Period relations for twisted Legendre equations

by

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MPI/90-32

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1. Introduction

Fix a square free polynomial $T \in \mathbb{C}[t]$ and let $L = L_T$ and $q = q_T$ be the parabolic cohomology group and the quadratic form which are associated as in §§ 3,6 below with the twisted Legendre equation over $\mathbb{C}(t)$

(i) $$y^2 = Tx(x-1)(x-t) .$$

In § 3 it is shown that L has rank $2d+e$ with $2d = \deg(T)$ if $\deg(T)$ is even and $2d = \deg(T)-1$ if $\deg(T)$ is odd and $e =$ the number of $a \neq 0,1$ such that $T(a) = 0$. The main purpose of this paper is to prove that there is a bijective isomorphism $\psi : \mathbb{Z}^{2d+e} \xrightarrow{\sim} L$ such that

(ii) $$q(\psi(x_1,\dots,x_{2d+e})) = \tfrac{1}{2}(x_1^2 + \dots + x_{2d}^2 - x_{2d+1}^2 - \dots - x_{2d+e}^2) .$$

The proof of (ii), which is completed in § 6, is based on general results of Endo [3] which imply that all elements of $L \otimes \mathbb{C}$ can be represented by periods $p(G)$ of suitable vector valued integrals of the second kind $G = \int dG$, that q can be defined by an integral $q(p(G)) = \int {}^tGPdG$, and that this integral for $q(p(G))$ has a $\mathbb{Z}$–bilinear expansion in terms of suitable values of G. Proofs of the results of [3] for the special case considered here are sketched in § 4 for the convenience of the reader; and explicit expansions for the integral for $q(p(G))$ are derived in §§ 6,7. In addition it is shown in § 5 that $d =$ the geometric genus of an associated elliptic surface $X_T \longrightarrow \mathbb{P}_1$, that the holomorphic

2–forms on X_T have the form $\omega = Rdt \wedge dx/y$ for some polynomial $R \in \mathbb{C}[t]$ with $\deg(R) \leq d-1$, and that each such ω determines a vector valued integral G_R of the first kind such that

$$\text{(iii)} \qquad \int_{X_T} \omega \wedge \overline{\omega} = 2q(p(G_R)) \ .$$

My earlier paper [6] contains an incorrect formula for q_T for the special case $T = (t-a_2)(t-a_3)(t-a_4)$; and my earlier paper [5] contains a formula for q_T on $2L_T$ (rather than L_T) for the special case $T = t(t-1)(t-a_2)$. The corrected formulas given here are needed for applications to problems which are described in [5, 6] and which concern variation of Hodge structure, Kuga–Satake varieties, and modular correspondences.

My work on this paper has been supported by the Faculty Academic Study Program of Rutgers University and by the Max–Planck–Institut für Mathematik in Bonn. I am very grateful to both for helping to make this work possible. I am also very grateful to Miss Grau for typing and helping to arrange my rough manuscript.

2. Preliminary definitions

(i) Let g_2, g_3, $\Delta = g_2^3 - 27g_3^2$, $j = 12^3 g_2^3/\Delta$ be well known modular forms of weights 4, 6, 12, 0 on the upper half plane $\mathfrak{H}$: Im $\tau > 0$; and let λ be the Legendre function, viewed as the universal cover of $\mathbb{P}_1 - \{\infty, 0, 1\}$, with $\Gamma(2)/\pm I$ acting as fundamental group, with (extended) values at cusps $\lambda(\pm 1) = \infty$, $\lambda(i\infty) = 0$, $\lambda(0) = 1$, and with $j = 2^8(\lambda^2 - \lambda + 1)/\lambda^2(\lambda - 1)^2$. Also for T as in § 1 let

$$\Sigma = \{\infty, 0, 1\} \cup \{\text{zeros of } T\} = \{a_\infty, a_0, \ldots, a_{e+1}\} \text{ with distinct } a_i \ ;$$

let $S = \mathbb{P}_1 - \Sigma$; and let $\varphi : U \longrightarrow S$ and $\pi_1(S)$ be the universal cover and fundamental group for S .

(ii) Let

$$\begin{aligned} w^2 &= 4z^3 - G_2 z - G_3 \text{ with} \\ G_2 &= 3(t^2 - t + 1)T^2 , \\ G_3 &= (t+1)(t-1/2)(t-2)T^3 , \end{aligned}$$

be the Weierstrass equation obtained from (i) by the substitution $(x,y) = (z/T + (t+1)/3, w/2T)$; and let $X_T \longrightarrow \mathbb{P}_1$ be the Neron model relative to $\mathbb{C}(t)$ for these equivalent equations.

(iii) There are holomorphic functions τ and h on U and a homomorphism $M : \pi_1(S) \longrightarrow SL_2(\mathbb{Z})$ such that

$$\mathrm{Im}(\tau) > \text{ on } U,$$
$$\varphi = \lambda \circ \tau,$$
$$G_2 \circ \varphi = (g_2 \circ \tau) h^{-4}$$
$$G_3 \circ \varphi = (g_3 \circ \tau) h^{-6}$$
$$h^2 = ((G_2/G_3)\circ\varphi)((g_3/g_2)\circ\tau$$
$$\tau \circ \alpha = M(\alpha)\tau = (a\tau+b)/(c\tau+d) \text{ and}$$
$$h \circ \alpha = (c\tau+d)h$$

for all $\alpha \in \pi_1(S)$ with $M(\alpha) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$.

(iv) One can choose a polygonal fundamental domain D for $\pi_1(S)$ on U as indicated in Figure (iv′) below, with successive vertices at cusps

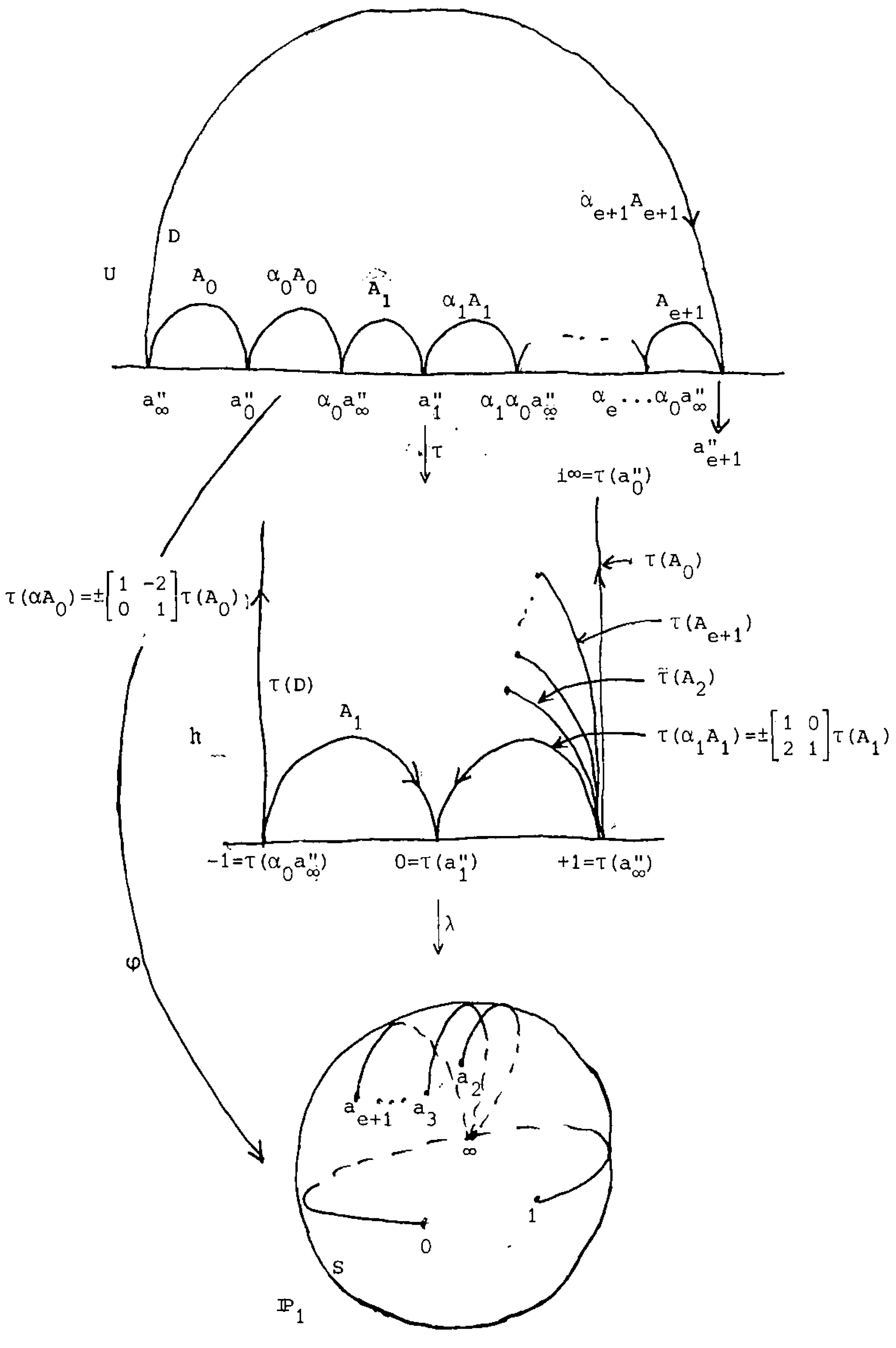
$$a_\infty'' = \infty'', a_0'', \alpha_0\infty'', \ldots, (\alpha_e \cdots \alpha_0)\infty'', a_{e+1}'', (\alpha_{e+1} \cdots \alpha_0)\infty'' = \infty''$$

which lie over points $a_\infty = \infty$ and $a_i \in \Sigma$, with boundary

$$\partial D = \sum_{i=0}^{e+1} (A_i - \alpha_i A_i)$$

consisting of pairs of congruent edges A_i, $\alpha_i A_i$ which lie over suitable arcs $\varphi(A_i) = \varphi(\alpha_i A_i)$ from ∞ to a_i in $\mathbb{P}_1$, with image $\tau(D)$ in $\mathfrak{h}$ which coincides with a standard fundamental domain for $\Gamma(2)$ on $\mathfrak{h}$ (except for deletion of points $\tau(a_2''), \ldots, \tau(a_{e+1}'')$ above points $a_2, \ldots, a_{e+1} \in \Sigma$ in case $e > 0$, and slight detours if necessary around such points on the boundary), and with generators $\alpha_\infty, \alpha_0, \ldots, \alpha_{e+1}$ for $\pi_1(S)$ which correspond to suitable clockwise loops about the points of Σ and satisfy

Figure (iv')



D
U
A_0
$\alpha_0 A_0$
A_1
$\alpha_1 A_1$
$\alpha_{e+1} A_{e+1}$
A_{e+1}
a''_∞
a''_0
$\alpha_0 a''_\infty$
a''_1
$\alpha_1 \alpha_0 a''_\infty$
$\alpha_e \cdots \alpha_0 a''_\infty$
a''_{e+1}
τ
$i\infty = \tau(a''_0)$
$\tau(A_0)$
$\tau(\alpha A_0) = \pm\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\tau(A_0)$
$\tau(A_{e+1})$
$\tau(A_2)$
$\tau(D)$
A_1
h
$\tau(\alpha_1 A_1) = \pm\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\tau(A_1)$
$-1 = \tau(\alpha_0 a''_\infty)$
$0 = \tau(a''_1)$
$+1 = \tau(a''_\infty)$
λ
φ
a_2
a_{e+1}
a_3
∞
1
0
S
$\mathbb{P}_1$

$$\alpha_{e+1} \cdots \alpha_0 \alpha_\infty = 1 \ .$$

(v) Consideration of Figure (iv′) and of types of singular fibers for X_T at a_i implies that the values $M_i = M(\alpha_i)$ are as follows:

$$M_i = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ for } i = 2,\dots,e+1 \text{ (if } e>0 \text{)},$$

$$M_1 = +\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \text{ resp. } -\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \text{ if } T(1)\neq 0 \text{ resp. } T(1)=0 \ ,$$

$$M_0 = +\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \text{ resp. } -\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \text{ if } T(0)\neq 0 \text{ resp. } T(0)=0 \ ,$$

$$M_\infty = +\begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \text{ resp. } -\begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \text{ if deg } T \text{ is odd resp. even.}$$

(vi) The universal cover of the complement of the singular fibers of X_T can be identified with the map

$$\Phi : U \times \mathbb{C} \longrightarrow S \times \mathbb{P}_2$$
$$(u,z) \longrightarrow (\varphi(u),(1),h(u)^{-2}\wp(z,\tau(u),h(u)^{-3}\wp'(z,\tau(u),1)))$$

with $\wp, \wp'$ the Weierstrass p–functions; and the fundamental group for Φ can be identified with semi–direct product $\pi_1(S) \ltimes \mathbb{Z}^2$ operating on $U \times \mathbb{C}$ by

$$(u,z) \longrightarrow (\alpha u,(c\tau(u)+d)^{-1}(z+m\tau(u)+n))$$

for each $\alpha \in \pi_1(S)$ with $M(\alpha) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and for each $(m,n) \in \mathbb{Z}^2$.

3. Parabolic cohomology

As in Shimura [7] the monodromy representation M in § 2 determines a group $Z^1_{par}(M)$ of parabolic cocycles, a subgroup $B^1(M)$ of coboundaries and a parabolic cohomology group

(i) $$L = Z^1_{par}(M)/B^1(M) .$$

By definition $Z^1_{par}(M)$ consists of all maps $Y : \pi_1(S) \longrightarrow \mathbb{Z}^2$ which satisfy the cocycle condition

(ii) $$Y(\alpha\beta) = Y(\alpha)+M(\alpha)Y(\beta) \text{ for all } \alpha,\beta \in \pi_1(S)$$

and also the following parabolic condition: For each $\alpha_v \in \pi_1(S)$ which stabilizes a cusp v for $\pi_1(S)$ on U there exists $V_v \in \mathbb{Q}^2$ such that

(iii) $$Y(\alpha_v) = (I-M(\alpha_v))V_v .$$

$B^1(M)$ consists of maps satisfying the coboundary condition: There exists $V_0 \in \mathbb{Q}^2$ such that

(iv) $$Y(\alpha) = (I-M(\alpha))V_0 \text{ for all } \alpha \in \pi_1(S) .$$

(v) Lemma. For fixed α_k as in § 2, let $Z^1_{par}(M,\alpha_k)$ consist of all $Y' \in Z^1_{par}(M)$ with $Y'(\alpha_k) = 0$. Also suppose that $\det(I-M(\alpha_k)) \neq 0$. Then there is a natural bijection

$$Z^1_{par}(M,\alpha_k) \xrightarrow{\sim} L .$$

Proof: For arbitrary $Y \in Z^1_{par}(M)$ there exists $V_k \in \mathbb{Q}^2$ such that $Y(\alpha_k) = (I-M(\alpha_k))V_k \in \mathbb{Z}^2$. By 2(v) the possible values of $I-M(\alpha_k)$ are $\begin{bmatrix}2&0\\0&2\end{bmatrix}$, $\begin{bmatrix}2&0\\2&2\end{bmatrix}$, $\begin{bmatrix}2&-2\\0&2\end{bmatrix}$ and $\begin{bmatrix}4&-2\\2&0\end{bmatrix}$. It follows that $2V_k \in \mathbb{Z}^2$, that $Y''(\alpha) = (I-M(\alpha))V_k \in \mathbb{Z}^2$ for all $\alpha \in \pi_1(S)$ since $M(\alpha) \in \Gamma(2)$ and $I-M(\alpha) \equiv 0 \bmod(2)$, and that Y'' is the unique element of $B^1(M)$ such that $Y-Y'' \in \mathbb{Z}^1_{par}(M,\alpha_k)$. Q.E.D.

(vi) Corollary. L can be identified with the set of elements

$$Y = \begin{bmatrix} Y_\infty \\ Y_0 \\ \vdots \\ Y_{e+1} \end{bmatrix} \in (\mathbb{Z}^2)^{(e+3)}$$

which satisfy the conditions

$$Y_{e+1} + M_{e+1}Y_e + \dots + (M_{e+1}\dots M_1)Y_0 + (M_{e+1}\dots M_1M_0)Y_\infty = 0$$

$Y_\infty = 0$ if $T=1$ or $T = t(t-1)$,

$Y_0 = 0$ if $T=t$,

$Y_1 = 0$ if $T = t-1$,

$Y_{e+1} = 0$ if $e > 0$,

$$Y_\infty = \begin{bmatrix} m_\infty \\ m_\infty \end{bmatrix} = \left(\begin{bmatrix}1&-0\\0&1\end{bmatrix} - \begin{bmatrix}3&-2\\2&-1\end{bmatrix}\right)V_\infty \text{ if } \deg T \text{ is odd}$$

$$Y_0 = \begin{bmatrix} m_0 \\ 0 \end{bmatrix} = \left(\begin{bmatrix}1&0\\0&1\end{bmatrix} - \begin{bmatrix}1&-2\\0&1\end{bmatrix}\right)V_0 \text{ if } T(0) \neq 0$$

$$Y_1 = \begin{bmatrix} 0 \\ n_1 \end{bmatrix} = \left(\begin{bmatrix}1&0\\0&1\end{bmatrix} - \begin{bmatrix}1&0\\2&1\end{bmatrix}\right)V_1 \text{ if } T(1) \neq 0$$

Proof: The first condition is obtained by applying the cocycle condition to the relation $Y(\alpha_{e+1}\cdots\alpha_0\alpha_\infty) = Y(1) = 0$ and the full set of conditions defines one of the submodules $Z^1_{par}(M,\alpha_i)$ with $i = \infty,0,1$ or $e+1$. Q.E.D.

(vii) Corollary (cf. Shioda [8]).

rank $L = 2(e+3)-4-\#\{I_2 \text{ fibers}\} = 2d+e$ with

$e = \#\{I_0^* \text{ fibers}\}$ and

$$2d = \#\{I_2^* \text{ fibers}\} + \#\{I_0^* \text{ fibers}\}-1 = \begin{cases} \deg T & \text{if } \deg T \text{ is even} \\ \deg T-1 & \text{if } \deg T \text{ is odd} \end{cases}$$

Proof: The total number of singular fibers is $e+3$ and the last three conditions in (vi) correspond to I_2 fibers. Q.E.D.

4. Results of Endo

The following results (i)–(iv) for $L = L_T$ are special cases of general results of Endo [3]. The results in [3] are valid for parabolic cohomology groups which are associated with general Weierstrass equation with coefficients in a function field over $\mathbb{C}$ and with non–constant J–invariant. Similar but less general results in Shioda [8], Hoyt [4], Cox and Zucker [1] explicitly exclude cases with I_0^* fibers. A general result of Stiller [9] provides a more natural proof of surjectivity of the period map p below but does not consider the period relations b and q_T. Preliminary arguments from Shimura [7] are used below to state (i)–(iv) in terms of vector valued integrals rather than scalar valued Eichler integrals as in [3]. For the convenience of the reader, proofs of (i)–(iv) are sketched below. Hypotheses and notation are as in §§ 1–3.

(i) Let $R \in \mathbb{C}(t)$ be such that the integral

$$G_R(u) = \int_{a''_\infty}^{u} R(t(t-1)/T)^{1/2}\Delta^{1/4}\begin{bmatrix}\tau\\1\end{bmatrix} d\tau ,$$

$$\text{with } \tau = \tau(u),\ t = \lambda(\tau(u)),\ \Delta = \Delta(\tau(u)) ,$$

is meromorphic on U and holomorphic on ∂D and convergent as u approaches cusps in D. Then G_R has periods

$$Y(\alpha) = G_R \circ \alpha - M(\alpha) G_R ,\ \ \alpha \in \pi_1(S) ,$$

which are independent of u and which determine an element $p(G_R) \in L \otimes \mathbb{C}$.

(ii) There is a symmetric bilinear form b on the space of such $p(G_R)$ which is defined by

$$b(p(G_Q),p(G_R)) = \int_{\partial D} {}^tG_Q P dG_R \text{ , with } P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .$$

(iii) There exist $2d+e = \text{rank } L$ such integrals $G_n = G_{R_n}$ which satisfy

$$\text{Matrix}(b(_p(G_i),_p(G_j))) = \begin{bmatrix} 0 & 0 & I_d \\ 0 & I_e & 0 \\ I_d & 0 & 0 \end{bmatrix} .$$

(iv) Consequently all elements of $L \otimes \mathbb{C}$ can be represented by periods of such G_R ; and $q = q_T$ can be defined on all of $L \otimes \mathbb{C}$ by

$$q_T(p(G_R)) = b(p(G_R),p(G_R)) \quad .$$

Proofs of (i) and (ii) follow easily from arguments in Eichler [2] and Shimura [7], combined with relations in (vi) below. Obviously (iii) implies (iv). (iii) can be proved by adapting arguments in [3] as follows: First note that periods $p(G_R) \in L \otimes \mathbb{C}$ can be defined as in (i) for more general integrals G_R, $R \in \mathbb{C}(t)$, which are meromorphic on U and convergent as u approaches the cusps a''_∞, a''_0, a''_1 in D but which are not necessarily convergent at a''_2, a''_{e+1} . Also note that such G_R are meromorphic at $a''_2,\ldots,a''_{e+1}$ in terms of parameters $\sigma_i = (\tau-\tau(a''_i))^{1/2}$, and that the bilinear form b can be extended to such $p(G_R)$ by defining

$$\text{(v)} \qquad b(p(G_Q),p(G_R)) = 2\pi i \sum_v \text{res}_v({}^tG_Q P dG_R) \quad ,$$

with the sum over representatives $v \in D \cup \{\text{cusps}\}$ for points of $\mathbb{P}_1$, with residues at $a''_2,\ldots,a''_{e+1}$ computed in terms of the parameters σ_i , and with residues $= 0$ at a''_∞, a''_0,

a_1'' since the G_R converge there.

The fact that (v) defines a symmetric bilinear form which depends only the classes of periods modulo coboundaries can be checked as in [3] by making use of local expansion for scalar valued Eichler integrals g_R and many valued modular forms g_R'' which are associated with the G_R and which satisfy

(vi)
$$g_R = -{}^tG_R P\begin{bmatrix}\tau\\1\end{bmatrix},$$
$$g_R'' = \frac{d^2}{d\tau^2}(g_R) = R(t(t-1)/T)^{1/2}\Delta^{1/4},$$
$$g_R''\circ\alpha = (c\tau+d)^3 g_R'' \text{ for each } \alpha \in \pi_1(S) \text{ with } M(\alpha) = \begin{bmatrix}a&b\\c&d\end{bmatrix},$$
$${}^tG_Q P dG_R = g_Q g_R'' d\tau \text{ and}$$
$$\mathrm{res}_v(g_Q g_R'' d\tau) = \mathrm{res}_v((g_R + A\tau + B)g_Q'' d\tau.$$

The next step, as in [3], is to show that for each choice of an auxiliary interior point $u_0 \in D$ there exist $2d+e$ linear combinations Q_n of the $2d+e+2$ functions

(vii)
$$(t-\varphi(u_0))^{i-1},\ (t-\alpha_{j+1})^{-1},\ (t-\varphi(u_0))^{-k},$$
$$1 \leq i \leq d,\ 1 \leq j \leq e,\ 1 \leq k \leq d+2,$$

with the following property: If $E_n = G_{Q_n}$, then the only nonzero residues of the form $\mathrm{res}_v({}^tE_m P dE_n)$ are

(viii)
$$\mathrm{res}_{u_0}({}^tE_\ell P dE_{d+e+\ell}) = \mathrm{res}_{v_0}({}^tE_{d+e+\ell} P dE_\ell) = \frac{1}{2\pi i},\ i \leq \ell \leq d,$$
$$\mathrm{res}''_{a_{j+1}}({}^tE_{d+j} P dE_{d+j}) = \frac{1}{2\pi i},\ 1 \leq j \leq e.$$

Because of the last relation in (vi), the desired vanishing and non–vanishing of residues in

(viii) becomes obvious if one chooses the linear combinations Q_n of the functions (vii) successively in such a way that the corresponding $e''_n = g''_{Q_n}$ have local expansions of the form (with suitable constants $c \neq 0$)

(ix)
$$\begin{aligned} e''_i &= c(\tau-\tau(u_0))^{d-i} + 0((\tau-\tau(u_0))^d), \quad 1 \leq i \leq d, \\ e''_{d+e+k} &= c(\tau-\tau(u_0))^{-1-(d-k)} + 0((\tau-\tau(u_0))^d), \quad 1 \leq k \leq d, \\ e''_{d+j} &= 0((\tau-\tau(u_0))^d) \\ &= c\sigma^{-3}_{j+1} + \sigma^{-1}_{j+1} o(1) \quad \text{for} \quad 1 \leq j \leq e, \\ e''_n &= \sigma^{-1}_{j+1} 0(1) \quad \text{for} \quad n \neq d+j. \end{aligned}$$

To complete the sketch of the proof of (iii) it suffices to check, as in [ ], that for each E_n there exists a rational function F_n such that

$$e''_n - \frac{d^2}{d\tau^2}\left[\frac{F_n}{h}\right]$$

vanishes at all cusps and has poles only on the orbit of u_0. It follows that

$$\frac{d^2}{d\tau^2}\left[\frac{F_n}{h}\right] = H_n(t(t-1)/T)^{1/2}\Delta^{1/4} \quad \text{for some} \quad H_n \in \mathbb{C}(t),$$

and that $G_{R_n} = E_n - G_{H_n}$ has the same periods as E_n, has poles only on the orbit of u_0, converges at cusps, and hence satisfies (iii). Q.E.D.

5. Meromorphic 2–forms

(i) Every meromorphic 2–form on X_T with poles only on fibers of $X_T \longrightarrow \mathbb{P}_1$ has the form $R dt \wedge dx/y$ for some $R \in \mathbb{C}(t)$. It can be checked that such a form is holomorphic on the fiber above $s \in \mathbb{P}_1$ if and only if $\mathrm{ord}_s R \geq 0$ if $s \neq \infty$ or $\mathrm{ord}_s R \geq -d+1$ if $s=\infty$. It follows that d = the geometric genus of X_T and that $\{t^i dt \wedge dx/y, 0 \leq i \leq d-1\}$ is a basis for holomorphic 2–forms on X_T if $d \geq 1$.

(ii) There is a relation

$$\pi \sqrt{-6}\, dt \wedge dx/y = (t(t-1)/T)^{1/2} \Delta^{1/4} d\tau \wedge dz$$

which can be obtained by identifying differentials with their pull backs along φ, τ and Φ and by combining classical relations with relations involving h, G_2, G_3 in § 2 as follows:

$$G_2^3 - 27G_3^2 = 3^6 2^{-2} t^2 (t-1)^2 T^6 = \Delta h^{-12},$$
$$dj/d\tau = 3^5 2^6 g_2^2 g_3 / \pi i \Delta = 3^5 2^6 G_2^2 G_3 h^{14} / \pi i \Delta,$$
$$\begin{aligned} dj \wedge dx/y &= (dj/d\tau) d\tau \wedge h dz \\ &= 3^5 2^6 G_2^2 G_3 h^{15} (\pi i \Delta)^{-1} d\tau \wedge dz \\ &= 3^5 2^6 G_2^2 G_3 (3^6 2^{-2} t^2 (t-1)^2 T^6)^{-5/4} \Delta^{1/4} (\pi i)^{-1} d\tau \wedge dz \end{aligned}$$
$$j = 2^8 (t^2-t+1)^3 / t^2 (t-1)^2,$$
$$\begin{aligned} dj/dt &= 2^9 (t^2-t+1)^2 (t+1)(t-1/2)(t-2)/t^3 (t-1)^3 \\ &= 2^9 3^{-2} G_2^2 G_3 T^{-7} / t^3 (t-1)^3, \end{aligned}$$
$$\begin{aligned} dt \wedge dx/y &= (dt/dj)(dj \wedge dx/y) \\ &= 2^{-1/2} 3^{-1/2} (\pi i)^{-1} T^{-1/2} (t(t-1))^{1/2} \Delta^{1/4} d\tau \wedge dz. \end{aligned}$$

(iii) If $\pi\sqrt{-6}\, R dt \wedge dx/y$ is holomorphic on X_T, then the integral G_R is holomorphic on U and convergent on $D \cup \{a''_\infty, a''_0, \ldots, a''_{e+1}\}$; and conversely.

Furthermore an argument in Shimura [7] implies that are relations

$$
\begin{aligned}
0 &< \int_{X_T} \omega \wedge \overline{\omega} \\
&= \int_{X_T} g''_R \, \overline{g}''_R \, dz \wedge d\overline{z} \wedge d\tau \wedge d\overline{\tau} \\
&= \int_D \int_{a=0}^{1} \int_{b=0}^{1} (\tau da+db) \wedge (\overline{\tau} da+db) g''_R \, \overline{g}''_R d\tau \wedge d\overline{\tau} \\
&= \int_D (\tau-\overline{\tau}) g''_R \, \overline{g}''_R d\tau \wedge d\overline{\tau} \\
&= \int_D {}^t dG_R \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \overline{dG_R} \\
&= 2\int_D {}^t\mathrm{Re}(dG_R) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathrm{Re}(dG_R) \\
&= 2\int_{\partial D} {}^t\mathrm{Re}(G_R) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathrm{Re}(dG_R) \\
&= 2q(\mathrm{Re}(p(G_R))) \, .
\end{aligned}
$$

6. Formulas for $q = q_T$

(i) For G_R as in 4 (i) let $G = G_R + V$ with constant $V \in \mathbb{C}^2$. Then an argument in Shimura [7] yields relations

$$\begin{aligned} q_T(p(G)) &= \int_{\partial D} {}^tGPdG \\ &= \sum_{i=0}^{e+1} \Big(\int_{A_i} - \int_{\alpha_i A_i}\Big) \\ &= -\sum_{i=0}^{e+1} {}^tY_iPM_i \int_{A_i} dG \\ &= -\sum_{i=0}^{e+1} {}^tY_iPM_i(G(a_i'') - G(\beta_i a_\infty'')) \end{aligned}$$

with $Y_i = Y(\alpha_i) = (I-M_i)G(a_i'')$, $M_i = M(\alpha_i)$, $\beta_0 = 1$ and $\beta_i = \alpha_{i-1}\beta_{i-1}$, $i \geq 1$.

(ii) The preceeding relations can be transformed as follows:

$$\begin{aligned} &q_T(p(G)) \\ &= -{}^tY_0PM_0(G_0 - G_\infty) - {}^tY_1PM_1(G_1 - Y_0 - M_0G_\infty) - \ldots - {}^tY_{e+1}PM_{e+1}(G_{e+1} - Y_e - \ldots \\ &\qquad\qquad \ldots - (M_e \ldots M_0)G_\infty) \\ &= -{}^t((M_{e+1}\ldots M_1)Y_0 + (M_{e+1}\ldots M_2)Y_1 + \ldots + Y_{e+1})P(M_{e+1}\ldots M_0)G_\infty \\ &\quad - {}^tY_0PM_0G_0 - {}^tY_1PM_1(G_1 - Y_0) - \ldots - {}^tY_{e+1}PM_{e+1}(G_{e+1} - Y_e - \ldots - (M_{e+1}\ldots M_1)Y_0) \\ &= -{}^tY_\infty'PG_\infty' - {}^tY_0'PG_0' - \sum_{i=1}^{e+1} {}^tY_i'P(G_i' - \sum_{j=1}^{i} Y_{j-1}') \\ &= +{}^tY_\infty'P_\infty'Y_\infty' + {}^tY_0'P_0'Y_0' + \sum_{i=1}^{e+1} {}^tY_i'(P_i'Y_i' + P\sum_{j=1}^{i} Y_{j-1}') \end{aligned}$$

with $G_i = G(a_i'')$, $Y_\infty' = (M_{e+1}\ldots M_0)Y_\infty$, $G_\infty' = (M_{e+1}\ldots M_0)G_\infty$, $Y_i' = (M_{e+1}\ldots M_{i+1})Y_i$ and $G_i' = (M_{e+1}\ldots M_{i+1}M_i)G_i$ for $0 \leq i \leq e+1$ and with P_i' as in (iii) below. To verify this, first use the relations $G(\beta_i a_\infty'') = Y_{i-1} + M_{i-1}G(\beta_{i-1}a_\infty'')$;

next regroup the terms involving G_∞ and use the relations ${}^tM_iPM_i = P$ and $(M_{e+1}\cdots M_0)Y_\infty + (M_{e+1}\cdots M_1)Y_0 + \cdots + Y_{e+1} = 0$; then replace Y_i and G_i by Y_i' and G_i' ; and finally use the relations $-{}^tY_i'PG_i' = {}^tY_i'P_i'Y_i'$ in (iii) below.

(iii) There are relations

$$-{}^tY_\infty PG_\infty = {}^tY_\infty P_\infty Y_\infty \text{ with } P_\infty = -\tfrac{1}{4}\begin{bmatrix}1&0\\0&1\end{bmatrix} \text{ resp. } \tfrac{1}{2}\begin{bmatrix}1&-2\\0&1\end{bmatrix} \text{ if } M_\infty = \begin{bmatrix}3&-2\\2&-1\end{bmatrix} \text{ resp. } -\begin{bmatrix}3&-2\\2&-1\end{bmatrix},$$

$$-{}^tY_0PM_0G_0 = {}^tY_0P_0Y_0 \text{ with } P_0 = -\tfrac{1}{2}\begin{bmatrix}1&0\\0&1\end{bmatrix} \text{ resp. } -\tfrac{1}{2}\begin{bmatrix}0&1\\-1&1\end{bmatrix} \text{ if } M_0 = \begin{bmatrix}1&-2\\0&1\end{bmatrix} \text{ resp. } -\begin{bmatrix}1&-2\\0&1\end{bmatrix},$$

$$-{}^tY_1PM_1G_1 = {}^tY_1P_1Y_1 \text{ with } P_1 = -\tfrac{1}{2}\begin{bmatrix}1&0\\0&1\end{bmatrix} \text{ resp. } \tfrac{1}{2}\begin{bmatrix}1&1\\-1&0\end{bmatrix} \text{ if } M_1 = \begin{bmatrix}1&0\\2&1\end{bmatrix} \text{ resp. } -\begin{bmatrix}1&0\\2&1\end{bmatrix},$$

$$-{}^tY_iPM_iG_i = {}^tY_iP_iY_i \text{ with } P_i = -\tfrac{1}{2}P \text{ for } 2 \leq i \leq e+1,$$

$$-{}^tY_i'PG_i' = {}^tY_i'P_i'Y_i' \text{ with } P_i' = {}^t(M_{e+1}\cdots M_{i+1})^{-1}P_i(M_{e+1}\cdots M_{i+1})^{-1} \text{ for } i=\infty,0,\ldots,e+1,$$

and so $P_i' = P_i = -\frac{1}{2}P$ and $P_i' + {}^tP_i' = 0$ for $2 \leq i \leq e+1$.

For example if $M_\infty = \begin{bmatrix}3&-2\\2&-1\end{bmatrix}$ and $G_\infty = \begin{bmatrix}w\\z\end{bmatrix}$, then

$$Y_\infty = \begin{bmatrix}-2&2\\-2&2\end{bmatrix}G_\infty = \begin{bmatrix}-2w+2z\\-2w+2z\end{bmatrix},$$

$${}^tY_\infty Y_\infty = 2(-2w+2z)^2 \text{ and}$$

$$-{}^tY_\infty PG_\infty = -[-2w+2z, -2w+2z]\begin{bmatrix}z\\-w\end{bmatrix} = +2(-w+z)^2 = \tfrac{1}{4}\,{}^tY_\infty Y_\infty . \text{ Etc.}$$

(iv) Lemma. If $e \geq 2$ and $Q'_i = P'_i + {}^tP'_i$, then

$$2q_T(p(G)) = {}^tY'(A+{}^tA)Y' = {}^tY''\,{}^tC\,{}^tB(A+{}^tA)BCY'' \text{ with}$$

$$Y' = \begin{bmatrix} Y'_\infty \\ Y'_0 \\ \vdots \\ Y'_{e+1} \end{bmatrix} = BCY'', \quad A = \begin{bmatrix} P'_\infty & 0 & 0 & 0 & 0 & \cdots \\ 0 & P'_0 & 0 & 0 & 0 & \\ 0 & P & P'_1 & 0 & 0 & \\ 0 & P & P & P'_2 & 0 & \\ 0 & P & P & P & P'_3 & \\ & & \vdots & & & \ddots \end{bmatrix}, \quad A+{}^tA = \begin{bmatrix} Q'_\infty & 0 & 0 & 0 & 0 & \cdots \\ 0 & Q'_1 & {}^tP & {}^tP & {}^tP & \\ 0 & P & Q'_0 & {}^tP & {}^tP & \\ 0 & P & P & 0 & {}^tP & \\ 0 & P & P & P & 0 & \\ & & \vdots & & & \ddots \end{bmatrix},$$

$$B = \begin{bmatrix} I & & & \\ & I & & \\ & & \ddots & \\ & 0 & & I \\ -I & -I & \cdots & -I \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad C = \begin{bmatrix} J_\infty & & & & \\ & J_0 & & 0 & \\ & & J_1 & & \\ & & & I & \\ & 0 & & & I \\ & & & & & \ddots \end{bmatrix}, \quad {}^tB(A+{}^tA)B = \begin{bmatrix} Q'_\infty & {}^tP & {}^tP & {}^tP & {}^tP & \cdots \\ P & Q'_0 & {}^tP & {}^tP & {}^tP & \\ P & P & Q'_1 & {}^tP & {}^tP & \\ P & P & P & 0 & P & \\ P & P & P & P & 0 & \\ & & \vdots & & & \ddots \end{bmatrix},$$

$$J_\infty = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ resp. } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ if } M_\infty = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \text{ resp. } -\begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$$

$$J_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ resp. } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ if } M_0 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \text{ resp. } -\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$$

$$J_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ resp. } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ if } M_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \text{ resp. } -\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

Proof: This follows from the final relation in (ii), the conditions on the Y_i in 3 (vi), and corresponding conditions on the Y'_i . For examples see the proof of (v), and § 7 below.

Q.E.D.

(v) Proposition. If $e \geq 4$ and if $T_1 = T/(t-a_e)(t-a_{e+1})$, then

$$2q_T \underset{\mathbb{Z}}{\cong} 2q_{T_1} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

Proof: For example for $T = \prod_{i=2}^{6}(t-a_i)$, $T_1 = \prod_{i=2}^{4}(t-a_i)$, $e=5$, there is a relation

$$2q_T(p(G)) = {}^tY^{*}\,{}^tE\,{}^tB(A+{}^tA)BEY^{*}$$

with factors as follows:

$$
\underset{A+{}^tA}{\begin{bmatrix}
Q'_\infty & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & Q'_0 & {}^tP & {}^tP & {}^tP & {}^tP & {}^tP & {}^tP\\
0 & P & Q'_1 & {}^tP & {}^tP & {}^tP & {}^tP & {}^tP\\
0 & P & P & 0 & {}^tP & {}^tP & {}^tP & {}^tP\\
0 & P & P & P & 0 & {}^tP & {}^tP & {}^tP\\
0 & P & P & P & P & 0 & {}^tP & {}^tP\\
0 & P & P & P & P & P & 0 & {}^tP\\
0 & P & P & P & P & P & P & 0
\end{bmatrix}}
\quad
\underset{B}{\begin{bmatrix}
I & 0 & 0 & 0 & 0 & 0\\
0 & I & 0 & 0 & 0 & 0\\
0 & 0 & I & 0 & 0 & 0\\
0 & 0 & 0 & I & 0 & 0\\
0 & 0 & 0 & 0 & I & 0\\
0 & 0 & 0 & 0 & 0 & I\\
-I & -I & -I & -I & -I & -I\\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}}
\quad
\underset{(A+{}^tA)B}{\begin{bmatrix}
Q'_\infty & 0 & 0 & 0 & 0 & 0\\
P & Q'_0+P & 0 & 0 & 0 & 0\\
P & 2P & Q_1+P & 0 & 0 & 0\\
P & 2P & 2P & P & 0 & 0\\
P & 2P & 2P & 2P & P & 0\\
P & 2P & 2P & 2P & 2P & P\\
0 & P & P & P & P & P\\
-P & 0 & 0 & 0 & 0 & 0
\end{bmatrix}}
$$

$$
\underset{{}^tB(A+{}^tA)B}{\begin{bmatrix}
Q'_\infty & {}^tP & {}^tP & {}^tP & {}^tP & {}^tP\\
P & Q'_0 & {}^tP & {}^tP & {}^tP & {}^tP\\
P & P & Q'_1 & {}^tP & {}^tP & {}^tP\\
P & P & P & 0 & {}^tP & {}^tP\\
P & P & P & P & 0 & {}^tP\\
P & P & P & P & P & 0
\end{bmatrix}}
\quad
\underset{Y'}{\begin{bmatrix}
Y'_\infty\\ Y'_0\\ Y'_1\\ Y'_2\\ Y'_3\\ Y'_4
\end{bmatrix}}
\quad
\underset{{}^tE{}^tB(A+{}^tA)BE}{\begin{bmatrix}
Q'_\infty & {}^tP & {}^tP & {}^tP & 0 & 0\\
P & Q'_0 & {}^tP & {}^tP & 0 & 0\\
P & P & Q'_1 & {}^tP & 0 & 0\\
P & P & P & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & P\\
0 & 0 & 0 & 0 & P & 0
\end{bmatrix}}
\quad
\underset{Y^*}{\begin{bmatrix}
Y'_\infty\\ Y'_0\\ Y'_1\\ Y'_2-Y'_3-Y'_4\\ Y'_3+Y'_2\\ Y'_4+Y'_3
\end{bmatrix}},
$$

Note that the $Y'_\infty, Y'_0, Y'_1, Y'_2$ block of ${}^tB(A+{}^tA)B$ is the analogue of ${}^tB(A+{}^tA)B$ for $2q_{T_1}$. Also note (in general) that J_∞, J_0, J_1 in (iv) are the same for both T and T_1: In the present example, $M_\infty = \begin{bmatrix} 3 & -2\\ 2 & -1\end{bmatrix}$, $J_\infty = \begin{bmatrix}1\\1\end{bmatrix}$ and $Y'_\infty = \begin{bmatrix}1\\1\end{bmatrix} m_\infty$ for both T and T_1. Since the row operations used to define E do not affect Y'_∞, Y'_0, Y'_1, it follows from 3 (vi) that $2q_T$ is completely determined by ${}^tEB(A+{}^tA)BE$, that $2q_{T_1}$ is completely determined by the $Y'_\infty, Y'_0, Y'_1, Y'_2$ block of ${}^tB(A+{}^tA)B$, and that

$$2q_T \cong 2q_{T_1} \oplus \begin{bmatrix} 0 & {}^tP \\ P & 0 \end{bmatrix} \cong 2q_{T_1} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

Clearly a similar argument works whenever $e \geq 5$. However for $e=4$ a modification of this argument (and of 3 (vi)) are needed to avoid row operations which affect Y_1' . For example if $T = \prod_{i=2}^{5}(t-a_i)$ with $e=4$, then the condition $Y_5 = 0$ in 3(vi) can be replaced by the condition $Y_\infty = 0$ with corresponding changes in Y_i', B, C and E . Details are omitted. Q.E.D.

(vi) Similar arguments, which stop short of replacing Y_i, G_i by Y_i', G_i' , show that $2q_T$ is determined up to isomorphism over $\mathbb{Z}$ by relations of the form

$$2q_T(p(G)) = {}^tY(D+{}^tD)Y = {}^tZ\,{}^tE(D+{}^tD)EZ$$

with $Z \in \mathbb{C}^{2d+e}$, with $p(G) \longrightarrow Z$ inducing an isomorphism $L_T \xrightarrow{\sim} \mathbb{Z}^{2d+e}$, with suitable E determined by conditions in 3 (vi), and with

$$D = \left[\begin{array}{ccc|ccc} P_\infty & 0 & 0 & & & \\ PM_0 & P_0 & 0 & & 0 & \\ PM_1M_0 & PM_1 & P_1 & & & \\ \hline & & & P_2 & 0 & 0 \ldots \\ & 0 & & -P & P_3 & 0 \\ & & & P & -P & P_4 \\ & & & & \vdots & \end{array}\right]$$

(vii) The relation 1 (ii) can now be verified as follows: First if $T = 1, t, t-1$, then $d=e=0$ and $q_T=0$. Next isomorphisms such as

$$X_{(t-1)(t-a)} \xrightarrow{\sim} X_{t(t-1+a)},\ (t,x,y) \longrightarrow (1-t,1-x,y) \ ,$$

which permute fibers above ∞, 0, 1 , can be used to transform all other cases with $e<4$ into one of the special cases for which the relations in (vi) above are calculated explicitly in § 7 below and for which elementary row and column operations can be used to verify 1 (ii). Finally the proposition in (v) above together with the relation

$$ {}^{t}E\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} E = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \mp 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} $$

reduce verification of 1 (ii) for other cases to cases with $e<4$. Q.E.D.

(viii) In each of the special cases tabulated in § 7, the vectors $Y \in \mathbb{Z}^{2(e+3)}$ and $Y_i \in \mathbb{Z}^2$ satisfying the relations specified in 3 (vi); and these relations determine (and can be explicitly determined from) the matrix E .

7. <u>Special cases with $e \leq 4$</u>..

T	$D + {}^tD$	Y	E	z	${}^tE(D + {}^tD)E$
$t-a_2$	$\begin{bmatrix} -\frac{1}{2} & 0 & 0 & -1 & 2 & -1 & \\ 0 & -\frac{1}{2} & 1 & 2 & -3 & 2 & 0 \\ 0 & 1 & -1 & 0 & 2 & -1 & \\ -1 & 2 & 0 & -1 & 1 & 0 & 0 \\ 2 & -3 & 2 & -1 & -1 & 0 & \\ -1 & 2 & 1 & 0 & 0 & -1 & 0 \\ 0 & & 0 & & 0 & & 0 \end{bmatrix}$	$\begin{bmatrix} m_\infty \\ m_\infty \\ m_0 \\ 0 \\ 0 \\ n_1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} m_n \end{bmatrix}$	$\begin{bmatrix} -1 \end{bmatrix}$
$t(t-1)$	$\begin{bmatrix} * & * & & * & \\ * & 0 & 0 & -2 & 1 \\ & 0 & 1 & -1 & 0 \\ * & -2 & -1 & 1 & 0 \\ & 1 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ m_0 \\ n_0 \\ m_1 \\ n_1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$	$\begin{bmatrix} m_0 \\ n_0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$t(t-a_2)$	$\begin{bmatrix} * & * & & * & & 0 \\ & 0 & 0 & 2 & -1 & \\ * & 0 & 1 & 1 & 0 & 0 \\ & 2 & 1 & -1 & 0 & \\ * & -1 & 0 & 0 & -1 & 0 \\ 0 & & 0 & & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ m_0 \\ n_0 \\ 0 \\ n_1 \\ m_2 \\ n_2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} m_0 \\ n_0 \\ n_1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
$(t-a_2)(t-a_3)$	$\begin{bmatrix} * & & & & & & & * \\ & -1 & 0 & 2 & -1 & & & \\ & 0 & -1 & 1 & 0 & & 0 & \\ & 2 & 1 & -1 & 0 & & & \\ & -1 & 0 & 0 & -1 & & & \\ & & & & & & 0 & 1 \\ & & 0 & & & 0 & -1 & 0 \\ & & & & 0 & -1 & & \\ * & & & & 1 & 0 & & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ m_0 \\ 0 \\ 0 \\ n_1 \\ m_2 \\ n_2 \\ m_2 - m_0 \\ n_2 - n_1 - 2m_0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} m_0 \\ n_1 \\ m_2 \\ n_2 \end{bmatrix}$	$\begin{bmatrix} -1 & -1 & 2 & -1 \\ -1 & -1 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \cong \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$(t-a_2)(t-a_3)(t-a_4)$	$\begin{bmatrix} -\frac{1}{2} & 0 & 0 & -1 & 2 & -1 & & & & & & \\ 0 & -\frac{1}{2} & 1 & 2 & -3 & 2 & & & & & & \\ 0 & 1 & -1 & 0 & 2 & -1 & & & 0 & & & \\ -1 & 2 & 0 & -1 & 1 & 0 & & & & & & \\ 2 & -3 & 2 & 1 & -1 & 0 & & & & & & \\ -1 & 2 & -1 & 0 & 0 & -1 & & & & & & \\ & & & & & & & & 0 & 1 & 0 & -1 \\ & & & & & & 0 & & -1 & 0 & 1 & 0 \\ & & & 0 & & & 0 & -1 & & & 0 & 1 \\ & & & & & & 1 & 0 & & 0 & -1 & 0 \\ & & & & & & 0 & 1 & 0 & -1 & & \\ & & & & & & -1 & 0 & 1 & 0 & & 0 \end{bmatrix}$	$\begin{bmatrix} m_\infty \\ m_\infty \\ m_0 \\ 0 \\ 0 \\ n_1 \\ m_2 \\ n_2 \\ m_3 \\ n_3 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 & 0 \\ -1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} m_\infty \\ m_0 \\ n_1 \\ m_2 \\ n_2 \end{bmatrix}$	$\begin{bmatrix} -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 2 & -1 \\ 1 & -1 & -1 & 1 & 0 \\ -1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{bmatrix} \cong \begin{bmatrix} -1 & & & & \\ & -1 & & & 0 \\ & & -1 & & \\ & 0 & & 1 & \\ & & & & 1 \end{bmatrix}$

$t(t-1)(t-a_2)(t-a_3)$

$$\begin{bmatrix} * & & & & & & & * \\ 0 & 0 & -2 & 1 & & & & \\ 0 & 1 & -1 & 0 & & & & \\ -2 & -1 & 1 & 0 & & & & \\ 1 & 0 & 0 & 0 & & & & \\ & & & & & 0 & 0 & -1 \\ & & & & & & 1 & 0 \\ & & & & 0 & 1 & & 0 \\ * & & & & -1 & 0 & & \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ m_0 \\ n_0 \\ m_1 \\ n_1 \\ m_2 \\ n_2 \\ m_3 \\ n_3 \end{bmatrix} \quad \begin{bmatrix} & & 0 & & & \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 \\ -2 & -1 & 0 & -1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} m_0 \\ n_0 \\ m_1 \\ n_1 \\ m_2 \\ n_2 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & -2 & 1 & 2 & -1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ -2 & -1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & -2 & 0 \end{bmatrix}$$

(4+,2-)

$t(t-a_2)(t-a_3)(t-a_4)$

$$\begin{bmatrix} * & 0 & 0 & 2 & -1 & & & & & * \\ & 0 & 1 & 1 & 0 & & & & & \\ & 2 & 1 & -1 & 0 & & & 0 & & \\ & -1 & 0 & 0 & -1 & & & & & \\ & & & & & 0 & 0 & -1 & 0 & 1 \\ & & & & & & 1 & 0 & -1 & 0 \\ & & & & 0 & 1 & & 0 & 0 & -1 \\ & 0 & & & -1 & 0 & & & 1 & 0 \\ & & & & 0 & 1 & 0 & 1 & & \\ * & & & & 1 & 0 & -1 & 0 & & 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ m_0 \\ n_0 \\ m_1 \\ n_1 \\ m_2 \\ n_2 \\ m_3 \\ n_3 \\ m_4 \\ n_4 \end{bmatrix} \quad \begin{bmatrix} & & & 0 & & & \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & -1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} m_0 \\ n_0 \\ n_1 \\ m_2 \\ n_2 \\ m_3 \\ n_3 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & -1 & 2 & -1 & -2 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 1 & 0 & -1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & -1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

(4+,3-)

References

1. D. Cox and S. Zucker, Intersection numbers of sections of elliptic surfaces, Inv. Math. 53 (1979), 1–44.
2. M. Eichler, Eine Verallgemeinerung der Abelschen Integrale, Math. Z. 67 (1959), 267–298.
3. Y. Endo, Parabolic cohomology and generalized cusp forms of weight three associated to Weierstrass equations over function fields, Thesis, Temple University (1985).
4. W. Hoyt, Parabolic cohomology and cusp forms of the second kind for extensions of fields of modular functions, in Proc. of Conf. on Automorphic Forms and Fuchsian Groups, T.A. Metzger, Ed. Univ. of Pittsburgh, 1978.
5. W. Hoyt, Notes on elliptic K3 surfaces, in LNM No. 1240, Springer–Verlag (1987), 196–213.
6. W. Hoyt, On twisted Legendre equations and Kummer surfaces, Theta Functions, Bowdoin 1987, AMS Proc. Symp. Pure Math. Vol. 49 Part 1, 695–707.
7. G. Shimura, Sur les integrales attachées aux formes automorphes, J. Math. Soc. Japan 11 (1959), 291–311.
8. T. Shioda, On elliptic modular surfaces, J. Math. Soc. Japan 29 (1972), 20–59.
9. P. Stiller, Special values of Dirichlet series, monodromy and the periods of automorphic forms, Memoirs of AMS, No. 299 (1984).