ON THE DIFFERENCE EQUATION OF THE POINCARÉ TYPE (Part 4).

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Dedicated to the memory of Professor A.O. Gelfond.

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§0. Foreword.

In [77]: was proved the following

Theorem 7. *Let* $s \in \mathbb{N} - 1, n \in \mathbb{N}$,

$$a_i^{\sim} \in \mathbb{C},\ a_i(\nu) \in \mathbb{C},$$

$$a_n(\nu) = 1,\ a_i(\nu) - a_i^{\sim} = O(1/(\nu + 1)) \tag{1}$$

for $\nu \in \mathbb{N} - 1$ *and* $i = 0, \ldots, n$. *Let us consider the following difference equation*

$$\sum_{k=0}^{n} a_k(\nu)y(\nu + k) = 0, \tag{2}$$

where $\nu \in \mathbb{N} - 1$.

For $m \in \mathbb{N}$ let V_m denotes the linear over $\mathbb{C}$ space of solutions $y = y(\nu)$ of the equation

$$\sum_{k=0}^{n} a_k(\nu)y(\nu + k) = 0, \tag{3}$$

where $\nu \in m + \mathbb{N} - 1$. Let the absolute values of all the roots of the characteristical polynomial

$$T(z) = \sum_{k=0}^{n} a_k^{\sim} z^k \tag{4}$$

are among the numbers $\{\rho_i \colon 1 \le i \le 1 + s\}$ such that $\rho_{s+1} = 0$ and $\rho_j < \rho_i$ for $1 \le i < j \le s + 1$. Let e_i and k_i denote respectively the sum and the maximum of the multiplicities of those roots, whose absolute value is equal to the number ρ_i, where $i = 1, \dots, s + 1$, and let $k^* = k_{s+1}$. We suppose that, if $s > 0$, then

$$e_i > 0 \tag{5}$$

for $i = 1 \dots, s$. For given $y = y(\nu)$ in $\mathbb{C}^{m-1+\mathbb{N}}$, let

$$\omega_{n,y}(\nu) = \max(|y(\nu)|, \dots, |y(\nu + n - 1)|).$$

Then there exist $A > 0$, $m \in \mathbb{N}$, $\alpha^{\wedge}(\nu) > 0$ with $\nu \in m + \mathbb{N} - 1$ and the subspaces $V^{\vee}_{m,1}, \dots, V^{\vee}_{m,s+1}$ such that

$$\lim_{\nu\to\infty} \alpha^{\wedge}(\nu) = 0,$$

$$V_m = V^{\vee}_{m,1} \oplus \dots, \oplus V^{\vee}_{m,s+1}, \ \dim_{\mathbb{C}}(V^{\vee}_{m,i}) = e_i, \ 1 \le i \le s + 1,$$

and, if $y \in V^{\vee}_{m,\theta}$ for some $\theta \in \{1, ..., s\}$, then

$$\exp(-A(ln(\nu) + \nu^{1-1/k_\theta}))(\rho_\theta)^\nu \omega_n(y)(m) \le \omega_{n,y}(\nu) \tag{6}$$

for $\nu \in m + \mathbb{N} - 1$; moreover, the spaces

$$V^{\wedge}_{m,j} = V^{\vee}_{m,j} \oplus \dots \oplus V^{\vee}_{m,s+1},$$

where $j = 1 \dots, s + 1$, and, if $s \ge 1$ natural projections π_j,

$$V^{\wedge}_{m,j} \mapsto V^{\vee}_{m,j},$$

where $j = 1 \dots, s$, have the following properties:

if $y \in V^{\wedge}_{m,\theta}$ for some $\theta \in \{1, ..., s\}$, then

$$\omega_{n,y}(\nu) \le \exp(A(\ln(\nu) + \nu^{1-1/k_\theta}))(\rho_\theta)^\nu \omega_{n,y}(m), \tag{7}$$

$$(8) \qquad (\omega_{n\pi_{\theta(y)}}(m) - \alpha(\nu)\omega_{n,y}(m))(\rho_\theta)^\nu$$

$$\exp(-A(\ln(\nu) + \nu^{1-1/k_\theta})) \le \omega_{n,y}(\nu),$$

where $\nu \in m + \mathbb{N} - 1$; *if*

$$(9) \qquad k^* > 0,$$

and $y \in V^\vee_{m,s+1} (= V^\wedge_{m,s+1})$, *then*

$$(10) \qquad |y(\nu)| \le (A/\nu)^{\nu/k^*}\omega_{n,y}(m),$$

where $\nu \in m + \mathbb{N} - 1$.

Remark 1. It follows from the Theorem 7 that the space $V^\wedge_{m,\theta}$, **where** $\theta = 1, \ldots, s+1$, **does not depend from the construction and is defined uniquelly by means of the equality**

$$V^\wedge_{m,\theta} = \{y \in V_m\colon \limsup |y(\nu)|^{1/\nu} \le \rho_\theta\}.$$

The presence of unkown $\alpha(\nu)$ (even tending to zero) in (8) constrict the possibilities of the application of this Theorem. Of course, in view of (6), in the case $e_{s+1} = 0$, $\theta = s$ this $\alpha(\nu)$ cannot play devil with us , because it vanishes then, but such happy case (see, for example, [75]-[76]) is rather an exeption from the rule. One may attempt to estimate the specified value but it would be better to get rid from it at all. With this goal I prove here the following

Theorem 10. *Let are fulfilled all the conditions of the Theorem 7. Let further* V *be an arbitrary linear subspace of* V_m *such that*

$$V \cap V_{m,\theta+1} = \{0\},$$

where $\theta \in \{1 \ldots, s\}$.

Then for this V *there exists a constant* $A^* = A^*(V) > 0$ *such that*

$$(11) \qquad \exp(-A^*(ln(\nu) + \nu^{1-1/k}))(\rho_\theta)^\nu\omega_n(y)(m) \le \omega_{n,y}(\nu)$$

where $y \in V$, $k = \max(k_1, \ldots k_s)$ *and* $\nu \in m + \mathbb{N} - 1$.

First I prove the vollowing

Theorem 8. *Let are fulfilled all the conditions of the Theorem 7. Let further*

$$V^*_{m,j} = V^\vee_{m,1} \oplus \ldots \oplus V^\vee_{m,j},$$

where $j = 1 \ldots, s+1$, *(and* $V^*_{m,1} = V^\vee_{m,1}$.*) Then vor* $V = V^*_{m,\theta}$ *with* $\theta \in \{1, ..., s\}$, *holds the assertion of the Theorem 10.*

Then I prove

Theorem 9. *Let for some* $\theta \in \{1 \ldots, s\}$ *is given a linear map* ξ_θ *of the space* $V^*_{m,\theta}$ *into* $V^\wedge_{m,\theta+1}$. *Let* I^*_θ *is the identity map* $V^*_{m,\theta} \to V^*_{m,\theta}$ *Then for*

$$V = (I^*_\theta + \xi_\theta)(V^*_{m,\theta})$$

holds the assertion of the Theorem 10.

In the section 4 I prove the Theorem 10.

And in the section 5 I discuss, what will take place, if instead (1) the following conditions hold:

$$\lim_{\nu\to\infty} a_i(\nu) = a_i^{\sim}, \tag{12}$$

where $i = 0, \ldots, n$,

$$a_n(\nu) = 1, \tag{13}$$

where $\nu \in \mathbb{N} - 1$.

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§1. Some preparatory results.

Lemma 1. *Let $a \in \mathbb{N}$, $b \in \mathbb{N} - 1 + a$, $C > 0$ Then*

$$0 < \sum_{\kappa=a}^{b} \ln(1 + C/\kappa) \leq \tag{14}$$

$$\ln(1 + C/a) + b\ln(1 + C/b) - a\ln(1 + C/a) + C\ln((b + C)/(a + C)).$$

Proof. See the proof of the Lemma 1 in [77]. ■

Corollary. *If $a \in \mathbb{N}$, $b \in \mathbb{N} + a - 1$, $b < 2a$, $C > 0$, then*

$$\sum_{\kappa=a}^{b} \ln(1 + C/\kappa) \leq 3C.$$

Proof. See the Proof of the Corollary of the Lemma 1 in [77]. ■

Lemma 2. *([51], Lemma 2, [44], Lemma 2, [72], Lemma 8.)*

Let $A \in Mat_n(\mathbb{C})$ an let k is a maximal order of its Jordan blocks. Then there exists a constante $\gamma^*(A) > 0$ with the following properties:

for any $\varepsilon > 0$ there exists a norm $p_{A,\varepsilon}$ on $\mathbb{C}^n$ such that

$$p_{A,\varepsilon} \leq \gamma^*(A)(\max(1, 1/\varepsilon)^{k-1}h, \tag{15}$$

$$h \leq \gamma^*(A)(\max(1, \varepsilon)^{k-1}p_{A,\varepsilon}, \tag{16}$$

$$(p_{A,\varepsilon})^{\sim} \leq (\gamma^*(A))^2(\max(\varepsilon, 1/\varepsilon)^{k-1}h^{\sim}, \tag{17}$$

$$h^{\sim} \leq (\gamma^*(A))^2(\max(\varepsilon, 1/\varepsilon)^{k-1}(p_{A,\varepsilon})^{\sim}, \tag{18}$$

$$\|A\|_{sp} \le (p_{A,\varepsilon})^{\sim} \le \|A\|_{sp} + (sign(k-1))\varepsilon, \tag{19}$$

where $\|A\|_{sp}$ *denotes the maximum of the absolute values of eigenvalues of the matrix* A*. If, moreover,*

$$\det(A) \ne 0,\ \left\|A^{-1}\right\|_{sp}^{-1} > (sign(k-1))\varepsilon, \tag{20}$$

then

$$\left\|A^{-1}\right\|_{sp} \le \tag{21}$$

$$(p_{A,\varepsilon})^{\sim}(A^{-1}) \le \left(\left\|A^{-1}\right\|_{sp}^{-1} - (sign(k-1))\varepsilon\right)^{-1} =$$

$$(sign(k-1))\left\|A^{-1}\right\|_{sp} + \varepsilon\left\|A^{-1}\right\|_{sp}\left(\left\|A^{-1}\right\|_{sp}^{-1} - (sign(k-1))\varepsilon\right)^{-1}.$$

Proof. See the proof of the Lemma 8 in [72]. ■

Corollary. *If all the eigenvalues of the matrix* A *are simple, then*

$$(p_{A,\varepsilon})^{\sim} = \|A\|_{sp}. \tag{22}$$

If, moreover,

$$\det(A) \ne 0, \tag{23}$$

then

$$(p_{A,\varepsilon})^{\sim}(A^{-1}) = \left(\left\|A^{-1}\right\|_{sp}\right)^{-1}. \tag{24}$$

Proof. See the proof of the Corollary of the Lemma 8 in [72]. ■

Lemma 3. *Let all the conditions of the Theorem 7 are fulfilled, and let*

$$k = \max(k_1, \ldots, k_s).$$

Then there exist $A > 0$, $m \in \mathbb{N}$ *such that*

$$\omega_{n,y}(\nu) \le \exp(A(ln(\nu) + \nu^{1-1/k}))(\rho_1)^{\nu}\omega_n(y)(m) \tag{25}$$

for any $y \in V_m$ *and* $\nu \in m + \mathbb{N} - 1$.

If, moreover, $k^* = 0$*, then there exist* $A > 0$, $m \in \mathbb{N}$ *such that*

$$\exp(-A(ln(\nu) + \nu^{1-1/k}))(\rho_s)^{\nu}\omega_{n,y}(m) \le \omega_{n,y}(\nu) \tag{26}$$

for any $y \in V_m$ *and* $\nu \in m + \mathbb{N} - 1$.

Proof. Since $V_m = V^{\wedge}_{m,1}$, it follows that the inequality (7) holds with $\theta = 1$ for any $y \in V_m$. For the full proof of the Lemma let us make some not large changes in the proof of the Lemma 2 in [77].

The condition $k^* = 0$ implies the inequality

$$a_0^{\sim} = T(0) \ne 0, \tag{27}$$

and for p in Theorem 6 of the paper [72] the equality $p = n$. Let

$$A(\nu) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0(\nu) & -a_1(\nu) & -a_2(\nu) & \dots & -a_{n-1}(\nu) \end{pmatrix}, \tag{28}$$

where $\nu \in \mathbb{N} - 1$, and let

$$A^{\sim} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0^{\sim} & -a_1^{\sim} & -a_2^{\sim} & \dots & -a_{n-1}^{\sim}. \end{pmatrix} \tag{29}$$

Let λ_j, where $j = 1, \dots, r$, is the sequence of all the mutually distinct roots of the polynomial (4) and k_j^* is the multiplicity of the root λ_j. Then

$$k = \sup\{k_j^* \colon j = 1, \dots, r\}.$$

Clearly,

$$\sum_{j=0}^{r} k_j^* = n,$$

$$\rho_s \le |\lambda_j| \le \rho_1 = \|A\|_{sp} < R = h^{\sim}(A^{\sim}) + 1,$$

where $j = 1, \dots, r$. In view of (1), there exists $C_1 > 0$ such that

$$h^{\sim}(A(\nu) - A^{\sim}) \le C_1/(\nu + 1). \tag{30}$$

Therefore, according to the Lemma 2, for any $\varepsilon > 0$ the following inequalities holds

$$p^{\sim}_{A^{\sim},\varepsilon}(A(\nu) - A^{\sim}) \le \tag{31}$$

$$(\gamma^*(A^{\sim}))^2(\max(\varepsilon, 1/\varepsilon))^{k-1} h^{\sim}(A(\nu) - A^{\sim}) \le$$
$$(\gamma^*(A^{\sim}))^2(\max(\varepsilon, 1/\varepsilon))^{k-1} C_1/(\nu + 1),$$

$$p^{\sim}_{A^{\sim},\varepsilon}(A(\nu)) \le \rho_1 + sign(k-1)\varepsilon + \tag{32}$$

$$(\gamma^*(A^{\sim}))^2(\max(\varepsilon, 1/\varepsilon))^{k-1} C_1/(\nu + 1) \le$$
$$(\rho_1 + (sign(k-1))\varepsilon)\left(1 + (\gamma^*(A^{\sim}))^2(\max(\varepsilon, 1/\varepsilon))^{k-1}\frac{C_1}{\rho_1(\nu + 1)}\right),$$

$$\ln(p^{\sim}_{A^{\sim},\varepsilon}(A(\nu))) \le \ln(\rho_1) + \ln(1 + (sign(k-1))C_{1,1}\varepsilon) + \tag{33}$$

$$\ln\left(1 + (\max(\varepsilon, 1/\varepsilon))^{k-1} C_{1,2}/(\nu + 1)\right),$$

where

$$C_{1,1} = 1/\rho_1,\ C_{1,2} = (\gamma^*(A^{\sim}))^2 C_1/\rho_1$$

and $\nu \in \mathbb{N}$.

We consider first the case $k > 1$. For given $\nu \in \mathbb{N}+1$ we take $d \in \mathbb{N}$ and $a_i \in [1, \nu] \cap \mathbb{N}$, where $i = 0, \dots, d$, in such a way that

$$a_0 = 1,\ a_d = \nu, a_{i-1} < a_i \leq 2a_{i-1}, \tag{34}$$

where $i = 1, \dots, d$, and

$$d \leq \frac{\ln(\kappa)}{\ln(2)} + 1. \tag{35}$$

According to the Corollary of the Lemma 1 and (34) – (35),

$$\sum_{\kappa=a_{i-1}}^{a_i-1} \ln(p^{\sim}_{A^{\sim},\varepsilon}(A(\kappa))) \leq \tag{36}$$

$$(a_i - a_{i-1})\ln(\rho_1) + (a_i - a_{i-1})\ln(1 + C_{1,1}\varepsilon) +$$
$$\sum_{\kappa=a_{i-1}}^{a_i-1} \ln\left(1 + C_{1,2}(\max(\varepsilon, 1/\varepsilon))^{k-1}/(\kappa+1)\right) \leq$$
$$(a_i - a_{i-1})(\ln(\rho_1) + \ln(1 + \varepsilon/\rho_1)) + 3C_{1,2}(\max(\varepsilon, 1/\varepsilon))^{k-1},$$

where $i = 1, \dots, d$. We take now in (36) $\varepsilon = \varepsilon_i = (a_{i-1})^{-1/k}$. Then we obtain the inequality

$$\sum_{\kappa=a_{i-1}}^{a_i-1} \ln(p^{\sim}_{A^{\sim},\varepsilon}(A(\kappa))) \leq$$
$$(a_i - a_{i-1})\ln(\rho_1) + (C_{1,1} + 3C_{1,2})(a_{i-1})^{1-1/k} =$$
$$(a_i - a_{i-1})\ln(\rho_1) + O(2^{(i-1)(1-1/k)}),$$

where $i = 1, \dots, d$. Therefore

$$\ln(p^{\sim}_{A^{\sim},\varepsilon_i}\left(\prod_{\kappa=1}^{a_i-a_{i-1}} A(a_i - \kappa)\right) \leq \tag{37}$$

$$(a_i - a_{i-1})\ln(\rho_1) + O(2^{(i-1)(1-1/k)}$$

and, in view of (18),

$$\ln(h^{\sim}\left(\prod_{\kappa=1}^{a_i-a_{i-1}} A(a_i - \kappa)\right) \leq \tag{38}$$

$$\ln((\gamma^*(A^{\sim}))^2)(\max(\varepsilon, 1/\varepsilon))^{k-1}) + (a_i - a_{i-1})\ln(\rho_1) + O(2^{(i-1)(1-1/k)}) \leq$$
$$2\ln(\gamma^*(A^{\sim})) + (i-1)(1-1/k)ln(2) + (a_i - a_{i-1})\ln(\rho_1) + O(2^{(i-1)(1-1/k)}) =$$
$$(i-1)(1-1/k)\ln(2) + (a_i - a_{i-1})\ln(\rho_1) + O(2^{(i-1)(1-1/k)}),$$

where $i = 1, \dots, d$. Consequently,

$$\ln\left(h^{\sim}\left(\prod_{\kappa=1}^{nu-1} A(\nu - \kappa)\right)\right) = \tag{39}$$

$$\ln\left(h^{\sim}\left(\left(\prod_{i=1}^{d}\prod_{\kappa=1}^{a_i-a_{i-1}} A(a_i-\kappa)\right)\right)\right)=$$
$$=\nu\ln(\rho_1)+O(\nu^{(1-1/k)}),$$

where $\nu\in\mathbb{N}$.

If $k=1$, then, according to (33),

$$\ln(p^{\sim}_{A^{\sim},1}(A(\nu)))\le\ln(\rho_1)+C_{1,2}/(\nu+1),$$

(40)
$$\ln\left(h^{\sim}\left(\prod_{\kappa=1}^{\nu-1}A(\nu-\kappa)\right)\right)\le$$
$$\ln\left((\gamma^*(A^{\sim}))^2 p^{\sim}_{A^{\sim},1}\left(\prod_{\kappa=1}^{\nu-1}A(\nu-\kappa)\right)\right)\le$$
$$\nu\ln(\rho_1)+O(\ln(e\nu)).$$

where $\nu\in\mathbb{N}$. In view of (39) – (40),

(41)
$$\ln\left(h^{\sim}\left(\prod_{\kappa=1}^{\nu-1}A(\nu-\kappa)\right)\right)=$$
$$=\nu\ln(\rho_1)+O((\nu+1)^{(1-1/k)})+O(\ln(e\nu))),$$

where $\nu\in\mathbb{N}$.

As in Section 3 of [72], let K denotes one of the fields $\mathbb{R}$ or $\mathbb{C}$ and L denotes a linear normed space over K with norm $p=p(x)$. If $L=K^n$, we fix as $p=p(x)$, wehre $x\in K^n$, the maximum of the absolute values of coordinates of the element x in the standard basis, i.e.

(42)
$$p(x)=h(x)=\sup(\{|x_1|,\ \dots\ ,|x_n|\}),$$

where

$$x=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}.$$

If L is a Banach space with the norm p, then $K-$algebra of all the linear continuous operators acting in L will be denoted by $\mathfrak{M}^{\wedge}(L)$, and the norm on $\mathfrak{M}^{\wedge}(L)$, associated with the norm p will be denoted by $p^{\sim}$. So,

$$p^{\sim}(A)=sup(\{p(AX)\colon X\in L,\,p(X)\le 1\}).$$

It is well known that the associciated with h norm on $Mat_n(\mathbb{C})$ is defined as follows

(43)
$$h^{\sim}(A)=sup\left(\left\{\sum_{k=1}^{n}|a_{i,j}|\colon i=1,\ \dots\ ,n\right\}\right),$$

where $A=(a_{i,k})\in Mat_n(\mathbb{C})$. The norms h and $h^{\sim}$ coincide respectiely with with the norms q_∞ and $q^{\sim}_\infty$ considered in section 6 of the paper [68].

Let $m \in \mathbb{N}$, and let $E_m(L)$ be the set $L^{m-1+\mathbb{N}}$ of all the maps of the set $m-1+\mathbb{N}$ into L. The set $E_m(L)$ is a linear space over K, where the muliplication of the elements by the number from K and addition of the elements are defined coordinate-wise. The subspace of $E_m(L)$ composed by all the constant maps is isomorphic to the space L, and we identify this subspace with L.

As in Section 4 of [72], for any $y \in E_m(\mathbb{C})$ and $n \in \mathbb{N}$ let $Y_{n,y}$ and $Y^{\#}_{n,y}$ denote the elements in the space $E_m(\mathbb{C}^n)$, wich are determined respectively by means the following equalities:

(44)
$$Y_{n,y}(\nu) = \begin{pmatrix} y(\nu) \\ \vdots \\ y(\nu+n-1) \end{pmatrix},$$

(45)
$$Y^{\#}_{n,y}(\nu) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ y(\nu) \end{pmatrix},$$

where $\nu \in m-1+\mathbb{N}$. Let is fixed $m \in \mathbb{N}$. If $y = y(\nu)$ is a solution of the equation (3) for $\nu \in \mathbb{N}+m-1$, then

$$Y_{n,y}(\nu) = \left(\prod_{\kappa=1}^{\nu-m} A(\nu-\kappa)\right) Y_{n,y}(m),$$

where $\nu \in \mathbb{N}$, and, in view of (41),

(46)
$$\omega_{n,y}(\nu) = h\left(Y_{n,y}(\nu)\right) \le$$
$$h^{\sim}\left(\prod_{\kappa=1}^{\nu-m} A(\nu-\kappa)\right) h\left(Y_{n,y}(m)\right) =$$
$$\exp\left(O(1)\left(\nu^{1-1/k} + \ln(\nu)\right)\right) (\rho_1)^{\nu}\, \omega_{n,y}(m),$$

where $\nu \in \mathbb{N}+m-1$. So, with $m=1$ the asserted by the Lemma the upper estimate (25) of the value $\omega_{n,y}(\nu)\,(\rho_\theta)^{-\nu} = \omega_{n,y}(\nu)\,(\rho_1)^{-\nu}$ is obtained. We shall take now

(47)
$$\varepsilon \in (0, \rho_s/2).$$

Then, in view of (21),

(48)
$$1/\rho_s \le (p_{A,\varepsilon})^{\sim}(A^{-1}) \le$$
$$1/\rho_s + 2(sign(k-1))\varepsilon/\rho_s^2 \le 2/\rho_s.$$

Let is fixed

(49)
$$m \in \max([(2/\rho_s)^k],\ [2C_1 h^{\sim}\left((A^{\sim})^{-1}\right)]) + \mathbb{N}.$$

Then

$$(50)\qquad h^{\sim}\left((A^{\sim})^{-1}(A(\nu)-A^{\sim})\right)\le h^{\sim}\left((A^{\sim})^{-1}\right)C_1/(\nu+1)\le$$

$$h^{\sim}\left((A^{\sim})^{-1}\right)C_1/([2C_1h^{\sim}\left((A^{\sim})^{-1}\right)]+1)\le 1/2,$$

if $\nu\in m-1+\mathbb{N}$, the matrix $E+((A^{\sim})^{-1}(A(\nu)-A^{\sim}))$ is invertible,

$$h^{\sim}\left(\left(E+((A^{\sim})^{-1}(A(\nu)-A^{\sim}))\right)^{-1}\right)\le 2,$$

if $\nu\in m-1+\mathbb{N}$, there exists the matix

$$(A(\nu))^{-1}=\left(E+\left((A^{\sim})^{-1}(A(\nu)-A^{\sim})\right)\right)^{-1}(A^{\sim})^{-1},$$

if $\nu\in m-1+\mathbb{N}$, moreover,

$$h^{\sim}\left((A(\nu))^{-1}\right)\le h^{\sim}\left(\left(E+\left((A^{\sim})^{-1}(A(\nu)-A^{\sim})\right)\right)^{-1}\right)h^{\sim}\left((A^{\sim})^{-1}\right)\le$$

$$2h^{\sim}\left((A^{\sim})^{-1}\right)$$

and, finally,

$$h^{\sim}\left((A(\nu))^{-1}-(A^{\sim})^{-1}\right)=$$

$$h^{\sim}\left((A(\nu))^{-1}(A^{\sim}-A(\nu))(A^{\sim})^{-1}\right)\le$$

$$h^{\sim}\left((A(\nu))^{-1}\right)h^{\sim}\left((A^{\sim}-A(\nu))\right)h^{\sim}\left((A^{\sim})^{-1}\right)\le C_2/(\nu+1),$$

where

$$C_2=\left(h^{\sim}\left((A^{\sim})^{-1}\right)\right)^2C_1$$

and $\nu\in m-1+\mathbb{N}$. Therefore

$$(51)\qquad p^{\sim}_{A^{\sim},\varepsilon}((A(\nu))^{-1}-(A^{\sim})^{-1})\le$$

$$(\gamma^*(A^{\sim}))^2(\max(\varepsilon,1/\varepsilon))^{k-1}h^{\sim}((A(\nu))^{-1}-(A^{\sim}))^{-1}\le$$

$$(\gamma^*(A^{\sim}))^2(\max(\varepsilon,1/\varepsilon))^{k-1}C_2/(\nu+1),$$

where $\nu\in m-1+\mathbb{N}$. In view of (51) and (21),

$$(52)\qquad p^{\sim}_{A^{\sim},\varepsilon}((A(\nu))^{-1})\le 1/\rho_s+$$

$$2(sign(k-1))\varepsilon/\rho_s^2\le 2/\rho_s+(\gamma^*(A^{\sim}))^2(\max(\varepsilon,1/\varepsilon))^{k-1}C_2/(\nu+1)\le$$

$$(1/\rho_s+2(sign(k-1))\varepsilon/\rho_s^2)\left(1+(\gamma^*(A^{\sim}))^2(\max(\varepsilon,1/\varepsilon))^{k-1}\rho_sC_2\rho_s/(\nu+1)\right),$$

$$(53)\qquad \ln(p^{\sim}_{A^{\sim},\varepsilon}((A(\nu))^{-1}))\le\ln(1/\rho_s)+$$

$$\ln(1+(sign(k-1))C_{2,1}\varepsilon)+\ln\left(1+(\max(\varepsilon,1/\varepsilon))^{k-1}C_{2,2}/(\nu+1)\right),$$

where

$$C_{2,1}=2/\rho_s,\ C_{2,2}=(\gamma^*(A^{\sim}))^2C_2\rho_s$$

and $\nu\in\mathbb{N}-1+m$. We take $\nu\in m-1+\mathbb{N}$.

We consider first the case $k > 1$ again now. For given $\nu \in \mathbb{N} + m$ we take $d \in \mathbb{N}$ and $a_i \in [m, \nu] \cap \mathbb{N}$, where $i = 0, \ldots, d$, in such a way that

$$a_0 = m,\ a_d = \nu, a_{i-1} < a_i \le 2a_{i-1} \le m2^i, \tag{54}$$

where $i = 1, \ldots, d$, and

$$d \le \frac{\ln(\nu)}{\ln(2)} + 1. \tag{55}$$

According to the Corollary of the Lemma 1 and (54) – (55),

$$\sum_{\kappa=a_{i-1}}^{a_i-1} \ln(p^{\sim}_{A^{\sim},\varepsilon}((A(\kappa))^{-1})) \le \tag{56}$$

$$(a_i - a_{i-1})\ln(1/\rho_s) + (a_i - a_{i-1})\ln(1 + C_{2,1}\varepsilon) +$$
$$\sum_{\kappa=a_{i-1}}^{a_i-1} \ln\left(1 + (\max(\varepsilon, 1/\varepsilon))^{k-1}C_{2,2}/(\kappa+1)\right) \le$$
$$(a_i - a_{i-1})(\ln(1/\rho_s) + \ln(1 + C_{2,1}\varepsilon)) + 3(\gamma^*(A^{\sim}))^2(\max(\varepsilon, 1/\varepsilon))^{k-1}C_{2,2},$$

where $i = 1, \ldots, d$. We take now in (56) $\varepsilon = \varepsilon_i = (a_{i-1})^{-1/k}$. Then, in view of (49),

$$a_{i-1} \ge m > (2/\rho_s)^k,\ (a_{i-1})^{1/k} > 2/\rho_s,\ \varepsilon_i < \rho_s/2,$$

where $i = 1, \ldots, d$, and therefore (47) and (56) hold. Consequently,

$$\sum_{\kappa=a_{i-1}}^{a_i-1} \ln(p^{\sim}_{A^{\sim},\varepsilon}((A(\kappa))^{-1})) \le \tag{57}$$

$$(a_i - a_{i-1})\ln(1/\rho_s) + (C_{2,1} + 3C_{2,2})(a_{i-1})^{1-1/k} =$$
$$(a_i - a_{i-1})\ln(1/\rho_s) + O(2^{(i-1)(1-1/k)}),$$

where $i = 1, \ldots, d$. Therefore

$$\ln(p^{\sim}_{A^{\sim},\varepsilon_i}\left(\prod_{\kappa=a_{i-1}}^{a_i-1}(A(\kappa))^{-1}\right) \le \tag{58}$$

$$(a_i - a_{i-1})\ln(\rho_1) + O(2^{(i-1)(1-1/k)}$$

and,in view of (18),

$$\ln(h^{\sim}\left(\prod_{\kappa=1}^{a_i-a_{i-1}}(A(\kappa))^{-11}\right) \le \tag{59}$$

$$\ln((\gamma^*(A^{\sim}))^2)(\max(\varepsilon_i, 1/\varepsilon_i))^{k-1}) + (a_i - a_{i-1})\ln(\rho_1) + O(2^{(i-1)(1-1/k)}) \le$$
$$2\ln(\gamma^*(A^{\sim})) + (i-1)(1-1/k)ln(2) + (a_i - a_{i-1})\ln(\rho_1) + O(2^{(i-1)(1-1/k)}) =$$
$$(i-1)(1-1/k)ln(2) + (a_i - a_{i-1})\ln(\rho_1) + O(2^{(i-1)(1-1/k)}),$$

where $i = 1, \dots, d$. Consequently,

(60)
$$\ln\left(h^{\sim}\left(\prod_{\kappa=m}^{\nu-1}(A(\kappa))^{-1}\right)\right) =$$
$$\ln\left(h^{\sim}\left(\left(\prod_{i=1}^{d}\prod_{\kappa=1}^{a_i-a_{i-1}}(A(\kappa))^{-1}\right)\right)\right) =$$
$$= \nu\ln(1/\rho_s) + O(\nu^{(1-1/k)}),$$

where $\nu \in \mathbb{N} + m$.

If $k = 1$, then, according to (53),

$$\ln(p^{\sim}_{A^{\sim},1}(A(\nu))) \le \ln(1\rho_s) + C_{2,2}/(\nu+1),$$

(61)
$$\ln\left(h^{\sim}\left(\prod_{\kappa=m}^{\nu-1}(A(\kappa))^{-1}\right)\right) \le$$
$$\ln\left((\gamma^*(A^{\sim}))^2 p^{\sim}_{A^{\sim},1}\left(\prod_{\kappa=m}^{\nu-1}(A(\kappa))^{-1}\right)\right) \le$$
$$\nu\ln(1/\rho_s) + O(\ln(e\nu)).$$

where $\nu \in \mathbb{N}$. In view of (60) – (61),

(62)
$$\ln\left(h^{\sim}\left(\prod_{\kappa=m}^{\nu-1}A(\nu-\kappa)\right)\right) =$$
$$= \nu\ln(1/\rho_s) + O(\nu^{(1-1/k)}) + O(\ln(e\nu))),$$

where $\nu \in \mathbb{N} + m$. Since

$$Y_{n,y}(m) = \left(\prod_{\kappa=1}^{\nu-m}A(\nu-\kappa)\right)^{-1} Y_{n,y}(\nu) =$$
$$\left(\prod_{\kappa=m}^{\nu-1}(A(\kappa))^{-1}\right) Y_{n,y}(\nu),$$

it follows that

(63)
$$\omega_{n,y}(m) = h\left(Y_{n,y}(m)\right) \le$$
$$h^{\sim}\left(\prod_{\kappa=m}^{\nu-1}(A(\kappa))^{-1}\right) h\left(Y_{n,y}(\nu)\right) =$$
$$\exp\left(O(1)\left(\nu^{1-1/k} + \ln(\nu)\right)\right)(\rho_1)^{-\nu}\,\omega_{n,y}(\nu),$$

where $\nu \in \mathbb{N}+m$. So, with m from (49) the asserted by the Lemma the lower estimate of the value $\omega_{n,y}(\nu)\,(\rho_s)^{-\nu}$ is obtained. ■

Remark 1. *In the case $s = 1$, $k^* = 0$ the assertions of the Lemma and Theorem 7 coincide.*

Lemma 4 (Perron's decomposition Lemma, [22], Hilfsatz 3)). *Let the characteristic polynomial (4) is represented as product*

$$T(z) = T_1(z)T_2(z), \tag{64}$$

where

$$T_1(z) = \sum_{\alpha=0}^{p} b_\alpha^{\sim} z^\alpha,\ T_2(z) = \sum_{\beta=0}^{q} u_\beta^{\sim} z^\beta, \tag{65}$$

with $b_p^{\sim} = u_q^{\sim} = a_n^{\sim} = 1$ *and absolute value of each root of* $T_1(z)$ *is greater than the absolute value of each root of* $T_2(z)$.

Then there exist $m \in \mathbb{N}$, $b_\alpha(\nu) \in \mathbb{C}, \alpha = 0, \dots, p, \nu \in \mathbb{N} + m - 1$, *and* $u_\beta(\nu) \in \mathbb{C}, \beta = 0, \dots, q, \nu \in \mathbb{N} + m - 1$, *such that*

$$\lim_{\nu\to\infty} b_\alpha(\nu) = b_\alpha^{\sim},\ \alpha = 0, \dots, p,\ b_p(\nu) = 1, b_0(\nu) \neq 0, \tag{66}$$

$$\lim_{\nu\to\infty} u_\beta(\nu) = u_\beta^{\sim}, \beta = 0, \dots, q,\ u_q(\nu) = 1, \tag{67}$$

where $\nu \in \mathbb{N} + m - 1$, *and, moreover, the equation (3) is equivalent to the equation*

$$\sum_{\alpha=0}^{p} b_\alpha(\nu) y(\nu + \alpha) = r(\nu),$$

where $\nu \in \mathbb{N} - 1 + m$ *and* $r(\nu)$ *satisfies to the equation*

$$\sum_{\beta=0}^{q} u_\beta(\nu) r(\nu + beta) = 0$$

with $\nu \in \mathbb{N} - 1 + m$.

Lemma 5. *Let all the conditions of the Perron's decomposition Lemma are fulfilled and* $a_k(\nu) - a_k^{\sim} = O(1/(\nu + 1)), k = 0, \dots, n,$ *when* $\nu \to \infty$.

Then for $b_\alpha(\nu)$ *with* $\alpha = 0, \dots, p$, *and* $u_\beta(\nu)$ *with* $\beta = 0, \dots, q$, *from the assertion of the Perron's decomposition Lemma the following conditions are fulfilled (cf. with (66) and (67)):*

$$b_\alpha(\nu) - b_\alpha^{\sim} = O(1/\nu),\ \alpha = 0, \dots, p, b_0(\nu) = 1,\ b_q(\nu) \neq 0,$$

$$u_\beta(\nu) - u_\beta^{\sim} = O(1/\nu),\ \beta = 0, \dots, q,\ u_0(\nu) = 1,$$

where $\nu \in \mathbb{N} - 1 + m$.

Proof. The Lemma is direct corollary of the Theorem 5 in [68]. ■

§2. Proof of the theorem 8.

We use below the notations of the section 3 in [77]. Let K be one of the fields $\mathbb{R}$ or $\mathbb{C}$. Let L be a linear normed space over K with a norm $p(x)$. In the case $L = K^n$ we fix as $p(x)$, wehre $x \in K^n$, the maximum of the absolute values ofcoordinates of x in the standard basis, i.e.

$$p(x) = h(x) = \sup(\{|x_1|, \dots, |x_n|\}), \tag{68}$$

where

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

If L is a Banach space with the norm p, then $K-$algebra of all the linear continuous operators acting in L will be denoted by $\mathfrak{M}^\wedge(L)$, and the norm on $\mathfrak{M}^\wedge(L)$, associated with the norm p will be denoted by $p^\sim$. So,

$$p^\sim(A) = sup(\{p(AX) \colon X \in L,\, p(X) \leq 1\}).$$

It is well known that the assocciated with h norm on $Mat_n(\mathbb{C})$ is defined as follows

$$h^\sim(A) = sup\left(\left\{\sum_{k=1}^{n} |a_{i,j}| \colon i = 1, \ldots, n\right\}\right), \tag{69}$$

where $A = (a_{i,k}) \in Mat_n(\mathbb{C})$. The norms h and $h^\sim$ coincide respectiely with the norms q_∞ and $q_\infty^\sim$ considered in [68], section 6. Let $m \in \mathbb{N} - 1$, and we denote by $E_m(L)$ the set $L^{m-1+\mathbb{N}}$ of all the maps of the set $m - 1 + \mathbb{N}$ into L. The set $E_m(L)$ is a linear space over K, where the muliplication of the elements by the number from K and addition of the elements is defined coordinate-wise. The subspace of $E_m(L)$ composed by all the constant maps is isomorphic to L, and we identify this subspace with L.

We denote by $\mathfrak{M}^\vee(L)$ the space of all the $K-$linear maps of the space L in L. If $\phi \in \mathfrak{M}^\vee(L)$ and $\psi \in \mathfrak{M}^\vee(L)$, then $\phi \circ \psi$ denotes the composition of operators ϕ and ψ, so that $(\phi \circ \psi)f = \phi((\psi f))$ for each $f \in L$. For $x \in E_m(L)$ let

$$p_{m,\infty}(x) = \sup(\{p(x(\nu)) \colon \nu \in m - 1 + \mathbb{N}\}.$$

Let further

$$E_{m,\infty}(L) = \{x \in E_m(L) \colon p_{m,\infty}(x) \neq \infty\},$$
$$E_{m,0}(L) = \{x \in E_m(L) \colon \lim_{\nu \to \infty} p(x(\nu)) = 0\},$$
$$E_m^\rightarrow(L) = L + E_{m,0}(L).$$

Clearly, the space $E_m^\rightarrow(L)$ consists of all the $y \in E_m(L)$, for which there exists

$$\lim(y) = \lim_{\nu \to \infty}(y(\nu)).$$

Let $m \in \mathbb{N} - 1$, $\mu \in m - 1 + \mathbb{N}$ and ler $r_{m,\mu}$ be the operator of restriction of the elements $y \in E_m(L)$ on te set $m - 1 + \mathbb{N}$. Clearly, the map $r_{m,\mu}$ is an epimorphism of the space $E_m(L)$ onto the space $E_\mu(L)$. If L is a K-algebra, then $E_m(L)$ is a K-algebra, where the muliplication and addition of the elements is defined coordinate-wise; so, in this case $r_{m,\mu}$ is an epimorphism of K-algebra $E_m(L)$ onto K-algebra $E_\mu(L)$. If L be an algebra with unity, let L^* denotes the group of all its invertible elements. Then

$$(L^*)^{m-1+\mathbb{N}} \subset L^{m-1+\mathbb{N}};$$

we denote below $(L^*)^{m-1+\mathbb{N}}$ by $E_m(L^*)$. Clearly,

$$E_m(L^*) = (E_m(L))^*.$$

Let $L = \mathbb{C}^n$, $y \in E_m(L)$, and let $y_i(\nu)$ denotes the i-th coordinate of the element $y(\nu)$, where $i = 1, \dots, n$, $\nu \in m-1+\mathbb{N}$; then the space $(E_m(\mathbb{C}))^n$ contains an element $\omega(y)$, which has $y_i(\nu)$ as the value of its i-th coordinate at the point $\nu \in m-1+\mathbb{N}$. So we obtain the natural isomorphism ω of the algebra $E_m(\mathbb{C}^n)$ onto $(E_m(\mathbb{C}))^n$. This map ω induces an isomorphism of the algebra $E_m(Mat_n(\mathbb{C}))$ onto $Mat_n(E_m(\mathbb{C}))$. If L is a $K-algebra$, then each element $a \in E_m(L)$ determines an acting on the space $E_m(L)$ K-linear operator $\mu_a \in \mathfrak{M}^\vee(E_m(L))$, which turns any $y \in E_m(L)$ into $\mu_a y = ay$. On the space $E_m(L)$ acts also K-linear operator $\triangledown \in \mathfrak{M}^\vee(L)$, which turns any element $y \in E_m(L)$ in the $\triangledown y \in E_m(L)$ such that

$$(\triangledown y)(\nu) = y(\nu+1)$$

for any $\nu \in m-1+\mathbb{N}$. Let us consider the subring $\mathfrak{A}_m(L)$ of the ring $\mathfrak{M}^\vee(\mathfrak{L})$ generated by the operator $\triangledown$ and by all the operators μ_a, where $a \in E_m(L)$. Clearly,

$$\mu_a \circ \triangledown^r \circ \mu_b \circ \triangledown^s = \mu_{a\triangledown^r b_k} \circ \triangledown^{r+s}, \tag{70}$$

where $\{r, s\} \subset \mathbb{N}-1$, $\{a, b\} \subset E_m(L)$. For each $\alpha \in \mathfrak{A}_m(L)\diagdown\{0_{\mathfrak{A}_m(L)}\}$ are uniquelly defined the number $\deg(\alpha)$ and representation of α in the form

$$\alpha = \sum_{k=0}^{\deg(\alpha)} \mu_{a_k} \circ \triangledown^k, \tag{71}$$

where $a_k \in E_m(L)$ for $k = 0, \dots, \deg(\alpha)$ and $a_{\deg(\alpha)} \neq 0_{E_m(L)}$. Clearly, (71) may be rewritten in the form

$$\alpha = \sum_{k=0}^{\infty} \mu_{a_k} \circ \triangledown^k, \tag{72}$$

where $a_k = 0_{E_m(L)}$ for $k \in \deg(\alpha)+\mathbb{N}$. It follows from (70) that $\mathfrak{A}_m(L)$ is a graduated algebra, and if

$$\beta = \sum_{r=0}^{p} \mu_{b_r} \circ \triangledown^r \in \mathfrak{A}_m(L), \tag{73}$$

$$\gamma = \sum_{s=0}^{q} \mu_{c_s} \circ \triangledown^s \in \mathfrak{A}_m(L), \tag{74}$$

then

$$\beta\gamma = \sum_{k=0}^{p+q} \sum_{\substack{\le r \le p \\ 0 \le s \le q \\ r+s=k}} \mu_{b_r \triangledown^r c_s} \circ \triangledown^{r+s}; \tag{75}$$

clearly, $\deg(\beta\gamma) = \deg(\beta)+\deg(\gamma)$, if $b_p(\nu)^r c_q(\nu+p)$ is different from 0 at least for one $\nu \in m-1+\mathbb{N}$. Let $\mathfrak{A}_m^{\rightarrow}(L)$ be the ring generated by the operator $\triangledown$ and by all the operators μ_a, where $a \in E_m^{\rightarrow}(L)$. Since $\triangledown a \in E_m^{\rightarrow}(L)$, if $a \in E_m^{\rightarrow}(L)$, it follows, in view of (70), that $\mathfrak{A}^{\to}_m(L)$ is a graduated subalgebra $\mathfrak{A}^{\to}_m(L)$ of the algebra $\mathfrak{A}_m(L)$, each $\alpha \in \mathfrak{A}_m(L)\diagdown\{0_{\mathfrak{A}_m(L)}\}$ admits a representation in the form (71) with $a_k \in E^{\to}_m(L)$ for $k = 0, \ldots, \deg(\alpha)$ and $a_{\deg(\alpha)} \neq 0_{E_m(L)}$; to each such α corresponds the limit operator

$$\lim(\alpha) = \sum_{k=0}^{\deg(\alpha)} \mu_{\lim(a_k)} \circ \bigtriangledown^k, \tag{76}$$

and polynomial

$$P(\alpha, z) = \sum_{k=0}^{\deg(\alpha)} \lim(a_k) z^k \in L[z]. \tag{77}$$

If $\alpha = 0_{\mathfrak{A}_m(L)}$, then we put

$$\lim(\alpha) = 0_{\mathfrak{A}_m(L)},\ P(\alpha, z) = 0_{L[z]}.$$

The equality (70) shows that the map

$$\alpha \to P(\alpha, z) \tag{78}$$

is an epimorphism of the algebra $\mathfrak{A}^{\to}_m(L)$ on the algebra $L[z]$ (if the algebra L is noncommutative, then we can treat the algebra $L[z]$ as a semigroup ring of the semigroup $(\mathbb{N}-1, +)$ over the algebra L). We note that, if $\alpha \in \mathfrak{A}_m(\mathbb{C})$, then, clearly, $Ker(\alpha)$ coincides with the linear space of all the solutions of the equation (3), and, moreover, if $\alpha \in \mathfrak{A}^{\to}_m(\mathbb{C})$, then the corresponding to α equation (3) is an equation of the Poincar'e type and $P(\alpha, z)$ is its characterictical polynomial.

Let L be an algebra with unity. The set of all $\alpha \in \mathfrak{A}_m(L)\diagdown\{0_{\mathfrak{A}_m(L)}\}$, which have the representation (71) with $a_{\deg(\alpha)} \in E_m(L^*)$ will be denoted further by $\mathfrak{A}_m(L)^\circ$. The set of all the the elements $\alpha \in \mathfrak{A}_m(L)\diagdown\{0_{\mathfrak{A}_m(L)}\}$, which have the representation (71) with $a_{\deg(\alpha)} = 1_{E_m(L)}$ will be denoted further by $\mathfrak{A}_m(L)^\vee$. The set of all the $\alpha \in \mathfrak{A}_m(L)\diagdown\{0_{\mathfrak{A}_m(L)}\}$, which have the representation (71) with $a_0 \in E_m(L^*)$, will be denoted further by $\mathfrak{A}_m(L)^\wedge$. Let further

$$\mathfrak{A}_m(L)^{\circ\wedge} = \mathfrak{A}_m(L)^\circ \bigcap \mathfrak{A}_m(L)^\wedge,\ \mathfrak{A}_m(L)^{\vee\wedge} = \mathfrak{A}_m(L)^\vee \bigcap \mathfrak{A}_m(L)^\wedge,$$

$$\mathfrak{A}_m(L)^{\circ\to} = \mathfrak{A}_m(L)^\circ \bigcap \mathfrak{A}_m(L)^\to,\ \mathfrak{A}_m(L)^{\vee\to} = \mathfrak{A}_m(L)^\vee \bigcap \mathfrak{A}_m(L)^\to,$$

$$\mathfrak{A}_m(L)^{\circ\wedge\to} = \mathfrak{A}_m(L)^{\circ\wedge} \bigcap \mathfrak{A}_m(L)^\to,\ \mathfrak{A}_m(L)^{\vee\wedge\to} = \mathfrak{A}_m(L)^{\vee\wedge} \bigcap \mathfrak{A}_m(L)^\to,$$

Clearly, $\mathfrak{A}_m(L)^\circ$ consists of epimorphisms of the space $E_m(L)$ onto $E_m(L)$. The above map $r_{m,\mu}$ induces epimorphism $r^{\triangleright}_{m,\mu}$ of the algebra $\mathfrak{A}_m(L)$ on the algebra $\mathfrak{A}_\mu(L)$ defined as follows:

if $\alpha \in \mathfrak{A}_m(L)$,

$$\alpha = \sum_{k=0}^{n} \mu_{a_k} \circ \bigtriangledown^k, \tag{79}$$

then

$$r^{\triangleright}_{m,\mu}(\alpha) = \sum_{k=0}^{n} \mu_{r_{m,\mu}(a_k)} \circ \bigtriangledown^k, \tag{80}$$

where the operator $\bigtriangledown$ in (79) acts in $E_m(L)$ and the operator $\bigtriangledown$ in (80) acts in $E_\mu(L)$. Clearly, $r^{\triangleright}_{m,\mu}$ surjectively maps
$\mathfrak{A}_m(L)^\circ$ onto $\mathfrak{A}_\mu(L)^\circ$, $\mathfrak{A}_m(L)^\vee$ onto $\mathfrak{A}_\mu(L)^\vee$,
$\mathfrak{A}_m(L)^\wedge$ onto $\mathfrak{A}_m u(L)^\wedge$, $\mathfrak{A}_m(L)^{\circ\wedge}$ onto $\mathfrak{A}_\mu(L)^{\circ\wedge}$,
$\mathfrak{A}_m(L)^{\vee\wedge}$ onto $\mathfrak{A}_\mu(L)^{\vee\wedge}$, $\mathfrak{A}_m(L)^{\circ\rightarrow}$ onto $\mathfrak{A}_\mu(L)^{\circ\rightarrow}$,
$\mathfrak{A}_m(L)^{\vee\rightarrow}$ onto $\mathfrak{A}_\mu(L)^{\vee\rightarrow}$, $\mathfrak{A}_m(L)^{\circ\wedge\rightarrow}$ onto $\mathfrak{A}_\mu(L)^{\circ\wedge\rightarrow}$,
$\mathfrak{A}_m(L)^{\vee\wedge\rightarrow}$ onto $\mathfrak{A}_\mu(L)^{\vee\wedge\rightarrow}$. Since the diagram

$$\begin{array}{ccc} E_m(L) & \xrightarrow{r_{m,\mu}} & E_\mu(L) \\ \alpha \downarrow & & \downarrow r^{\triangleright}_{m,\mu}(\alpha) \\ E_m(L) & \xrightarrow[r_{m,\mu}]{} & E_\mu(L) \end{array}$$

is commuative and therefore

$$r_{m,\mu}\alpha = r^{\triangleright}_{m,\mu}(\alpha) r_{m,\mu}, \tag{81}$$

it follows that $r_{m,\mu}$ surjectively maps $Ker(r_{m,\mu}\alpha)$ onto

$$Ker(r^{\triangleright}_{m,\mu}(\alpha)) \supset r_{m,\mu}Ker(\alpha).$$

Lemma 6. *If $\mu \in m-1+\mathbb{N}$ and $\alpha \in \mathfrak{A}_m(L)^\wedge$, then the operator α bijectively maps $Ker(r_{m,\mu})$ onto $Ker(r_{m,\mu})$.*

Proof. Proof is given in [72], Lemma 3.

Corollary 1. *Let $\mu \in m-1+\mathbb{N}$ and let $\alpha \in \mathfrak{A}_m(L)^\wedge$. If*

$$g \in E_m(L),\ x \in E_\mu(L),\ m \le \mu,\ \alpha \in \mathfrak{A}_m(L)^\wedge,$$

$$r_{m,\mu}(g) = (r^{\triangleright}_{m,\mu}(\alpha))(x),$$

then there exists a unique $y \in E_m(L)$ such that

$$\alpha(y) = g,\ r_{m,\mu}(y) = x;$$

Proof. Proof is given in [72], Corollary 1 to the Lemma 3.

Corollary 2. *Let $\mu \in m-1+\mathbb{N}$ and $\alpha \in \mathfrak{A}_m(L)^\wedge$. Then and $r_{m,\mu}$ bijectively maps $Ker(\alpha)$ onto $Ker(r^{\triangleright}_{m,\mu}(\alpha)) = r_{m,\mu}(Ker(\alpha))$.*

Proof. Proof is given in [72], Corollary 2 to the Lemma 3.

If for the equation (2) are fulfilled the conditions (1) then

$$a_k = (a_k(0),\ a_k(1),\ \dots,\ a_k(\nu),\ \dots) \in E_0^{\rightarrow}(\mathbb{C}), \tag{82}$$

where $k = 0, \dots, n$. Moreover, $a_n = 1_{E_0(\mathbb{C})}$, for α in (79) $Ker(\alpha)$ coincides with the linear over $\mathbb{C}$ space of all the solutions of the equation (2), polynomial (4) is equal to the polynomial $P(\alpha, z) = P(r^{\triangleright}_{0,m}(\alpha), z)$, where $m \in \mathbb{N}$, and the set $Ker(r^{\triangleright}_{0,m}(\alpha))$ coincides with linear over $\mathbb{C}$ space V_m of all the solutions of the equation (3).

Let $\mathfrak{v}$ be the element in $E_{0,0}$, for which

$$\mathfrak{v}(\nu) = \frac{1}{\nu+1},$$

where $\nu \in \mathbb{N}-1$. Clearly, $r_{0,m}(\mathfrak{v})E_{m,\infty}(\mathbb{C}) \subset E_{m,0}(\mathbb{C})$ for any $m \in \mathbb{N}-1$. Let

$$E^{\succ}_{m,0}(L) = r_{0,m}(\mathfrak{v})E_{m,\infty}(L),\ E^{\succ}_{m}(L) = L + E^{\succ}_{m,0}(L).$$

Let us consider the ring $\mathfrak{A}^{\succ}_m(L)$ generated by the operator $\bigtriangledown$ and by all the operators μ_a, where $a \in E_m(L)^{\succ}$. The Lemma 5 may be reformulated as follows:

Lemma 7. *Let $\alpha \in \mathfrak{A}^{\succ}_0(\mathbb{C})\bigcap\mathfrak{A}^{\vee}_0(\mathbb{C})$, and $P(\alpha, z)$ coincides with the polynomial $T(z)$ in (4) and (64).*

Then there exist $m \in \mathbb{N}$ and representation of the operator $r^{\triangleright}_{0,m}(\alpha)$ in the form

$$r^{\triangleright}_{0,m}(\alpha) = \eta\beta \tag{83}$$

such that

$$\eta \in \mathfrak{A}^{\succ}_m(\mathbb{C})\bigcap\mathfrak{A}^{\vee}_m(\mathbb{C}),\ deg(\eta) = q \tag{84}$$

$$\beta \in \mathfrak{A}^{\succ}_m(\mathbb{C})\bigcap\mathfrak{A}^{\vee\wedge}_m(\mathbb{C}),\ \deg(\beta) = p = n - q, \tag{85}$$

and the polynomials $P(\beta, z)$, $P(\eta, z)$ coincide respectively with the polynomials $T_1(z)$, $T_2(z)$ in (65) .

Lemma 8. *Let are fulfilled all the conditions of the Theorem 7. Let $\alpha \in \mathfrak{A}^{\succ}_0(\mathbb{C})\bigcap\mathfrak{A}^{\vee}_0(\mathbb{C})$ corresponds to the equation (2), i.e. (with $m = 0$) (79) holds*

$$\alpha = \sum_{k=0}^{n} \mu_{a_k} \circ \bigtriangledown^k, \tag{86}$$

where $a_n = 1_{E_0(\mathbb{C})}$ and

$$a_k = (a_k(0),\, a_k(1),\, \dots ,\, a_k(\nu),\, \dots) \in E^{\succ}_0(\mathbb{C}), \tag{87}$$

for $k = 0, \dots , n$.

Let the characteristic polynomial (4) is represented as product

$$P(\alpha, z) = T(z) = \prod_{k=1}^{s+1} T_i(z), \tag{88}$$

where

$$T_i(z) = \sum_{\alpha=0}^{e_i} b^{\sim}_{i,\alpha} z^{\alpha}, \tag{89}$$

with $b_{i,e_i} = a^{\sim}_n = 1$ and absolute value of each root of the polynomial $T_i(z)$ is equal to ρ_i.

Then there exist $m \in \mathbb{N}$ and representation of the operator $r^{\triangleright}_{0,m}(\alpha)$ in the form

$$r^{\triangleright}_{0,m}(\alpha) = \prod_{i=0}^{s} \beta_{s+1-i} \tag{90}$$

such that

$$\beta_i \in \mathfrak{A}^{\succ}_m(\mathbb{C}) \bigcap \mathfrak{A}^{\vee}_m(\mathbb{C}),\ \deg(\beta_i) = e_i, \tag{91}$$

$$P(\beta_i, z) = T_i(z),$$

where $i = 1, \dots, s+1$ and

$$\beta_i \in \mathfrak{A}^{\wedge}_m(\mathbb{C}), \tag{92}$$

where $i = 1, \dots, s$.

Proof. The assertion of the Lemma may be obtained by means of the sequentielly applying of the Lemma 7. ■

Let

$$\beta^*_\theta = \prod_{i=0}^{\theta-1} \beta_{\theta-i},\ \beta^{\wedge}_\theta = \prod_{i=s+1-\theta}^{s} \beta_{s+1-i}, \tag{93}$$

where $\theta = 1, \dots s+1$. In view of (refeq:2cdj),

$$r^{\triangleright}_{0,m}(\alpha) = \beta^{\wedge}_{\theta+1}\beta^*_\theta, \tag{94}$$

$\theta = 1, \dots s$.

Let $C > 0$, $n \in \mathbb{N}$,

$$w_{C,n}(\nu) = \left(\frac{C}{\nu}\right)^{\nu/n}, \tag{95}$$

where $\nu \in \mathbb{N}$; let further $\rho > 0$ and

$$v_{C,n,\rho}(\nu) = \rho^\nu \exp\left(C\left((\nu)^{1-1/n} + \ln(\nu)\right)\right), \tag{96}$$

where $\nu \in \mathbb{N}$.

Let $\theta_1 \in \{1, \dots s\}$.Replacing m by some bigger m and applying to $\beta^{\wedge}_{\theta_1}$ in (93) the Theorem 7, we see that there exist $A^{\vee} > 0$, $m \in \mathbb{N}$, $\alpha^{\vee}(\nu) > 0$ with $\nu \in m + \mathbb{N} - 1$ and the subspaces $R^{\vee}_{m,\theta_1,\theta_1}, \dots, R^{\vee}_{m,\theta_1,s+1}$ of the space $R_m = Ker(\beta^{\wedge}_{\theta_1})$ such that

$$\lim_{\nu\to\infty} \alpha^{\vee}(\nu) = 0,$$

$$R_m = R^{\vee}_{m,\theta_1,\theta_1} \oplus \dots, \oplus R^{\vee}_{m,\theta_1,s+1},\ \dim_{\mathbb{C}}(R^{\vee}_{m,\theta_1,i}) = e_i,\ \theta_1, \le i \le s+1,$$

if

$$r \in R^{\vee}_{m,\theta_1,\theta} \tag{97}$$

for some $\theta \in \{\theta_1, , ..., s\}$, then

$$\exp(-A^{\vee}(\ln(\nu) + \nu^{1-1/k_\theta}))\omega_{q,r}(m)(\rho_\theta)^\nu \le \omega_{q,r)}(\nu), \tag{98}$$

where $\nu \in \mathbb{N}+m-1$; moreover, the spaces

$$R^{\wedge}_{m,\theta_1,j} = R^{\vee}_{m,\theta_1,j} \oplus \ldots \oplus R^{\vee}_{m,\theta_1,s+1},$$

where $j = \theta_1, , \ldots , s+1$ and natural projections $\pi^{\vee}_{\theta_1,j}$ of the space $R^{\wedge}_{m,\theta_1,j}$ onto the space $R^{\vee}_{m,\theta_1,j}$, where $j = \theta_1, , \ldots , s$, have the following properties:

if

$$r \in R^{\wedge}_{m,\theta_1,\theta} \tag{99}$$

for some $\theta \in \{\theta_1, , ..., s\}$, then

$$\omega_{q,r}(\nu) \le (\rho_\theta)^\nu \exp(A^{\vee}(\ln(\nu) + \nu^{1-1/k_\theta}))\omega_{q,r}(m), \tag{100}$$

$$\omega_{q,r}(\nu) \ge (\omega_{q,\pi^{\vee}_{\theta_1,\theta}(r)}(m) - \alpha^{\vee}(\nu)\omega_{q,r)}(m)) \times \tag{101}$$

$$(\rho_\theta)^\nu \exp(-A^{\vee}(\ln(\nu) + \nu^{1-1/k_\theta})),$$

where $\nu \in \mathbb{N}+m-1$; if $e_{s+1} > 0$, and

$$r \in R^{\vee}_{m,\theta_1,s+1} (= R^{\wedge}_{m,\theta_1,s+1}), \tag{102}$$

then

$$|r(\nu)| \le (A^{\vee}/\nu)^{\nu/k^*}\omega_{q,r}(m), \tag{103}$$

where $\nu \in \mathbb{N}+m-1$.

Let

$$\beta = \beta^*_1 = \beta_1, \, \eta = \beta^{\wedge}_2. \tag{104}$$

Then, in view of (94), (83) holds.

Lemma 9. *Let all the conditions of the Theorem 7 are fulfilled. For sufficient big $m \in \mathbb{N}$ there exists the splitting monomorphism*

$$\psi^{\wedge}_2 \colon Ker(\eta) \to Ker(\eta\beta)$$

with following properties:

(a)

$$Ker(\beta) = V^{\vee}_{m,1}, \psi^{\wedge}_2(Ker(\eta)) = V^{\wedge}_{m,2},$$

where $V^{\vee}_{m,1}$ and $V^{\wedge}_{m,2}$ are defined in the assertion of the Theorem 7;

(b) the monomorphism $\psi^{\wedge}_2$ maps isomorphically the space $R^{\vee}_{m,2,j}$, where $j = 2, , \ldots , s+1$, onto the space $V^{\vee}_{m,j}$;

(c) the map $\beta_1\psi^wedge_2$ coincides with identiti map $Ker(\eta) \to Ker(\eta)$;

(d) the restriction of the map $\psi^wedge_2\beta_1$ on the space $V^{\wedge}_{m,2}$ coincides with the identity map $V^{\wedge}_{m,2} \to V^{\wedge}_{m,2}$;

(e) let I be the identiti map $V_m \to V_m$; then the natural projection π_1 of the space $V^{\wedge}_{m,1} = V_m$ onto $V^{\vee}_{m,1}$ coincides with restriction of the map $I - \psi^{\wedge}_2\beta_1$ on the space $V^{\vee}_{m,1}$;

(f) if $j \in [2, s] \cap \mathbb{N}$, then natural projection π_j of the space $V^{\wedge}_{m,j}$ onto the space$V^{\vee}_{m,j}$ coincides with the restriction of the map $\psi^{\wedge}_2\pi^{\vee}_{2,j}\beta_1$ on the space $V^{\wedge}_{m,j}$

and the projection $\pi^{\vee}_{2,j}$ coincides with the restriction of the map $\beta_1\pi_j, \psi^{\wedge}_2$ on the space $R^{\wedge}_{2,j}$.

Proof. See the proof of the Theorem 7 in [77], especially the section 2 and section 3. ■.

Remark 2. I use this opportunity to make a corrections in [77]. On the page 13, third line from the bottom must stand $\psi_m\pi^{\vee}_j\phi_m$ instead $\psi_m\pi^{\vee}_j$; on the second line from the bottom must stand $\pi^{\vee}_j = \phi_m\pi_j\psi_m$ instead $\pi^{\vee}_j = \phi_m\pi_j$.

Lemma 10. *Let all the conditions of the Lemma 9 are fulfilled. Then (for sufficient big m)*

$$V^*_{m,\theta} = Ker(\beta^*_{1,\theta}), \tag{105}$$

where $\theta = 1, \ldots, s$.

Proof. We use induction on s For $s = 1$ or $\theta = 1$ the assertion of the Lemma directly follows from the Lemma 9.

Let $s > 1$ and assertion of the lemma is true for $s - 1$. Let $\theta > 1$. Since the monomorphism $\psi^{\wedge}_2$ maps isomorphically the space $R^{\vee}_{m,2,j}$, where $j = 2, , \ldots, \theta$, onto the space $V^{\vee}_{m,j}$ and the map $\beta_1\psi^{\wedge}_2$ coincides with identity map $Ker(\eta) \to Ker(\eta)$, it follows that the restriction of the map β_1 on the space $V^{\vee}_{m,2} \oplus \ldots \oplus V^{\vee}_{m,\theta}$ is an isomorphism of $V^{\vee}_{m,2} \oplus \ldots \oplus V^{\vee}_{m,\theta}$ onto $R^*_{m,2,\theta}$. According to the inductive hypothesis,

$$R^*_{m,2,\theta} = Ker(\beta^*_{2,\theta}).$$

Therefore, if $y \in V^{\vee}_{m,2} \oplus \ldots \oplus V^{\vee}_{m,\theta}$ then $\beta^*_{1,\theta}(y) = \beta^*_{2,\theta}(\beta_1(y)) = 0$; moreover, since also $V^{\vee}_{m,1} = Ker(\beta_1)$, it follows that $V^*_{m,\theta} \subset Ker(\beta^*_{1,\theta})$. Since, according to the Theorem 7,

$$\dim_{\mathbb{C}}(V_{m,\theta}) = \sum_{i=1}^{\theta} e_i = \tag{106}$$

$$\dim_{\mathbb{C}}(Ker(\beta^*_{1,\theta})) = n_\theta := \deg(\beta^*_{1,\theta}),$$

it follows that (105) holds. ■.

Proof of the Theorem 8. In view of (105), (106), we apply to $\beta^*_{1,\theta}$ the Lemma 3. Since θ plays the role of s in the lemma 3 now and

$$\max(k_1, \ldots, k_\theta) \le k = \max(k_1, \ldots, k_s),$$

it follows that there exist $A = A_\theta > 0$, $m \in \mathbb{N}$ such that

$$\exp(-A(ln(\nu) + \nu^{1-1/k}))(\rho_s)^\nu\omega_{n_\theta,y}(m) \le \tag{107}$$

$$\omega_{n_\theta,y}(\nu) \le \omega_{n,y}(\nu)$$

for any $y \in V^*_{m,\theta}$. Since both the functions $y \to \omega_{n_theta,y}(m), y \in V^*_{m,\theta}$, and $y \to \omega_{n,y}(m), y \in V^*_{m,\theta}$ are two norms on the n_θ-dimensional space $V^*_{m,\theta}$ there exists $B = B_\theta > 0$ such that

$$\omega_{n,y}(m)\exp(-B) \le \omega_{n_\theta,y}(m)$$

for any $y \in V^*_{m,\theta}$. Then

$$\exp(-A(ln(\nu) + \nu^{1-1/k}) - B)(\rho_\theta)^\nu\omega_{n,y}(m) \le \omega_{n,y}(\nu) \tag{108}$$

for any $y \in V^*_{m,\theta}$. ■

§3. Proof of the theorem 9.

Theorem 9. *Let for some $\theta \in \{1 \dots , s\}$ is given a linear map ξ_θ of the space $V^*_{m,\theta}$ into $V^\wedge_{m,\theta+1}$. Let I^*_θ is the identity map $V^*_{m,\theta} \to V^*_{m,\theta}$ Then for*

$$V = (I^*_\theta + \xi_\theta)(V^*_{m,\theta})$$

holds the assertion of the Theorem 10.

Proof of the theorem 9. Since $V^*_{m,\theta}$ is finite-dimensional linear space over $\mathbb{C}$, it follows that the map ξ_θ is continuous. Therevore there exists $C > 0$ such that

$$\omega_{n_\theta,\xi_\theta(y)}(m) \le \exp(C)\omega_{n_\theta,\xi_\theta(y)}(m) \tag{109}$$

for any $y \in V^*_{m,\theta}$. According to the Theorem 7 and Theorem 8, there exist a numbers $A > 0$, $m \in \mathbb{N}$, such that, if $y \in V^*_{m,\theta}$, $\nu \in m - 1 + \mathbb{N}$, then

$$\omega_{n,\xi_\theta(y)}(\nu) \le \exp(A(\ln(\nu) + \nu^{1-1/k_{\theta+1}}))(\rho_\theta)^\nu \omega_{n,\xi_\theta(y)}(m) \tag{110}$$

for $\theta < s$,

$$\omega_{n_\theta,\xi_\theta(y)}(m) = 0 \tag{111}$$

for $\theta = s$ and $k^* = k_{s+1} = 0$,

$$|\xi_\theta(y)(\nu)| \le (A/\nu)^{\nu/k^*}\omega_{n,\xi_\theta(y)}(m) \tag{112}$$

for $\theta = s$ and $k^* = k_{s+1} > 0$,

$$\exp(-A(ln(\nu) + \nu^{1-1/k}))(\rho_\theta)^\nu \omega_{n,y}(m) \le \omega_{n,y}(\nu). \tag{113}$$

Therefore

(114)

$$\omega_{n,y+\xi_\theta(y)}(\nu) \ge \omega_{n,y}(\nu) - \omega_{n,\xi_\theta(y)}(\nu) \ge \exp(-A(ln(\nu) + \nu^{1-1/k}))(\rho_\theta)^\nu \omega_{n,y}(m)$$

for $\theta = s$ and $k^* = k_{s+1} = 0$,

$$\begin{gathered} \omega_{n,y+\xi_\theta(y)}(\nu) \ge \omega_{n,y}(\nu) - \omega_{n,\xi_\theta(y)}(\nu) \ge \\ \exp(-A(ln(\nu) + \nu^{1-1/k}))(\rho_\theta)^\nu \omega_{n,y}(m) - \\ |\xi_\theta(y)(\nu)| \le (A/\nu)^{\nu/k^*} \exp(C)\omega_{n,y}(m) = \\ \exp(-A(ln(\nu) + \nu^{1-1/k}))(\rho_\theta)^\nu \omega_{n,y}(m) \times \\ \big(1 + \exp(A(ln(\nu) + \nu^{1-1/k}))(\rho_\theta)^{-\nu}(A/\nu)^{\nu/k^*} \exp(C)\big) \end{gathered} \tag{115}$$

for $\theta = s$ and $k^* = k_{s+1} > 0$,

$$\begin{gathered} \omega_{n,y+\xi_\theta(y)}(\nu) \ge \omega_{n,y}(\nu) - \omega_{n,\xi_\theta(y)}(\nu) \ge \\ \exp(-A(ln(\nu) + \nu^{1-1/k}))(\rho_\theta)^\nu \omega_{n,y}(m) - \\ (A/\nu)^{\nu/k^*} \exp(C)\omega_{n,y}(m) = \\ \exp(-A(ln(\nu) + \nu^{1-1/k}))(\rho_\theta)^\nu \omega_{n,y}(m) \times \end{gathered} \tag{116}$$

$$\big(1 - \exp(A(ln(\nu) + \nu^{1-1/k}))(\rho_\theta)^{-\nu}(A/\nu)^{\nu/k^*}\exp(C)\big)$$

for $\theta = s$ and $k^* = k_{s+1} > 0$,

$$\omega_{n,y+\xi_\theta(y)}(\nu) \geq \omega_{n,y}(\nu) - \omega_{n,\xi_\theta(y)}(\nu) \geq \tag{117}$$

$$\exp(-A(ln(\nu) + \nu^{1-1/k}))(\rho_\theta)^\nu\omega_{n,y}(m) -$$
$$\exp(A(\ln(\nu) + \nu^{1-1/k}))(\rho_\theta)^\nu\exp(C)\omega_{n,y}(m) =$$
$$\exp(-A(ln(\nu) + \nu^{1-1/k}))(\rho_\theta)^\nu\omega_{n,y}(m)\times$$
$$\big(1 - \exp(2A(ln(\nu) + \nu^{1-1/k}))(\rho_{\theta+1}/\rho_\theta)^\nu\exp(C)\big)$$

for $\theta < s$.■

Remark 3. The value of $\omega_{n,y}(m)$ in (8) may be much bigger than the value of $\omega_{n\pi_{\theta(y)}}(m)$. Therefore we cannot (as show a simple examples) for fixed value of A in the inequality (11) to replace the linear space V of the Theorem 10 by the set $V_m \diagdown V^\wedge_{m,\theta+1}$. But in (116) – (117) $\omega_{n,y}(m)$ is carried out the brackets, and the value in the brackets tends to 1, when $\nu \to +\infty$, being greather than 1/2 for $\nu \in m - 1 + \mathbb{N}$, with sufficient big $m \in \mathbb{N}$, which depends only from the equation.

§4. Proof of the theorem 10.

Proof of the theorem 10. Let π^* and $\pi^\wedge$ are restrictions on V of the natural endomorphisms of the space V_m onto respectively $V^*_{m,\theta}$ and $V^\wedge_{m,\theta+1}$ and let I_0 be the identity map $V \to V$. Then

$$I_0 = \pi^* + \pi^\wedge \tag{118}$$

Since $Ker(\pi^*) \subset V \cap V^\wedge_{m,\theta+1} = \{0\}$, it follows that π^* is an isomorphism of the space V onto linear subspace V' of the space $V^*_{m,\theta}$. Let τ be the inverse isomorphism to the isomorphism π^*. Then $\pi^*\tau$ is identity map $V' \to V'$ and it follows from (118) that

$$\tau = \pi^*\tau + \pi^\wedge\tau. \tag{119}$$

Clearly, the linear map $\pi^\wedge\tau : V' \to V^\wedge_{m,\theta+1}$ have an extension

$$\xi_\theta : V^*_{m,\theta} \to V^\wedge_{m,\theta+1}. \tag{120}$$

It follows from (119) that

$$V = \tau(V') \subset (I + \xi_\theta)V^*_{m,\theta},$$

where I is identity map $V_{m,\theta} \to V_{m,\theta}$ and ξ_θ is a linear map in (120).

So, the Theorem 10 is Corollary of the Theorem 9. ■

§5. The case of the general differrence equation of the Poincaré type.

If (12) and (13) hold instead of (1), then making use of the above arguments we obtain the following chainges in the Theorem 7 and theorem 10. For any $\varepsilon > 0$ there exists a constant$A^\wedge = A^\wedge(\varepsilon) > 0$ such that, if $y \in V^\wedge_{m,\theta}$ with some $\theta \in \{1, ..., s\}$, then (instead of (7) the following inequality holds)

$$\omega_{n,y}(\nu) \le \exp(A^\wedge(\rho_\theta \exp(\varepsilon))^\nu \omega_{n,y}(m), \tag{121}$$

where $\nu \in m + \mathbb{N} - 1$; if $y \in V^\vee_{m,s+1} (= V^\wedge_{m,s+1})$, then (instead of (10) the following inequality holds)

$$\omega_{n,y}(\nu) \le \exp(A^\wedge(\exp(-\varepsilon))^\nu \omega_{n,y}(m), \tag{122}$$

where $\nu \in m + \mathbb{N} - 1$.

Let further V be an arbitrary linear subspace of V_m such that

$$V \cap V_{m,\theta+1} = \{0\},$$

where $\theta \in \{1 \ldots , s\}$. Then for this subspace V and any $\varepsilon > 0$ there exists a constant $A^\vee = A^\vee(V, \varepsilon) > 0$ such that (instead of (11) the following inequality holds)

$$\exp(-A^\vee)(\rho_\theta \exp(-\varepsilon))^\nu \omega_n(y)(m) \le \omega_{n,y}(\nu) \tag{123}$$

where $y \in V$ and $\nu \in m + \mathbb{N} - 1$.

Corollary. (See [51], Theorem 3 and [75], Lemma 16). *Let as before (12) and (13) hold instead of (1). Let V be a r-dimensional subspace of V_m, let*

$$V \cap V^\vee_{m,s+1} = \{0\}$$

and let $\{y_1(\nu) , \ldots , y_r(\nu)\}$ be a basis of the space V. Let

$$k_3(V) = \max\{k \in \mathbb{Z} \colon 1 \le k \le s,\ V \subset V^\wedge_{m,k}\},$$

and

$$k_4(V) = \min\{k \in \mathbb{Z} \colon 1 \le k \le s,\ V \cap V^\wedge_{m,k+1} = \{0\}\}.$$

For $X = (x_1 , \ldots , x_r),\ X \in \mathbb{C}^r$, let

$$h(X) = max\{|x_1| , \ldots , |x_r|\},$$

$$y = y^\vee(X, \nu) = x_1 y^\vee_1(\nu) + \ldots + x_r y^\vee_r(\nu).$$

Then for every $\varepsilon \in (0, 1)$ there exist $C_3(\varepsilon) > 0$ and $C_4(\varepsilon) > 0$ such that

$$C_4(\varepsilon)(\rho_{k_4}(1 - \varepsilon))^\nu h(X) \le \omega_{n,y}(\nu) \le C_3(\varepsilon)(\rho_{k_3} + \varepsilon)^\nu h(X).$$

Proof The functions $h(X)$ and the restriction of $\omega_{n,y}(m)$ on V are two norms on the r-dimensional over $\mathbb{C}$ linear space V. Therefore there exists a constant $C_5 > 0$ such that $h(X) \le C_5 \omega_{n,y}(m)$ and $\omega_{n,y}(m) \le C_5 h(X)$ The assertion of the Corollary directly follows from (121) and (123) now. ■

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