A Generalization of the Goldfeld-Sarnak Estimate on Selberg's Kloosterman Zeta-function

Wladimir de Azevedo Pribitkin

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1 Introduction

Let Γ denote a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ which has a fundamental region with finite hyperbolic area but which is not compact. For convenience we assume that ∞ is cuspidal and also that Γ contains $-I$. Let $S_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, $\lambda > 0$, be the minimal translation in Γ and let Γ_∞ denote the subgroup generated by S_λ and $-I$. It is well-known that Γ supports automorphic forms and that the Fourier coefficients of these forms (and certain automorphic integrals), when expanded at ∞, involve the *generalized Kloosterman sum*

$$A_c(m,n) = A(c;m,n,k,v,\Gamma) = \sum_{M \in \Gamma_\infty \backslash \Gamma / \Gamma_\infty} \overline{v}(M) e\Big(\frac{\tilde{m}a + \tilde{n}d}{c}\Big),$$

where $M = \begin{pmatrix} a & * \\ c & d \end{pmatrix}$, $c > 0$, $\tilde{m} = \frac{m+\kappa}{\lambda}$, $\tilde{n} = \frac{n+\kappa}{\lambda}$, and m, n are arbitrary integers. For brevity we use the usual notation $e(z) = e^{2\pi i z}$. Here v is a *multipier system* (MS) for Γ in the weight k. This means that v is a function from Γ into the unit circle which, for z in the complex upper half-plane, must satisfy the "consistency condition"

$$v(M_3)(c_3 z + d_3)^k = v(M_1)v(M_2)(c_1 M_2 z + d_1)^k (c_2 z + d_2)^k,$$

where $M_3 = M_1 M_2$, and $M_j = \begin{pmatrix} * & * \\ c_j & d_j \end{pmatrix}$, $j = 1, 2, 3$. We also suppose that $v(-I)(-1)^k = 1$ (otherwise nontrivial automorphic forms would not exist). The parameter κ is determined from v by $v(S_\lambda) = e(\kappa)$, $0 \le \kappa < 1$. Finally, note that although k can be any real number, we could suppose that $0 \le k < 2$, because a MS in weight k' is also a MS in weight $k \equiv k' (\mathrm{mod}\ 2)$.

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Atle Selberg, who wrote a seminal paper [9] which discusses the problem of estimating the order of magnitude of Fourier coefficients of cusp forms, introduced the Dirichlet series

$$Z_{m,n}(s) = Z(s; m, n, k, v, \Gamma) = \sum_{c>0} \frac{A_c(m,n)}{c^{2s}},$$

known today as *Selberg's Kloosterman zeta-function.* Observe that $Z_{m,n}(s)$ converges absolutely for $\sigma > 1$, $s = \sigma + it$, and is analytic in this half-plane. (This is obvious when Γ is a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ of finite index but is nontrivial for general Γ.) Selberg proved that $Z_{m,n}(s)$ has an analytic continuation to a function which is meromorphic in the whole s-plane.

Good estimates on the growth of $Z_{m,n}(s)$, however, were not obtained until the early 80's, when Goldfeld and Sarnak [1] jointly established that, for $\tilde{m}\tilde{n} \neq 0$,

$$Z_{m,n}(s) = O\left(|(m+1)(n+1)| \cdot \frac{|s|^{1/2}}{\sigma - \frac{1}{2}} \right),$$

for $\sigma > \frac{1}{2}$ as $|t| \to \infty$. The implied constant depends at most upon k, v and Γ. Actually, both Selberg and Goldfeld-Sarnak assumed that Γ is a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ of finite index, but (and this was mentioned in [1]) there is no need for this restriction and so we are working in a more general setting.

In this note we provide a minor improvement of the above bound (but not in the s-aspect) and present estimates on $|Z_{m,n}(s)|$ which are valid for certain pole-free lacunary strips throughout the whole s-plane. We achieve the latter at the cost of placing a restriction on v which guarantees that the spectrum of the hyperbolic Laplacian in weight k is discrete.

Theorem *Let k be any real number and m, n be any integers.*

(a) Assume that $\tilde{m}\tilde{n} \neq 0$. If $0 < \varepsilon \leq \frac{1}{2}$ and $\frac{1}{2} < \sigma \leq 1 + \varepsilon$, $|t| \geq 1$, then

$$Z_{m,n}(s) = O\left(|(m+\kappa)(n+\kappa)|^{1-\sigma+\varepsilon} \cdot \frac{|t|^{1-\sigma+\varepsilon}}{\sigma - \frac{1}{2}} \right),$$

where the implied constant depends at most upon ε, v and Γ.

(b) Assume that $\kappa \neq 0$ (and also make the analogous assumption on v in case there are other parabolic cusps inequivalent to ∞). If $0 < \varepsilon_j \leq 1$ and $-j + \frac{1}{2} < \sigma < -j + \frac{3}{2}$, $|t| \geq 1$, then

$$Z_{m,n}(s) = O\left(|(m+\kappa)(n+\kappa)|^{2j+\varepsilon_j} \cdot \frac{|t|^{1/2}}{(\sigma + j - \frac{1}{2})(-j + \frac{3}{2} - \sigma)} \right),$$

where $j = 1, 2, 3, \ldots$ and the implied constant depends at most upon j, ε_j, v and Γ.

For comments about the implied constants and also some refinements see the remarks concluding this paper.

2 Proof of Theorem

As usual (see [9] and [1]) we shall exploit the fact that $Z_{m,n}(s)$ occurs in the Fourier expansion of the *nonanalytic Poincaré series*

$$P_m(z|s) = P_m(z|s;k,v,\Gamma) = \sum_{M \in \Gamma_\infty \backslash \Gamma} \overline{v}(M) \left(\frac{cz+d}{|cz+d|} \right)^{-k} e(\tilde{m} Mz) [\mathrm{Im}(Mz)]^s.$$

Here k is real, m is an integer, $z = x + iy \in \mathcal{H}$ (the complex upper half-plane) and $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$. Observe that $P_m(z|s)$ is holomorphic (in s) for $\sigma > 1$ and that

$$P_m(Vz|s) = v(V) \left(\frac{\gamma z + \delta}{|\gamma z + \delta|} \right)^k P_m(z|s)$$

$\forall\, V = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix} \in \Gamma$. By employing spectral theory Selberg [9] proved that $P_m(z|s)$ has an analytic continuation to a function which is meromorphic in the whole s-plane. In fact, he then used this to extract the aforementioned analytic behavior of $Z_{m,n}(s)$. In particular he showed that the pole sets of both these functions consist of points on the real ray $-\infty < \sigma \leq 1$, points on the vertical lines $\sigma = -j + \frac{1}{2}$, $j = 0, 1, 2, \ldots$ and (in case the MS v permits Eisenstein series) points in $\sigma < \frac{1}{2}$.

We first consider the case where $\tilde{m} > 0$ and $\tilde{n} > 0$. Our lemmas are generalizations of those found in [1]. The first lemma, which provides a fundamental identity, shows how the inner product of two nonanalytic Poincaré series that are "out of phase" can give us direct access to Selberg's Kloosterman zeta-function.

Lemma 1 *For $\tilde{m} > 0$, $\tilde{n} > 0$ and $\mathrm{Re}(w) - L > \sigma > 1$, $L = 0, 1, 2, \ldots$ we have that*

$$\begin{aligned} \langle P_m(z|s), P_n(z|w) \rangle \;=\; & \delta_{m,n} \cdot \lambda (4\pi\tilde{n})^{1-s-\overline{w}} \Gamma(s + \overline{w} - 1) \\ & + (-i)^k 4^{-\overline{w}+1} \pi^{s-\overline{w}+1} \tilde{n}^{s-\overline{w}} \Gamma(s + \overline{w} - 1) \\ & \times \sum_{p=0}^{L} \frac{(-4\pi^2 \tilde{m}\tilde{n})^p \Gamma(\overline{w} - s - p)}{p! \Gamma(s + k/2 + p) \Gamma(\overline{w} - k/2 - p)} \cdot Z_{m,n}(s+p) \\ & + R_{m,n}(s|w;L), \end{aligned}$$

where $\delta_{m,n}$ is the Kronecker symbol and

$$\begin{aligned} R_{m,n}(s|w;L) \;=\; & \sum_{c>0} \frac{A_c(m,n)}{c^{2s}} \int_0^\infty y^{\overline{w}-s-1} e^{-2\pi\tilde{n}y} \\ & \times \int_{-\infty}^{\infty} \frac{(x+i)^{-k}}{(x^2+1)^{s-k/2}} e(-\tilde{n}yx) \sum_{p=L+1}^{\infty} \frac{\left(\frac{-2\pi i \tilde{m}}{c^2 y (x+i)} \right)^p}{p!} \, dx dy. \end{aligned}$$

Proof: A standard computation which uses unfolding (note that $\mathcal{R}_\Gamma$ (resp. $\mathcal{R}_{\Gamma_\infty}$) is a fundamental region for Γ (resp. Γ_∞)) shows that

$$\begin{aligned}
\langle P_m(z|s), P_n(z|w)\rangle &:= \iint\limits_{\mathcal{R}_\Gamma} P_m(z|s)\overline{P_n(z|w)}\,\frac{dxdy}{y^2} \\
&= \iint\limits_{\mathcal{R}_{\Gamma_\infty}} P_m(z|s)e(-\tilde{n}\overline{z})y^{\overline{w}}\,\frac{dxdy}{y^2} \\
&= \delta_{m,n}\cdot\lambda(4\pi\tilde{n})^{1-s-\overline{w}}\Gamma(s+\overline{w}-1) \\
&\quad + \sum_{c>0}\frac{A_c(m,n)}{c^{2s}}\int_0^\infty\int_{-\infty}^\infty \frac{y^{\overline{w}-s-1}}{(x^2+1)^s}\left[\frac{x+i}{(x^2+1)^{1/2}}\right]^{-k} \\
&\qquad\qquad \times e\left(\frac{-\tilde{m}}{yc^2(x+i)}-\tilde{n}(y(x-i))\right)\,dxdy \\
&= \delta_{m,n}\cdot\lambda(4\pi\tilde{n})^{1-s-\overline{w}}\Gamma(s+\overline{w}-1) \\
&\quad + \sum_{p=0}^{L}\frac{(-2\pi i\tilde{m})^p}{p!}\sum_{c>0}\frac{A_c(m,n)}{c^{2s+2p}}\int_0^\infty y^{\overline{w}-s-1-p}e^{-2\pi\tilde{n}y} \\
&\quad \times\int_{-\infty}^\infty \frac{(x+i)^{-k-p}}{(x^2+1)^{s-k/2}}e(-\tilde{n}yx)\,dxdy + R_{m,n}(s|w;L),
\end{aligned}$$

where $R_{m,n}$ is defined in the lemma. Now, we know that [5, pp. 208–209]

$$\int_{-\infty}^\infty \frac{e(-Yx)}{(x+i)^\alpha(x-i)^\beta}\,dx = \frac{\pi(-i)^{\alpha-\beta}(\pi Y)^{\frac{\alpha+\beta}{2}-1}}{\Gamma(\alpha)}W_{\frac{\alpha-\beta}{2},\frac{\alpha+\beta-1}{2}}(4\pi Y),$$

for $Y>0$, $\mathrm{Re}(\alpha+\beta)>1$, and where $W_{\cdot,\cdot}(Y)$ is the Whittaker function that decays exponentially in Y as $Y\to\infty$. So the middle term above equals

$$\begin{aligned}
4(-i)^k\tilde{n}^{s-1}\pi^s&\sum_{p=0}^{L}\frac{(-2\pi^{3/2}\tilde{m}\tilde{n}^{1/2})^p}{p!\Gamma(s+k/2+p)}\cdot Z_{m,n}(s+p) \\
&\times\int_0^\infty y^{\overline{w}-p/2-2}e^{-2\pi\tilde{n}y}W_{\frac{k+p}{2},\frac{2s+p-1}{2}}(4\pi\tilde{n}y)\,dy.
\end{aligned}$$

The lemma now follows from the formula

$$\int_0^\infty W_{\beta,\mu}(y)e^{-y/2}y^{s-1}\,dy = \frac{\Gamma(s+1/2+\mu)\Gamma(s+1/2-\mu)}{\Gamma(s-\beta+1)},$$

valid for $\mathrm{Re}(s+1/2\pm\mu)>0$, which can be found by using the integral representations for $W_{\beta,\mu}$ and the Γ-function (see, e.g., [7, p. 32]). □

The case $L=0$ in Lemma 1 was considered by Goldfeld and Sarnak [1]. Note that the factor of λ which immediately follows $\delta_{m,n}$ is missing in [1, p. 248]. The next two lemmas establish bounds on $R_{m,n}$ and $||P_m||$.

Lemma 2 *Assume that $\tilde{m} > 0$ and $\tilde{n} > 0$.*

(a) $R_{m,n}(s|\overline{s}+2;0)$ is analytic for $\sigma > 0$. For $\varepsilon > 0$ and $\varepsilon \le \sigma \le 1+\varepsilon$ we have that

$$R_{m,n}(s|\overline{s}+2;0) = O\bigg(\frac{(\tilde{m}\tilde{n})^{1-\sigma+\varepsilon}}{\tilde{n}^2}\bigg),$$

where the implied constant depends upon ε and Γ.

(b) $R_{m,n}(s|\overline{s}+2j+2;2j)$ is analytic for $\sigma > -j$, $j = 1,2,3,\ldots$ For $\sigma > -j+\frac{1}{2}$ and $0 < \varepsilon_j \le 1$ we have that

$$R_{m,n}(s|\overline{s}+2j+2;2j) = O\bigg(\frac{\tilde{m}^{2j+\varepsilon_j}}{\tilde{n}^{2-\varepsilon_j}}\bigg),$$

where the implied constant depends at most upon j, ε_j and Γ.

Proof: The definition of $R_{m,n}$ gives us that

$$\begin{aligned} |R_{m,n}(s|w;L)| \quad \le \quad & \sum_{c>0} \frac{|A_c(m,n)|}{c^{2\sigma}} \int_0^\infty y^{\mathrm{Re}(\overline{w})-\sigma-1} e^{-2\pi\tilde{n}y} \\ & \times \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^\sigma} \bigg| \sum_{p=L+1}^{\infty} \frac{\big(\frac{-2\pi i\tilde{m}}{c^2 y(x+i)}\big)^p}{p!} \bigg| \, dxdy. \end{aligned}$$

But

$$\bigg| e^z - \sum_{p=0}^{L} \frac{z^p}{p!} \bigg| < B_r |z|^r, \qquad L \le r \le L+1,$$

for $\mathrm{Re}(z) \le 0$, $z \ne 0$, and where

$$B_r = \begin{cases} \dfrac{1}{r!} & \text{if } r = L+1, \\ e+1 & \text{if } L \le r < L+1. \end{cases}$$

As an aside we remark that this inequality is a generalized version of the fact that e^z is "enveloped" by its Maclaurin series for $\mathrm{Re}(z) \le 0$, $z \ne 0$. Since $\tilde{m} > 0$ we can apply the above basic inequality to produce

$$\begin{aligned} |R_{m,n}(s|w;L)| \quad \le \quad & B_r (2\pi\tilde{m})^r \sum_{c>0} \frac{|A_c(m,n)|}{c^{2\sigma+2r}} \cdot \frac{\Gamma(\mathrm{Re}(\overline{w})-\sigma-r)}{(2\pi\tilde{n})^{\mathrm{Re}(\overline{w})-\sigma-r}} \\ & \times \sqrt{\pi}\, \frac{\Gamma(\sigma+r/2-1/2)}{\Gamma(\sigma+r/2)}. \end{aligned}$$

To show part (a) we set $w = \overline{s}+2$, $L = 0$, $r = 1-\sigma+\varepsilon$, where $\varepsilon > 0$, and use the fact that the series of absolute values associated with $Z_{m,n}(1+\varepsilon)$ is $O_{\varepsilon,\Gamma}(1)$. To prove part (b) set $w = \overline{s}+2j+2$, $L = 2j$, $r = 2j+\varepsilon_j$, where $0 < \varepsilon_j \le 1$ and again appeal to the absolute convergence properties of $Z_{m,n}$. The case $\varepsilon_j = 1$ establishes analyticity for $\sigma > -j$. In this special case the implied constant does not depend upon j. $\square$

Lemma 3 *Assume that* $\tilde{m} > 0$.

(a) For $0 < \varepsilon \le \frac{1}{2}$ *and* $\frac{3}{2} \le \sigma \le B$, $B \ge \frac{3}{2}$, *we have that*

$$||P_m(z|s)|| = O(\tilde{m}^{1-\sigma+\varepsilon}),$$

where the implied constant depends at most upon ε, B, v *and* Γ.

(b) For $0 < \varepsilon \le \frac{1}{2}$ *and* $\frac{1}{2} < \sigma \le \frac{3}{2}$, $|t| \ge 1$, *we have that*

$$||P_m(z|s)|| = O\left(\frac{\tilde{m}^{1-\sigma+\varepsilon}}{\sigma - \frac{1}{2}}\right),$$

where the implied constant depends at most upon ε, k, v *and* Γ.

(c) Assume that $\kappa \ne 0$ *(and also make the analogous assumption on* v *in case there are other parabolic cusps inequivalent to* ∞*). For* $0 < \varepsilon_j \le \frac{1}{2}$ *and* $-j + \frac{1}{2} < \sigma < -j + \frac{3}{2}$, $|t| \ge 1$, *we have that*

$$||P_m(z|s)|| = O\left(\frac{\tilde{m}^{1-\sigma+\varepsilon_j}}{(\sigma + j - \frac{1}{2})(-j + \frac{3}{2} - \sigma)}\right),$$

for $j = 1, 2, 3, \ldots$ *and where the implied constant depends at most upon* j, ε_j, k, v *and* Γ.

Proof: To prove part (a) we first observe that

$$||P_m(z|s)||^2 \le \int_0^\infty \int_0^\lambda |P_m(z|s)| e^{-2\pi\tilde{m}y} y^{\sigma-2}\, dxdy.$$

But we also have that

$$\begin{aligned} |P_m(z|s)| &\le \sum_{M \in \Gamma_\infty \backslash \Gamma} e^{-2\pi\tilde{m}\mathrm{Im}(Mz)} [\mathrm{Im}(Mz)]^\sigma \\ &\le \frac{\lceil r \rceil !}{(\pi\tilde{m})^r} \sum_{M \in \Gamma_\infty \backslash \Gamma} e^{-\pi\tilde{m}\mathrm{Im}(Mz)} [\mathrm{Im}(Mz)]^{\sigma - r}, \end{aligned}$$

where $r \ge 0$. Here we used our assumption that $\tilde{m} > 0$ and the simple inequality $e^{-u} < \frac{\lceil r \rceil !}{u^r}$, $u > 0$, $r \ge 0$. (Note that $\lceil \cdot \rceil$ denotes the "ceiling" function.) The proof of part (a) follows from letting $r = \sigma - 1 - \varepsilon$, $\varepsilon > 0$, and recalling the fact that the series of absolute values connected with $P_m(z|1+\varepsilon)$ is $O_{\varepsilon,v,\Gamma}(1)$ throughout $\mathcal{H}$.

Part (b) is a consequence of part (a) and the recursion relation (first observed by Selberg [9])

$$P_m(z|s) = -4\pi\tilde{m}(s - k/2) R_{s(1-s)}(P_m(z|s+1)), \qquad \sigma > \frac{1}{2},\ s \notin \mathcal{D}_0,$$

where $\mathcal{D}_0 = \mathcal{D}_0(k, v, \Gamma)$ is a finite subset of the real segment $\frac{1}{2} < \sigma \le 1$. (Actually, this relation is first computed for $\sigma > 1$ and then extended by analytic continuation.) Here $R_\mu = (\Delta_k + \mu)^{-1}$ is the resolvent at μ of

$$\Delta_k = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - iky\frac{\partial}{\partial x},$$

the hyperbolic Laplacian in weight k. It is known that Δ_k admits a self-adjoint extension to $L^2(\Gamma\backslash\mathcal{H}, k, v)$, the Hilbert space of functions $f : \mathcal{H} \to \mathbb{C}$ which satisfy

$$f(Vz) = v(V)\left(\frac{\gamma z + \delta}{|\gamma z + \delta|}\right)^k f(z)$$

$\forall\, V = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix} \in \Gamma$ and

$$||f|| < \infty.$$

Because Δ_k is self-adjoint spectral theory tells us that spectrum$(\Delta_k) \subset \mathbb{R}$ and that

$$||R_\mu|| \leq \frac{1}{\text{distance}(\mu, \text{spectrum}(\Delta_k))},$$

for $\mu \notin$ spectrum(Δ_k). Therefore we get that

$$||R_{s(1-s)}|| \leq \frac{1}{|t|(2\sigma - 1)}$$

for $\sigma > \frac{1}{2}$, $t \neq 0$. Combining this with the recursion relation and part (a) (the case $B = \frac{5}{2}$) we obtain

$$||P_m(z|s)|| = O\left(\frac{\tilde{m}^{1-\sigma+\varepsilon}|s - k/2|}{|t|(\sigma - \frac{1}{2})}\right),$$

for $\frac{1}{2} < \sigma \leq \frac{3}{2}$, $t \neq 0$, where the implied constant depends at most upon v, Γ and ε, $0 < \varepsilon \leq 1/2$. In the lacunary strip $\frac{1}{2} < \sigma \leq \frac{3}{2}$, $|t| \geq 1$ we clearly have that $\frac{|s-k/2|}{|t|} = O_k(1)$. This proves part (b).

To show part (c) we need to extend the domain of validity for Selberg's recursion relation. In fact, because of our assumption on v (and this is crucial) we know that the spectrum of Δ_k is discrete in $L^2(\Gamma\backslash\mathcal{H}, k, v)$. The absence of continuous spectra means that $R_{s(1-s)}$ has no "bad" singularities (i.e., it has only poles) and analytic continuation gives us that, for $j = 1, 2, 3 \ldots,$

$$P_m(z|s) = -4\pi\tilde{m}(s - k/2)R_{s(1-s)}(P_m(z|s+1)), \qquad \sigma > -j + \frac{1}{2},\ s \notin \mathcal{D}_j,$$

where $\mathcal{D}_j = \mathcal{D}_j(k, v, \Gamma)$ is the union of a finite set of points on the real segment $-j + \frac{1}{2} < \sigma \leq 1$ and a discrete set of points on the vertical lines $\sigma = -j + \frac{1}{2} + p$, $p = 1, \ldots, j$. From this formula we infer

$$||P_m(z|s)|| \leq (2\pi)^{j+1}\tilde{m}^{j+1}\left(\prod_{p=1}^{j+1}\frac{|s + p - 1 - k/2|}{|t||\sigma + p - \frac{3}{2}|}\right)||P_m(z|s + j + 1)||,$$

for $-j + \frac{1}{2} < \sigma < -j + \frac{3}{2}$, $|t| \neq 0$, where $j = 1, 2, 3, \ldots$ The proof now follows from obvious considerations of (euclidean) distance coupled with part (a). $\square$

We can now finish the proof of the Theorem in the case $\tilde{m} > 0$ and $\tilde{n} > 0$. Let $w = \overline{s} + 2j + 2$ and $L = 2j$, $j = 0, 1, 2, \ldots$ in the identity of Lemma 1 and use analytic continuation to obtain

$$\begin{aligned} Z_{m,n}(s) &= (-i)^{-k} 4^{s+2j+1} \pi^{2j+1} \tilde{n}^{2j+2} \frac{\Gamma(s+k/2)\Gamma(s+2j+2-k/2)}{\Gamma(2j+2)\Gamma(2s+2j+1)} \\ &\quad \times \left\{ \langle P_m(z|s), P_n(z|\overline{s}+2j+2) \rangle - R_{m,n}(s|\overline{s}+2j+2; 2j) \right\} \\ &\quad - \frac{\Gamma(s+k/2)\Gamma(s+2j+2-k/2)}{\Gamma(2j+2)} \left\{ \delta_{m,n} \cdot \lambda (-i)^{-k} 4^{-s} \pi^{-2s} \tilde{n}^{-2s+1} \right. \\ &\quad \left. + \sum_{p=1}^{2j} \frac{(-4\pi^2 \tilde{m}\tilde{n})^p \Gamma(2j+2-p)}{p!\Gamma(s+k/2+p)\Gamma(s+2j+2-k/2-p)} \cdot Z_{m,n}(s+p) \right\}, \end{aligned}$$

where $\sigma > -j + \frac{1}{2}$, $s \notin \mathcal{D}_j$. The Theorem now follows by induction from the estimates in Lemma 2, Lemma 3 (use the Cauchy-Bunyakovsky-Schwarz inequality), Stirling's formula and the boundedness of $Z_{m,n}(u)$ for $\mathrm{Re}(u) \geq \frac{3}{2}$. For part (a) we also invoked the convexity principle of Phragmén-Lindelöf. Note that the estimates in the Theorem do not depend on k because, as was mentioned in our introduction, we can assume that $0 \leq k < 2$.

Because the proof for the case $\tilde{m} > 0$ and $\tilde{n} < 0$ parallels the one just completed, we shall omit it. We only mention that whereas before we considered $\langle P_m(z|s; k, v, \Gamma), P_n(z|w; k, v, \Gamma) \rangle$, here it is preferable to examine

$$\langle P_m(z|s; k, v, \Gamma), \overline{P_{n^*}(z|w; -k, \overline{v}, \Gamma)} \rangle,$$

where

$$n^* = \begin{cases} -n-1 & \text{if } 0 < \kappa < 1, \\ -n & \text{if } \kappa = 0. \end{cases}$$

Recall that κ is the parameter associated with v and note that if v is a MS in weight k, then $\overline{v}$ is a MS in weight $-k$.

In order to conclude the proof altogether, we must still analyse the case when $\tilde{m} < 0$. But this is an immediate implication of the case $\tilde{m} > 0$ coupled with the nice observation that

$$\overline{Z(\overline{w}; m, n, k, v, \Gamma)} = \begin{cases} Z(w; -m-1, -n-1, -k, \overline{v}, \Gamma) & \text{if } 0 < \kappa < 1, \\ Z(w; -m, -n, -k, \overline{v}, \Gamma) & \text{if } \kappa = 0. \end{cases}$$

Remark 1 *Although the implied constants in our estimates do depend upon v and Γ, this dependence is at most upon κ, λ and $c_{\min}$, the minimal positive element which occurs in the left column and lower row of a matrix in Γ. In particular, and this is in contrast with the Goldfeld-Sarnak estimate, our implied constants never depend upon* $\mathrm{Area}(\mathcal{R}_\Gamma)$*, the hyperbolic area of $\mathcal{R}_\Gamma$. This is clear from our proof of Lemma 3, part (a), and is not surprising because the inner product of two classical holomorphic Poincaré series (which are cusp forms) also does not depend upon* $\mathrm{Area}(\mathcal{R}_\Gamma)$*, but does depend upon λ. We are in a* *generalized situation. Also note that for certain families of groups the area of the fundamental region can grow without bound while λ remains constant. This is the case for the Hecke congruence groups of level N, $\Gamma_0(N)$, $N = 1, 2, 3, \ldots$, where $\mathrm{Area}(\mathcal{R}_{\Gamma_0(N)}) \to \infty$ as $N \to \infty$ and yet $\lambda = 1 \,\forall\, N$.*

Remark 2 *We do not claim that our estimates are the best possible in the m-aspect or the n-aspect. For certain groups and MS the bound for $\frac{1}{2} < \sigma \leq 1 + \varepsilon$, $|t| \geq 1$ could be improved by making use of stronger estimates on the generalized Kloosterman sum. Moreover, for general Γ one ought to be able to improve the estimate for $-j + \frac{1}{2} < \sigma < -j + \frac{3}{2}$, $|t| \geq 1$ by replacing the exponent $2j + \varepsilon_j$, $0 < \varepsilon_j \leq 1$, with $j + \frac{1}{2} + \varepsilon_j$, $0 < \varepsilon_j \leq \frac{1}{2}$, where $j = 1, 2, 3, \ldots$. Although we cannot prove this, we can show (with the same assumption on the MS v stated in part (b) of the Theorem) that for $0 < \varepsilon_j \leq \frac{1}{2}$ and any compact subset S_j of $\sigma > -j + \frac{1}{2}$, $s \notin \mathcal{D}_j$, we have that*

$$Z_{m,n}(s) = O(|(m+\kappa)(n+\kappa)|^{j+1/2+\varepsilon_j}),$$

where $j = 1, 2, 3, \ldots$ and the implied constant depends at most upon j, S_j, ε_j, v and Γ. Our proof of this relies on the fact (stated here for $\tilde{m} > 0$ and $\tilde{n} > 0$) that $R_{m,n}(s|\overline{s} + j + 2; j)$ is actually analytic for $\sigma > -j$, $j = 1, 2, 3, \ldots$ and for $0 < \varepsilon_j \leq \frac{1}{2}$ we have that

$$R_{m,n}(s|\overline{s} + j + 2; j) = O\left(\frac{\tilde{m}^{j+1/2+\varepsilon_j}}{\tilde{n}^{3/2-\varepsilon_j}}\right),$$

in any compact subset T_j of $\sigma > -j + \frac{1}{2}$, where the implied constant depends at most upon j, T_j, ε_j and Γ.

Remark 3 *If we lift the restriction on the MS v in part (b) of the Theorem, then the situation becomes more delicate. Certainly estimates of the kind stated in part (b) are in general impossible because of the existence of poles which occur to the left of $\sigma = \frac{1}{2}$ but are not in $\mathcal{D}_j$. That is, they are in sets of the form $-j + \frac{1}{2} < \sigma < -j + \frac{3}{2}$, $t \neq 0$, where $j = 1, 2, 3, \ldots$. Nonetheless, with suitable restrictions on s (and perhaps on Γ), there ought to be estimates. We hope to return to this problem at a later time.*

Remark 4 *The estimate in part (b) can be used to find explicit formulas for the Fourier coefficients of modular forms (and certain modular integrals) on $SL(2, \mathbb{Z})$ of weight k, $0 < k \leq 1$, and arbitrary MS v. We shall present this in a forthcoming article.*

Remark 5 *For a nice discussion of the Goldfeld-Sarnak estimate (in the case where Γ is the full modular group and $A_c(m,n)$ is the classical Kloosterman sum) as well as references to applications see Sarnak's book [8, pp. 33–41]. For a rather comprehensive proof of this estimate for general Γ see Hejhal's volume [3, pp. 665–709]. For more information on the spectral theory of automorphic forms refer to the two tomes by Hejhal [2, 3] and the more recent books by Venkov [10] and Iwaniec [4]. Lastly, for the pretty notion of enveloping series consult the classic by Pólya and Szegö [6, pp. 32–36].*

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Max-Planck-Institut für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn
Germany

E-mail: pribitki@mpim-bonn.mpg.de; w_pribitkin@msn.com