EULER CHARACTERISTIC AND DIMCA–NÉMETHI FORMULA

by

A.Parusiński

Max-Planck-Institut für Mathematik,
Gottfried-Claren-Strasse 26
D-5300 Bonn 3,
West Germany

Institute of Mathematics,
Gdańsk Universtity,
ul.Wita Stwosza 57,
PL-80-952 Gdańsk, Poland

MPI/89-15

EULER CHARACTERISTIC AND DIMCA–NÉMETHI FORMULA

A.Parusiński[1]

Max-Planck-Institut für Mathematik,

Gottfried-Claren-Strasse 26

D-5300 Bonn 3, West Germany[2]

Abstract. In this paper, using the additivity of the topological Euler-Poincaré characteristic of a complex stratification, some elementary properties of the behaviour of the Euler-Poincaré characteristic in linear systems of divisors are established. As a corollary a new simple proof of the Dimca-Némethi formula for the multiplicity of the dual variety is presented. The method of the proof allows to extend the formula for the case of any codimension of the dual variety and to give a general formula for the degree of dual variety.

[1] Supported by the Alexander von Humboldt Stiftung

[2] On leave from Institute of Mathematics, Gdańsk Universtity, ul.Wita Stwosza 57, PL-80-952 Gdańsk, Poland.

Let M be a connected compact complex manifold and let E be a holomorphic vector bundle over M. Let $v \in H^0(E) \setminus 0$ be a holomorphic section of E and let $X = v^{-1}(0)$. Then, we define the number $\mu(M, X)$ (or just $\mu(X)$, if M and E are obvious) as

$$\mu(X) = (-1)^{dimM - dimE + 1}(\chi(X) - \chi(M, E)),$$

where $\chi(X)$ denotes the topological Euler-Poincaré characteristic of X and $\chi(M, E)$ is the Euler-Poincaré characteristic excpected for a smooth zero set of a section (i.e. of a section transversal to the zero section), which can be expressed in terms of Chern classes $c(E), c(M)$ of E and M as follows

$$(1) \qquad \chi(M, E) = \langle c_{dimE}(E) \cdot c(M)/c(E), [M] \rangle,$$

where $[M]$ is a fundamental class of M (for a more general formula for the Euler-Poincaré characteristic of smooth degeneracy loci see **[P]**). One may prove (1) in the same way as in the case of line bundle (see **[H]**). The normal bundle of X (provided it is smooth) equals $E|_X$ and consequently the tangent bundle equals $TM|_X/E|_X$. Therefore, $\chi(X) = \langle c(X), [X] \rangle = \langle c(M)|_X/c(E)|_X), [X] \rangle$ and (1) follows from the fact that $i_*([X]) = [M] \cap c_{dimE}(E)$ (here i denotes the inclusion $X \subset M$). For the case $dimE = 1$ see **[P]** for the discusion of the properties of $\mu(X)$. In particular, in this case, if X has only isolated singularities, then

$$(2) \qquad \mu(X) = \sum_{p \in Sing(X)} \mu(X; p),$$

where $\mu(X; p)$ denotes the Milnor number of X at p (see **[M]** for the definition).

The aim of this paper is to study the behaviour of this number (i.e. in fact the behaviour of the topological Euler-Poincaré characteristic) in a system of linearly dependent divisors (Proposition 1) and using its properties to prove a formula (Formula 2) for the multiplicity of the dual variety. This formula generalizes the Dimca-Némethi formula (Formula 1, see also **[D1]** [N]) to the case of any codimension of the dual variety. As a corollary we reprove (Proposition 2) the generalized Plcker formulas for the degree of the dual variety (see e.g. **[Ho]**, **[Ka]**, **[K1]**,**[K2]**).

Let L be a holomorphic line bundle over M (M as above) and let V be a k-dimensional vector subspace of $\mathbf{P}(H^0(L))$. Consider

$$T = \{(x, v) \in M \times V; v(x) = 0\}$$

and the canonical projections p_1, p_2 of T onto M and V respectively. Note that there exists a stratification $\mathcal{S}$ of V such that $\chi(p_2)^{-1}(v)$ is constant along each stratum of $\mathcal{S}$. For example we may take a stratification such that p_2 is topologically locally trivial along each stratum. The existence of such a stratification follows from the existence of Whitney stratification of a complex analytic set (see e.g. **[L-T]**). In the case $k = 1$ this stratification consist of a finite set and its complement.

PROPOSITION 1. *Let V be a k-dimensional linear subspace of $\mathbf{P}(H^0(L))$ such that a generic section in V has a smooth zero set X_g. Let Y denote the base points set of V and let $\mathcal{S}$ be a stratification of $\check{V}$ such that $\chi(p_2)^{-1}(v)$ (p_2 defined above) is constant along each stratum of $\mathcal{S}$. Then the number*

$$\gamma(V) = \sum_{S \in \mathcal{S}} \chi(S)\mu(X_S) + (-1)^{k+1}\mu(Y),$$

where $\mu(X_S) = (-1)^{dim M}(\chi(X_g) - \chi(X_S))$ and X_S denotes a generic fibre of p over $S \in \mathcal{S}$, does not depend on V but only on L and k and equals

$$(-1)^{dim M}(k \cdot \chi(M) - (k+1) \cdot \chi(M, L) + \chi(M, k \cdot L))$$

$$=(-1)^{dim M}\langle(k \cdot c(L)^{k+1} - (k+1) \cdot c_1(L) \cdot c(L)^k + c_1(L)^{k+1})c(M)/c(L)^{k+1}, [M]\rangle.$$

(in particular for $k = 1$ it equals $(-1)^{dim M}\langle c(M)/c(L)^2, [M]\rangle$)

NOTATION. *We will denote the number from Proposition 2 by* $\gamma_k(L)$.

Proof The main tool we will use in the proof is the good behaviour of the Euler-Poincaré characteristic of a complex stratified set (see e.g. **[L.T]**) and of a fibration. First we compute $\chi(T)$ using p_1 and next p_2. So,

$$
\begin{aligned}
\chi(T) &= \chi(\tilde{M} \setminus p_1^{-1}(Y)) + \chi(p_1^{-1}(Y)) \\
&= \chi(M \setminus Y)\cdot(\chi(V) - 1) + \chi(Y)\cdot\chi(V) \qquad (3)\\
&= k\cdot\chi(M) + \chi(Y).
\end{aligned}
$$

On the other hand we have

$$\chi(T) = \sum_{S\in\mathcal{S}} \chi(S)\cdot\chi(X_S), \qquad (4)$$

where $\chi(X_S)$ denotes the Euler-Poincaré characteristic of a fibre of p_2 over $S \in \mathcal{S}$. By comparing (3) and (4) we obtain

$$
\begin{aligned}
&\sum_{S\in\mathcal{S}} \chi(S)\mu(X_S) + (-1)^{k+1}\mu(Y) \\
=&(-1)^{dim M}(k\cdot\chi(M) - (k+1)\cdot\chi(M, L) + \chi(M, k\cdot L))
\end{aligned}
$$

The last equality of the statement of the proposition follows directly from (1).

□

Assume that M is imbeded into $\mathbf{P}^N$ in such a way that it is not contained in any projective subspace of $\mathbf{P}^N$. Consider on M the bundle L associated to the restriction to M of $\mathcal{O}_{\mathbf{P}^N}(1)$. Then we can treat $\check{\mathbf{P}}^N$-the space of hyperplanes of $\mathbf{P}^N$ as a subspace of $\mathbf{P}(H^0(L))$. The set of hyperplanes H for which $H \cap M$ is singular is a proper subvariety of $\check{\mathbf{P}}^N$ and is called the dual variety of M and denoted by $\check{M}$ (see e.g. **[K1]**, **[K2]** for more information). We will use Proposition 1 for studying the properties of $H \cap M$. First we need the following lemma.

Lemma 1. *Let $M \subset \mathbf{P}^N$ be a smooth irreducible subvariety. Then, for $H \in Reg(\check{M})$ the set of singular points of $M \cap H$ is a linear subspace $(T_H\check{M})\check{}$ of $\mathbf{P}^N$. Moreover, at any its singular point $M \cap H$ has a transversal singularity of the type A_1. For an arbitrary $H \in \check{M}$ the dimension of the set of singular points of $M \cap H$ can not be smaller than for generic one i.e. $codim\check{M} - 1$.*

Proof The first statement follows from Biduality Theorem (see e.g. [**K1**]) which says that $\check{\check{M}} = M$. In fact, let $C(M) \subset \mathbf{P}^N \times \check{\mathbf{P}}^N$ be a projective conormal space i.e. $C(M) = \{(x, H); T_xM \subset H\}$ and π_1, π_2 denote the projection of $C(M)$ into the factors. Then $\check{M}$ equals the image of $C(M)$ by π_2. By Biduality Theorem $Sing(M \cap H) = \pi_1(\pi_2^{-1}(H)) = (T_H\check{M})\check{}$ and H is a regular value of π_2 iff $H \in Reg(\check{M})$. To examine $Sing(M \cap H)$ locally we follow the notation of [**D1**] . Take $x_0 = 0 \in \mathbf{C}^N \subset \mathbf{P}^N$ and assume that M near x_0 is given by

$$h : (\mathbf{C}^n, 0) \to (\mathbf{C}^N, 0);\ h(t) = (t_1, \dots, t_n, f_{n+1}(t), \dots, f_N(t)),$$

for some germs of analytic functions $f_j : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$. We may assume $\frac{\partial f_j}{\partial t_m}(0) = 0$ for each $j = n+1, \dots, N; m = 1, \dots, n$. We parametrize affine hyperplanes H : $a_0 + a_1u_1 + \ldots + a_Nu_N = 0$ (normalizing $a_N \equiv 1$) in $\mathbf{C}^N$ and tangent to M near x_0 by

$$a_m(t, \alpha) = \begin{cases} -\sum_{j=1}^N a_jh_j(t) & \text{if } m = 0 \\ -\sum_{j=n+1}^N \alpha_j \frac{\partial f_j}{\partial t_m}(t) & \text{if } m = 1, \dots, n \\ \alpha_m & \text{if } m = n+1, \dots, N \end{cases},$$

where $\alpha = (\alpha_{n+1}, \dots, \alpha_{N-1}) \in \mathbf{C}^{N-n-1}$ and $\alpha_N \equiv 1$. Assume that $H_0 \in \check{M}$ given by $\{a_0 = a_1 = \ldots = a_{N-1} = 0\}$ belongs to $Reg(\check{M})$ and is a regular value of $\pi_2 : C(M) \to \check{M}$. Consider the points of $\pi_2^{-1}(H_0)$. They are defined by the equations $\frac{\partial f_N}{\partial t_1} = \ldots = \frac{\partial f_N}{\partial t_n} = 0$, $\alpha = 0$.

Take the point of $\pi_2^{-1}(H_0)$ corresponding to x_0 (i.e. $t = 0, \alpha = 0$). It is a regular point of π_2, so the differential of π_2

$$\begin{pmatrix} (A(t,\alpha)) & (*) \\ \left(-\sum_{j=n+1}^{N} \alpha_j \frac{\partial^2 f_j}{\partial t_i \partial t_l}\right)_{i,l=1,\dots,n} & (*) \\ (0) & (Id) \end{pmatrix}$$

has constant rank, say s, $0 \leq s \leq n$, in the neighbourhood of this point. Since $A(0,0) = 0$, the matrix

$$\left(-\sum_{j=n+1}^{N} \alpha_j \frac{\partial^2 f_j}{\partial t_i \partial t_l}\right)_{i,l=1,\dots,n}$$

has also constant rank s. In particular we obtain that at the points of $H_0 \cap M$ the Hessian

$$\mathcal{H}(t) = \left(-\frac{\partial^2 f_N}{\partial t_i \partial t_l}\right)_{i,l=1,\dots,n}.$$

has also constant rank s. After a linear change of variables we can assume that at x_0 (i.e. $t = 0$)

$$\mathcal{H}(x_0) = \begin{pmatrix} (Id_{\mathbb{C}^s}) & (0) \\ (0) & (0) \end{pmatrix}.$$

We know that $Sing(H_0 \cap M)$ is (near x_0) the zero set of $\frac{\partial f_N}{\partial t_1}, \dots, \frac{\partial f_N}{\partial t_n}$. We claim that it is defined by the first s of them. In fact, they have independent differentials near x_0 and, by biduality, $Sing(H_0 \cap M)$ is a submanifold of M of codimension $n - s$, so the claim follows. It is now obvious that the transversal singularity of f_N at x_0 is a nondegenerate one. The last statement of the lemma is obvious.

□

REMARK 1.. *When M is a hypersurface or a complete intersection, then it is not difficult to see that $\check{M}$ is a hypersurface (see e.g* [**K2**]*). Moreover, then π_2 is finite (see* [**I**], [**D2**] *or* [**F-L**]*), so the intersection $M \cap H$ has always only isolated singularities. Generaly, we have trivially $dim\check{M} \geq codimM - 1$ and, for M smooth as above, $dim\check{M} \geq dimM$ (see* [**Z**],[**F-L**] *and also* [**K1**]*).*

Assume $codim\check{M} = 1$. Then the following formula for the multiplicity of the dual variety holds.

FORMULA 1. (Dimca, Némethi)
Let $codim\check{M} = 1$ and $H \in \check{\mathbf{P}}^N$. Then,

$$m_H\check{M} = \mu(M \cap H) + \mu(M \cap H \cap H_g),$$

where H_g is a generic hyperplane of $\mathbf{P}^N$.

The formula above was first proved by Némethi in [N] as a corollary of his Affine Lefschetz Theorem. In the case when $Sing(M \cap H)$ is finite it was also proved, by elementary methods, by Dimca in [**D1**]. Then $M \cap H \cap H_g$ is smooth and the fomula has a simple form

$$m_H\check{M} = \sum_{p \in Sing(M \cap H)} \mu(M \cap H; p).$$

Now we generalize the formula for the case of any codimension of $\check{M}$.

FORMULA 2. *Let $k = codim\check{M}$ and $H \in \check{\mathbf{P}}^N$. Then,*

$$m_H\check{M} = \mu(M \cap H) + (-1)^{k-1}\mu(M \cap H \cap W_g),$$

where W_g is a generic $dim\check{M}$-dimensional linear subspace of $\mathbf{P}^N$.

Proof For $H \notin \check{M}$, it is obvious. If $H \in \check{M}_{reg}$, then by Lemma 1 $M \cap H \cap W_g$ is smooth and the formula follows from the following lemma.

Lemma 2. *For $H \in \check{M}_{reg}$*

$$\mu(M \cap H) = (-1)^{k-1}.$$

Proof This follows, for example, from Proposition1.5 in **[P]** and Lemma 1. We can prove it also using Proposition 1. First note that $\gamma_{k-1}(L) = 0$. Let V' be a generic $(k-1)$-dimensional linear subspace of $\check{\mathbf{P}}^N$ going through H. Then, by Proposition

$$(-1)^k \mu(\check{V}' \cap M) + \mu(M \cap H) = 0.$$

But by Lemma 1 $\check{V}' \cap M$ has only one nondegenerate singular point and so the lemma follows from (2).

□

Take arbitrary $H_0 \in \check{M}$ and consider a generic k-dimensional linear subspace V of $\check{\mathbf{P}}^N$ going through H_0 and crossing $\check{M} \setminus H$ only at regular points $H_1, H_2, \dots, H_s$ with transversal crossings. We can assume also that l contains a generic hyperplane H_g (in particular V is not tangent to the normal cone to $\check{M}$ at H). Move V a little to obtain a linear subspace V' crossing $\check{M}$ only at regular points $:H_1', H_2', \dots, H_s'$ coresponding to $H_1, H_2, \dots, H_s$; and $H_{s+1}', \dots, H_{s+m}'$ coresponding to H_0, where $m = m_H \check{M}$ and $s + m = deg\check{M}$. Then, $V' \pitchfork \check{M}$ and by Proposition 1

$$\gamma_k(L) = \mu(M \cap H) + s + (-1)^{k+1} \mu(M \cap \check{V}) \tag{5}$$
$$= m + s + (-1)^{k+1} \mu(M \cap \check{V}').$$

Since V' is general $M \cap \check{V}'$ is smooth and consequently $\mu(M \cap \check{V}') = 0$. The formula follows from (5).

□

Next using Proposition 1 we prove the formula for the degree of dual variety which is due to Holme **[Ho]** (for $k = 2$ to Katz **[Ka]**, see also **[K1]**).

Proposition 2. *Let $k = dim\check{M}$. Then,*

$$\gamma_1(L) = \dots = \gamma_{k-1}(L) = 0, \gamma_k(0) > 0$$

and

$$\begin{aligned}
deg\check{M} &= \gamma_k(L) \\
&= (-1)^{dimM}\big(k\cdot\chi(M) - (k+1)\cdot\chi(M,L) + \chi(M,k\cdot L)\big) \\
&= (-1)^{dimM}\langle h^{k-1}\cdot c(M)/(1+h)^{k+1}, [M]\rangle \\
&= (-1)^{dimM}\langle \sum_{i=k-1}^{dimM} \binom{i+1}{k}(-h)^i c_{dimM-i}(M), [M]\rangle,
\end{aligned}$$

where H_g is a generic hypersurface of $\mathbf{P}^N$, W_g a generic linear subspace of codmension k and h is the hyperplane class.

Proof The first statement is obvious. Take a generic k-dimensional linear subspace V of $\check{\mathbf{P}}^N$. Then V intersects $\check{M}$ in exactly $deg\check{M}$ regular points. Because V is generic, its base points set is smooth (Bertini Theorem) and consequently by Proposition and Lemma 1

$$\gamma_k(L) = \sum_{H\in\check{M}\cap V} \mu(m\cap H) = deg\check{M}.$$

To end the proof of the proposition it sufficies to prove that

$$\begin{aligned}
&(-1)^{dimM}\langle (k(1+h)^{k+1} - (k+1)h(1+h)^k + h^{k+1})c(M)/(1+h)^{k+1}, [M]\rangle \\
&= (-1)^{dimM}\langle h^{k-1}\cdot c(M)/(1+h)^{k+1}, [M]\rangle.
\end{aligned}$$

This follows from the following lemma.

Lemma 3. *Let M be a connected compact complex manifold and let L be a holomorphic line bundle over M. Then for each $s\in\mathbf{N}$ we have*

$$\begin{aligned}
&\gamma_s + 2\cdot\gamma_{s-1} + \gamma_{k-2} \\
&= (-1)^{dimM}\langle c_1(L)^{k-1}\cdot c(M)/c(L)^{k+1}, [M]\rangle,
\end{aligned}$$

where $\gamma_{-1} = \gamma_0 = 0$.

Proof This follows directly from Proposition 2 and the following formula

$$S_k(t) - 2S_{k-1}(t) + S_{k-2}(t) = t^{k-1}/(1+t)^{k+1},$$

where $S_i(t) = \big(i(t+1)^{i+1} - (i+1)t(t+1)^i + t^{i+1}\big)/(t+1)^{i+1}$, applied to $t = c_1(L)$. This ends the proof of the lemma and of the corollary.

□

REFERENCES

[D1] A. Dimca, "Milnor numbers and multiplicities of dual varieties", *Rev. Roumaine Math. Pures Appl.,* **31**, (1986), 535-538

[D2] A. Dimca, "Topics on Real and Complex Singularities" Vieweg Advanced Lectures, Braunschweig/Wiesbaden 1987

[F-L] W. Fulton and R. Lazarsfeld "Connectivity and its applications in algebraic geometry" Lect. Note in Math., **862** Springer-Verlag, (1980), 26-92

[H] F. Hirzebruch, "Topological methods in algebraic geometry", New York, Springer 1966

[Ho] A.Holme, "On the dual of smooth variety", *Algebraic geometry* (Proc., Copenhagen, 1978), Lecture Notes in Math., **732**, Springer-Verlag, 1979, 144-156

[Ka] N. Katz, "Penceaux de Leschetz. theoreme d'existence", SGA 7 II, Expose XVII, Springer Lecture Notes, **340**, (1973), 212-253

[K1] S.L. Kleiman, "Tangency and duality", *Can. Math. Soc. Conf. Proc.* , **6**, (1986), 163-225

[K2] S.L. Kleiman, "The enumerative theory of singularities", Real and Complex Singularities *Proc. Nordic Summer School/NAVF Symp. Math., Oslo, 1976* , Sijthoff and Noordhoff, Alphen aan den Rijn. (1977), 297-396

[I] S. Ishii, "A Characterization of Hyperplane Cuts of Smooth Complete Intersections", *Proc. Japan Acad.*, **58**, Ser.A (1982), 309-311

[L-T] Lê D.T. and B. Teissier, "Cycles evanescentes, sections planes et conditions de Whitney.II", *Proc. Symp. Pure Math.* , **40**, (1983), Part 2, 65-102

[M] J. Milnor, "Singular points of complex hypersurfaces", *Ann. of Math. Studies*, **61.** Princeton: Princeton Univ. Press 1968

[N] A. Némethi, "Lefschetz theory for complex affine varieties", *Rev. Roumaine Math. Pures Appl.*, **33**, (1988), 233-250

[P] A. Parusiński, "A generalization of the Milnor number", *Math. Ann.*, **281**, (1988), 247-254

[Pr] P. Pragacz, "Enumerative geometry of degeneracy loci", *Ann. Sci. Éc. Norm. Sup.* , **21**, (1988), 413-454

[Z] F.L. Zak, "Projections of algebraic varieties", *Math. USSR Sbornik*, **44**, (1983), 535-544