functor $\mathsf{Tor}^{\odot}{}_k(M,-)$ preserves finite products for all $k \geq 0$. Moreover the natural map*

$$\mathsf{Tor}^{\odot}{}_k(M, A \vee B) \to \mathsf{Tor}^{\odot}{}_k(M, A) \times \mathsf{Tor}^{\odot}{}_k(M, B)$$

is an isomorphism provided $k \geq 2$. Here A and B are square groups. If $k = 1$, then one has a short exact sequence of square groups

$$0 \to (\mathsf{Tor}_1^{\mathbb{Z}}(M_{\mathrm{ee}}, \mathsf{Coker}(P^A) \otimes \mathsf{Coker}(P^B)))^{\otimes} \to \mathsf{Tor}^{\odot}{}_1(M, A \vee B) \to \mathsf{Tor}^{\odot}{}_1(M, A) \times \mathsf{Tor}^{\odot}{}_1(M, B) \to 0$$

Proof. Since

$$\pi_*(M^c \underline{\circledcirc} (A \times B)) = \pi_*(M^c \underline{\circledcirc} A) \times \pi_*(M^c \underline{\circledcirc} B)$$

we have

$$\mathsf{Tor}^{\odot}{}_*(M, A \times B) = \mathsf{Tor}^{\odot}{}_*(M, A) \times \mathsf{Tor}^{\odot}{}_*(M, B)$$

and the statement on finite products follows. For the rest one apllies the long exact sequence for $\mathsf{Tor}^{\odot}$-functors to the short exact sequence in Proposition 5 and uses Corollary 38. $\square$

12. Quadratic rings

A monoid in the monoidal category $(\mathsf{SG}, \underline{\circledcirc})$ is termed *quadratic ring*. More explicitly, a quadratic ring structure on a square group R is given by a multiplicative monoid structure on R_{e} and a ring structure on R_{ee}. The multiplicative unit of R_{e} is denoted by 1. One requires that these structures satisfy the following additional properties. First of all we have

(i) $x(y + z) = xy + xz$,
(ii) $(x + y)z = xz + yz + P((y \mid x)_H \, H(z))$.

Thus $\mathsf{Coker}(P)$ is a ring. Moreover the maps

$$-T : R_{\mathrm{ee}} \to R_{\mathrm{ee}}$$
$$(- \mid -)_H : \mathsf{Coker}(P_R) \otimes \mathsf{Coker}(P_R) \to R_{\mathrm{ee}}$$

are ring homomorphisms, in other words one has

(iii) $(x \mid y)_H (u \mid v)_H = (xu \mid yv)_H$,
(iv) $T(ab) + T(a)T(b) = 0$.

Let us observe that $T(abc) = T(a)T(b)T(c)$.

Furthermore the following equations hold

(v) $P(a\Delta(x)) = P(a)x$,
(vi) $P((x \mid x)_H a) = xP(a)$,
(vii) $H(xy) = (x \mid x)_H H(y) + H(x)\Delta(y)$.

It follows from Lemma 27 that

$$\Delta : \mathsf{Coker}(P_R) \to R_{\mathrm{ee}}$$

is a ring homomorphism.

Let QR denote the category of quadratic rings. We have a full embedding of categories

$$\mathsf{Rings} \subset \mathsf{QR}$$

which identifies rings with quadratic rings R satisfying $R_{\mathrm{ee}} = 0$. This inclusion has a left adjoint given by $R \mapsto \mathsf{Coker}(P_R)$.

Let R be a quadratic ring. A *right R-quadratic module* is an object $M \in \mathsf{SG}$ together with a right action of the monoid R in the monoidal category $(\mathsf{SG}, \underline{\circledcirc})$. Equivalently, it is given by a square group M together with a right R_{ee}-module structure on M_{ee} and a right action of the multiplicative monoid R_{e} on M_{e} such that the following holds ($m, n \in M_{\mathrm{e}}$, $c \in M_{\mathrm{ee}}$)

(i) $m(x + y) = mx + my$,
(ii) $(m + n)x = mx + nx + P((n \mid m)_H\, H(x))$.
(iii) $(m \mid n)_H\, (x \mid y)_H = (mx \mid ny)_H$,
(iv) $T(ca) + T(c)T(a) = 0$,
(v) $P(c\Delta(x)) = P(c)x$,
(vi) $P((m \mid m)_H\, a) = mP(a)$,
(vii) $H(mx) = (m \mid m)_H\, H(x) + H(m)\Delta(x)$.

The category of right R-quadratic modules is denoted by $\mathsf{QM}\text{-}R$. In a similar manner one can define the notion of a left R-quadratic module.

12.1. **Monoid quadratic rings.** Thanks to Proposition 34 the functor $\mathbb{Z}_{\mathrm{nil}}[-]$ is a monoidal functor from the monoidal category of sets to square groups $(\mathsf{SG}, \underline{\circledcirc})$. It follows that for any monoid M one gets a quadratic ring $\mathbb{Z}_{\mathrm{nil}}[M]$ and thus the functor

$$\mathsf{Monoids} \to \mathsf{QR}.$$

Now we observe that for a quadratic ring R the set of linear elements $\mathbb{L}(R)$ is a multiplicative submonoid of R_{e}. This follows directly from the formula $H(xy) = (x \mid x)_H\, H(y) + H(x)\Delta(y)$. Thus we obtain the functor

$$\mathbb{L} : \mathsf{QR} \to \mathsf{Monoids}$$

The following result is a multiplicative version of Proposition 17.

Proposition 40. *The functor $\mathbb{Z}_{\mathrm{nil}}[-] : \mathsf{Monoids} \to \mathsf{QR}$ is left adjoint to the functor $\mathbb{L}$.*

Proof. Let M be a monoid and let R be a quadratic ring. Given a morphism $f : \mathbb{Z}_{\mathrm{nil}}[M] \to R$ of quadratic rings, the corresponding map $f_0 : M \to \mathbb{L}(R)$ is just the composite $M \subset \mathbb{Z}_{\mathrm{nil}}[M] \to R$ and hence is multiplicative.

Conversely we must show that any homomorphism $f_0 : M \to \mathbb{L}(R) \subset R_{\mathrm{e}}$ of monoids extends uniquely to a quadratic ring morphism $f : \mathbb{Z}_{\mathrm{nil}}[M] \to R$. In the proof of Proposition 17 we already constructed the morphism of square groups $f : \mathbb{Z}_{\mathrm{nil}}[M] \to R$ extending f_0. It remains to show that f_{e} and f_{ee} are multiplicative. First note that the induced map

$$f_* : \mathbb{Z}[M] = \mathsf{Coker}(P_{\mathbb{Z}_{\mathrm{nil}}[M]}) \to \mathsf{Coker}(P_R)$$

is multiplicative as it is the unique additive extension of the composite monoid homomorphism

$$M \to \mathbb{L}(R) \to R_{\mathrm{e}} \twoheadrightarrow \mathsf{Coker}(P_R).$$

It follows that f_{ee} is multiplicative since by definition it factors into the composition of multiplicative maps as follows.

$$\mathbb{Z}[M]\otimes\mathbb{Z}[M] = \mathsf{Coker}(P_{\mathbb{Z}_{\mathrm{nil}}[M]})\otimes\mathsf{Coker}(P_{\mathbb{Z}_{\mathrm{nil}}[M]}) \xrightarrow{f_*\otimes f_*} \mathsf{Coker}(P_R)\otimes\mathsf{Coker}(P_R) \xrightarrow{(\text{-}|\text{-})_H} R_{\mathrm{ee}}$$

Finally the function Ξ of two variables on $\mathbb{Z}_{\mathrm{nil}}[M]_{\mathrm{e}}$ given by

$$\Xi(x,y) = f_{\mathrm{e}}(xy) - f_{\mathrm{e}}(x)f_{\mathrm{e}}(y)$$

is central in R_{e}. This function vanishes if both x and y are in M, so if we show that it is biadditive it will follow that f_{e} is multiplicative. Indeed

$$\begin{aligned}\Xi(x,y+y') = f_{\mathrm{e}}(xy+xy') - f_{\mathrm{e}}(x)(f_{\mathrm{e}}(y)+f_{\mathrm{e}}(y')) = f_{\mathrm{e}}(xy)+f_{\mathrm{e}}(xy')-f_{\mathrm{e}}(x)f_{\mathrm{e}}(y')\\ = \Xi(xy)+\Xi(xy')\end{aligned}$$

and

$$\begin{aligned}&\Xi(x+x',y) = f_{\mathrm{e}}(xy+x'y+P((\bar{x}\otimes\bar{x}')H(y))) - f_{\mathrm{e}}(x+x')f_{\mathrm{e}}(y) =\\ &f_{\mathrm{e}}(xy)+f_{\mathrm{e}}(x'y)+Pf_{\mathrm{ee}}((\bar{x}\otimes\bar{x}')H(y)) - (f_{\mathrm{e}}(x)f_{\mathrm{e}}(y)+f_{\mathrm{e}}(x')f_{\mathrm{e}}(y)+P((\overline{f_{\mathrm{e}}(x)}\otimes\overline{f_{\mathrm{e}}(x')})Hf_{\mathrm{e}}(y)))\\ &= \Xi(x,y)+\Xi(x',y)+P(f_{\mathrm{ee}}(\bar{x}\otimes\bar{x}')f_{\mathrm{ee}}H(y)) - (\overline{f_{\mathrm{e}}(x)}\otimes\overline{f_{\mathrm{e}}(x')})Hf_{\mathrm{e}}(y)) = \Xi(x,y)+\Xi(x',y)\end{aligned}$$

and we are done. ☐

12.2. **Commutative quadratic rings.** Since $\circledcirc$ defines a symmetric monoidal category structure on SG, one can talk about commutative monoids in this monoidal category. We call them commutative quadratic rings. Hence by definition a quadratic ring R is commutative if the following equations hold

$$\begin{aligned} ba &= ab, && a,b\in R_{\mathrm{ee}};\\ yx &= xy - P(H(x)TH(y)), && x,y\in R_{\mathrm{e}}.\end{aligned}$$

In the following result we obtain a kind of *cup_1-product.* We refer to Remark 44 below for the homotopy theoretic meaning of the groups involving it. The elements of the form $k^M(x)$ are the same as in Corollary 15.

Theorem 41. *Let R be a commutative quadratic ring, then the map $R_{\mathrm{e}} \to \mathsf{Ker}(P)$ of degree 4 given by $x \mapsto H(x)TH(x) \in \mathsf{Ker}(P)$ yields a well-defined quadratic map*

$$\psi : \mathsf{Coker}(P_R) \to \mathsf{Ker}(P : R_{\mathrm{ee}}/(\mathsf{Id}-T) \to R_{\mathrm{e}})$$

satisfying

$$\psi(\bar{x}+\bar{y}) = \psi(\bar{x})+\psi(\bar{y}) - k^R(x)k^R(y)$$

and

$$\psi(\bar{x}\bar{y}) = k^R(x^2)\psi(\bar{y}) + \psi(\bar{x})k^R(y^2).$$

Proof. We have

$$\begin{aligned}
\psi(\bar{x}+\bar{y}) =& H(x+y)TH(x+y)\\
=&(H(x)+H(y)+(x|y)_H)(TH(x)+TH(y)-(y|x)_H)\\
=&H(x)TH(x)+H(y)TH(y)+H(x)TH(y)+H(y)TH(x)\\
&-H(x)(y|x)_H+(x|y)_HTH(x)-H(y)(y|x)_H+(x|y)_HTH(x)-(xy|yx)_H\\
=&\psi(\bar{x})+\psi(\bar{y})+H(x)TH(y)-T(TH(y)H(x))\\
&-H(x)(y|x)_H+T((y|x)_HH(x))-H(y)(y|x)_H+T((y|x)_HH(y))-(xy|yx)_H;
\end{aligned}$$

using commutativity of R_{ee}, this is equal to

$$\psi(\bar{x})+\psi(\bar{y})-(xy|yx)_H+(1-T)(H(x)TH(y)-H(x)(y|x)_H-H(y)(y|x)_H).$$

On the other hand,

$$\begin{aligned}
\Delta(x)\Delta(y) =&((x|x)_H-H(x)+TH(x))((y|y)_H-H(y)+TH(y))\\
=&(xy|xy)_H-(x|x)_HH(y)+(x|x)_HTH(y)-H(x)(y|y)_H+TH(x)(y|y)_H\\
&+H(x)H(y)+TH(x)TH(y)-H(x)TH(y)-TH(x)H(y)\\
=&(xy|xy)_H-(x|x)_HH(y)+T((x|x)_HH(y))\\
&-H(x)(y|y)_H+T(H(x)(y|y)_H)+H(x)H(y)-T(H(x)H(y))\\
&-H(x)TH(y)+T(H(x)TH(y))\\
=&(xy|xy)_H\\
&+(1-T)(-(x|x)_HH(y)-H(x)(y|y)_H+H(x)H(y)-H(x)TH(y)).
\end{aligned}$$

Thus it remains to prove

$$(xy|yx)_H=(xy|xy)_H;$$

but by the hypothesis yx is equal to xy modulo image of P, and $(\_|\_)_H$ vanishes if one of the operands is in the image of P.

For the second equality in the theorem one calculates

$$\begin{aligned}
\psi(xy) =&H(xy)TH(xy)\\
=&((x|x)_HH(y)+H(x)\Delta(y))T((x|x)_HH(y)+H(x)\Delta(y))\\
=&((x|x)_HH(y)+H(x)\Delta(y))((x|x)_HTH(y)+TH(x)\Delta(y))\\
=&(x^2|x^2)_HH(y)TH(y)+H(x)TH(x)\Delta(y)^2\\
&+(x|x)_HH(y)TH(x)\Delta(y)+H(x)\Delta(y)(x|x)_HTH(y)\\
=&(x^2|x^2)_H\psi(y)+\psi(x)\Delta(y^2)\\
&+(x|x)_HH(y)TH(x)\Delta(y)-T((x|x)_HH(y)TH(x)\Delta(y)).
\end{aligned}$$

□

13. Comparison of $\underline{\circledcirc}$, $\circledcirc$ and $\odot$

In this section we prove Proposition 8. Actually, we only show that $\underline{\circledcirc}$ and $\odot$ are isomorphic. The rest is similar. Given the product $A\underline{\circledcirc}B$ we define

$$x \circledcirc y = x\underline{\circledcirc}y - H(x)\bar{\otimes}TH(y).$$

We then must check that the equalities (3) and (5) hold in $A\underline{\circledcirc}B$. Indeed we have

$$\begin{aligned}(x_1+x_2)\circledcirc y &= (x_1+x_2)\underline{\circledcirc}y - H(x_1+x_2)\bar{\otimes}TH(y)\\ &= x_1\underline{\circledcirc}y + x_2\underline{\circledcirc}y + (x_2|x_1)_H\bar{\otimes}H(y) - (H(x_1)+H(x_2)+(x_1|x_2)_H)\bar{\otimes}TH(y)\\ &= x_1\circledcirc y + x_2\circledcirc y + (x_2|x_1)_H\bar{\otimes}H(y) - (x_1|x_2)_H\bar{\otimes}TH(y).\end{aligned}$$

Since m

$$(x_2|x_1)_H\bar{\otimes}H(y) = TT((x_2|x_1)_H)\bar{\otimes}TTH(y) = -T((x_2|x_1)_H)\bar{\otimes}TH(y) = (x_1|x_2)_H\bar{\otimes}TH(y),$$

We see that (3) indeed holds. For (5) one considers

$$\begin{aligned}x\circledcirc P(b) &= x\underline{\circledcirc}P(b) - H(x)\bar{\otimes}THP(b) = (x|x)_H\bar{\otimes}b - H(x)\bar{\otimes}HP(b)\\ &= (x|x)_H\bar{\otimes}b - H(x)\bar{\otimes}b - H(x)\bar{\otimes}T(b) = (x|x)_H\bar{\otimes}b - H(x)\bar{\otimes}b - TTH(x)\bar{\otimes}T(b)\\ &= (x|x)_H\bar{\otimes}b - H(x)\bar{\otimes}b + TH(x)\bar{\otimes}b = \Delta(x)\bar{\otimes}b.\end{aligned}$$

Moreover we have

$$\begin{aligned}H(x\circledcirc y) &= H(x\underline{\circledcirc}y - H(x)\bar{\otimes}TH(y))\\ &= H(x\underline{\circledcirc}y) + H(-H(x)\bar{\otimes}TH(y)) - (x\underline{\circledcirc}y|H(x)\bar{\otimes}TH(y))_H\\ &= H(x\underline{\circledcirc}y) - H(H(x)\bar{\otimes}TH(y)) + (H(x)\bar{\otimes}TH(y)|H(x)\bar{\otimes}TH(y))_H\\ &= (x|x)_H\otimes H(y) + H(x)\otimes\Delta(y) - H(x)\otimes TH(y) + TH(x)\otimes H(y)\\ &= ((x|x)_H + TH(x))\otimes H(y) + H(x)\otimes(\Delta(y) - TH(y))\\ &= (\Delta(x)+H(x))\otimes H(y) + H(x)\otimes((y|y)_H - H(y)) = \Delta(x)\otimes H(y) + H(x)\otimes(y|y)_H\end{aligned}$$

We thus see that $A\underline{\circledcirc}B$ with $x\circledcirc y$ defined as above has all properties required of $A\odot B$ in Definition 7. Conversely, we must show that the relations from Definition 7 imply all the relations for $A\underline{\circledcirc}B$. We have

$$\begin{aligned}(x_1+x_2)\underline{\circledcirc}y =&(x_1+x_2)\circledcirc y + H(x_1+x_2)\bar{\otimes}TH(y)\\ =&x_1\circledcirc y + x_2\circledcirc y + (H(x_1)+H(x_2)+(x_1|x_2)_H)\bar{\otimes}TH(y)\\ =&x_1\underline{\circledcirc}y - H(x_1)\bar{\otimes}TH(y) + x_2\underline{\circledcirc}y - H(x_2)\bar{\otimes}TH(y)\\ &+ H(x_1)\bar{\otimes}TH(y) + H(x_2)\bar{\otimes}TH(y) + (x_1|x_2)_H\bar{\otimes}TH(y)\\ =&x_1\underline{\circledcirc}y + x_2\underline{\circledcirc}y + (x_1|x_2)_H\bar{\otimes}TH(y).\end{aligned}$$

Moreover

$$(x_1|x_2)_H\bar{\otimes}TH(y) = -T(x_2|x_1)_H\bar{\otimes}TH(y) = (x_2|x_1)_H\bar{\otimes}H(y).$$

Next, we check

$$\begin{aligned} x\underline{\circledcirc}P(b) &= x \circledcirc P(b) + H(x)\bar{\otimes}THP(b) = \Delta(x)\bar{\otimes}b + H(x)\bar{\otimes}THP(b) \\ &= (x|x)_H\bar{\otimes}b - H(x)\bar{\otimes}b + TH(x)\bar{\otimes}b + H(x)\bar{\otimes}HP(b) \\ &= (x|x)_H\bar{\otimes}b - H(x)\bar{\otimes}b - TTH(x)\bar{\otimes}Tb + H(x)\bar{\otimes}T(b) + H(x)\bar{\otimes}b = (x|x)_H\bar{\otimes}b. \end{aligned}$$

The remaining conditions of $A\underline{\circledcirc}B$ are trivially satisfied.

14. Bilinear maps for square groups

The tensor product of square groups has a universal property similar to that of abelian groups. To formulate it we need an analog of the notion of a bilinear map for square groups.

Definition 42. For square groups A, B, C a *bilinear map*

$$\phi : (A, B) \to C$$

consists of three maps

$$\phi_{\mathrm{l}}, \phi_{\mathrm{r}} : A_{\mathrm{e}} \times B_{\mathrm{e}} \to C_{\mathrm{e}}$$

and

$$\phi_{\mathrm{ee}} : A_{\mathrm{ee}} \times B_{\mathrm{ee}} \to C_{\mathrm{ee}}$$

such that ϕ_{l} is left linear, ϕ_{r} is right linear, ϕ_{ee} is bilinear and moreover one has

$$\begin{aligned} \phi_{\mathrm{l}}(P(a), y) &= P\phi_{\mathrm{ee}}(a, \Delta(y)), \\ \phi_{\mathrm{r}}(x, P(b)) &= P\phi_{\mathrm{ee}}(\Delta(x), b), \\ H\phi_{\mathrm{l}}(x, y) &= \phi_{\mathrm{ee}}((x|x)_H, H(y)) + \phi_{\mathrm{ee}}(H(x), \Delta(y)), \\ H\phi_{\mathrm{r}}(x, y) &= \phi_{\mathrm{ee}}(\Delta(x), H(y)) + \phi_{\mathrm{ee}}(H(x), (y|y)_H), \\ P\phi_{\mathrm{ee}}(T(a), T(b)) &= -P\phi_{\mathrm{ee}}(a, b), \\ \phi_{\mathrm{l}}(x, y) - \phi_{\mathrm{r}}(x, y) &= P\phi_{\mathrm{ee}}(H(x), TH(y)). \end{aligned}$$

For a bilinear map ϕ as above and a morphism $f : C \to C'$ of square groups the composite bilinear map $f\phi : A \times B \to C'$ is defined by the obvious equalities

$$\begin{aligned} (f\phi)_{\mathrm{l}}(x, y) &= f_{\mathrm{e}}\phi_{\mathrm{l}}(x, y), \\ (f\phi)_{\mathrm{r}}(x, y) &= f_{\mathrm{e}}\phi_{\mathrm{r}}(x, y), \\ (f\phi)_{\mathrm{ee}}(x, y) &= f_{\mathrm{ee}}\phi_{\mathrm{ee}}(x, y). \end{aligned}$$

With this definition we then have

Theorem 43. *For square groups A, B the rules*

$$\begin{aligned} \upsilon_{\mathrm{l}}(x, y) &= x\underline{\circledcirc}y, \\ \upsilon_{\mathrm{r}}(x, y) &= x \circledcirc y, \\ \upsilon_{\mathrm{ee}}(a, b) &= a \otimes b \end{aligned}$$

define a bilinear map

$$\upsilon : (A, B) \to A \odot B$$

which is universal; this means that for any bilinear map $\phi : (A, B) \to C$ there exists a unique morphism $f^\phi : A \odot B \to C$ of square groups with $\phi = f^\phi \upsilon$.

Proof. It is straightforward to check that υ indeed is a bilinear map. Moreover given a bilinear map ϕ we define

$$\begin{aligned} f^\phi_{\mathrm{e}}(x \underline{\circledcirc} y) &= \phi_{\mathrm{l}}(x, y), \\ f^\phi_{\mathrm{e}}(x \circledcirc y) &= \phi_{\mathrm{r}}(x, y), \\ f^\phi_{\mathrm{e}}(a \bar{\otimes} b) &= P\phi_{\mathrm{ee}}(a, b), \\ f^\phi_{\mathrm{ee}}(a \otimes b) &= \phi_{\mathrm{ee}}(a, b). \end{aligned}$$

One checks easily that this indeed defines a morphism of square groups. Uniqueness is then clear because of the relation $a \bar{\otimes} b = P(a \otimes b)$. □

15. Quadratic functors

As mentioned in the introduction square groups correspond to quadratic endofunctors of the category of groups. We now make this correspondence explicit.

15.1. **Quadratic functors.** Let $\mathscr{C}$ be a category with a zero object and finite coproducts. We choose a zero object $0 \in \mathscr{C}$ and for any objects X and Y we choose a coproduct $X \vee Y$ in $\mathscr{C}$. We let $i_1 : X \to X \vee Y$ and $i_2 : Y \to X \vee Y$ be the structural inclusions. The set $\mathsf{Hom}_{\mathscr{C}}(X, Y)$ has a distinguished element, the zero morphism, which is the composite $X \to 0 \to Y$ and which is denoted by 0 again. For morphisms $f : X \to Z$, $g : Y \to Z$ let $(f, g) : X \vee Y \to Z$ be the unique morphism with $(f, g)i_1 = f$ and $(f, g)i_2 = g$. In particular, we have the morphisms $r_1 : X \vee Y \to X$ and $r_2 : X \vee Y \to Y$ given respectively by $r_1 = (\mathsf{Id}_X, 0)$ and $r_2 = (0, \mathsf{Id}_Y)$. By definition one has $r_1 i_1 = \mathsf{Id}_X$, $r_2 i_2 = \mathsf{Id}_Y$, $r_1 i_2 = 0$ and $r_2 i_1 = 0$.

We consider functors $F : \mathscr{C} \to \mathsf{Groups}$ to the category of groups with $F(0) = 0$. For a morphism $f : X \to Y$ in $\mathscr{C}$ let f_* denote the induced morphism $F(f) : F(X) \to F(Y)$. Then the canonical homomorphism

$$(r_{1*}, r_{2*}) : F(X \vee Y) \to F(X) \times F(Y)$$

is always surjective. The kernel of this map is denoted by $F(X|Y)$ and is called the *second cross effect* of F. A functor F is called *linear* provided the second cross effect vanishes, thus F is linear iff

$$(r_{1*}, r_{2*}) : F(X \vee Y) \to F(X) \times F(Y)$$

is an isomorphism. Moreover, F is called *quadratic* provided $F(X|Y)$ is linear in X and Y. Any linear functor has values in the subcategory Ab of abelian groups, while values of any quadratic functor lie in the subcategory Nil of groups of nilpotence class two [7]. Moreover, for any quadratic functor F the group $F(X|Y)$ is a central subgroup of $F(X \vee Y)$ [7].

15.2. **Pre-square groups and quadratic functors on the category of finite pointed sets.** Let Γ be the category of finite pointed sets. For any $n \geqslant 0$ we denote by $[n]$ the set $\{0, \cdots, n\}$. We consider $[n]$ as an object of Γ with basepoint 0. Let $\mathsf{Quad}(\Gamma)$ be the category of all quadratic functors from Γ to Groups. There is an equivalence of categories

$$\mathsf{PSG} \cong \mathsf{Quad}(\Gamma) \tag{8}$$

between the category of pre-square groups and the category of quadratic functors from Γ to Groups, which is a particular case of results obtained in [20]. We now discuss functors involved in this equivalence.

There is a bifunctor

$$\Gamma \times \mathsf{PSG} \to \mathsf{Groups}$$

given as follows. Let M be a pre-square group and S a pointed set with basepoint $*$. Then $S \odot M$ is the group generated by the symbols $s \odot x$ and $[s,t] \odot a$ with $s,t \in S$, $x \in M_{\mathrm{e}}$, $a \in M_{\mathrm{ee}}$ subject to the relations

$$\begin{aligned}
&[s,s] \odot a = s \odot P(a) \\
&* \odot x = 0 = [*,s] \odot a \\
&[s,t] \odot a = [t,s] \odot T(a) \\
&[s,t] \odot \{x,y\} = -t \odot x - s \odot y + t \odot x + s \odot y.
\end{aligned}$$

Here $s \odot x$ is linear in x and $[s,t] \odot a$ is central and linear in a. For such M the functor $(-) \odot M : \Gamma \to \mathsf{Groups}$ is quadratic. In this way one obtains a functor $\mathsf{PSG} \to \mathsf{Quad}(\Gamma)$. A functor cr in the opposite direction has the following description. Let $F : \Gamma \to \mathsf{Groups}$ be a quadratic functor. Then we set

$$\mathrm{cr}(F)_{\mathrm{e}} := F([1]), \quad \mathrm{cr}(F)_{\mathrm{ee}} := F([1]|[1]).$$

Hence one gets a central extension of groups

$$0 \to \mathrm{cr}(F)_{\mathrm{ee}} \to F([2]) \to \mathrm{cr}(F)_{\mathrm{e}} \times \mathrm{cr}(F)_{\mathrm{e}} \to 0.$$

The bracket $\{-,-\} : \mathrm{cr}(F)_{\mathrm{e}} \times \mathrm{cr}(F)_{\mathrm{e}} \to \mathrm{cr}(F)_{\mathrm{ee}}$ is defined by

$$\{x,y\} := [F(i_1)(x), F(i_2)(y)] \in F([2])$$

where $x, y \in F([1]) = \mathrm{cr}(F)_{\mathrm{e}}$ and $i_1, i_2 : [1] \to [2]$ are pointed maps given by $i_1(1) = 1$ and $i_2(1) = 2$ respectively. Moreover, the pointed involution $[2] \to [2]$ given by $1 \mapsto 2$, $2 \mapsto 1$ yields an involution on $\mathrm{cr}(F)_{\mathrm{ee}}$ which is denoted by T. We have also the homomorphism $P : \mathrm{cr}(F)_{\mathrm{ee}} \to \mathrm{cr}(T)_{\mathrm{e}}$ induced by the pointed map $[2] \to [1]$ given by $1, 2 \mapsto 1$. One checks that in this way one obtains a well-defined pre-square group $\mathrm{cr}(F)$. Then the functor $\mathrm{cr} : \mathsf{Quad}(\Gamma) \to \mathsf{PSG}$ is an equivalence of categories whose quasi-inverse is the functor $M \mapsto (-) \odot M$.

Linear functors correspond to pre-square groups with $M_{\mathrm{ee}} = 0$. Any such functor is isomorphic to a functor $S \mapsto A \otimes \bar{\mathbb{Z}}(S)$, where A is an abelian group. Here S is a pointed set with base point $*$, while $\bar{\mathbb{Z}}(S)$ is the free abelian group generated by S modulo the relation $* = 0$.

15.3. **Square groups and quadratic endofunctors of the category of groups.** We restrict ourselves to endofunctors $F : \mathsf{Groups} \to \mathsf{Groups}$ preserving filtered colimits and reflexive coequalizers. The last condition means that for any simplicial group G_* the canonical homomorphism $\pi_0(F(G_*)) \to F(\pi_0(G_*))$ is an isomorphism. Such a functor F is completely determined by the restriction of F to the subcategory of finitely generated free groups. Let $\mathsf{Lin}(\mathsf{Groups})$ (resp. $\mathsf{Quad}(\mathsf{Groups})$) be the category of such linear (resp. quadratic) endofunctors.

A composite of linear endofunctors is linear, therefore $\mathsf{Lin}(\mathsf{Groups})$ is actually a monoidal category, where the monoidal structure is induced by composition of endofunctors. Any endofunctor in $\mathsf{Lin}(\mathsf{Groups})$ is isomorphic to a functor T of the form

$$T(X) = A \otimes X^{\mathrm{ab}}$$

where A is an abelian group. Therefore there is a monoidal equivalence of monoidal categories

$$(\mathsf{Lin}(\mathsf{Groups}), \circ) \simeq (\mathsf{Ab}, \otimes).$$

If T_i, $i = 1, 2$ are quadratic endofunctors of the category of groups, then the composite $T_1 \circ T_2$ in general is not quadratic, but it has a maximal quadratic quotient which is denoted by $T_2 \square T_1$. Then $\square$ defines a (highly nonsymmetric) monoidal category structure on the category $\mathsf{Quad}(\mathsf{Groups})$ [7].

We now describe an equivalence of categories

$$\mathsf{SG} \cong \mathsf{Quad}(\mathsf{Groups}).$$

For any square group M and any group G one defines the group $G \otimes M$ [7] by the generators $g \otimes x$ and $[g, h] \otimes a$ with $g, h \in G$, $x \in M_{\mathrm{e}}$ and $a \in M_{\mathrm{ee}}$ subject to the relations

$$\begin{aligned}(g + h) \otimes x &= g \otimes x + h \otimes x + [g, h] \otimes H(x),\\ [g, g] \otimes a &= g \otimes P(a).\end{aligned}$$

Here $g \otimes x$ is linear in x and $[g, h] \otimes a$ is central and linear in each variable g, h and a. One can check that the functor

$$(-) \otimes M : \mathsf{Groups} \to \mathsf{Groups}$$

preserves filtered colimits and reflexive coequalizers. For any groups X and Y one has by [7] the following short exact sequence.

$$0 \to X^{\mathrm{ab}} \otimes Y^{\mathrm{ab}} \otimes M_{\mathrm{ee}} \to (X \vee Y) \otimes M \to (X \otimes M) \times (Y \otimes M) \to 0$$

Therefore $(-) \otimes M : \mathsf{Groups} \to \mathsf{Groups}$ is a quadratic functor and hence this functor is in the category $\mathsf{Quad}(\mathsf{Groups})$. In this way one obtains a functor $\mathsf{SG} \to \mathsf{Quad}(\mathsf{Groups})$. The functor in the opposite direction has the following description. If $F : \mathsf{Groups} \to \mathsf{Groups}$ is a quadratic functor we put

$$\mathrm{cr}(F)_{\mathrm{e}} = F(\mathbb{Z}), \quad \mathrm{cr}(F)_{\mathrm{ee}} = F(\mathbb{Z} \mid \mathbb{Z}).$$

The homomorphism P of the square group $\mathrm{cr}(F)$ is the restriction of the homomorphism $(\mathsf{Id},\mathsf{Id})_* : F(\mathbb{Z}*\mathbb{Z}) \to F(\mathbb{Z})$. Here $*$ is the coproduct in the category of groups, so that $\mathbb{Z}*\mathbb{Z}$ is the free group on two generators e_1 and e_2 . The map H is given by

$$H(x) = \mu_*(x) - p_2(\mu_* x) - p_1(\mu_* x)$$

Here $\mu : \mathbb{Z} \to \mathbb{Z}*\mathbb{Z}$ is the unique homomorphism which sends 1 to e_1+e_2, while p_1 and p_2 are endomorphisms of $\mathbb{Z}*\mathbb{Z} \to \mathbb{Z}*\mathbb{Z}$ such that $p_i(e_i)=e_i$, $i=1,2$ and $p_i(e_j)=0$, if $i\neq j$.

For example the quadratic functor corresponding to the square group $A^{\otimes}$ is given by

$$G \mapsto A\otimes G^{\mathrm{ab}} \otimes G^{\mathrm{ab}}.$$

Here $A\in \mathsf{Ab}$. Similarly, the quadratic functor corresponding to $\mathbb{Z}_{\mathrm{nil}}[S]$ is given by

$$G \mapsto \bigvee_{s\in S} G^{\mathrm{nil}}$$

where the coproduct is taken in the category Nil.

The main result of [7] shows that

$$(\mathsf{SG},\square) \cong (\mathsf{Quad}(\mathsf{Groups}),\square) \tag{9}$$

is a monoidal equivalence of monoidal categories. Here $\square : \mathsf{SG}\times\mathsf{SG} \to \mathsf{SG}$ is defined as follows [7]. If M and N are square groups, then

$$M\square N = (\ (M\square N)_{\mathrm{e}} \xrightarrow{H} (M\square N)_{\mathrm{ee}} \xrightarrow{P} (M\square N)_{\mathrm{e}}\ )$$

where

$$(M\square N)_{\mathrm{e}} = M_{\mathrm{e}}\otimes N/([x,Pa]\otimes c\sim 0), \quad x\in M_{\mathrm{e}}, a\in M_{\mathrm{ee}}, c\in N_{\mathrm{ee}}$$

and

$$(M\square N)_{\mathrm{ee}} = ((M_{\mathrm{ee}}\otimes \mathsf{Coker}(P_N))\oplus(\mathsf{Coker}(P_M)\otimes\mathsf{Coker}(P_M)\otimes N_{\mathrm{ee}}))/\sim$$

where one uses the equivalence relation

$$(x|y)_{H_M}\otimes z \sim \bar{y}\otimes\bar{x}\otimes\Delta(z)$$

Moreover, the homomorphism $P_{M\square N}$ is given by

$$P_{M\square N}(a\otimes\bar{z}) = (Pa)\otimes z, \quad P_{M\square N}(\bar{x}\otimes\bar{y}\otimes c) = [x,y]\otimes c$$

while $H_{M\square N}$ is the unique quadratic map satisfying

$$\begin{aligned}(\bar{x}\otimes\bar{z}|\bar{x'}\otimes\bar{z'})_{H_{M\square N}} &= \bar{x'}\otimes\bar{x}\otimes(z|z')_{H_N}\\ H_{M\square N}(x\otimes z) &= H(x)\otimes\bar{z}+\bar{x}\otimes\bar{x}\otimes H(z),\\ H_{M\square N}([x,y]\otimes c) &= \bar{x}\otimes\bar{y}\otimes c+\bar{y}\otimes\bar{x}\otimes T(c).\end{aligned}$$

The unit object in the monoidal category $(\mathsf{SG},\square)$ is $\mathbb{Z}_{\mathrm{nil}}$ from Section 1.

15.4. **The functor $\wp : \mathsf{SG} \to \mathsf{PSG}$ in terms of quadratic functors.** Recall that we have the functor

$$\wp : \mathsf{SG} \to \mathsf{PSG}$$

defined by

$$\wp(M) = (M_{\mathrm{e}}, M_{\mathrm{ee}}, T = HP - \mathsf{Id}, (-,-)_H, P),$$

where M is a square group.

For a pointed set S let $\langle S\rangle$ be the free group generated by S modulo the relation $* = 0$, where $*$ is the base point of S. Then one has a natural isomorphism

$$S \odot \wp(M) \ \cong\ \langle S\rangle \otimes M.$$

In other words the following diagram commutes.

$$\begin{tikzcd} \Gamma \arrow[r, "\langle - \rangle"] \arrow[dr, "- \odot \wp(M)"'] & \mathsf{Groups} \arrow[d, "- \otimes M"] \\ & \mathsf{Groups} \end{tikzcd}$$

This fact has the following interpretation in the language of functors. Suppose $F : \Gamma \to \mathsf{Groups}$ and $T : \mathsf{Groups} \to \mathsf{Groups}$ are functors, then the composite $T \circ F : \Gamma \to \mathsf{Groups}$ is a well-defined functor. Moreover, if $F = \langle - \rangle$ is the free group functor, then $T \circ F$ is quadratic provided T is quadratic. This follows from the fact that $\langle - \rangle$ respects coproducts. Thus one obtains the functor $\mathsf{Quad}(\mathsf{Groups}) \to \mathsf{Quad}(\Gamma)$ which corresponds to $\wp$ under the equivalences (8) and (9).

If F and T are quadratic functors then $T \circ F$ in general is not quadratic (it is of degree $\leqslant 4$), but has a maximal quadratic quotient denoted by $F\square T$. In terms of (pre)square groups this means that there is a well-defined bifunctor

$$\square : \mathsf{PSG} \times \mathsf{SG} \to \mathsf{PSG}$$

with the property that for square groups M, N

$$\wp(M)\square N = \wp(M\square N).$$

More precisely, if M is a pre-square group and N is a square group, then the definition of $M\square N$ mimics the previous definition in Section 15.3. The groups $(M\square N)_{\mathrm{e}}$ and $(M\square N)_{\mathrm{ee}}$ and the homomorphism $P_{M\square N}$ are defined by the same formulæ , of course now we have to consider the relation $\{x,y\}_M \otimes z \sim \bar{y} \otimes \bar{x} \otimes \Delta(z)$. The bilinear map $\{-,-\}_{M\square N}$ is defined by

$$\{\bar{x} \otimes \bar{z}, \bar{x'} \otimes \bar{z'}\}_{M\square N} = \bar{x'} \otimes \bar{x} \otimes (z \mid z')_{H_N}$$

and the involution $T_{M\square N}$ is given by

$$T_{M\square N}(a \otimes z) = T_M(a) \otimes z, \quad T_{M\square N}(\bar{x} \otimes \bar{y} \otimes c) = \bar{y} \otimes \bar{x} \otimes T_N(c).$$

Remark 44. It is well known that any functor $\Gamma \to \mathsf{Spaces}$ gives rise to a spectrum (see [24] and [11]). If $F : \mathsf{Groups} \to \mathsf{Groups}$ is a functor, then it yield a functor $\tilde{F} : \Gamma \to \mathsf{Spaces}$ given by $S \mapsto B(F(\langle S\rangle))$. Here $B(G)$ is the classifying space of a group G. In particular F gives rise to a spectrum $\mathsf{sp}(F)$, This construction can be applied to $F = (-) \otimes M$, where

$M \in \mathsf{SG}$. According to [8] the homotopy groups of this connective spectrum are isomorphic to the homology groups of the complex

$$\cdots \xrightarrow{\mathsf{Id}-T} M_{\mathrm{ee}} \xrightarrow{\mathsf{Id}+T} M_{\mathrm{ee}} \xrightarrow{\mathrm{Id}-T} M_{\mathrm{ee}} \xrightarrow{\mathsf{Id}+T} M_{\mathrm{ee}} \xrightarrow{\mathrm{Id}-T} M_{\mathrm{ee}} \xrightarrow{P} M_{\mathrm{e}}.$$

Thus in Corollary 15 we constructed for any square group the homomorphism $k^M : \pi_0(\mathsf{sp}(F)) \to \pi_1(\mathsf{sp}(F))$ for $F = (-) \otimes M$, which coincides with the first Postnikov invariant of the spectrum $\mathsf{sp}(F)$ (see [8]). Similarly, for any commutative quadratic ring R in Theorem 41 we constructed a nontrivial cup$_1$-quadratic map $\psi : \pi_0(\mathsf{sp}(F)) \to \pi_1(\mathsf{sp}(F))$ for $F = (-) \otimes R$. Here ψ is not an invariant of the homotopy type of the spectrum $\mathsf{sp}(F)$ but depends on the structure of $\mathsf{sp}(F)$ given by the commutative quadratic ring R.

15.5. **Square rings.** A monoid in the monoidal category $(\mathsf{SG}, \square)$ is termed a *square ring*. More explicitly a square ring can be defined as follows (see [4], [7], [5]). A square ring Q is a square group such that Q_{e} has additionally a multiplicative monoid structure. The multiplicative unit of Q_{e} is denoted by 1. One requires that this monoid structure induces a ring structure on the abelian group $\mathsf{Coker}(P_Q)$ through the canonical projection

$$Q_{\mathrm{e}} \to \mathsf{Coker}(P_Q), \quad a \mapsto \bar{a}.$$

Moreover the abelian group Q_{ee} is a $\mathsf{Coker}(P_Q) \otimes \mathsf{Coker}(P_Q) \otimes (\mathsf{Coker}(P_Q))^{\mathrm{op}}$-module with action denoted by $(\bar{t} \otimes \bar{s}) \cdot u \cdot \bar{r} \in Q_{\mathrm{ee}}$ for $\bar{t}, \bar{s}, \bar{r} \in \mathsf{Coker}(P_Q)$, $u \in Q_{\mathrm{ee}}$. In addition the following identities must be satisfied, where $H(2) = H(1+1)$:

(i) $(x \mid y)_H = (\bar{y} \otimes \bar{x}) \cdot H(2)$,
(ii) $T((\bar{x} \otimes \bar{y}) \cdot a \cdot \bar{z}) = (\bar{y} \otimes \bar{x}) \cdot T(a) \cdot \bar{z}$,
(iii) $P(a)x = P(a \cdot x)$,
(iv) $xP(a) = P((\bar{x} \otimes \bar{x}) \cdot a)$,
(v) $H(xy) = (\bar{x} \otimes \bar{x}) \cdot H(\bar{y}) + H(x) \cdot \bar{y}$,
(vi) $(x+y)z = xz + yz + P((\bar{x} \otimes \bar{y}) \cdot H(z))$,
(vii) $x(y+z) = xy + xz$.

The category of square rings is denoted by SR. Square rings Q with $Q_{\mathrm{ee}} = 0$ are precisely rings. Thus we have a full embedding

$$\mathsf{Rings} \to \mathsf{SR}$$

which has a left adjoint given by $Q \mapsto \mathsf{Coker}(P_Q)$. The initial object in the category of SR is $\mathbb{Z}_{\mathrm{nil}}$. Observe that in any square ring Q one has $\Delta(a) = H(2) \cdot \bar{a}$ for any $a \in Q_{\mathrm{e}}$.

15.6. **Relation between the $\square$ and $\underline{\circledcirc}$ products.**

Proposition 45. *For square groups M and N there is a well-defined morphism of square groups*

$$\sigma = \sigma_{M,N} : M \square N \to M \underline{\circledcirc} N$$

given by

$$\begin{aligned}\sigma_{\mathrm{e}}(x\otimes z)&=x\underline{\circledcirc}z,\\ \sigma_{\mathrm{e}}([x,y]\otimes c)&=(y|x)_H\bar{\otimes}c,\\ \sigma_{\mathrm{ee}}(a\otimes\bar{z})&=a\otimes\Delta(z),\\ \sigma_{\mathrm{ee}}(\bar{x}\otimes\bar{y}\otimes c)&=(y|x)_H\otimes c.\end{aligned}$$

Here $x,y\in M_{\mathrm{e}}$,$z\in N_{\mathrm{e}}$ and $c\in M_{\mathrm{ee}}$. Moreover, σ equips the identity functor

$$\mathsf{Id}:(\mathsf{SG},\underline{\circledcirc})\to(\mathsf{SG},\square)$$

with the structure of a lax monoidal functor.

Proof. First we have to check that σ_{e} is well-defined. In other words σ_{e} respects all relations of $M_{\mathrm{e}}\otimes N$ and σ vanishes on $[x,Pa]\otimes c$. The last assertion is easy to check:

$$\sigma_{\mathrm{e}}([x,Pa]\otimes c)=(Pa|x)_H\bar{\otimes}c=0.$$

We have $\sigma_{\mathrm{e}}((x+y)\otimes z)=(x+y)\underline{\circledcirc}z=x\underline{\circledcirc}z+y\underline{\circledcirc}z+(y|x)_H\bar{\otimes}H(z)$. On the other hand $\sigma_{\mathrm{e}}(x\otimes z+y\otimes z+[x,y]\otimes Hz)=x\underline{\circledcirc}z+y\underline{\circledcirc}z+(y|x)_H\bar{\otimes}H(z)$. Thus σ_{e} respects the corresponding relation of the definition of the tensor product $M_{\mathrm{e}}\otimes N$. Similarly

$$\sigma_{\mathrm{e}}([x,x]\otimes c)=(x|x)_H\bar{\otimes}c=x\underline{\circledcirc}P(c),$$

therefore σ_{e} respects another relation of the definition of the tensor product $M_{\mathrm{e}}\otimes N$. Other relations are even easier to check and therefore we omit them. Next we have to show that σ is a morphism of square groups. We have

$$\sigma_{\mathrm{e}}P(a\otimes z)=\sigma_{\mathrm{e}}(Pa\otimes z)=Pa\underline{\circledcirc}z=a\bar{\otimes}\Delta(z)=P\sigma_{\mathrm{ee}}(a\otimes z)$$

and

$$\sigma_{\mathrm{e}}P(x\otimes y\otimes c)=\sigma_{\mathrm{e}}([x,y]\otimes c)=(y|x)_H\bar{\otimes}c=P\sigma(x\otimes y\otimes c)$$

which shows compatibility with P. To show compatibility with H, first we have to check it for cross-effects

$$\begin{aligned}(\sigma_{\mathrm{e}}(x\otimes z)\mid\sigma_{\mathrm{e}}(y\otimes z'))_H&=(x\underline{\circledcirc}z\mid y\underline{\circledcirc}z')\\ &=(x\mid y)_H\underline{\circledcirc}(z\mid z')_H=\sigma_{\mathrm{ee}}(y\otimes x\otimes(z\mid z')_H)=\sigma_{\mathrm{ee}}((x\otimes z\mid y\otimes z')_H).\end{aligned}$$

Then we have

$$\begin{aligned}H(\sigma_{\mathrm{e}}(x\otimes z))=H(x\underline{\circledcirc}z)&=(x\mid x)_H\otimes Hz+Hx\otimes\Delta(z)\\ &=\sigma_{\mathrm{ee}}(x\otimes x\otimes Hz+Hx\otimes z)=\sigma_{\mathrm{ee}}(H(x\otimes z))\end{aligned}$$

as well as

$$\begin{aligned}H\sigma_{\mathrm{e}}([x,y]\otimes c)&=H((y\mid x)_H\bar{\otimes}c)\\ &=(y\mid x)_H\bar{\otimes}c-T(y\mid x)_H\bar{\otimes}Tc\\ &=\sigma_{\mathrm{ee}}(x\otimes y\otimes c+y\otimes x\otimes Tc)\\ &=\sigma_{\mathrm{ee}}H([x,y]\otimes c),\end{aligned}$$

hence σ is compatible with H. $\square$

Theorem 51 below describes an important case, when the transformation σ is an isomorphism.

Corollary 46. *Any quadratic ring R gives rise to a square ring, whose underlying square group is the same, while the $\mathsf{Coker}(P) \otimes \mathsf{Coker}(P) \otimes \mathsf{Coker}(P)^{\mathrm{op}}$-module structure on M_{ee} is given by*

$$(\bar{x} \otimes \bar{y})az = (x|y)_H a\Delta(z).$$

In this way one obtains a functor

$$U : \mathsf{QR} \to \mathsf{SR}.$$

16. Abelian groups with cosymmetry as a full monoidal subcategory

In this section we introduce "abelian groups with cosymmetry" which form a subcategory of the category of square groups. The corresponding quadratic endofunctors of the category of groups of nilpotence class 2 are coproduct preserving.

16.1. **Symmetric monoidal category Cos.** A pair (A, ∂) is called an *abelian group with cosymmetry* if A is an abelian group and $\partial : A \to \mathrm{Sym}^2(A)$ is a map from A to the second symmetric power of A satisfying

$$\partial(a + b) = \partial(a) + \partial(b) + ab, \quad a, b \in A.$$

Let Cos be the category of abelian groups with cosymmetry. We equip this category with a symmetric monoidal structure. To this end we need the maps

$$* : A \times \mathrm{Sym}^2(B) \to \mathrm{Sym}^2(A \otimes B),$$
$$* : \mathrm{Sym}^2(A) \times B \to \mathrm{Sym}^2(A \otimes B)$$

and the homomorphism

$$* : \mathrm{Sym}^2(A) \otimes \mathrm{Sym}^2(B) \to \mathrm{Sym}^2(A \otimes B)$$

defined respectively by

$$(a, bb') \mapsto (a \otimes b)(a \otimes b'),$$
$$(aa', b) \mapsto (a \otimes b)(a' \otimes b),$$

and

$$(aa') \otimes (bb') \mapsto (a \otimes b)(a' \otimes b') + (a' \otimes b)(a \otimes b').$$

Now we define

$$(A, \partial_A) \otimes (B, \partial_B) = (A \otimes B, \partial_{A \otimes B})$$

where $\partial_{A \otimes B} : A \otimes B \to \mathrm{Sym}^2(A \otimes B)$ is given by

$$\partial_{A \otimes B}(a \otimes b) = a * \partial_B(b) + \partial_A(a) * b - \partial_A(a) * \partial_B(b).$$

It is straightforward to check that in this way we really get a symmetric monoidal category $(\mathsf{Cos}, \otimes)$, with unit object given by $(\mathbb{Z}, \binom{-}{2})$

16.2. **Coproduct preserving endofunctors of the category Nil.** Let $T \in \mathsf{Quad}(\mathsf{Groups})$ be a functor. By definition T is a quadratic endofunctor of the category of groups which preserves filtered colimits and reflexive coequalizers. According to [7] values of T lie in the category Nil of groups of nilpotence class two and moreover for any group G there is an isomorphism

$$T(G) \cong T(G^{\mathrm{nil}})$$

This implies that

$$\mathsf{Quad}(\mathsf{Groups}) \cong \mathsf{Quad}(\mathsf{Nil})$$

and we identify any object of $\mathsf{Quad}(\mathsf{Groups})$ with a quadratic endofunctor of the category Nil which preserves filtered colimits and reflexive coequalizers.

Proposition 47. *If $F : \mathsf{Nil} \to \mathsf{Nil}$ preserves finite coproducts then F is quadratic.*

Proof. Since $F(X \vee Y) \cong F(X) \vee F(Y)$ it follows from the exact sequence (2) that one has the following exact sequence

$$0 \to F(X)^{\mathrm{ab}} \otimes F(Y)^{\mathrm{ab}} \to F(X \vee Y) \to F(X) \times F(Y) \to 0.$$

Thus we have to show that the functor G given by $G(X) = F(X)^{\mathrm{ab}}$ is linear. Since $(-)^{\mathrm{ab}} : \mathsf{Nil} \to \mathsf{Ab}$ commutes with coproducts, we see that the functor $G : \mathsf{Nil} \to \mathsf{Ab}$ commutes with finite coproducts, and therefore G is linear, because finite coproducts in Ab are finite products as well. $\square$

Let Quad^{Σ} be the full subcategory of the category $\mathsf{Quad}(\mathsf{Nil})$ with finite coproduct preserving functors as objects. By Proposition 47 the category Quad^{Σ} consists of all endofunctors $T : \mathsf{Nil} \to \mathsf{Nil}$ which commute with all colimits. Therefore we get the following result.

Corollary 48. *If $F_i \in \mathsf{Quad}^{\Sigma}$, $i = 1, 2$ then the composite $F_1 \circ F_2 \in \mathsf{Quad}^{\Sigma}$. Therefore the canonical projection*

$$F_2 \square F_1 \to F_1 \circ F_2$$

is an isomorphism.

It follows that $(\mathsf{Quad}^{\Sigma}, \circ)$ is a monoidal category. Below we show that it is equivalent as a monoidal category to $(\mathsf{Cos}, \otimes)$. In particular $(\mathsf{Quad}^{\Sigma}, \circ)$ is a symmetric monoidal category.

Lemma 49. *Let M be a square group and let $F = (-) \otimes M : \mathsf{Nil} \to \mathsf{Nil}$ be the corresponding functor. Then $F \in \mathsf{Quad}^{\Sigma}$ if and only if the homomorphism of abelian groups*

$$(-|-)_H : \mathsf{Coker}(P) \otimes \mathsf{Coker}(P) \to M_{\mathrm{ee}}$$

is an isomorphism.

Proof. For the groups $M_{\mathrm{e}} = F(\mathbb{Z})$, M_{ee} and $F(\mathbb{Z}) \vee F(\mathbb{Z})$ we have the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & M_{\mathrm{ee}} \arrow[r] & F(\mathbb{Z} \vee \mathbb{Z}) \arrow[r] & F(\mathbb{Z}) \times F(\mathbb{Z}) \arrow[r] & 0 \\ 0 \arrow[r] & F(\mathbb{Z})^{\mathrm{ab}} \otimes F(\mathbb{Z})^{\mathrm{ab}} \arrow[u, "(-|-)_H"] \arrow[r, "{[-,-]}"'] & F(\mathbb{Z}) \vee F(\mathbb{Z}) \arrow[u, "\alpha"] \arrow[r] & F(\mathbb{Z}) \times F(\mathbb{Z}) \arrow[u, "\mathsf{Id}"'] \arrow[r] & 0 \end{tikzcd}$$

It follows that the canonical map $\alpha : F(\mathbb{Z}) \vee F(\mathbb{Z}) \to F(\mathbb{Z} \vee \mathbb{Z})$ is an isomorphism iff $(- \mid -)_H : F(\mathbb{Z})^{\mathrm{ab}} \otimes F(\mathbb{Z})^{\mathrm{ab}} \to M_{\mathrm{ee}}$ is an isomorphism. Since this map always factors through $\mathsf{Coker}(P) \otimes \mathsf{Coker}(P)$ it follows also that the quotient map $F(\mathbb{Z})^{\mathrm{ab}} \otimes F(\mathbb{Z})^{\mathrm{ab}} \to \mathsf{Coker}(P) \otimes \mathsf{Coker}(P)$ is an isomorphism as well. Thus we have proved the "if" part. Assume $(- \mid -)_H : \mathsf{Coker}(P) \otimes \mathsf{Coker}(P) \to M_{\mathrm{ee}}$ is an isomorphism. Then we have the following exact sequence

$$0 \to M_{\mathrm{e}} \otimes M_{\mathrm{e}} \otimes X^{\mathrm{ab}} \otimes Y^{\mathrm{ab}} \to F(X \vee Y) \to F(X) \times F(Y) \to 0.$$

We have to show that $F(X) \vee F(Y) \to F(X \vee Y)$ is an isomorphism for all $X, Y \in \mathsf{Nil}$. Since F respects reflexive coequalizers, it suffices to assume that X, Y are free in Nil. Since F preserves filtered colimits we can assume that X and Y are finitely generated free. So it suffices to show that for all n the natural map from the n-th copower of $F(\mathbb{Z})$ to $F(X)$ is an isomorphism, where $X = \mathbb{Z} \vee \cdots \vee \mathbb{Z}$ (n-fold coproduct). We already proved the statement for $n = 2$. Since it is also clear for $n = 0$ or $n = 1$ we can proceed by induction. Let Y be the $(n-1)$-st copower of $\mathbb{Z}$. Then we have the following exact sequence

$$0 \to M_{\mathrm{e}}^{\mathrm{ab}} \otimes M_{\mathrm{e}}^{\mathrm{ab}} \otimes Y^{\mathrm{ab}} \to F(Y \vee \mathbb{Z}) \to F(Y) \times M_{\mathrm{e}} \to 0.$$

We also have the following exact sequence

$$0 \to F(Y)^{\mathrm{ab}} \otimes M_{\mathrm{e}}^{\mathrm{ab}} \to F(Y) \vee M_{\mathrm{e}} \to F(Y) \times M_{\mathrm{e}} \to 0.$$

Since $Y^{\mathrm{ab}} = \mathbb{Z}^{n-1}$ and since by the inductive assumption $F(Y)^{\mathrm{ab}} = (M_{\mathrm{e}} \vee \cdots \vee M_{\mathrm{e}})^{\mathrm{ab}} = (M_{\mathrm{ee}}^{\mathrm{ab}})^{n-1}$ the result follows. □

Corollary 50. *The category* Quad^{Σ} *is equivalent to the full subcategory* SG_{Σ} *of* SG *consisting of square groups* M *for which*

$$(- \mid -)_H : \mathsf{Coker}(P_M) \otimes \mathsf{Coker}(P_M) \to M_{\mathrm{ee}}$$

is an isomorphism.

Proof. It is enough to notice that $\mathsf{Quad}^{\Sigma} \subset \mathsf{Quad}(\mathsf{Nil})$ and therefore an object $F \in \mathsf{Quad}^{\Sigma}$ is isomorphic to a functor $F = (-) \otimes M$, with a square group M. The rest follows from Lemma 49. □

Now we define a functor

$$\mathscr{J} : \mathsf{Cos} \to \mathsf{SG}_{\Sigma}$$

which is based on a construction from [13]. Let ∂ be a cosymmetry on an abelian group A. The square group $\mathscr{J}(A,\partial)$ is defined as follows. Consider the pullback diagram in the category of sets:

$$\begin{tikzcd}
(\mathscr{J}(A,\partial))_{\mathrm{e}} \arrow[r, "H"] \arrow[d, "p"'] & A\otimes A \arrow[d, "\pi"] \\
A \arrow[r, "\partial"'] & \mathrm{Sym}^2(A)
\end{tikzcd}$$

where $\pi(a\otimes b)=ab$. Then $\mathscr{J}(A,\partial)_{\mathrm{e}}$ is a group via

$$(a,x)+(b,y)=(a+b,x+y+a\otimes b)$$

where $a,b\in A$ and $x,y\in A\otimes A$ are such elements that $\pi(x)=\partial(a)$ and $\pi(y)=b$. Let $\mathscr{J}(A,\partial)_{\mathrm{ee}}$ be $A\otimes A$, with P given by $P(a\otimes b)=(0,a\otimes b-b\otimes a)\in\mathscr{J}(A,\partial)_{\mathrm{e}}$. One easily shows that

$$\mathscr{J}(A,\partial)=(\mathscr{J}(A,\partial)_{\mathrm{e}}\xrightarrow{H}\mathscr{J}(A,\partial)_{\mathrm{ee}}\xrightarrow{P}\mathscr{J}(A,\partial)_{\mathrm{e}})\in\mathsf{SG}.$$

Let ∂ be a cosymmetry on an abelian group A. It follows from the construction of the square group $M=\mathscr{J}(A,d)$ that

$$\mathsf{Coker}(P_M)\cong A\cong(M_{\mathrm{e}})^{\mathrm{ab}}$$

and the commutator map $\Lambda^2(A)\to M_{\mathrm{e}}$ is a monomorphism. Moreover, the map

$$(-\mid-)_H: A\otimes A\to M_{\mathrm{ee}}$$

is an isomorphism. Hence we get in fact the functor

$$\mathscr{J}:\mathsf{Cos}\to\mathsf{SG}_\Sigma.$$

For example, for a set S there is a unique cosymmetry structure on $A=\mathbb{Z}[S]$ such that $\partial(s)=0$ for all $s\in S$. In this case one has

$$\mathscr{J}(A,\partial)\cong\mathbb{Z}_{\mathrm{nil}}[S].$$

The following theorem is a reformulation of a result from [13].

Theorem 51.

i) *The above functor*

$$\mathscr{J}:\mathsf{Cos}\to\mathsf{SG}_\Sigma$$

is an equivalence of categories.

ii) *A morphism $f:M\to N$ in SG_Σ is an isomorphism iff the induced homomorphism $\mathsf{Coker}(P_M)\to\mathsf{Coker}(P_N)$ is an isomorphism.*

iii) *Let M and N be square groups. If $M,N\in\mathsf{SG}_\Sigma$, then $M\underline{\circledcirc}N\in\mathsf{SG}_\Sigma$ and $M\square N\in\mathsf{SG}_\Sigma$ and the canonical morphism $\sigma:M\square N\to M\underline{\circledcirc}N$ is an isomorphism.*

iv) *Let (A,∂) and (A',∂') be abelian groups with cosymmetry. Then there is a natural isomorphism*

$$\mathscr{J}(A,\partial)\underline{\circledcirc}\mathscr{J}(A',\partial')\cong\mathscr{J}((A,\partial)\otimes(A',\partial'))$$

Thus the functor

$$\mathscr{J} : (\mathsf{Cos}, \otimes) \to (\mathsf{SG}, \underline{\circledcirc})$$

is symmetric monoidal; moreover it is full.

Proof. By Corollary 50 we know that Cos is equivalent to the category of square groups for which $M_{\mathrm{ee}} = \mathsf{Coker}(P) \otimes \mathsf{Coker}(P)$ and $\{-|-\} : \mathsf{Coker}(P) \otimes \mathsf{Coker}(P) \to M_{\mathrm{ee}}$ is the identity map. It follows from the definition of square groups that

$$P(a \otimes b) = a \otimes b - b \otimes a$$

and

$$H(x + y - x - y) = -b \otimes a + a \otimes b.$$

Thus we have the following commutative diagram with exact rows

$$\begin{tikzcd}
 & \Lambda^2 A \arrow[r, "\mathsf{c}"] \arrow[d, equal] & M_{\mathrm{e}} \arrow[r] \arrow[d, "H"] & A \arrow[r] \arrow[d, dashed, "\delta"] & 0 \\
0 \arrow[r] & \Lambda^2 A \arrow[r, "\mathsf{d}"'] & A \otimes A \arrow[r, "\pi"'] & \mathrm{Sym}^2(A) \arrow[r] & 0
\end{tikzcd}$$

which yields a cosymmetry $\delta : A \to \mathrm{Sym}^2(A)$. Here c is the commutator map $\mathsf{c}(\bar{x} \wedge \bar{y}) = x+y-x-y$ and $\mathsf{d}(a \wedge b) = a \otimes b - b \otimes a$. It follows that $\mathsf{c} : \Lambda^2 A \to M_{\mathrm{e}}$ is a monomorphism and $M \cong \mathscr{J}(A, \partial)$. In this way we have constructed a functor Ψ in the opposite direction, hence the equivalence of categories in i) is proved. Explicitly, we have $\Psi(M) = (\mathsf{Coker}(P_M), \delta_M)$, where δ_M is induced from the composite

$$H : M_{\mathrm{e}} \to M_{\mathrm{ee}} \cong \mathsf{Coker}(P_M) \otimes \mathsf{Coker}(P_M) \to \mathrm{Sym}^2(\mathsf{Coker}(P_M)).$$

ii) follows from i) because f is an isomorphism iff $\Psi(f)$ is an isomorphism. iii) First we consider the case of the $\underline{\circledcirc}$-product. It suffices to check that $H_{M \underline{\circledcirc} N}$ yields an isomorphism $\mathsf{Coker}(P_{M \underline{\circledcirc} N}) \otimes \mathsf{Coker}(P_{M \underline{\circledcirc} N}) \to (M \underline{\circledcirc} N)_{\mathrm{ee}}$. But this is clear, since $\mathsf{Coker}(P_{M \underline{\circledcirc} N})$ is $\mathsf{Coker}(P_M) \otimes \mathsf{Coker}(P_N)$. For the $\square$-product this follows from Corollary 48 and Corollary 50. It remains to prove iv), which claims compatibility with the $\underline{\circledcirc}$ product. This follows from the explicit description of this operation. $\square$

16.3. **An open question.** Let $\mathbf{T}$ be an algebraic theory, and consider the category $\mathsf{End}^{\Sigma}(\mathbf{T})$ of all endofunctors of the category of models of $\mathbf{T}$ preserving all colimits. Then $\mathsf{End}(\mathbf{T})$ is closed with respect to composition. Under what conditions is the monoidal category $(\mathsf{End}^{\Sigma}(\mathbf{T}), \circ)$ symmetric? Does this happen if $\mathbf{T}$ is the theory of nilpotent groups of any class n? It is obviously so if $n = 1$ and it follows from our results that the same is true for $n = 2$. A classical result of Kan [17] implies that this also holds for the algebraic theory of groups.

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