Scalar product of Hecke L-functions

and its application

B.Z. Moroz

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

West Germany

MPI/87-12

Table of contents

Introduction.

This is a slightly revised version of my notes prepared for the lectures given in Genova (Italy) during the second fortnight of February 1987.

Let us start by recalling a few classical notions to fix our notations. Let k be an algebraic number field of finite degree over $\mathbb{Q}$. The (absolute) Weil group $W(k)$ is defined, [23], [24], as the projective limit

$$W(k) = \varprojlim W(E|k)$$

of relative Weil groups $W(E|k)$, where E varies over all the finite Galois extensions of k. Each of the relative Weil groups $W(E|k)$ is *quasi-compact*:

$$W(E|k) \cong \mathbb{R}_+ \times W_1(E|k) ,$$

where $\mathbb{R}_+$ is the multiplicative group of positive real numbers and $W_1(E|k)$ is a compact group, being an extension of the finite Galois group $G(E|k)$ of E over k by a compact group C_E^1 of idèle-classes having unit volume. Any *continuous* finite dimensional representation

$$\rho : W(k) \longrightarrow GL(\ell, \mathbb{C}) \qquad (1)$$

factors through $W(E|k)$ for some E. We say that ρ is *normalised* if it factors through $W_1(E|k)$ for a finite Galois extension $E|k$. Let $S_0(F)$, $S_1(F)$, and $S_2(F)$ denote the sets of prime ideals, of real places and of complex places of a number field F, respectively; sometimes we write $S_\infty(F) := S_1(F) \cup S_2(F)$, and $S(F) := S_0(F) \cup S_\infty(F)$. For $\mathfrak{p} \in S(F)$, let $F_\mathfrak{p}$ be the completion of F at $\mathfrak{p}$ and let $F^a_\mathfrak{p}$ denote the maximal abelian extension of $F_\mathfrak{p}$. With a finite Galois extension $E|k$ and a pair of primes $\mathfrak{p} \in S_0(E)$, $p \in S_0(k)$ such that $\mathfrak{p}|p$ one associates the inertia subgroup

$$I_\mathfrak{p} = \{\tau \mid \tau \in G(E^a_\mathfrak{p}|k_p)\ ,\ \tau\alpha \equiv \alpha (\mathrm{mod}\ \mathfrak{p}) \quad \text{for} \quad \alpha \in \mathfrak{o}_\mathfrak{p}\}$$

and the Frobenius class

$$\sigma_\mathfrak{p} = \{\tau \mid \tau \in G(E^a_\mathfrak{p}|k_p)\ ,\ \tau\alpha \equiv \alpha^{|p|} (\mathrm{mod}\ \mathfrak{p}) \quad \text{for} \quad \alpha \in \mathfrak{o}_\mathfrak{p}\}$$

in $G(E^a_\mathfrak{p}|k_p)$, the Galois group of $E^a_\mathfrak{p}$ over k_p, here $\mathfrak{o}_\mathfrak{p}$ stands for the ring of integers in $E^a_\mathfrak{p}$ and we write, for brevity,

$$|\mathfrak{a}| := N_{F/\mathbb{Q}}\mathfrak{a} \ ,$$

where $\mathfrak{a}$ ranges over the fractional ideals of a number field F. The Galois group $G(E^a_\mathfrak{p}|k_p)$ is regarded as a subgroup of $W(E|k)$. Let V be the representation space of ρ and suppose that ρ factors through $W(E|k)$; one defines a vector space

$$V_p = \{x \mid x \in V\ ,\ \tau x = x \quad \text{for} \quad \tau \in I_\mathfrak{p}\}$$

and proves (cf., e.g., [14, p. 21]) that the set

$$\{p \mid p \in S_0(k),\ V_p \neq V\}$$

is finite. Let

$$L(s,\chi) = \prod_{p \in S_0(k)} \det(I - \rho(\sigma_{\mathfrak{p}})|p|^{-s})^{-1} , \tag{2}$$

where $\chi := \operatorname{tr} \rho$, $\rho(\sigma_{\mathfrak{p}}) = \rho(\tau)|_{V_p}$ for $\tau \in \sigma_{\mathfrak{p}}$ (the restriction $\rho(\tau)|_{V_p}$ of the operator $\rho(\tau)$ to V_p is easily seen to depend only on $\sigma_{\mathfrak{p}}$ but not on the choice of τ ; moreover, the Euler factor

$$\det(I - \rho(\sigma_{\mathfrak{p}})|p|^{-s})^{-1}$$

depends only on p but not on the choice of $\mathfrak{p}$ above p). If ρ is normalised (or if it is unitary) then the Euler product (2) converges absolutely in the half-plane $\operatorname{Re} s > 1$. Continuing the function

$$s \longmapsto L(s,\chi)$$

meromorphically to $\mathbb{C}$, one defines the Weil L-function, [24], associated to ρ . If the image of ρ is *finite*, it can be shown to factor through the Galois group $G(F|k)$ of a finite Galois extension $F|k$; on the other hand, since

$$W(k|k) \cong C_k ,$$

any one-dimensional representation of $W(k)$ may be identified with a Grössencharakter of k (here C_k denotes the idèle-class group of k). Thus the class of Weil L-functions contains both any Artin L-function and any Hecke L-function "mit Grössencharakteren". Let $X(k)$ be the set of all the continuous finite-dimensional normalised representations of the form (1) and let $gr(F)$ be the group of normalised grossencharacters of an algebraic number field F . We shall view an element of $gr(F)$ both as a character of C_F (trivial on $\mathbb{R}_+$ embedded diagonally in the connected component of C_F) and as a multiplicative function on the monoid $I_\circ(F)$ of integral ideals of F . By a theorem of R. Brauer, [1], the Weil L-function (2) may be decomposed in a product of abelian L-functions (cf. [24]):

$$L(s,\chi) \doteq \prod_{i=1}^{\mu} L(s,\varphi_i)^{e_i} , \quad e_i \in \{-1,1\} , \quad \varphi_i \in gr(E_i) , \qquad (2')$$

where $k \subseteq E_i \subseteq E$ (assuming that ρ factors through $W_1(E|k)$, say). It has been conjectured, [24], that the function

$$s \longmapsto (s-1)^{g(\chi)} L(s,\chi) ,$$

where $g(\chi)$ denotes the multiplicity of the identical representation in ρ , is holomorphic in $\mathbb{C}$ (*Artin-Weil conjecture*).

Lecture 1. *Estimates for character sums in number fields.*

Let $\lambda \in gr(F)$ and let $F(\lambda)$ denote the conductor of λ, then

$$\lambda((\alpha)) = \prod_{p \in S_\infty(F)} |\alpha_p|^{it_p(\lambda)} \left(\frac{\alpha_p}{|\alpha_p|}\right)^{a_p(\lambda)} \quad \text{for} \quad \alpha \equiv 1(F(\lambda)), \quad \alpha \in \mathfrak{o}_F ,$$

where $\mathfrak{o}_F$ denotes the ring of integers of a number field F and $\alpha_p = \sigma_p(\alpha)$, $\sigma_p : F \to F_p$ being the natural embedding of F into its completion F_p at the place p. Moreover,

$$t_p(\lambda) \in \mathbb{R},\ a_p(\lambda) \in \mathbb{Z},\ a_p(\lambda) \in \{0,1\} \quad \text{for} \quad p \in S_1(F) .$$

We write

$$a(\lambda) = \prod_{p \in S_1} (2+|t_p(\lambda)|)^{1/2} \prod_{p \in S_2} \left(2+\frac{|t_p(\lambda)|+|a_p(\lambda)|}{2}\right) ,$$

and

$$b(\lambda) = (|D_F|\ N_{F/\mathbb{Q}}F(\lambda))^{1/2} ,$$

where D_F is the discriminant of F. One develops the Weil L-function (2) in a Dirichlet series:

$$L(s,\chi) = \sum_{a \in I_o(k)} c(a,\chi)\,|a|^{-s}$$

and remarks that

$$c(a,\chi) = \chi(a) \quad \text{for} \quad \chi \in gr(k) \ .$$

Let

$$\rho_j : W(k) \longrightarrow GL(d_j, \mathbb{C}) , \ 1 \leq j \leq r ,$$

be a continuous normalised representation of the Weil group of k and let

$$L(s,\vec{\chi}) = \sum_{a \in I_\circ(k)} |a|^{-s} \prod_{j=1}^{r} c(a,\chi_j) , \qquad (3)$$

where $\vec{\chi} := (\chi_1,\ldots,\chi_r)$, $\chi_j := \operatorname{tr} \rho_j$. On the other hand , let k_j be a finite extension of k and let $d_j = [k_j:k]$, $1 \leq j \leq r$. For $\lambda_j \in gr(k_j)$, let

$$L(s,\lambda_j) = \sum_{a \in I_\circ(k)} |a|^{-s} c_a(\lambda_j) ,$$

so that

$$c_a(\lambda_j) = \sum_{\mathfrak{b} \in I_\circ(k_j) ; \ N_{k_j/k}\mathfrak{b} = a} \lambda_j(\mathfrak{b}) ,$$

and let

$$L(s,\vec{\lambda}) = \sum_{a \in I_\circ(k)} |a|^{-s} \prod_{j=1}^{r} c_a(\lambda_j) , \qquad (4)$$

where $\vec{\lambda} := (\lambda_1,\ldots,\lambda_r)$. We call the function (3) the scalar product of Weil L-functions $L(s,\chi_j)$, $1 \leq j \leq r$, while the function (4) is called the scalar product *over* k of the Hecke functions $L(s,\lambda_j)$, $1 \le j \le r$. Regarding λ_j as an one-dimensional representation of $W(k_j)$ we deduce from the basic properties of the Weil L-functions that

$$c_a(\lambda_j) = c(a,\chi_j) \quad \text{when} \quad \rho_j = \mathrm{Ind}_{W(k_j)}^{W(k)} \lambda_j$$

and, in particular,

$$L(s,\vec{\lambda}) = L(s,\vec{\chi}) \quad \text{when} \quad \rho_j = \mathrm{Ind}_{W(k_j)}^{W(k)} \lambda_j \ , \ 1 \le j \le r \ . \qquad (5)$$

<u>Proposition 1</u>. *Let* $\rho = \rho_1 \otimes \ldots \otimes \rho_r$, $\chi = \mathrm{tr}\,\rho$. *Then*

$$L(s,\vec{\chi}) = L(s,\chi) \prod_{p \in S_0(k)} \Phi_p(|p|^{-s}) \prod_{p \in S_0(\vec{\chi})} l_p(|p|^{-s}) \ , \qquad (6)$$

where $\Phi_p(t) \in \mathbb{C}[t]$, $\Phi_p(t) = 1 (\mathrm{mod}\ t^2)$, $S_0(\vec{\chi})$ *is a finite set*, $l_p(t) \in \mathbb{C}[t]$. *Moreover, the degrees of* $\Phi_p(t)$ *and* $l_p(t)$ *are bounded by* $d-1$, $d := \prod_{j=1}^{d} d_j$.

Equation (6) defines a meromorphic continuation of the function $s \longmapsto L(s,\vec{\chi})$ to the half-plane $\mathrm{Re}\ s > \frac{1}{2}$ since $L(s,\chi)$ is meromorphic in $\mathbb{C}$, $S_0(\vec{\chi})$ is finite and the product

$$L(s,\Phi) = \prod_{p \in S_0(k)} \Phi_p(|p|^{-s})^{-1}$$

converges absolutely for $\mathrm{Re}\, s > \frac{1}{2}$. Thus, by (5), we obtain a meromorphic continuation of the scalar product $L(s,\vec{\lambda})$ of Hecke L-functions to the half-plane $\mathrm{Re}\ s > \frac{1}{2}$. Moreover, by a theorem of G. Mackey, [12], a tensor product of monomial representations, is equivalent to a direct sum of monomial representations, so that $L(s,\chi)$ satisfies the Artin-Weil conjecture when each of ρ_j, $1 \leq j \leq r$, is induced by a λ_j. To be more precise, we have the following result.

Corollary 1. *There are number fields* E_i, $1 \leq i \leq \nu$, *and grossen-characters* ψ_i *such that*

$$L(s,\vec{\lambda}) = \prod_{i=1}^{\nu} L(s,\psi_i)\; L(s,\Phi)^{-1}\; L_\circ(\vec{\lambda},s) \quad \textit{for} \quad \operatorname{Re} s > \frac{1}{2}, \qquad (7)$$

where $L_\circ(\vec{\lambda},s) = \prod_{p \in S_\circ(\vec{\chi})} l_p(|p|^{-s})$ *in notations of* (6), $\psi_i \in gr(E_i)$, $k \subseteq E_i \subseteq K$, K *being the smallest Galois extension of* k *containing each of the fields* k_j, $1 \leq j \leq r$.

In particular, one remarks that

$$L(s,\vec{\lambda}) = \frac{\omega(\vec{\lambda},s)}{(s-1)^w} + f(s),\ \omega(\vec{\lambda},s) \in \mathbb{C}[s],$$

where $f(s)$ is holomorphic for $s = 1$ and

$$w = \operatorname{card}\{i \mid \psi_i = 1\}.$$

The methods of classical analytic number theory, [10], lead to an asymptotic estimate for the sum

$$A(x,\vec{\lambda}) := \sum_{|a|<x} \prod_{j=1}^{r} c_a(\lambda_j),$$

and, more generally, for the sum

$$A(x,\vec{\chi}) := \sum_{|a|<x} \prod_{j=1}^{r} c(a,\chi_j) .$$

We write

$$A(x,\vec{\lambda}) = x\, P_{\vec{\lambda}}(\log x) + R(\vec{\lambda},x) \tag{8}$$

and

$$A(x,\vec{\chi}) = x\, P_{\vec{\chi}}(\log x) + R(\vec{\chi},x) , \tag{9}$$

where $P_{\vec{\lambda}}(t)$ (respectively, $P_{\vec{\chi}}(t)$) is a polynomial in $\mathbb{C}[t]$ of degree $w-1$ (respectively, $g(\chi)-1$) whose coefficients are effectively computable in terms of the behaviour of $L(s,\vec{\lambda})$ (respectively, $L(s,\vec{\chi})$) in the neighbourhood of the point $s = 1$. If $w = 0$ (respectively, $g(\chi) = 0$) we let $P_{\vec{\lambda}}(t) = 0$ (respectively, $P_{\vec{\chi}}(t) = 0$).

<u>Theorem 1</u>. *Estimate* (9) *holds with*

$$R(\vec{\chi},x) = O(x \exp(-c_1(\vec{\chi})\sqrt{\log x})) , \quad c_1(\vec{\chi}) > 0 ,$$

as $x \to \infty$ *with effectively computable (in terms of* $\vec{\chi}$ *) constants. Moreover, if* $L(s,\chi)$ *in* (6) *satisfies the Artin-Weil conjecture, then*

$$R(\vec{\chi},x) = O(C_2(\varepsilon;\delta_1)B(\chi)x^{1-\frac{2}{N+2}+\varepsilon} (\log x)^{n\,d}), \quad \varepsilon > 0 ,$$

where, in notations of (2'),

$$N = \sum_{j=1}^{\mu} [E_j:\mathbb{Q}] \frac{e_j+1}{2}, \quad \delta_1 = d \operatorname{card}(S_\circ(\vec{\chi})), \quad n = [k:\mathbb{Q}] ,$$

$$B(\chi) = \prod_{i=1}^{\mu} a(\varphi_i)^{e_i+1} b(\varphi_i)^2 B_\circ$$

with $B_\circ$ *depending on* $g(\chi)$ *and* $\sum_{j=1}^{\mu}(e_j-1)$ *only. Finally, if each of the functions* $L(s,\varphi_i)$ *in* (2') *satisfies Riemann hypothesis we have*

$$R(\vec{\chi},x) = O_\varepsilon(x^{1/2+\varepsilon} \prod_{i=1}^{\mu} a(\varphi_i)^\varepsilon b(\varphi_i)^\varepsilon), \quad \varepsilon > 0 .$$

In view of Corollary 1, one obtains as a consequence of Theorem 1 the following result.

<u>Corollary 2</u>. *Estimate* (8) *holds with*

$$R(\vec{\lambda},x) = O(C_2(\varepsilon;\delta_1)(\prod_{i=1}^{\nu} a(\psi_i)^2 b(\psi_i)^2)6^W x^{1-\frac{2}{N+2}+\varepsilon}(\log x)^{nd}) ,$$

where $N = \sum_{j=1}^{\nu}[E_j:\mathbb{Q}]$, $\varepsilon > 0$.

We remark that an estimate of the form

$$R(\chi,x) = O(x^{1/2-\gamma}) \quad \text{with} \quad \gamma > 0$$

would imply the Artin-Weil conjecture for $L(s,\chi)$ and that, on the other hand, the well-known Ω-theorem, [2], for grossencharacters gives

$$R(\chi,x) = O(x^\gamma) \Rightarrow \gamma \geq \frac{1}{2} - \frac{1}{2n}$$

when $\chi \in gr(k)$.

Corollary 2 will be used in the last lecture to obtain an equidistribution theorem for integral ideals having equal norms.

Theorem 2. *Suppose* ρ *is normalised and* $L(s,\chi)$ *satisfies* (2'). Then

$$\sum_{|p|<x} \chi(p) = g(\chi) \int_2^x \frac{du}{\log u} + O\Big(\sum_{j=1}^{\mu} \big(x^{\alpha_j} + \sqrt{x}\,[E_j:\mathbb{Q}] +$$

$$+ x \exp\big(-c_3 \frac{\log x}{\log(a(\varphi_j)b(\varphi_j)) + \sqrt{[E_j:\mathbb{Q}]\log x}}\big)\big)\Big) \quad \textit{with} \quad c_3 > 0 ,$$

where α_j *denotes the possible (real) Siegel's zero of* $L(s,\varphi_j)$ *(when* $\varphi_j^2 = 1$ *) and where* $\chi(p) := \operatorname{tr} \rho(\sigma_p)$ *for* $p \in S_0(k)$. *If each of the* $L(s,\varphi_j)$, $1 \le j \le \mu$, *satisfies Riemann hypothesis, then*

$$\sum_{|p|<x} \chi(p) = g(\chi) \int_2^x \frac{du}{\log u} + O\Big(\sqrt{x} \sum_{j=1}^{\mu} [E_j:\mathbb{Q}] \log[x \sum_{j=1}^{\mu} a(\varphi_j) b(\varphi_j)]\Big) .$$

This theorem may be proved along the classical lines (cf., for instance, [15]); it will be a starting point of our considerations in the next lecture.

Lecture 2. *A new prime number theorem.*

Let $|S_j(k)| = r_j$, $j = 1,2$, so that $n = r_1 + 2r_2$, and let E_k be the group of units in k . By the Dirichlet's unit theorem, we can write

$$E_k \cong \mathbb{Z}^{r_1+r_2-1} \times W ,$$

where W is a finite cyclic group. Let

$$X = \prod_{p \in S_\infty} k_p \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$$

and let X^* be the group of invertible elements in the $\mathbb{R}$-algebra X ; let i denote the diagonal embedding of k^* in X^* . Obviously,

$$X^*/_{i(E_k)} \cong \mathbb{R}_+ \times T ,$$

where T is an (n-1)-dimensional real torus. Finally, consider a homomorphism

$$f : I(k) \longrightarrow T$$

subject to the condition

$$f((\alpha)) = \pi(i(\alpha)) \quad \text{for} \quad \alpha \in k^* ,$$

where π denotes the natural projection of X^* on T and $I(k)$ stands for the group of fractional ideals of k .

Theorem 3. *Let* $\mathcal{U}$ *be a "smooth" subset* (* of T *and let* A *be a conjugacy class in the Galois group of a finite Galois extension* E/k *. The following formula holds:*

$$\operatorname{card}\{p \mid p \in S_0(k),\ |p| < x,\ f(p) \in \mathcal{U},\ \sigma_p = A\} =$$

$$\frac{\mu(\mathcal{U})\,|A|}{[E:k]} \int_2^x \frac{du}{\log u} + O\,(x \exp(-C(\mathcal{U})\sqrt{\log x})),\ C(\mathcal{U}) > 0\ ,$$

where μ *is the Haar measure on* T *normalised by the condition* $\mu(T) = 1$.

As a special case of this theorem, one obtains Chebotarev's density theorem (just take $\mathcal{U} = T$); on the other hand, if E is chosen to be Abelian over k theorem 3 reduces to an equidistribution theorem in the spirit of Hecke's multidimensional arithmetic, [4], [5], [13]. Theorem 3 is an easy consequence of the estimates of theorem 2 (see, for instance, [14, p. 68]).

To state the main theorem of this paragraph consider a *finite* subset $\mathfrak{N}$ of $X(k)$ and let

$$\mathfrak{m} = \{\chi \mid \chi = \operatorname{tr} \boldsymbol{\rho},\ \boldsymbol{\rho} \in \mathfrak{N}\}$$

be the corresponding set of characters.

Theorem 4. *Let* $g_0 \in W(k)$ *and let* $0 < \varepsilon < 1$ *. The following asymptotic formula holds:*

(* This notion will be analysed in the fourth lecture. For the time being one may picture $\mathcal{U}$ as a rectangular subset of T .

$$\operatorname{card}\,\{p \mid p \in S_{\circ}(k),\ |\chi(p)-\chi(g_{\circ})| < \varepsilon \ \text{ for each } \chi \text{ in } \mathfrak{M}\} =$$

$$\alpha(\mathfrak{M},g_{\circ},\varepsilon)\int_2^x \frac{du}{\log u} + O\,(x\exp(-c_4\sqrt{\log x})),\ c_4 > 0\ . \tag{10}$$

Moreover, $\alpha(\mathfrak{M},g_{\circ},\varepsilon) > c_5\varepsilon^{c_6}$, $c_5 > 0$, $c_6 > 0$. *Here the constants* c_j, $j = 4,5,6$, *depend at most on* $\mathfrak{M}$ *(but not on* ε, $g_{\circ}$, x *).*

Theorem 4 can also be viewed as a generalisation of both Cheboratev's density theorem and Hecke's type equidistribution theorems: one obtains the former one when each ρ in $\mathfrak{N}$ has a finite image, while the latter ones follow if each ρ in $\mathfrak{N}$ is one-dimensional. To sketch the proof of theorem 4 let us note first that there is a finite Galois extension $E|k$ such that each ρ in $\mathfrak{N}$ factors through $W_1(E|k)$ since $\mathfrak{N}$ is finite. Consider a subset

$$A(\mathfrak{M},g_{\circ},\varepsilon) = \{g \mid g \in W_1(E|k),\ |\chi(g)-\chi(g_{\circ})| < \varepsilon \ \text{ for } \chi \in \mathfrak{M}\}$$

of $W_1(E|k)$ and let μ be the Haar measure on $W_1(E|k)$ normalised by the condition $\mu(W_1(E|k)) = 1$. One can actually take

$$\alpha(\mathfrak{M},g_{\circ},\varepsilon) = \mu(A(\mathfrak{M},g_{\circ},\varepsilon)) \tag{11}$$

and prove that

$$\mu(A(\mathfrak{M},g_{\circ},\varepsilon)) > c_5\varepsilon^{m}\ ,\ m := [E:\mathbb{Q}] - 1\ .$$

To deduce the asymptotic formula (10) with α defined by (11) one notes that, in fact, each ρ in $\mathfrak{N}$ factors through a certain group G which fits in the exact sequence

$$1 \to T \to G \to H \to 1 ,$$

where T is a finite-dimensional real torus and H is a compact group. Moreover, conditions $|\chi(g)-\chi(g_0)| < \varepsilon$, $\chi \in \mathfrak{M}$, define a semialgebraic subset $\mathcal{U}$ of G. The asymptotic formula to be proved would follow now from theorem 2 and the general equidistribution principles if one could estimate from above the volume of the δ-neighbourhood of the boundary of $\mathcal{U}$ uniformly in the interval, say, $0 < \delta < 1$. Such an estimate can indeed be proved as a consequence of recent results on volumes of tubes around semialgebraic sets, [25] (cf. also [3]). This concludes our sketch of the proof of theorem 4 (cf., however, [16] and [17, § 5] where this proof has been carried out in detail). We close this lecture by suggesting an open problem: can one prove a general theorem on equidistribution of Frobenius classes in a Weil group (cf. [14, p. 69-71]) that, in particular, would imply both theorem 3 and theorem 4 ?

Lecture 3. *Analytic continuation and the natural boundary of scalar products.*

The object of this lecture is the following theorem.

Theorem 5.

(i) *The function* $L(s,\vec{\chi})$ *defined by* (3) *can be meromorphically continued to the half-plane* $\mathbb{C}_+ = \{s \mid \mathrm{Re}\ s > 0\}$.

(ii) *Suppose that* $r \geq 2$ *and* $d_1 \geq \ldots \geq d_r \geq 2$. *The line* $\mathbb{C}^\circ = \{s \mid \mathrm{Re}\ s = 0\}$ *is the natural boundary of* $L(s,\vec{\chi})$ *unless* $r = d_1 = d_2 = 2$.

(iii) *If* $r = d_1 = d_2 = 2$, *then the function* $L(s,\vec{\chi})$ *is equal to a ratio of two Weil L-functions (up to a finite number of Euler factors) and therefore it is meromorphic in* $\mathbb{C}$.

Identity (5) shows that assertions (i)-(iii) hold true when one replaces $L(s,\vec{\chi})$ by $L(s,\vec{\lambda})$. Let us describe now the main steps in the proof of theorem 5. In view of (6), the function $L(s,\vec{\chi})L(s,\Phi)^{-1}$ is meromorphic in $\mathbb{C}$; moreover, the polynomials $\Phi_p(t)$ can be explicitely evaluated when $r = d_1 = d_2 = 2$ and this evaluation proves (iii). In general, the sequence of polynomials $\{\Phi_p(t) \mid p \in S_\circ(k)\}$ can be parametrized as follows. Let $\mathcal{V}$ be the ring of virtual characters of $W(k)$ and let

$$H(t) = \sum_{j=0}^{l} t^j a_j \ , \ a_j \in \mathcal{V} \ ,$$

be a polynomial with coefficients in this ring. We extend the definition $\chi(p) = \operatorname{tr} \rho(\sigma_p)$ by linearity to $\mathcal{Y}$ and write

$$H_p(t) = \sum_{j=0}^{l} t^j a_j(p) \quad \text{for} \quad p \in S_o(k) ,$$

and

$$H_g(t) = \sum_{j=0}^{l} t^j a_j(g) \quad \text{for} \quad g \in W(k) .$$

Lemma 1. *There are a polynomial* $H(t)$ *in* $\mathcal{Y}[t]$ *and a finite subset* $S_o^!(\vec{\chi})$ *of* $S_o(k)$ *such that*

$$H_p(t) = \Phi_p(t) \quad \text{for} \quad p \in S_o(k) \setminus S_o^!(\vec{\chi}) . \tag{12}$$

This lemma shows that it suffices to investigate the analytic properties of the Euler product

$$L(s,H) = \prod_{p \in S_o(k)} H_p(|p|^{-s})^{-1} \tag{13}$$

for $H(t) \in \mathcal{Y}[t]$.

Definition. *Let* $H(t) \in \mathcal{Y}[t]$ *and suppose that* $H(0) = 1$. *We say that* H *is unitary if*

$$H_g(\alpha) = 0 \Rightarrow |\alpha| = 1$$

for each g *in* $W(k)$.

Lemma 2. *If* $2 \leq r \leq d_r \leq \ldots \leq d_1$ *and* $d_1 d_2 r > 8$ *, then the polynomial* H *defined by* (12) *and* (6) *is not unitary.*
In view of Lemma 2, the statements (i) and (ii) follow from the following proposition.

Proposition 2. *Let* $H(t) \in \mathcal{V}[t]$ *and* $H(0) = 1$ *. The function*

$$s \longmapsto L(s,H)$$

defined by (13) *in the half-plane* $\mathrm{Re}\ s > 1$ *can be meromorphically continued to* $\mathbb{C}_+$ *; if* H *is not unitary, then* $\mathbb{C}^\circ$ *is the natural boundary of this function.*

We sketch the proof of Proposition 2. Without loss of generality, we may assume that

$$H(t) = 1 + \sum_{j=1}^{\ell} a_j t^j \ , \ a_j \in \mathcal{V}_0 \ ,$$

where $\mathcal{V}_0$ denotes the ring of virtual characters of $W_1(E|k)$ for a finite Galois extension $E|k$. Let

$$H_g(t) = \prod_{i=1}^{\ell} (1-\alpha_i(g)t), \ g \in W_1(E|k) \ ,$$

and let

$$\gamma = \sup \{|\alpha_i(g)| \ \Big| \ 1 \leq i \leq \ell, \ g \in G\} \ .$$

Lemma 3. (i) *Formally, in* $Y_\circ[[t]]$, *we have*

$$H(t) = \prod_{n=1}^{\infty} \prod_{\varphi \in X_n(H)} \det(1-t^n\varphi)^{b_n(\varphi)} , \quad b_n(\varphi) \in \mathbf{Z} ,$$

where $X_n(H)$ *is a finite subset of* X . *Moreover,*

$$\Big| \sum_{\varphi \in X_n(H)} b_n(\varphi) \operatorname{tr} \varphi(g) \Big| \leq \ell\gamma^n \frac{1}{n} \sum_{d|n} 1 \quad \text{for} \quad g \in W_1(E|k) .$$

Making use of lemma 3, one can prove the following statement.

Lemma 4. *There is* $M_\circ$ *such that if* $M > M_\circ$ *and* $N > (\gamma+1)^M$ *then*

$$L(s,H) = Z_N(s)U_M(s)g(s) , \tag{14}$$

where

$$U_M(s) = \prod_{1 \leq n < M} \prod_{\varphi \in X_n(H)} L(ns, \operatorname{tr}\varphi)^{-b_n(\varphi)} ,$$

$$Z_N(s) = \prod_{|p|<N} H_p(|p|^{-s})^{-1} ,$$

and $g(s)$ *is holomorphic and has no zeros in*

$$\mathbb{C}_{1/M} = \{s \mid \operatorname{Re} s > \tfrac{1}{M}\} .$$

Since both $Z_N(s)$ and $U_M(s)$ are meromorphic in $\mathbb{C}$ and since

$$\mathbb{C}_+ = \bigcup_{M>M_\circ} \mathbb{C}_{1/M} ,$$

equations (14) provide a meromorphic continuation of $L(s,H)$ to $\mathbb{C}_+$. Moreover, as an easy consequence of Lemma 3 one proves that $L(s,H)$ is a meromorphic function in $\mathbb{C}$ when H is unitary (that is, if $\gamma = 1$). If $\gamma > 1$ one can prove that the closure of the set

$$\{s \mid \operatorname{Re} s > 0,\ Z_N(s)^{-1} = 0 \text{ for some } N\}$$

contains the line $\mathbb{C}^\circ$. To show that this line is the natural boundary of $L(s,H)$ it remains to prove that the poles of this function coming from the first factor in (14) cannot be cancelled by the zeros of the second factor. Making use of theorem 4 one can estimate the number of poles of $Z_N(s)$ in a neighbourhood of a fixed point in $\mathbb{C}^\circ$. On the other hand, a careful analysis of the structure of the zero-set of $U_M(s)$ provides an upper bound for the number of such zeros which shows that complete cancellation cannot occur. We have to refer to [17] for the details of this argument; an alternative proof of theorem 5 may be found in [9].

Theorem 5 is of considerable interest for the general theory of L-functions having Euler product. For the history of its proof and for some related results we refer the reader to two short notes, [21], [22], and our final report on this problem, [17]. This exposition (as well as the article [17]) owes much to the early work of N. Kurokawa, [6]-[8], where Theorem 5 has been proved for representations ρ_j, $1 \leq j \leq r$, of Galois type.

Lecture 4. *Integral points on algebraic sets defined by a system of norm-forms.*

The following problem has stimulated much research in number theory and arithmetic algebraic geometry. Let $f_j(x_j) \in \mathbb{Z}[x_j]$, $1 \leq j \leq r$, and consider the algebraic set U given by the system of equations $f_j(x_j) = 0,\ 1 \leq j \leq r$. One chooses a compact subset of $U(\mathbb{R})$ and asks for an estimate of the number of integer points in this subset (here $x_j = (x_{j1},\dots,x_{jn_j})$ is an array of n_j variables). When the number of variables $n = \sum_{j=1}^{r} n_j$ is relatively small compared to the degrees and the number of equations the analytic methods must be supplemented by arithmetical considerations. We study here the simplest problem of this type: dealing with norm-form equations allows one to avoid algebro-geometric considerations and to work in the framework of classical algebraic number theory. To give a precise statement of our results we need a notion of "smoothness" generalising the notion of a plane domain with a boundary satisfying Lipschitz condition. Consider a triple (W,E,μ) consisting of a set W , a Borel measure μ and a system E of measurable subsets of W ; a measurable subset V of W is said to be n-*smooth* if for each Δ in the interval $0 < \Delta < 1$ one can find a finite subset $E_\circ(\Delta)$ of E satisfying the following conditions:

(i) $\operatorname{card} E_\circ(\Delta) \leq \Delta^{-n}$;

(ii) $\rho \cap \rho' = \phi$ when $\rho \neq \rho'$, $\rho \in E_\circ(\Delta)$, $\rho' \in E_\circ(\Delta)$;

(iii) let $V_+ = \bigcup_{\mathfrak{p} \in E_\circ(\Delta)} \mathfrak{p}$, $V_- = \bigcup_{\substack{\mathfrak{p} \in E_\circ(\Delta) \\ \mathfrak{p} \subseteq V}} \mathfrak{p}$, then

$V_+ \supseteq V$ and $\mu(V_+ \setminus V_-) < C(V)\Delta$ (for some constant $C(V)$ independent of Δ and called the *smoothness* constant of V). The elements of E are refered to as *elementary sets*. Consider, secondly, a triple (S, π, N) consisting of a set S , a map $\pi : S \to W$ and a map $N : S \to \mathbb{R}_+$; this triple is said to be (E,μ)-*equidistributed* if

$$\operatorname{card}\{s \mid s \in S,\ \pi(s) \in \mathfrak{p},\ Ns < x\} = \mu(\mathfrak{p})a(x) + \mathcal{O}(b(x))$$

for $\mathfrak{p} \in E$, where $\frac{b(x)}{a(x)} \to 0$ as $x \to \infty$.

<u>Lemma 5</u>. *If* (S, π, N) *is* (E,μ)-*equidistributed, then*

$$\operatorname{card}\{s \mid s \in S,\ \pi(s) \in V,\ Ns < x\} = \mu(V)a(x) + O(b_1(x))$$

with $b_1(x)/a(x) \to 0$ *(and* b_1 *exactly expressible in terms of* b, n, $C(V)$) *for any* n-*smooth set* V .

Let $k_j|\mathbb{Q}$, $1 \leq j \leq r$, be a finite Galois extension of degree d_j and let $K = k_1 \ldots k_r$ be the composite field of k_j , $[K:\mathbb{Q}] =: d$. To simplify our exposition we impose the following condition on these fields (cf. [11]):

$$(e_i(p),\ e_j(p)) = 1 \quad \text{for} \quad i \neq j,\ p \in S_\circ(\mathbb{Q}) \ , \tag{15}$$

where $e_i(p)$ denotes the ramification index of p in k_i .

Condition (15) implies, in particular, that $d = \prod_{j=1}^{r} d_j$, so that the fields $k_1,\dots,k_r$ are linearly disjoint over $\mathbb{Q}$. Let $T = T_1 \times \dots \times T_r$ and let $H = H_1 \times \dots \times H_r$, where T_j denotes the (d_j-1)-dimensional torus assigned to k_j as in lecture 2 and where H_j is the ideal class group of k_j ; let $f : I(k_1)\times \dots \times I(k_r) \to T$ be the product of homomorphisms $f_j : I(k_j) \to T_j$ defined in lecture 2 and consider two sets of r-tuples of ideals

$$I_\circ = \{\vec{a} \mid \vec{a} = (a_1,\dots,a_r),\ a_j \in I_\circ(k_j),\ Na_1 = \dots = Na_r\}$$

and

$$S_\circ = \{\vec{p} \mid \vec{p} = (p_1,\dots,p_r),\ p_j \in S_\circ(k_j),\ Np_1 = \dots = Np_r\} .$$

<u>Theorem 6</u>. *Take the rectangular subsets of* T *as elementary and let* μ *be the Haar measure on* T *normalised by the condition* $\mu(T) = 1$. *Let* $A \in H$ *and suppose that* τ *is a* $\sum_{j=1}^{r}(d_j-1)$*-smooth subset of* T . *Then*

$$\mathrm{card}\ \{\vec{p} \mid \vec{p} \in S_\circ \cap \vec{A},\ f(\vec{p}) \in \tau\} = \frac{\mu(\tau)}{|H|}\int_2^x \frac{du}{\log u} + O\ (x\ \exp(-c\sqrt{\log x}))$$

with $c > 0$, *and*

$$\mathrm{card}\ \{\vec{a} \mid \vec{a} \in I_\circ \cap \vec{A},\ f(\vec{a}) \in \tau\} = \frac{\mu(\tau)w_\circ}{|H|}\ x + O\ (x^{1-\gamma})$$

with $\gamma > 0$, *where* $w_\circ = w(K)L(1,\Phi)^{-1}$, $w(K)$ *denotes the residue* *of the ζ-function* $Z_K(s)$ *of* K *at* $s = 1$ *and* $L(s,\Phi)$ *is defined by the equation* $\sum_{n=1}^{\infty} n^{-s} \prod_{j=1}^{r} a_n^{(j)} = Z_K(s)L(s,\Phi)^{-1}$,

$a_n^{(j)} := \mathrm{card}\,\{\mathfrak{a} \mid \mathfrak{a} \in I_\circ(k_j),\ N\mathfrak{a} = n\}$.

Theorem 6 shows that both integral and prime divisors with equal norms are equidistributed in the sense of E. Hecke; it can be deduced form the estimates (8) and the prime number theorem for grossencharacters with the help of lemma 5 and equation (6). Choose an integral ideal $\mathfrak{a}_{\circ j}$ in A_j and let $\{w_{ji} \mid 1 \le i \le d_j\}$ be a $\mathbb{Z}$-basis of $\mathfrak{a}_{\circ j}$; one defines a norm-form f_j associated to A_j as follows:

$$f_j(x_j) = N_{k_j(x_j)/\mathbb{Q}(x_j)}\Big(\sum_{1 \le i \le d_j} x_{ji} w_{ji}\Big) N\mathfrak{a}_{\circ j}^{-1} ;$$

obviously, $f_j(x_j) \in \mathbb{Z}[x_j]$. Consider the algebraic set W defined by a system of equations:

$$f_1(x_1) = \dots = f_r(x_r)$$

and let $W_\circ = W_{\circ 1} \times \dots \times W_{\circ r}$ with $W_{\circ j}$, $1 \le j \le r$, be defined by the equation $f_j(x_j) = 1$. In what follows we assume, for simplicity, that k_j is *totally complex*, then f_j is positive definite and we can define a projection

$$\pi : W(\mathbb{R}) \to W_\circ(\mathbb{R}),\ \pi : a_j \longmapsto a_j f_j(a_j)^{-1/d_j},\ a_j \in \mathbb{R}^{d_j}$$

(this map is not defined on the subset $f_j(x_j) = 0$, $1 \le j \le r$, of smaller dimension and containing no integer points except the origin). Moreover, one can define a natural projection map $h : W_o(\mathbb{R}) \to T$ of $W_o(\mathbb{R})$ on T and take as elementary the subsets of $W(\mathbb{R})$ of the shape:

$$\{\vec{a} \mid \vec{a} = (a_1, \ldots, a_r),\ t_1 < f_j(a_j) \leq t_2,\ \pi(\vec{a}) \in U\} ,$$

where $0 \leq t_1 < t_2$ and where $h(U)$ is a smooth subset of T.

<u>Theorem 7</u>. *There is a Borel measure* μ *on* $W(\mathbb{R})$ *such that*

$$\operatorname{card}\ (V \cap W(\mathbb{Z})) = \mu(V)\ \frac{w(K)}{|H|L(1,\Phi)} + O\ (C(V)t(V)^{1-\gamma})$$

with $\gamma > 0$, *where* $t(V) = \max\ \{f_j(a_j) \mid \vec{a} \in V\}$, *for any smooth subset* V *of* $W(\mathbb{R})$ *(i.e. smooth with respect to the system of elementary sets we have just described).*

<u>Corollary 3</u>. *The following asymptotic formula holds true:*

$$\operatorname{card}\ \{\vec{a} \mid \vec{a} \in W(\mathbb{Z}),\ \| a_j \| < x^{1/d_j},\ 1 \leq j \leq r\} = \frac{m(x)w(K)}{|H|L(1,\Phi)} +$$

$$O(x^{1-\gamma}) \quad \textit{with} \ \ \gamma > 0 \ , \ \textit{where} \ \ \| a_j \| := \max_{1 \leq i \leq d_j} |a_{ij}| \quad \textit{and}$$

where $m(x) := \mu(\{\vec{y} \mid \vec{y} \in W(\mathbb{R}),\ \| y_j \| < x^{1/d_j},\ 1 \leq j \leq r\})$.
Moreover, there two constants C_1, C_2 *such that* $C_1 x \leq m(x) \leq C_2 x$.
Theorem 7 can be deduced (with the help of lemma 5) from theorem 6. The results described in this lecture appear in [18]-[20] (cf. also [11] and [14, Ch. III]).

Concluding remarks and acknowlegdements.

These lectures are meant as a summary of our work carried out during the latest years. We refer for the details omitted here to the articles [15]-[20] and to our recent monograph, [14]. It is my pleasant duty to thank Professor A. Perelli for inviting me to visit Genova; we are grateful to our colleagues at the Mathematical Institute (University of Genova) for their kind hospitality. The author acknowledges financial assistance of the University of Genova which made this visit possible.

Literature cited.

[1] R. Brauer, On Artin's L-series with general group characters, Annals of Mathematics, 48 (1947), p. 502-514.

[2] K. Chandrasekharan, R. Narasimhan, Functional equations with multiple gamma factors and the average order of arithmetical functions, Annals of Mathematics, 76 (1962), p. 93-136.

[3] M. Gromov, Entropy, homology and semialgebraic geometry (after Y. Yomdin), Séminaire Bourbaki, n° 663, Astérisque, 145 (198), p.

[4] E. Hecke, Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen (zweite Mitteilung), Mathematische Zeitschrift, 6 (1920), p. 11-51.

[5] I.P. Kubilus, On some problems in geometry of prime numbers (in Russian), Mathematical Sbornik of the USSR, 31 (1952) No 3, p. 507-542.

[6] N. Kurokawa, On the meromorphy of Euler products, Part I: Artin type, Tokyo Institute of Technology, Preprint, 1977.

[7] N. Kurokawa, On the meromorphy of Euler products, Proceedings of Japan Academy, 54A (1978), p. 163-166.

[8] N. Kurokawa, On Linnik's problem, ibid., p. 167-169.

[9] N. Kurokawa, On the meromorphy of Euler products, I, II, Proceedings of the London Mathematical Society, 53 (1986), p. 1-47 and p. 209-236.

[10] E. Landau, Über Ideale und Primideale in Idealklassen, Mathematische Zeitschrift, 2 (1918), p. 52-154.

[11] W.-Ch.W. Li, B.Z. Moroz, On ideal classes of number fields containing integral ideals of equal norms, Journal of Number Theory, 21 (1985), p. 185-203.

[12] G.W. Mackey, Induced representations of locally compact groups, I, Annals of Mathematics, 55 (1952), p. 101-139.

[13] T. Mitsui, Generalised prime number theorem, Japanese Journal of Mathematics, 26 (1956), p. 1-42.

[14] B.Z. Moroz, Analytic arithmetic in algebraic number fields, Lecture Notes in Mathematics 1205, Springer-Verlag, 1986.

[15] B.Z. Moroz, Estimates for character sums in number fields, Israel Journal of Mathematics, to appear.

[16] B.Z. Moroz, Equidistribution of Frobenius classes and the volumes of tubes, Acta Arithmetica, 51 (1988), to appear.

[17] B.Z. Moroz, On a class of Dirichlet series associated to the ring of representations of a Weil group, M.P.I. für Math. (Bonn), Preprint 86-33 (submitted for publication).

[18] B.Z. Moroz, On the distribution of integral and prime divisors with equal norms, Annales de l'Institut Fourier (Grenoble), 34 (1984), fasc. 4, p. 1-17.

[19] B.Z. Moroz, Integral points on norm-form varieties, Journal of Number Theory, 24 (1986), p. 272-283.

[20] B.Z. Moroz, On the number of integral points on a norm-form variety in a cube-like domain, Journal of Number Theory, to appear.

[21] B.Z. Moroz, Euler products (variation on a theme of Kurokawa's), Astérisque, 94 (1982), p. 143-151.

[22] B.Z. Moroz, Produits eulériens sur les corps de nombres, Comptes Rendus de l'Académie des Sciences (Paris), 301 (1985), Série I, no. 10, p. 459-462.

[23] J. Tate, Number theoretic background, Proceedings of Symposia in Pure Mathematics (American Mathematical Society), 33 (1979), Part II, p. 3-26.

[24] A. Weil, Sur la théorie du corps de classes, Journal of the Mathematical Society of Japan, 3 (1951), p. 1-35.

[25] Y. Yomdin, Metric properties of semialgebraic sets and mappings and their applications in smooth analysis, Ben-Gurion University of Negev, Preprint, 1985.