Noncommutative Grassmannians and related constructions.

Maxim Kontsevich, Alexander Rosenberg

Introduction.

Commutative schemes were first realized as geometric spaces. Then it was observed that, for construction of new schemes and establishing their general categorical properties, it is much more convenient to replace schemes by presheaves of sets on the category of affine schemes they represent. The presheaves representable by schemes are sheaves for a whole bunch of non-trivial topologies on the category of affine schemes, starting with the Zariski topology, with the flat (**fpqc**) topology being a preferable choice.

¿From the point of view of noncommutative algebraic geometry, the main invariant of a 'space' is the category of quasi-coherent (pre)sheaves on this 'space'. The category of quasi-coherent presheaves is defined for any functor as the category opposite to the category of *cartesian sections* of this functor (cf. [KR4]). In particular, we have the category $Qcoh(F)$ of quasi-coherent presheaves on any fibred category $\mathfrak{F} \xrightarrow{F} \mathcal{B}$. For a topology τ on the base $\mathcal{B}$, it is defined (in [KR4]) the category $Qcoh(F,\tau)$ of quasi-coherent sheaves on (F,τ) – a full subcategory of the category $Qcoh(F)$. For any presheaf of sets X on the category $\mathcal{B}$, we consider the category $\mathcal{B}/X$ of objects of $\mathcal{B}$ over X and the fibred category $\mathfrak{F}^X \xrightarrow{F_X} \mathcal{B}/X$ with the base $\mathcal{B}/X$ obtained via pulling-back the fibred category along the canonical functor $\mathcal{B}/X \longrightarrow \mathcal{B}$. We denote by $Qcoh(X)$ the category of quasi-coherent presehaves on the fibred category $\mathfrak{F}^X \xrightarrow{F_X} \mathcal{B}/X$ and by $Qcoh(X,\tau)$ its full subcategory generated by quasi-coherent sheaves on (F_X,τ_X), where τ_X is the topology on $\mathcal{B}/X$ induced by τ.

The first application of this formalism is as follows. We take as a base $\mathcal{B}$ the category $\mathbf{Aff}_k$ of *noncommutative affine k-schemes* which is, by definition, the category opposite to the category Alg_k of associative unital algebras. So that presheaves of sets on $\mathbf{Aff}_k$ are functors from Alg_k to **Sets**. Consider the fibred category $\widetilde{\mathbf{Aff}}_k$ with the base $\mathbf{Aff}_k$ whose fibers are categories of left modules over corresponding algebras. For every presheaf of sets X on $\mathbf{Aff}_k$, we have the fibred category $\widetilde{\mathbf{Aff}}_k/X$ which is the pull-back of $\widetilde{\mathbf{Aff}}_k$ along the canonical functor $\mathbf{Aff}_k/X \longrightarrow \mathbf{Aff}_k$. The category $Qcoh(X)$ of quasi-coherent sheaves on the presheaf X is the category opposite to the category of cartesian sections of the fibred category $\widetilde{\mathbf{Aff}}_k/X$. For a topology τ on $\mathbf{Aff}_k$, we have (for any presheaf of sets X on $\mathbf{Aff}_k$) the subcategory $Qcoh(X,\tau)$ of *quasi-coherent sheaves* on $(\mathbf{Aff}_k/X,\tau)$.

Theorem. *(a) A topology τ on the category $\mathbf{Aff}_k$ is subcanonical (that is all representable presheaves of sets are sheaves) iff $Qcoh(X,\tau) = Qcoh(X)$ (in other words, 'descent' topologies on $\mathbf{Aff}_k$ are precisely subcanonical topologies).*

In this case, $Qcoh(X) = Qcoh(X,\tau) \hookrightarrow Qcoh(X^\tau) = Qcoh(X^\tau,\tau)$, where X^τ is the sheaf on $(\mathbf{Aff}_k,\tau)$ associated with the presheaf X and $\hookrightarrow$ is a natural full embedding.

(b) If τ is a topology of effective descent, then the embedding $Qcoh(X) \hookrightarrow Qcoh(X^\tau)$ is a category equivalence.

This theorem says that, roughly speaking, the category $Qcoh(X)$ of quasi-coherent presheaves knows itself which topologies to choose. It also indicates where one should look for a correct noncommutative version of the category Esp (of sheaves of sets on the **fpqc** site of commutative affine schemes): this should be the category $NEsp_\tau$ of sheaves of sets on the presite $(\mathbf{Aff}_k, \tau)$, where τ is a topology of *effective descent*. ¿From the minimalistic point of view, the best choice would be the (finest) topology of effective descent. But, there is a more important consideration. The main role of a topology is that it is used for glueing new 'spaces'. The topology that seems to be the most relevant for Grassmannians, in particular, for noncommutative projective 'spaces', and a number of other smooth noncommutative spaces constructed in this work, is the *smooth* topology introduced in [KR2].

The theorem is quite useful on pragmatical level. Namely, if $\mathfrak{X}$ is a sheaf of sets on $(\mathbf{Aff}_k, \tau)$ for an appropriate topology of effective descent and X is a presheaf of sets on Aff_k such that its associated sheaf is isomorphic to $\mathfrak{X}$, and $\mathfrak{R} \underset{\mathfrak{p}_2}{\overset{\mathfrak{p}_1}{\rightrightarrows}} \mathfrak{U} \xrightarrow{\pi} X$ is an exact sequence of presheaves with $\mathfrak{R}$ and $\mathfrak{U}$ representable, then the category $Qcoh(X)$ (hence the category $Qcoh(\mathfrak{X})$) is constructively and canonically described via the pair $\mathcal{A} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathcal{R}$ of k-algebra morphisms representing $\mathfrak{R} \underset{\mathfrak{p}_2}{\overset{\mathfrak{p}_1}{\rightrightarrows}} \mathfrak{U}$. This consideration is used to describe the categories of quasi-coherent sheaves on noncommutative 'spaces'.

There is another important aspect of noncommutative geometry which we take into consideration here (see also [R2]). Local objects of commutative algebraic geometry are commutative rings which might be regarded as commutative algebras in the simmetric-monoidal category of $\mathbb{Z}$-modules. One of particularities of noncommutative geometry is that some interesting noncommutative spaces'live' in non-trivial monoidal categories. For instance, the quantumflag variety of a simple Lie algebra $\mathfrak{g}$ is a scheme in the monoidal category of $\mathbb{Z}^r$-graded vector spaces endowed with a braidingdetermined by the Cartan matrix of $\mathfrak{g}$ (cf [LR2]). Therefore, we choose here a framework which allows to take this fact into consideration and gives to our constructions an appropriate level of generality. This framework is as follows: instead of the monoidal category of modules over a commutative unital ring k, we take an arbitrary monoidal category $\mathcal{A}^\sim$ and define the category $\mathbf{Aff}_{\mathcal{A}^\sim}$ of *affine schemes in* $\mathcal{A}^\sim$ as the category opposite to the category $Alg\mathcal{A}^\sim$ of algebras in $\mathcal{A}^\sim$. Given an action, $\mathfrak{C}$, of the monoidal category $\mathcal{A}^\sim$ on a category C, we define, for any algebra S in $\mathcal{A}^\sim$, the category $S - mod_C$ *of S-modules in C* otherwise called the category of *quasi-coherent modules on* $\mathfrak{Sp}(S)/\mathfrak{C}$. A standard noncommutative example is the monoidal category $R-bimod^\sim = (R-bimod, \otimes_R, R)$ of bimodules over an associative ring R acting (by tensoring over R) on the category $R-mod$ of left R-modules. For any algebra S in the monoidal category $R-bimod^\sim$ (that is a ring morphism $R \longrightarrow S$), the category of quasi-coherent modules on $\mathfrak{Sp}(S)/\mathfrak{C}$ is naturally equivalent to the category $S-mod$ of left S-modules. A non-trivial example is given by the monoidal category $\mathcal{A}^\sim = \mathbf{S} - Vec_k^\sim$ of $\mathbf{S}$-*spaces* whose objects are families of representations of all symmetric groups, $\mathbf{S}_n,\ n \geq 1$, in vector spaces over a field k, and the tensor product is the so called *plethysm product*. Algebras in the monoidal category $\mathbf{S} - Vec_k^{\sim}$ are called *k-linear operads*. The monoidal category of $\mathbf{S}$-spaces acts canonically on the category $C = Vec_k$ of k-vector spaces. For each k-linear operad $\mathcal{P}$, the corresponding category of quasi-coherent sheaves on $\mathfrak{Sp}(\mathcal{P})$ is traditionally called *the category of $\mathcal{P}$-algebras*.

The quoted above theorem about quasi-coherent sheaves and subcanonical topologies extends to this, much more general, setting. We do not give here the proof of this statement (which is an adaptation of the argument of 2.7.3.1 in [KR4]), in spite of its importance as a background of this work, because the purpose of this text is to describe certain classically looking constructions of noncommutative 'spaces' avoiding depths of general theory.

Note that some basic notions of *commutative* algebraic geometry in symmetric monoidal categories (starting with the site of affine schemes with **fpqc** topology) were sketched by P. Deligne in connection with the characterization of Tannakian categories [Dl]. Our starting point is similar, only monoidal categories are not necessarily symmetric and algebras are usually not commutative.

The paper is organized as follows. In Section 1, we review affine schemes in monoidal categories and categories of quasi-coherent modules on them. In Section 2, we give examples of affine schemes. Section 3 contains the construction of Grassmannians and generic Grassmannians. Section 4 is dedicated to a futher study of Grassmannians. In Section 5, generic flag varieties are introduced and their relation with universal Stiefel schemes is discussed. In Section 6, we introduce a construction of generalized Grassmannian type spaces. Grassmannians of Section 3 and generic flag variety of Section 5 are obtained as special cases of this construction.

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1. Affine schemes in a monoidal category. Quasi-coherent modules.

1.0. Preliminaries: algebras and (bi)modules in monoidal categories. Fix a monoidal category $\mathcal{A}^{\sim} = (\mathcal{A}, \otimes, 1, a, l, r)$. Here $\mathcal{A}$ is a category, $\otimes$ is a functor from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$, a is a functor isomorphism $\otimes \circ (Id_{\mathcal{A}} \times \otimes) \longrightarrow \otimes \circ (\otimes \times Id_{\mathcal{A}})$ (*associativity constraint*), and $Id_{\mathcal{A}} \xrightarrow{l} 1 \otimes Id_{\mathcal{A}}$, $Id_{\mathcal{A}} \xrightarrow{r} Id_{\mathcal{A}} \otimes 1$ are functor isomorphisms compatible with the associativity constraint a. An *algebra* (or *monoid*) in $\mathcal{A}^{\sim}$ is a pair (R, μ) where $R \in Ob\mathcal{A}$ and μ is a morphism $R \otimes R \longrightarrow R$ such that $\mu \circ (\mu \otimes id_R) \circ a_{R,R,R} = \mu \circ (id_R \otimes \mu)$. The *unit* of an algebra (R, μ) is a morphism $1 \xrightarrow{\eta} R$ such that $\mu \circ \eta \otimes id_R \circ l_R = id_R = \mu \circ id_R \otimes \eta \circ r_R$. The unit (if it exists) is unique. We assume that all algebras considered here are unital. Algebras in $\mathcal{A}^{\sim}$ form a category which we denote by $Alg\mathcal{A}^{\sim}$.

A left module over an algebra (R, μ) is a pair (M, m), where $M \in Ob\mathcal{A}$, m is a morphism $R \otimes M \longrightarrow M$ such that $m \circ id_R \otimes m = m \circ \mu \otimes id_M \circ a_{R,R,M}$ and $m \circ \eta \otimes id_M = id_M$. Left modules over $R^{\sim} = (R, \mu)$ form a category $R^{\sim} - mod$.

The category $mod - R^{\sim}$ of right $R^{\sim}$-modules is defined in an obvious way. Note that right modules are just left modules in the opposite monoidal category. A triple (m, M, m'), where (m, M) and (M, m') are resp. left and right $R^{\sim}$-modules is called an $R^{\sim}$*-bimodule* if $m \circ id_R \otimes m' = m' \circ m \otimes id_R \circ a_{R,M,R}$.

Suppose that for any $\mathcal{M} \in Ob\mathcal{A}$, the functor $\mathcal{M} \longmapsto \mathcal{M} \otimes -$ preserves cokernels of reflexive pairs of arrows. Then there is a functor

$$mod - R^{\sim} \times R^{\sim} - mod \xrightarrow{\otimes_R} \mathcal{A}$$

which assigns to any pair of resp. right and left $R^{\sim}$-modules $(M, m), (\nu, N)$ the cokernel of the pair of morphisms $id_M \otimes \nu, m \otimes \nu \circ a_{M,R,N} : M \otimes (R \otimes N) \longrightarrow M \otimes N$. The functor $\otimes_R$ induces a structure of a monoidal category on the category $R^{\sim} - bimod$ of $R^{\sim}$-bimodules.

Let β be a symmetry of the monoidal category $\mathcal{A}^{\sim}$. An algebra $R^{\sim} = (R, \mu)$ is called β*-commutative* (or *commutative* if β is fixed) if $\mu \circ \beta_{R,R} = \mu$. The full subcategory of $Alg\mathcal{A}^{\sim}$ formed by β-commutative algebras will be denoted by $Alg_{\beta}\mathcal{A}^{\sim}$.

For any β-commutative algebra $R^{\sim}$, the map $(m, M) \longmapsto (m, M, m \circ \beta_{M,R})$ defines a functor, Δ_{β}, identifying the category $R^{\sim} - mod$ of left $R^{\sim}$-modules with a full subcategory of the category $R^{\sim} - bimod$ of $R^{\sim}$-bimodules.

Suppose the functor $\mathcal{M} \longmapsto \mathcal{M} \otimes -$ is right exact for any $\mathcal{M} \in Ob\mathcal{A}$. Then the functor Δ_{β} identifies $R^{\sim} - mod$ with a monoidal subcategory of $R^{\sim} - bimod$; and the symmetry β induces a symmetry on $R^{\sim} - mod$.

1.1. Noncommutative affine schemes in a monoidal category. Fix a monoidal category $\mathcal{A}^{\sim} = (\mathcal{A}, \otimes, \mathbf{1}, a)$. We define the category $\mathbf{Aff}_{\mathcal{A}^{\sim}}$ of *affine schemes in* $\mathcal{A}^{\sim}$ as the opposite category to the category $Alg\mathcal{A}^{\sim}$. We denote by $\mathfrak{Sp}(R^{\sim})$, or by $\mathfrak{Sp}_{\mathcal{A}^{\sim}}(R^{\sim})$ the functor represented by the algebra $R^{\sim}$.

1.1.1. Monoidal functors and affine schemes. Recall that a monoidal functor from a monoidal category $\mathcal{A}^{\sim} = (\mathcal{A}, \otimes, \mathbf{1}, a, l, r)$ to a monoidal category $\mathcal{A}^{'\sim} = (\mathcal{A}', \otimes', \mathbf{1}', a', l', r')$ is a triple $\Phi^{\sim} = (\Phi, \phi, \phi_0)$, where Φ is a functor $\mathcal{A} \longrightarrow \mathcal{A}'$, ϕ is a functor morphism $\{\phi_{X,Y} : \Phi(X) \otimes' \Phi(Y) \longrightarrow \Phi(X \otimes Y)$, ϕ_0 a morphism $\mathbf{1}' \longrightarrow \Phi(\mathbf{1})$ satisfying natural compatibility conditions.

Any monoidal functor $\Phi^{\sim} : \mathcal{A}^{\sim} \longrightarrow \mathcal{A}'$ induces a functor

$$\Phi^{\sim}_{Alg} : Alg\mathcal{A}^{\sim} \longrightarrow Alg\mathcal{A}^{'\sim}, \quad (R, m) \longmapsto \Phi^{\sim}(R, m) = (\Phi(R), \Phi(m) \circ \phi_{R,R}), \qquad (1)$$

hence a functor

$$\Phi^{\sim}_{\mathbf{Aff}} : \mathbf{Aff}_{\mathcal{A}^{\sim}} \longrightarrow \mathbf{Aff}_{\mathcal{A}'^{\sim}}$$

1.2. Coproducts and fiber products of affine schemes. Fix a monoidal category $\mathcal{A}^{\sim} = (\mathcal{A}, \otimes, \mathbf{1}, a)$.

1.2.1. Lemma. *Suppose that the category $\mathcal{A}$ has products of $|J|$ objects. Then the category $Alg\mathcal{A}^{\sim}$ has products of $|J|$ objects.*

Proof. Let $R_i^{\sim} = (R_i, \mu_i)$, $i \in J$, be a set of algebras in $\mathcal{A}^{\sim}$. By assumption, there exists a product $\prod_{i\in J} R_i$ of the set of objects $\{R_i|\ i \in J\}$, and thecanonical projections $\prod_{i\in J} R_i \xrightarrow{p_j} R_j$ together with multiplications μ_j provide morphisms

$$\prod_{i\in J} R_i \otimes \prod_{i\in J} R_i \xrightarrow{p_j\otimes p_j} R_j \otimes R_j \xrightarrow{\mu_j} R_j, \quad j \in J. \tag{1}$$

By the universal property of products, there exists a unique morphism

$$\prod_{i\in J} R_i \otimes \prod_{i\in J} R_i \xrightarrow{\mu} \prod_{i\in J} R_i$$

such that $p_i \circ \mu = \mu_i \circ p_i \otimes p_i$ for any $i \in J$. We leave to the reader to check that thus defined morphism μ is a structure of an associative unital algebra on $\prod_{i\in J} R_i$ and that the algebra $(\prod_{i\in J} R_i, \mu)$ is a product of the set of algebras $\{R_i^{\sim} = (R_i, \mu_i),\ i \in J\}$. ■

1.2.2. Proposition. *Suppose that $\mathcal{A}^{\sim} = (\mathcal{A}, \otimes, \mathbf{1}, a)$ is a monoidal category with countable coproducts. Then, for any algebras $R^{\sim} = (R, \mu)$ and $S^{\sim} = (S, \nu)$ in $\mathcal{A}^{\sim}$ such that the functors $R \otimes -$ and $S \otimes -$ are compatible with countable coproducts, there exists a free product $R^{\sim} \star S^{\sim}$ which is a coproduct in the category $Alg\mathcal{A}^{\sim}$.*

In particular, $\mathfrak{Sp}_{\mathcal{A}^{\sim}}(R^{\sim} \star S^{\sim})$ is a product of $\mathfrak{Sp}_{\mathcal{A}^{\sim}}(R^{\sim})$ and $\mathfrak{Sp}_{\mathcal{A}^{\sim}}(S^{\sim})$ in the category $\mathbf{Aff}_{\mathcal{A}^{\sim}}$ of affine schemes in $\mathcal{A}^{\sim}$.

Proof. Let $\mathfrak{L}^{\sim}_{\mathcal{A}^{\sim}} = (\mathfrak{L}, a)$ be the canonical monoidal functor "of left multiplication"

$$\mathcal{A}^{\sim} \longrightarrow End^{\sim}\mathcal{A}, \quad M \longmapsto M \otimes -.$$

The monoidal functor $\mathfrak{L}^{\sim}_{\mathcal{A}^{\sim}}$ induces a functor from $Alg\mathcal{A}^{\sim}$ to $\mathfrak{Mon}\mathcal{A}$ which assigns to algebras $R^{\sim} = (R, \mu)$ and $S^{\sim} = (S, \nu)$ the monads $(\mathfrak{L}(R), \mu_R)$ and $(\mathfrak{L}(S), \nu_S)$ such that the functors $\mathfrak{L}(R)$ and $\mathfrak{L}(S)$ preserve countable coproducts. The assertion follows now from [R2, Proposition 2.6.2.3]. ■

1.2.3. Proposition. *Let $\mathcal{A}$ be a category with countable colimits. And suppose that for any $M \in Ob\mathcal{A}$, the functor $M \otimes - : \mathcal{A} \longrightarrow \mathcal{A}$ is compatible with countable colimits. Then the category $\mathbf{Aff}_{\mathcal{A}^{\sim}}$ of affine schemes in $\mathcal{A}^{\sim}$ has fiber products.*

Proof. Let $R^{\sim} \longleftarrow S^{\sim} \longrightarrow T^{\sim}$ be morphisms of the category $Alg\mathcal{A}^{\sim}$. Since the functor $M \longmapsto M \otimes -$ preserves cokernels of coreflexive pairs of arrows, we have monads $\mathbb{F} = R^{\sim} \otimes_{S^{\sim}} -$ and $\mathbb{G} = T^{\sim} \otimes_{S^{\sim}} -$ on the category $S^{\sim} - mod$ of left $S^{\sim}$-modules.

Because the functors $R \otimes -$ and $T \otimes -$ preserve countable coproducts, the functors $R \otimes_{S^{\sim}} -$ and $T \otimes_{S^{\sim}} -$ from $S^{\sim} - mod$ to $S^{\sim} - mod$ have the same property.

By [R2, 2.6.2.3], there exists the coproduct $\mathbb{F} \star \mathbb{G}$ which we denote by (F', μ'). To the monad (F', μ') on $S^{\sim} - mod$, there corresponds a monad $(f_* F' f^*, \mu'_f)$ on $\mathcal{A}$. Here f_*

denotes the forgetful functor $S^\sim - mod \longrightarrow \mathcal{A}$ and f^* is its left adjoint, $L \longmapsto (S \otimes L, m_L)$, where m_L is the canonical action induced by the multiplication on S. The multiplication μ'_f is the composition of the canonical morphism $f_*F'f^*f_*F'f^* \longrightarrow f_*F'^2f^*$ induced by the adjunction morphism $f^*f_* \longrightarrow Id$ and the morphism $f_*\mu'f^* : f_*F'^2f^* \longrightarrow f_*F'f^*$. One can associate with the monad $(f_*F'f^*, \mu'_f)$ the algebra $(f_*F'f^*(\mathbf{1}), \mu'') = (f_*F'(S), \mu'')$, where μ'' is a naturally defined multiplication. This algebra is a fiber coproduct of $R^\sim$ and $T^\sim$ over $S^\sim$. A less formal (and a more constructive) argument uses the explicit description of $f_*F'(S)$ in terms of tensor product of copies of $R^\sim$ and $S^\sim$ over $S^\sim$ (following [R2, 2.6.2]) in which the multiplication and the universal property are evident. This argument is left to the reader. ■

1.2.4. Example. Let k be a commutative associative ring and $\mathcal{A}^\sim = (k - mod^\sim) = (k - mod, \otimes_k, k)$. Algebras in $k - mod^\sim$ are k-algebras, and the coproduct of algebras A and B is their free product (otherwise called *$\star$-product*). For instance, the coproduct of two copies of polynomial algebra in one variable is isomorphic to the free algebra in two variables: $k[x] \star k[y] \simeq k\langle x, y\rangle$.

1.3. Quasi-coherent modules. An action of the monoidal category $\mathcal{A}^\sim$ on a category C gives rise to a fibered category over the category $\mathbf{Aff}_{\mathcal{A}^\sim}$. For any presheaf of sets X on $\mathbf{Aff}_{\mathcal{A}^\sim}$, we have the induced fibered category over $\mathbf{Aff}_{\mathcal{A}^\sim}/X$. Cartesian sections of the latter fibered category are quasi-coherent modules on X. Details are below.

1.3.1. Actions of a monoidal category. An action of the monoidal category $\mathcal{A}^\sim = (\mathcal{A}, \otimes, \mathbf{1}, a)$ on a category $\mathcal{C}$ is a triple $(\mathcal{A}^\sim, C, \Phi^\sim)$, where $\mathcal{C}$ is a category, $\Phi^\sim = (\Phi, \phi)$ is a monoidal functor $\mathcal{A}^\sim \longrightarrow End_2^\sim C$. In this work, we assume that the functor Φ takes values in the full subcategory End_2C of the category $EndC$ whose objects are functors which preserve cokernels of reflexive pairs of arrows. In other words, $\Phi^\sim$ is a unital action of the monoidal category $\mathcal{A}^\sim$ on a category $\mathcal{C}$ which preserves cokernels of reflexive pairs of arrows. Recall that a pair of arrows $L \underset{v}{\overset{u}{\rightrightarrows}} M$ is called *reflexive* if there exists an arrow $M \xrightarrow{g} L$ such that $u \circ g = id_L = v \circ g$.

The monoidal functor $\Phi^\sim$ induces a functor, $R^\sim \longmapsto \Phi^\sim(R^\sim)$, from the category $Alg\mathcal{A}^\sim$ to the category of monads on the category $\mathcal{C}$ (more precisely, to the category of algebras in $End_2^\sim C$).

1.3.2. A fibered category associated with an action. Fix an action $\mathfrak{C} = (\mathcal{A}^\sim, \mathcal{C}, \Phi^\sim)$. We associate with $\mathfrak{C}$ a cofibered category $\mathcal{F}^{\mathfrak{C}} \xrightarrow{\pi} (Alg\mathcal{A}^\sim)^{op} = \mathbf{Aff}_{\mathcal{A}^\sim}$ in a natural way: objects of the category $\mathcal{F}^{\mathfrak{C}}$ are pairs $(R^\sim, M)$, where $R^\sim = (R, \mu)$ is an algebra in $\mathcal{A}^\sim$, M is a $\Phi^\sim(R^\sim)$-module. A morphism from $(R^\sim, M)$ to $(S^\sim, L)$ is a pair (ψ, ξ), where ψ is an algebra morphism $R^\sim \longrightarrow S^\sim$, ξ is a morphism $M \longrightarrow \psi_*(L)$, where ψ_* is the pull-back functor $R^\sim - mod_C \longrightarrow S^\sim - mod_C$ induced by the algebra morphism ψ. The composition is defined in a standard way. The projection π assigns to any object $(R^\sim, M)$ the algebra $R^\sim$ and to any morphism (ψ, ξ) the algebra morphism ψ. It follows from the assumptions and [R2, 1.0.6.1] that the functors ψ_* have left adjoints, hence $\mathcal{F}^{\mathfrak{C}} \xrightarrow{\pi} \mathbf{Aff}_{\mathcal{A}^\sim}$ is a bifibered, in particular fibered, category.

1.3.3. Quasi-coherent modules. For any fibered category (more generally, for any category over a category) $\mathfrak{B}$ over a base $\mathcal{E}$, the category $Qcoh(\mathfrak{B})$ of quasi-coherent modules on $\mathfrak{B}$ is defined as the category opposite to the category $Lim\mathfrak{B}$ of cartesian sections of $\mathfrak{B}$ (cf. [R2, 1.0.1, 1.0.2], or in more general context, [KR2, 11.1]). For any presheaf of sets X on the base $\mathcal{E}$, we have a canonical functor $\mathcal{E}/X \longrightarrow \mathcal{E}$, hence the induced fibered category $\mathfrak{B}/X$ over $\mathcal{E}/X$. The category $Qcoh(\mathfrak{B}/X)$ of quasi-coherent modules will be called the category of quasi-coherent modules on X and sometimes denoted simply by $Qcoh_X$. If the presheaf X is representable by an object x of $\mathcal{E}$, then the category $Qcoh(\mathfrak{B}/X)$ is equivalent to the category opposite to the fiber of $\mathfrak{B}$ over x.

Applying this general nonsense to the fibered category $\mathcal{F}^{\mathfrak{C}}$ associated with the action $\mathfrak{C} = (\mathcal{A}^\sim, \Phi^\sim, \mathcal{C})$ (cf. 1.3.2), we obtain for any presheaf of sets X on $\mathbf{Aff}_{\mathcal{A}^\sim}$ the category of quasi-coherent modules on X.

1.3.3.1. Quasi-coherent modules on an affine scheme. If the presheaf X representable, i.e. it is isomorphic to $\mathfrak{Sp}(R^\sim)$ for some algebra $R^\sim$, then the category $Qcoh_X$ of quasi-coherent modules on X is equivalent to the category $R^\sim - mod_C$ of $\Phi^\sim(R^\sim)$-modules which we call simply $R^\sim$-modules in $\mathcal{C}$.

1.4. Examples.

1.4.1. Bimodules and modules. A standard noncommutative example is the monoidal category $R-bimod^\sim = (R-bimod, \otimes_R, R)$ of bimodules over an associative ring R acting (by tensoring over R) on the category of left R-modules. For any algebra S in the monoidal category $R - bimod^\sim$ (that is a ring morphism $R \longrightarrow S$), the category of quasi-coherent modules on $\mathfrak{Sp}(S)$ is naturally equivalent to the category $S - mod$ of left S-modules.

1.4.2. The left and right canonical actions of a monoidal category. Let $\mathcal{A}^\sim = (\mathcal{A}, \otimes, \mathbf{1})$ be a monoidal category such that for any $\mathcal{M} \in Ob\mathcal{A}$, the functor

$$\mathcal{A} \xrightarrow{\mathcal{M}\otimes -} \mathcal{A}, \quad V \longmapsto \mathcal{M} \otimes V,$$

preserves cokernels of reflexive pairs of arrows. The left canonical action of $\mathcal{A}^\sim$ on the category $\mathcal{A}$ is the triple $(\mathcal{A}^\sim, \mathcal{A}, (\mathfrak{L}, a))$, where $\mathfrak{L} = \mathfrak{L}_{\mathcal{A}^\sim}$ denotes the functor

$$\mathcal{A} \longrightarrow End\mathcal{A}, \quad \mathcal{M} \longmapsto \mathcal{M} \otimes -,$$

and a is the associativity constraint. The functor $\Phi^\sim$ is faithful, because $\mathcal{M} \longmapsto \mathcal{M} \otimes \mathbf{1}$ is a faithful functor. Replacing the action $\mathcal{M} \longmapsto \mathcal{M} \otimes -$ by $\mathcal{M} \longmapsto - \otimes \mathcal{M}$, we obtain the right canonical action of the monoidal category $\mathcal{A}^\sim$ on $\mathcal{A}$.

1.4.3. Operads and algebras over operads. Fix a symmetric additive monoidal category $\mathcal{C}^\sim = (\mathcal{C}, \otimes, \mathbf{1}, a, l, r; \beta)$ (here β is a symmetry, $\beta_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$). Let $\mathbf{S}$ denote the category whose objects are sets $[n] = \{1, ..., n\}$, $n \geq 1$, and $[0] = \emptyset$ and morphisms are bijections. Denote by $\mathcal{C}^{\mathbf{S}}$ the category of functors $\mathbf{S}^{op} \longrightarrow \mathcal{C}$. In other words, objects of $\mathcal{C}^{\mathbf{S}}$ are collections $\mathcal{M} = (M(n)|\ n \geq 0)$, where M_n is an object of $\mathcal{C}$ with an action of the symmetric group S_n.

The category $\mathcal{C}^{\mathbf{S}}$ acts on the category $\mathcal{C}$ by *polynomial functors*:

$$M : V \longmapsto M(V) = \bigoplus_{n\geq 0} M(n) \otimes_{S_n} V^{\otimes n} \tag{1}$$

The composition of polynomial functors is again a polynomial functor. This defines a tensor product, $\odot$, on $\mathcal{C}^{\mathbf{S}}$ called the *plethism product.* We denote the corresponding monoidal category $(\mathcal{C}^{\mathbf{S}}, \odot, \mathbf{1_S})$ by $\mathcal{C}^{\sim\mathbf{S}}$. Here $\mathbf{1_S}$ is the unit object $\mathbf{1_S}$. One can see that $\mathbf{1_S}(n) = 0$ if $n \neq 1$ and $\mathbf{1_S}(1)$ is the unit object of the category $\mathcal{C}^{\sim}$. Thus we have an action $\mathfrak{C}$ of the monoidal category $\mathcal{C}^{\mathbf{S}}$ on the category $\mathcal{C}$.

Algebras in the monoidal category $\mathcal{C}^{\mathbf{S}}$ are called *operads*, or $\mathcal{C}^{\sim}$*-operads.* For each operad $\mathcal{R}$, the corresponding category of quasi-coherent sheaves on $\mathfrak{Sp}(\mathcal{R})$ is usually called *the category of* $\mathcal{R}$*-algebras.*

1.5. Affine schemes and relative affine schemes. We shall use here the terminology of [KR3]. Fix an action $\mathfrak{C} = (\mathcal{A}^{\sim}, C, \Phi^{\sim})$. Let $\mathbf{Sp}_{\mathfrak{C}}$ denote the functor $(Alg\mathcal{A}^{\sim})^{op} = \mathbf{Aff}_{\mathcal{A}^{\sim}} \longrightarrow Cat^{op}/\mathcal{C}$ which assigns to any affine scheme $\mathbf{X} = \mathfrak{Sp}(R^{\sim})$ in $\mathcal{A}^{\sim}$ the object $(\mathbf{Sp}_{\mathcal{C}}(R^{\sim}), f_{\mathcal{C}})$ of $Cat^{op}/\mathcal{C}$. Here $\mathbf{Sp}_{\mathcal{C}}(R^{\sim})$ is the object of Cat^{op} corresponding to the category $R^{\sim} - mod_{\mathcal{C}}$, and $f_{\mathcal{C}}$ is the canonical morphism having the forgetful functor $R^{\sim} - mod_{\mathcal{C}} \longrightarrow \mathcal{C}$ as a direct image functor. To any affine morphism $\mathfrak{Sp}(R^{\sim}) \longrightarrow \mathfrak{Sp}(S^{\sim})$, the functor $\mathbf{Sp}_{\mathfrak{C}}$ assigns the morphism direct image of which is the corresponding pull-back functor. There is a canonical morphism from the fibered category $\mathcal{F}^{\mathfrak{C}} \xrightarrow{\pi} (Alg\mathcal{A}^{\sim})^{op}$ to the fibered category $Cat^{op}/\mathcal{C}$ which assigns to any object $(R^{\sim}, M)$ of $\mathcal{F}^{\mathfrak{C}}$ the object $(\mathbf{Sp}_{\mathfrak{C}}(R^{\sim}), M)$.

1.5.1. $\mathfrak{C}$-affine morphisms. Morphisms of the form $\mathbf{Sp}_{\mathfrak{C}}(\psi)$ will be called $\mathfrak{C}$-*affine*, or, loosely, *affine*. They are not usually affine in the sense of [R1], 2.3, as one can see taking as $\mathcal{A}^{\sim}$ the monoidal category $End_2^{\sim} C$ and as $\Phi^{\sim}$ the identical monoidal functor. Let $\mathfrak{C}$ be the left canonical action $(\mathcal{A}^{\sim}, \mathfrak{L}^{\sim})$ of a monoidal category $\mathcal{A}^{\sim}$ satisfying the conditions of 1.4.2.

1.5.2. Lemma. *Let $\mathcal{R} = (R, \mu)$ and $\mathcal{S} = (S, \nu)$ be algebras in $\mathcal{A}^{\sim}$ and ψ an algebra morphism $\mathcal{S} \longrightarrow \mathcal{R}$. The morphism $\mathfrak{Sp}_{\mathfrak{C}}(\psi) : \mathcal{R} - mod \longrightarrow \mathcal{S} - mod$ is affine iff the inner hom $\mathbf{Hom}_{\mathcal{S}}(R, M)$ exists for all left $\mathcal{S}$-modules M.*

Proof. Suppose the inner hom $\mathbf{Hom}_{\mathcal{S}}(R, M)$ exists for all left $\mathcal{S}$-modules M. It has a canonical left action of $\mathcal{R}$. For any left $\mathcal{R}$-module L, we have a canonical functorial isomorphism

$$Hom_{\mathcal{R}}(L, \mathbf{Hom}_{\mathcal{S}}(\psi_*(\mathcal{R}), M)) \xrightarrow{\sim} Hom_{\mathcal{S}}(\psi_*(\mathcal{R} \otimes_{\mathcal{R}} L), M) = Hom_{\mathcal{S}}(\psi_*(L), M) \tag{1}$$

which shows that the functor $M \longmapsto \mathbf{Hom}_{\mathcal{S}}(\psi_*(\mathcal{R}, M)$ is a right adjoint to the direct image functor ψ_*. The isomorphism (1) can be regarded as a definition of $\mathbf{Hom}_{\mathcal{S}}(\psi_*(\mathcal{R}, M)$. In particular, the existence of a right adjoint to ψ_* implies that of $\mathbf{Hom}_{\mathcal{S}}(\psi_*(\mathcal{R}), \mathbf{M})$. ■

1.5.2.1. Note. In the case $\mathcal{A}^{\sim} = k - mod^{\sim} = (k - mod, \otimes_k, k)$ for a commutative ring k, algebras in $\mathcal{A}^{\sim}$ are k-algebras and, given an algebra morphism $\mathcal{S} \longrightarrow \mathcal{R} = (R, \mu)$ and an $\mathcal{S}$-module M, the inner hom, $\mathbf{Hom}_{\mathcal{S}}(R, M)$, coincides with $Hom_{\mathcal{S}}(R, M)$.

1.6. The change of action. A morphism from an action $\mathfrak{C}' = (\mathcal{A}'^{\sim}, \mathcal{C}', \Phi'^{\sim})$ to an action $\mathfrak{C} = (\mathcal{A}^{\sim}, \mathcal{C}, \Phi^{\sim})$ is a triple $(\Psi^{\sim}, g^*; \lambda)$, where $\Psi^{\sim}$ is a monoidal functor $\mathcal{A}'^{\sim} \longrightarrow \mathcal{A}^{\sim}$ and g^* an inverse image functor of a morphism $\mathcal{C}' \longrightarrow \mathcal{C}$, and λ a functorial isomorphism $\mathcal{C}(M, \Phi\Psi(E')(L)) \xrightarrow{\sim} \mathcal{C}'(g^*(M), \Phi'(E')(g^*(L)))$. We leave to the reader defining the composition of morphisms. The main example is as follows.

1.6.1. Fix an action $\mathfrak{C} = (\mathcal{A}^{\sim}, C, \Phi^{\sim})$ such that the category $\mathcal{C}$ has cokernels of reflexive pairs of arrows. Let $R^{\sim} = (R, \mu)$ be an algebra in the monoidal category $\mathcal{A}^{\sim}$. We assume that for any $M \in Ob\mathcal{A}$, the functors $M \otimes -$ and $- \otimes M$ preserve cokernels of reflexive pairs of arrows. In this case we have a well defined monoidal category $R^{\sim} - bimod^{\sim} = (R^{\sim} - bimod, \otimes_{R^{\sim}} -, R^{\sim})$. Moreover, there is a naturally defined action of the monoidal category $R^{\sim} - bimod^{\sim}$ on the category $\Phi^{\sim}(R^{\sim}) - mod_C$.

In fact, for any $R^{\sim}$-bimodule $\mathcal{M} = (m', M, m'')$ and for any $\Phi^{\sim}(R^{\sim})$-module (L, ν), the action of $\mathcal{M}$ on (L, ν) assigns to (L, ν) the $\Phi^{\sim}(R^{\sim})$-module $\Phi^{\sim}(\mathcal{M}) \otimes_{\Phi^{\sim}(R^{\sim})} L$. The latter is the cokernel of the pair of arrows $\Phi^{\sim}(\mathcal{M})\Phi^{\sim}(R^{\sim})(L) \rightrightarrows \Phi^{\sim}(\mathcal{M})(L)$, where one arrow is the composition of the constraint

$$\phi_{\mathcal{M},R}(L) : \Phi^{\sim}(\mathcal{M})\Phi^{\sim}(R^{\sim})(L) \longrightarrow \Phi^{\sim}(\mathcal{M} \otimes R^{\sim})(L)$$

and the morphism $\Phi^{\sim}(m'')(L)$ induced by the right action of $R^{\sim}$ on $\mathcal{M}$. The other arrow is the image $\Phi^{\sim}(\mathcal{M})(\nu)$ of the left action on L.

We have the forgetful functor $R^{\sim} - mod \xrightarrow{g_*} C = \mathcal{A}$ with a left adjoint $L \overset{g^*}{\longmapsto} R^{\sim} \otimes L$ and the functor $\mathcal{A} \xrightarrow{\Psi} R^{\sim} - bimod$, $M \longmapsto R^{\sim} \otimes M \otimes R^{\sim}$. The latter extends canonically to a monoidal functor $\mathcal{A}^{\sim} \xrightarrow{\Psi^{\sim}} R^{\sim} - bimod^{\sim}$.

1.7. The fibered category of bimodules. Fix a monoidal category $\mathcal{A}^{\sim}$ such that for all $M \in Ob\mathcal{A}$, the functors $M \otimes -$ preserve colimits of reflexive pairs of arrows. We associate with $\mathcal{A}^{\sim}$ a cofibered category $Bi^{\mathcal{A}^{\sim}} \xrightarrow{\pi} (Alg\mathcal{A}^{\sim})^{op} = \mathbf{Aff}_{\mathcal{A}^{\sim}}$ defined as follows. Objects of the category $Bi^{\mathcal{A}^{\sim}}$ are pairs $(R^{\sim}, M)$, where $R^{\sim} = (R, \mu)$ is an algebra in $\mathcal{A}^{\sim}$, M is an $R^{\sim}$-bimodule. A morphism from $(R^{\sim}, M)$ to $(S^{\sim}, L)$ is a pair (ψ, ξ), where ψ is an algebra morphism $R^{\sim} \longrightarrow S^{\sim}$, ξ is an $S^{\sim}$-bimodule morphism $M \longrightarrow \psi_*(L)$, where ψ_* is the pull-back functor $S^{\sim} - bimod \longrightarrow R^{\sim} - bimod$ induced by the algebra morphism ψ. The composition is defined in a standard way. The projection π assigns to any object $(R^{\sim}, M)$ the algebra $R^{\sim}$ and to any morphism (ψ, ξ) the algebra morphism ψ. It follows from the assumptions (and [R2, 1.0.6.1]) that $Bi^{\mathcal{A}^{\sim}} \xrightarrow{\pi} \mathbf{Aff}_{\mathcal{A}^{\sim}}$ is a bifibered category: the functors ψ_* have left adjoints. For any $R^{\sim} \in ObAlg\mathcal{A}^{\sim}$, the fiber $R^{\sim} - bimod$ over $\mathfrak{Sp}(R^{\sim})$ will be regarded as a monoidal category with respect to $\otimes_{R^{\sim}}$. For any algebra morphism $\psi : R^{\sim} \longrightarrow S^{\sim}$, the corresponding direct image functor ψ_* is a monoidal functor (in a week sense). This monoidal structure is inherited by the fibered category $\mathfrak{Qcoh}(Bi^{\mathcal{A}^{\sim}})$ of quasi-coherent morphisms of $Bi^{\mathcal{A}^{\sim}}$ (cf. 1.0.1) and, therefore, by the category $Qcoh(Bi^{\mathcal{A}^{\sim}})$ of quasi-coherent presheaves of bimodules (see 1.0.2).

Fix an action $\mathfrak{C} = (\mathcal{A}^{\sim}, \mathcal{C}, \Phi^{\sim})$ such that the category $\mathcal{C}$ has cokernels of reflexive pairs of arrows. It follows from 1.6 that the action $\Phi^{\sim}$ of the monoidal category $\mathcal{A}^{\sim}$ on the category $\mathcal{C}$ induces an action of the fibered category $Bi^{\mathcal{A}^{\sim}}$ on the fibered category $\mathcal{F}^{\mathfrak{C}}$ defined in 1.4. This action induces an action of $\mathfrak{Qcoh}(Bi^{\mathcal{A}^{\sim}})$ on the corresponding fibered category $\mathfrak{Qcoh}(\mathcal{F}^{\mathfrak{C}})$ of quasi-coherent morphisms.

1.8. Fibered categories and quasi-coherent presheaves associated with a functor from $\mathbf{Aff}_{\mathcal{A}^\sim}$. Let G be a functor $\mathbf{Aff}_{\mathcal{A}^\sim} = (Alg\mathcal{A}^\sim)^{op} \longrightarrow \mathcal{S}$. For any $X \in Ob\mathcal{S}$, we have a natural functor $G/X \longrightarrow \mathbf{Aff}_{\mathcal{A}^\sim}$ and the corresponding pull-back, $\mathcal{F}^{\mathfrak{C},G/X}$ (see [R2, 1.0.4]), of the fibered category $\mathcal{F}^{\mathfrak{C}}$ (cf. 1.4). In particular, we have the category $Qcoh(\mathcal{F}^{\mathfrak{C},G/X})$ of quasi-coherent presheaves on X (cf. 1.0.4). Since the construction is functorial in X, it gives a rise to a fibered category over $\mathcal{S}$ having $Qcoh(\mathcal{F}^{\mathfrak{C},G/X})$ as a fiber at an object X. Similarly, one can define the monoidal *category of quasi-coherent bimodules* on each object X of the category S as the category of quasi-coherent presheaves of the fibered category $\pi_{G/X} : Bi^{\mathcal{A}^\sim,G/X} \longrightarrow \mathbf{Aff}_{\mathcal{A}^\sim}$ obtained as the pull-back by the canonical functor $G/X \longrightarrow \mathbf{Aff}_{\mathcal{A}^\sim}$.

1.8.1. Quasi-coherent presheaves on a presheaf of sets. The case of a particular interest is when the functor G is the canonical embedding

$$h : \mathbf{Aff}_{\mathcal{A}^\sim} \longrightarrow Fun(\mathbf{Aff}_{\mathcal{A}^\sim}^{op}, \mathbf{Sets}), \quad X \longmapsto \mathbf{Aff}_{\mathcal{A}^\sim}(-, X)$$

of the category $\mathbf{Aff}_{\mathcal{A}^\sim}$ to the category of presheaves of sets on $\mathbf{Aff}_{\mathcal{A}^\sim}$. We have then the notion of the category $PQcoh_X$ of quasi-coherent presheaves on a presheaf of sets X and the monoidal category $Qcoh(Bi_X)$ of quasi-coherent bimodules on X together with an action of $Qcoh(Bi_X)$ on $Qcoh_X$.

2. Examples of affine schemes.

2.1. Vector fiber of an object. Let $\mathcal{A}^\sim$ be a monoidal category, and let $\mathcal{F}$ denote the forgetful functor

$$Alg\mathcal{A}^\sim \longrightarrow \mathcal{A}, \quad (R, \mu) \longmapsto R.$$

2.1.1. Lemma. *Let $E \in Ob\mathcal{A}$ be such that the functors $E \otimes - : \mathcal{A} \longrightarrow \mathcal{A}$ and $- \otimes E$ preserve countable coproducts. And let there exist a coproduct $\coprod_{n\geq 0} E^{\otimes n}$. Then the functor $\mathcal{A}(E, \mathcal{F}-) : Alg\mathcal{A}^\sim \longrightarrow \mathbf{Sets}$ is corepresentable.*

Proof. Denote by $T(E)$ the algebra $(\coprod_{n\geq 0} E^{\otimes n}, \mu_E)$, where the multiplication μ_E is given by the identical morphisms $E^{\otimes n} \otimes E^{\otimes m} \longrightarrow E^{\otimes(m+n)}$. For any algebra (R, μ) in $\mathcal{A}^\sim$, the natural map $\mathcal{A}(E, R) \longrightarrow Alg\mathcal{A}^\sim(T(E), (R, \mu))$ is a functorial isomorphism. ∎

2.1.2. Corollary. *Assume that $\mathcal{A}$ has countable coproducts and $\otimes : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ preserves countable coproducts in both arguments. Then the functor*

$$\mathcal{F} : Alg\mathcal{A}^\sim \longrightarrow \mathcal{A}, \quad (R, \mu) \longmapsto R$$

has a left adjoint.

2.1.3. Corollary. *Let $E_1, E_2 \in Ob\mathcal{A}$ be such that the functors $E_i \otimes -$ and $- \otimes E_i$, $i = 1, 2$, preserve countable coproducts. Assume that $T(E_1)$, $T(E_2)$, and $T(E_1 \coprod E_2)$ exist. Then there is a natural algebra isomorphism $T(E_1 \coprod E_2) \simeq T(E_1) \star T(E_2)$.*

2.1.4. Vector fibers. Let $E \in Ob\mathcal{A}$ satisfy the conditions of 2.1.1. The *vector fiber*, $\mathbf{V}(E)$, *associated with* E is the affine scheme $\mathfrak{Sp}_{\mathcal{A}^\sim}(T(E))$.

It follows that if the objects E_1, E_2 of $\mathcal{A}$ satisfy the assumptions of 2.1.3, then the product $\mathbf{V}(E_1) \times \mathbf{V}(E_2)$ exists and is isomorphic to $\mathbf{V}(E_1 \coprod E_2)$.

2.1.5. Proposition. *Let $E, E' \in Ob\mathcal{A}$ satisfy the assumptions of 2.1.1, and let E' $\phi : E \longrightarrow E'$ be a strict epimorphism. If $\Phi(E')$ is right exact, then the corresponding morphism $\mathbf{V}(E') \longrightarrow \mathbf{V}(E)$ is a closed immersion.*

Proof. Since $\phi : E \longrightarrow E'$ is a strict epimorphism and the functor $E \longmapsto T(E)$ is right exact (compatible with all colimits as any functor having a right adjoint), the corresponding algebra morphism $T(\phi) : T(E) \longrightarrow T(E')$ is a strict epimorphism, hence the assertion. ■

2.2. Quasi-coherent modules on a vector fiber. Fix an action $\mathfrak{C} = (\mathcal{A}^{\sim}, C, \Phi)$. Let an object E of $\mathcal{A}$ is such that $\Phi^{\sim}(T(E))$ is naturally isomorphic to $T(\Phi(E))$ (which is the case if the functors $E \otimes -$ and Φ are compatible with countable coproducts). Then the category $Qcoh_{\mathbf{V}(E)/\mathfrak{C}}$ of quasi-coherent $\mathbf{V}(E)/\mathfrak{C}$-modules is isomorphic to the category of actions of $\Phi(E)$: its objects are pairs $(M, \Phi(E)(M) \xrightarrow{\xi} M)$, and morphisms and their composition are defined in an obvious way.

2.3. Admissible pairs of objects. Finite objects. We say that a pair (M, L) of objects of $\mathcal{C}$ is $\mathfrak{C}$-*admissible* if the functor

$$\mathcal{A} \longrightarrow \mathbf{Sets}, \quad F \longmapsto C(M, \Phi(F)(L)) \tag{1}$$

is corepresentable, i.e. there is an object $L^{\wedge}M$ of $\mathcal{A}$ (defined uniquely up to isomorphism) and a functorial (in F) isomorphism $\mathcal{C}(M, \Phi(F)(L)) \simeq \mathcal{A}(L^{\wedge}M, F)$. We call an object L of the category $\mathcal{C}$ *finite* if (M, L) is an admissible pair for any $M \in ObC$.

2.3.1.1. Example. Let $\mathcal{C} = R - mod$ for an associative ring R, $\mathcal{A}^{\sim}$ is the monoidal category of R-bimodules. Then finite objects of the category $\mathcal{C}$ are projective R-modules of finite type. If L is a projective R-module of finite type and M an arbitrary left R-module, then $L^{\wedge}M \simeq M \otimes L^{\vee}$, where $L^{\vee}$ denotes the dual to L (right) R-module: $L^{\vee} \simeq R - mod(L, R)$. In particular, if $L = R$, then $L^{\wedge}M$ is isomorphic to the R-bimodule $M \otimes R_r$, where R_r is R regarded as a right R-module.

2.3.1.2. Example. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{A}^{\sim}, \Phi)$ be the left standard action of $\mathcal{A}^{\sim}$ on $\mathcal{A}$ (cf. 1.4.2.). Let $L, M \in Ob\mathcal{A}$. By definition, the pair (M, L) is admissible iff the functor $\mathcal{A} \longrightarrow \mathbf{Sets}$, $F \longmapsto C(M, F \otimes L))$ is corepresentable. Suppose that L is a *finite object*, i.e. there exists an object $L^!$ such that the functor $L^! \otimes -$ is a right adjoint to $L \otimes -$. Then $F \otimes L \simeq \mathbf{Hom}(L^!, F)$, hence

$$\mathcal{C}(M, F \otimes L)) \simeq C(M, \mathbf{Hom}(L^!, F)) \simeq C(M \otimes L^!, F)$$

functorially in F. In other words, L is finite object of the monoidal category $\mathcal{A}^{\sim}$ iff it is a finite object (in the sense of 2.3.1) of the left base.

2.3.2. Lemma. *Let $(\Psi^{\sim}, g^*; \lambda) : \mathfrak{C}' = (\mathcal{A}'^{\sim}, C', \Phi'^{\sim}) \longrightarrow \mathfrak{C} = (\mathcal{A}^{\sim}, C, \Phi^{\sim})$ be a morphism (cf. 3.6) such that the functor Ψ has a left adjoint, Ψ^*. Then for any $\mathfrak{C}$-admissible pair (L, M), the pair $(g^*(L), g^*(M))$ is $\mathfrak{C}'$-admissible.*

Proof. In fact, we have functorial isomorphisms

$$\mathcal{A}'(\Psi^*(L^\wedge M), E') \simeq \mathcal{A}(L^\wedge M, \Psi(E')) \simeq$$

$$\mathcal{C}(M, \Phi\Psi(E')(L)) \simeq C'(g^*(M), \Phi'(E')(g^*(L)))$$

Hence the assertion. ∎

2.3.3. Corollary. *Let $\mathcal{A}^\sim$ (resp. $\mathcal{A}'^\sim$) be the category of continuous (i.e. having a right adjoint) endofunctors of a category $\mathcal{C}$ (resp. $\mathcal{C}'$). Let $\mathfrak{C} = (\mathcal{A}^\sim, C, \Phi^\sim)$ and $\mathfrak{C}' = (\mathcal{A}'^\sim, C', \Phi'^\sim)$ be the corresponding actions, and let $g : C' \longrightarrow C$ be a continuous morphism such that g_* has a right adjoint. Then for any $\mathfrak{C}$-admissible pair (L, M), the pair $(g^*(L), g^*(M))$ is $\mathfrak{C}'$-admissible. In particular, $(g^*(L), g^*(M))$ is admissible for any affine morphism g.*

Proof. 1) The morphism g induces a morphism $(\Psi_g^\sim, g) : \mathfrak{C}' = (\mathcal{A}'^\sim, C', \Phi'^\sim) \longrightarrow \mathfrak{C} = (\mathcal{A}^\sim, C, \Phi^\sim)$, and the functor Ψ_g maps every continuous functor $F' : C' \longrightarrow C'$ (– an object of $\mathcal{A}'$) to $g_*F'g^*$. It has a left adjoint, Ψ_g^*, which maps any continuous functor $F : C \longrightarrow C$ to g^*Fg_*. By 2.3.2, the pair $(g^*(L), g^*(M))$ is admissible and (by the argument of 2.3.2) $(g^*(L))^\wedge(g^*(M)) = \Psi_g^*(L^\wedge M) = g^*(L^\wedge M)g_*$.

2) One can give an independent (on 2.3.2) argument. For any continuous functor $F : C \longrightarrow C$, the functor g_*Fg^* is continuous, and we have canonical isomorphisms

$$\mathcal{C}'(g^*(L), Fg^*(M)) \simeq C(L, g_*Fg^*(M)) \simeq \mathcal{A}(L^\wedge M, g_*Fg^*(M)) \simeq \mathcal{A}'(g^*(L^\wedge M)g_*, F)$$

Since these isomorphisms are functorial in F, it follows that the pair $(g^*(L), g^*(M))$ is admissible and $g^*(L)^\wedge g^*(M) \simeq g^*(L^\wedge M)g_*$. ∎

2.4. Vector fibers associated with admissible pairs. Adjoining maps and imposing relations. The following proposition is a generalization of Theorem 3.1 in [B].

2.4.1. Proposition. *(a) Let (M, P) be an admissible pair of objects of $\mathcal{C}$. Then the functor*

$$Alg\mathcal{A}^\sim \longrightarrow Sets, \quad S \longmapsto Hom_S(S(M), S(P)), \tag{1}$$

is (representable by) an affine $\mathcal{A}^\sim$-scheme.

(b) More generally, given a family of admissible pairs of $\{(M_i, P_i),\ i \in J\}$ of objects of $\mathcal{C}$, for which there exists a coproduct of $\{P_i^\wedge M_i \mid i \in J\}$, there exists an algebra B and a uniquely defined family of morphisms $f_i : B(M_i) \longrightarrow B(P_i)$ with the universal property: for any algebra T and any family of T-module morphisms $g_i : T(M_i) \longrightarrow T(P_i),\ i \in J$, there exists a unique algebra morphism $B \longrightarrow T$ such that $g_i = T \otimes_B f_i$ for all $i \in J$.

Proof. (a) Let ψ_* denote the forgetful functor $S - mod_C \longrightarrow C$. We have the following isomorphisms functorial in S:

$$Hom_S(S(M), S(P)) \simeq \mathcal{A}(M, \psi_*(S(P))) \simeq \mathcal{A}(P^\wedge M, \psi_*(S)) \simeq$$

$$Alg\mathcal{A}^\sim(\mathbf{T}(P^\wedge M), S) \simeq \mathbf{Aff}_{\mathcal{A}^\sim}(\mathfrak{Sp}(S), \mathbf{V}(P^\wedge M)),$$

i.e. the functor (1) is representable by the vector fiber $\mathbf{V}(P^\wedge M)$.

(b) By (a), we have functorial isomorphisms:

$$\prod_{i\in J} Hom_S(S(M_i), S(P_i)) \simeq \prod_{i\in J} \mathcal{A}(P_i^\wedge M_i, \psi_*(S)) \simeq \mathcal{A}(\coprod_{i\in J}(P_i^\wedge M_i), \psi_*(S))$$

$$\simeq Alg\mathcal{A}^\sim(\mathbf{T}(\coprod_{i\in J} P_i^\wedge M_i), S) \simeq \mathbf{Aff}_{\mathcal{A}^\sim}(\mathfrak{Sp}(S), \mathbf{V}(\coprod_{i\in J} P_i^\wedge M_i)),$$

hence the assertion. ■

The next assertion is a generalization of Theorem 3.2 in [B].

2.4.2. Proposition. (Imposing relations) *(a) Let (M,P) be an admissible pair of objects of $\mathcal{C}$, and $M \underset{g}{\overset{f}{\rightrightarrows}} P$ a pair of arrows. There exists a unique up to isomorphism algebra R such that $R(f) = R(g)$ and universal for the property: given any algebra S with $S(f) = S(g)$, there exists a unique algebra morphism $R \longrightarrow S$.*

(b) More generally, given a family of pairs of morphisms, $M_i \underset{g_i}{\overset{f_i}{\rightrightarrows}} P_i$, $i \in J$, such that all pairs (M_i, P_i) are admissible and there exists a coproduct of $\{P_i^\wedge M_i |\ i \in J\}$, there exists an algebra R universal for the property $R(f_i) = R(g_i)$ for all $i \in J$.

Proof. (a) Since $Hom_S(S(M), S(P)) \simeq Alg\mathcal{A}^\sim(\mathbf{T}(P^\wedge M), S)$ (see the argument of 2.4.1(a)), the universal algebra R is a cokernel of the pair of algebra morphisms

$$\mathbf{T}(P^\wedge M) \underset{g^a}{\overset{f^a}{\rightrightarrows}} \mathbf{1}$$

corresponding to the morphisms f and g.

(b) It follows from the functorial isomorphism

$$\prod_{i\in J} Hom_S(S(M_i), S(P_i)) \simeq Alg\mathcal{A}^\sim(\mathbf{T}(\coprod_{i\in J}(P_i^\wedge M_i), S)$$

(see the argument of 2.4.1(b)) that the universal algebra R is a cokernel of the pair of algebra morphisms

$$\mathbf{T}(\coprod_{i\in J} P_i^\wedge M_i) \underset{g^a}{\overset{f^a}{\rightrightarrows}} \mathbf{1}$$

corresponding to the family of pairs of morphisms $\{f_i, g_i,\ i \in J\}$. ■

2.5. The group scheme GL_V. Fix objects V, W of the category $\mathcal{C}$. We have a functor

$$Iso_{V,W} : \mathbf{Aff}^{op}_{\mathcal{A}^\sim} \longrightarrow \mathbf{Sets}, \quad \mathbf{X} \longmapsto Aut_{Qcoh_{\mathbf{X}/\mathfrak{C}}}(f^*_{\mathbf{X}}(V), f^*_{\mathbf{X}}(W)) \qquad (1)$$

2.5.1. Proposition. *Let V, W be objects of the category $\mathcal{C}$ such that the pairs (V,W), (W,V), (V,V), and (W,W) are admissible. Then the functor $Iso_{V,W}$ is representable by an affine $\mathcal{A}^\sim$-scheme.*

Proof. (a) Let (V,W) and (W,Z) be admissible pairs of objects of $\mathcal{C}$. Consider the functor $G_{V,W,Z} : \mathbf{Aff}^{op}_{\mathcal{A}^\sim} \longrightarrow \mathbf{Sets}$ which assigns to any affine scheme $\mathbf{X} = \mathfrak{Sp}_{\mathcal{A}^\sim}(R)$ the pair of morphisms $f^*_{\mathbf{X}}(V) \xrightarrow{v} f^*_{\mathbf{X}}(W) \xrightarrow{w} f^*_{\mathbf{X}}(Z)$. The functor $G_{V,W,Z}$ is representable by $\mathbf{V}(W^\wedge V) \times \mathbf{V}(Z^\wedge W)$.

(b) Let (V,W), (W,V), and (V,V) be admissible pairs of objects of $\mathcal{C}$. Denote by $\Phi_{V,W}$ the subfunctor of the functor $G = G_{V,W,V}$ which assigns to any affine scheme $\mathbf{X}$ the subset of all pairs $(v,u) \in G_{V,W,V}(\mathbf{X})$ such that $u \circ v = id$. We claim that the functor $\Phi_{V,W}$ is representable by an affine scheme.

Consider two functorial maps

$$G_{V,W,V}(\mathbf{X}) \underset{\beta_{\mathbf{X}}}{\overset{\alpha_{\mathbf{X}}}{\rightrightarrows}} Qcoh_{\mathbf{X}/\mathfrak{C}}(f^*_{\mathbf{X}}(V), f^*_{\mathbf{X}}(V))$$

defined by $\alpha_{\mathbf{X}}(v,u) = u \circ v$ and $\beta_{\mathbf{X}}(v,u) = id_{f^*_{\mathbf{X}}(V)}$. Since the pair (V,V) is admissible,

$$Qcoh_{\mathbf{X}/\mathfrak{C}}(f^*_{\mathbf{X}}(V), f^*_{\mathbf{X}}(V)) \simeq \mathbf{Aff}_{\mathcal{A}^\sim}(\mathbf{X}, \mathbf{V}(V^\wedge V))$$

Let α', β' denote the corresponding morphisms

$$\mathbf{V}(W^\wedge V \coprod V^\wedge W) \simeq \mathbf{V}(W^\wedge V) \times \mathbf{V}(V^\wedge W) \longrightarrow \mathbf{V}(V^\wedge V)$$

The functor $\Phi_{V,W}$ is representable by the kernel of the pair of morphisms α', β'.

(c) Let (V,W), (W,V), (V,V), and (W,W) be admissible pairs of objects of $\mathcal{C}$. And let $\mathbf{X}$ be an affine $\mathcal{A}^\sim$-scheme. The set $Iso(f^*_{\mathbf{X}}(V), f^*_{\mathbf{X}}(W))$ is naturally isomorphic to the set of pairs (u,v), where $f^*_{\mathbf{X}}(V) \xrightarrow{u} f^*_{\mathbf{X}}(W)$ and $f^*_{\mathbf{X}}(W) \xrightarrow{v} f^*_{\mathbf{X}}(V)$ are quasi-coherent module morphisms such that $u \circ v = id$ and $v \circ u = id$. So $Iso(f^*_{\mathbf{X}}(V), f^*_{\mathbf{X}}(W))$ is identified with the fiber product of the morphisms

$$\Phi_{V,W}(\mathbf{X}) \xrightarrow{\phi} G_{V,W,V}(\mathbf{X}) \xleftarrow{\psi} \Phi_{W,V}(\mathbf{X}),$$

where $\phi = \phi_{V,W}$ is the natural embedding, ψ is the composition of the natural embedding $\phi_{W,V} : \Phi_{W,V} \longrightarrow G_{W,V,W}$ and the functorial isomorphism

$$G_{W,V,W} \longrightarrow G_{V,W,V}, \quad (u,v) \longmapsto (v,u)$$

Thus the map $Iso_{V,W}$ is extended to the functor which is the kernel of the pair of functor morphisms (ϕ, ψ). Since both source and target of the arrows ϕ, ψ, the functors $\Phi_{V,W}$ and $G_{V,W,V}$, are representable, the functor $Iso_{V,W}$ is representable too. ∎

2.5.2. Corollary. *Let V be an object of $\mathcal{C}$ such that the pair (V,V) is admissible. The functor Aut_V is representable by an affine $\mathcal{A}^\sim$-scheme in groups.*

The following proposition is a generalization of Theorem 3.3 in [B].

2.6. Proposition. *Let B and D be small categories, G a functor $B \longrightarrow C$ which is a bijection on objects. Let $F : B \longrightarrow \mathcal{C}$ be a functor having the following property:*

(†) *the pair of objects* $(F(X), F(Y))$ *is admissible if* $D(G(X), G(Y))$ *is not empty.*

Suppose the category $\mathcal{A}$ *has small colimits. Then there exist an algebra* $R_{F,G}$ *and a functor* $H_{F.,G} : D \longrightarrow R_{F,G} - mod_{\mathcal{C}}$ *which make the following diagram commute*

$$\begin{array}{ccc} B & \xrightarrow{F} & \mathcal{C} \\ G\big\downarrow & & \big\downarrow R_{F,G}\otimes \\ D & \xrightarrow{H_{F,G}} & R_{F,G} - mod_{\mathcal{C}} \end{array} \tag{1}$$

and which are universal for this property.

Proof. (a) Suppose first that D coincides with the image of G. Then applying 2.4.2(b), we obtain an algebra R (which is a quotient of $\mathbf{1}$) and a functor $H : D \longrightarrow R - mod_{\mathcal{C}}$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{F} & \mathcal{C} \\ G\big\downarrow & & \big\downarrow R\otimes \\ D & \xrightarrow{H} & R - mod_{\mathcal{C}} \end{array}$$

commutes and the pair (H, R) is universal for this property.

(b) If $G : HomB \longrightarrow HomD$ is not surjective, (a) gives a reduction to the case when B is a subcategory of D with the same set of objects and G is the inclusion functor. We apply 2.4.1(b) to obtain all morphisms needed, and then apply 2.4.2(b) for relations. Details are left to the reader. ∎

It is convenient to have a slight modification of 2.6:

2.6.1. Proposition. *Let* B *and* D *be small categories, and let* $G : B \longrightarrow D$ *be a functor injective on objects and such that every object of* D *is isomorphic to an object of the image of* G*. Suppose the category* $\mathcal{A}$ *has small colimits. Then for any functor* $F : B \longrightarrow \mathcal{C}$ *satisfying the condition* (†) *of 2.6, there exist an algebra* $R_{F,G}$ *and a functor* $H_{F.,G} : C \longrightarrow R_{F,G} - mod_{\mathcal{C}}$ *which make the following diagram commute*

$$\begin{array}{ccc} B & \xrightarrow{F} & \mathcal{C} \\ G\big\downarrow & & \big\downarrow R_{F,G}\otimes \\ D & \xrightarrow{H_{F,G}} & R_{F,G} - mod_{\mathcal{C}} \end{array} \tag{1}$$

and which are universal for this property. The functor $H_{F,G}$ *is defined uniquely up to isomorphism.*

Proof. Let D' be the full subcategory of D defined by $ObD' = G(ObB)$, and let $G' : B \longrightarrow D'$ be the corestriction of the functor G to D'. By 2.6, there exist an algebra $R_{F,G'}$ and a functor $H_{F,G'} : D' \longrightarrow \mathcal{C}_{F,G'}$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{F} & \mathcal{C} \\ G'\big\downarrow & & \big\downarrow R_{F,G'}\otimes \\ D' & \xrightarrow{H_{F,G'}} & R_{F,G'} - mod_{\mathcal{C}} \end{array} \tag{2}$$

commutes and which are universal for this property. The embedding $J_* : D' \longrightarrow D$ is an equivalence of categories. Let $J^* : C \longrightarrow D'$ denote a left adjoint (a quasi-inverse) to J_* such that $J^* \circ J_* = Id_{D'}$. Then $H_{F,G} = H_{F,G'} \circ J^*$ is the desired universal functor. ■

2.7. Localizations and universal localizations. We have the following corollary of Proposition 2.6:

2.7.1. Proposition. *Let $F : B \longrightarrow \mathcal{C}$ be a functor satisfying the condition*

($\dagger^*$) *The pair $(F(X), F(Y))$ is admissible if $B(X,Y)$ is non-empty.*

Let Σ be a class of morphisms of the category B and $G = Q_\Sigma$ the localization functor $B \longrightarrow \Sigma^{-1}B$. Then there exists a unique (up to isomorphism) algebra R_Σ and a unique functor $F_\Sigma : \Sigma^{-1}B \longrightarrow R_\Sigma - mod_{\mathcal{C}}$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{F} & \mathcal{C} \\ {\scriptstyle Q_\Sigma}\downarrow & & \downarrow{\scriptstyle R_\Sigma\otimes} \\ \Sigma^{-1}B & \xrightarrow{F_\Sigma} & R_\Sigma - mod_{\mathcal{C}} \end{array} \tag{3}$$

commutes and which are universal for this property.

2.7.2. Note. Suppose the conditions of 2.6.1 on functors $D \xleftarrow{G} B \xrightarrow{F} \mathcal{C}$ and the category $\mathcal{A}$ hold. Consider a map which assigns to any algebra S in $\mathcal{A}^\sim$ the set $\mathcal{H}_{G,F}(S)$ of all functors $H : D \longrightarrow S - mod_{\mathcal{C}}$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{F} & \mathcal{C} \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle S\otimes} \\ D & \xrightarrow{H} & S - mod_{\mathcal{C}} \end{array}$$

commutes. It is easy to see that the map is functorial. The assertion 2.6.1 means that this functor is (representable by) an affine $\mathcal{A}^\sim$-scheme.

In the case when the functor $G : B \longrightarrow D$ is a localization, the set $\mathcal{H}_{G,F}(S)$ is either empty, or has only one element.

2.4.5. Proposition (base change). *Let conditions of 2.6 hold. Let S be an algebra in $\mathcal{A}^\sim$ and F_S the composition of $F : B \longrightarrow \mathcal{C}$ and $S\otimes : \mathcal{C} \longrightarrow S - mod_{\mathcal{C}}$, $M \longmapsto S(M)$. Then the universal algebra $R_{F',G}$ is naturally identified with $R_{F,G} \coprod S$ and the canonical functor $H_{F',G}$ with the composition of the functor $H_{F,G} : D \longrightarrow R_{F,G} - mod_{\mathcal{C}}$ and the canonical functor $R_{F,G} - mod_{\mathcal{C}} \longrightarrow R_{F,G} \coprod S - mod_{\mathcal{C}}$.*

Proof. It follows from the commutative diagram

$$\begin{array}{ccccc} B & \xrightarrow{F} & \mathcal{C} & \longrightarrow & S - mod_{\mathcal{C}} \\ {\scriptstyle G}\downarrow & & \downarrow & & \downarrow \\ D & \xrightarrow{H_{F,G}} & R_{F,G} - mod_{\mathcal{C}} & \xrightarrow{\Phi} & R_{F,G} \coprod S - mod_{\mathcal{C}} \end{array} \tag{4}$$

that there exists a unique morphism $\psi : R_{F',G} \longrightarrow R_{F,G} \coprod S$ such that $\Phi \circ H_{F,G}$ is the composition of $H_{F',G}$ and the natural functor $R_{F',G} - mod_{\mathcal{C}} \longrightarrow R_{F,G} \coprod S - mod_{\mathcal{C}}$. ∎

3. Grassmannians.

Given a quasi-site (A, τ) and a presheaf of sets X on A, we denote by X^τ a sheaf associated with X. In what follows, A is the category $\mathbf{Aff}_{\mathcal{A}^\sim}$ of affine schemes in a monoidal category $\mathcal{A}^\sim$.

3.1. The functor $Gr_{M,V}$. Fix a pair, (M, V), of objects of the category $\mathcal{C}$. Consider the functor, $Gr_{M,V} : Alg\mathcal{A}^\sim \longrightarrow Sets$, which assigns to any algebra R the set of isomorphism classes of split epimorphisms $R(M) \longrightarrow R(V)$ and to any R-ring morphism $R \xrightarrow{\phi} S$ the map $Gr_{M,V}(R) \longrightarrow Gr_{M,V}(S)$ induced by the corresponding inverse image functor $R - mod_{\mathcal{C}} \xrightarrow{\phi^*} S - mod_{\mathcal{C}}$, $\mathcal{N} \longmapsto T \otimes_S \mathcal{N}$.

3.1.1. The functor $G_{M,V}$. Denote by $G_{M,V}$ the functor $Alg\mathcal{A}^\sim \longrightarrow Sets$ which assigns to any algebra R the set of pairs of morphisms $R(V) \xrightarrow{v} R(M) \xrightarrow{u} R(V)$ such that $u \circ v = id_{R(V)}$ and acts naturally on morphisms. The map

$$\pi = \pi_{M,V} : G_{M,V} \longrightarrow Gr_{M,V}, \ (v, u) \longmapsto [u], \tag{1}$$

is a functor epimorphism.

3.1.2. Relations. Denote by $\mathfrak{R}_{M,V}$ the "functor of relations" $G_{M,V} \times_{Gr_{M,V}} G_{M,V}$. By definition, $\mathfrak{R}_{M,V}$ is a subfunctor of $G_{M,V} \times G_{M,V}$ which assigns to each algebra R the set of all 4-tuples $(u_1, v_1; u_2, v_2) \in G_{M,V} \times G_{M,V}$ such that the epimorphisms $u_1, u_2 : R(M) \longrightarrow R(V)$ are equivalent. The latter means that there exists an isomorphism $\varphi : R(V) \longrightarrow R(V)$ such that $u_2 = \varphi \circ u_1$, or, equivalently, $\varphi^{-1} \circ u_2 = u_1$. Since $u_i \circ v_i = id$, $i = 1, 2$, these equalities imply that $\varphi = u_2 \circ v_1$ and $\varphi^{-1} = u_1 \circ v_2$. Thus $\mathfrak{R}_{M,V}(R)$ is a subset of all $(u_1, v_1; u_2, v_2) \in G_{M,V}(R) \times G_{M,V}(R)$ satisfying the following relations:

$$u_2 = (u_2 \circ v_1) \circ u_1, \quad u_1 = (u_1 \circ v_2) \circ u_2 \tag{2}$$

in addition to the relations describing $G_{M,V}(R) \times G_{M,V}(R)$:

$$u_1 \circ v_1 = id = u_2 \circ v_2 \tag{3}$$

Denote by p_1, p_2 the canonical projections $\mathfrak{R}_{M,V} \rightrightarrows G_{M,V}$. It follows from the surjectivity of $G_{M,V} \longrightarrow Gr_{M,V}$ that the diagram

$$\mathfrak{R}_{M,V} \rightrightarrows G_{M,V} \xrightarrow{\pi} Gr_{M,V} \tag{4}$$

is exact.

3.1.3. Proposition. *If the pairs of objects (M, V) and (V, M) are admissible, the functors $G_{M,V}$ and $\mathfrak{R}_{M,V}$ are corepresentable.*

Proof. The representability of the functor $G_{M,V}$ is a consequence of 2.6 applied to the following setting: D is a category with two objects, x and y, and arrows $f : x \to y$, $g : y \to x$ such that $f \circ g = id_y$; B is the discrete subcategory of D (i.e. it has only identical morphisms) with objects x and y; $F : B \longrightarrow \mathcal{C}$ maps the object x to M and the object y to V.

The representability of the functor $\mathfrak{R}_{M,V}$ follows from the representability of $G_{M,V}$, the description of $\mathfrak{R}_{M,V}$ in terms of relations (cf. 3.1.2), and Proposition 2.4.2. Details are left to te reader. ∎

3.1.4. τ-Grassmannians. Let τ be a topology on the category $\mathbf{Aff}_{\mathcal{A}^\sim}$ of affine schemes in the monoidal category $\mathcal{A}^\sim$. We denote by $Gr^\tau_{M,V}$ the τ-space (i.e. a τ-sheaf) associated with $Gr_{M,V}$. We call the functor $Gr^\tau_{M,V}$ a *τ-Grassmannian of the type* (M, V).

Suppose the pairs (M, V) and (V, M) are admissible. Let $\mathcal{G}_{M,V}$ and $\mathcal{R}_{M,V}$ be algebras corepresenting the functors resp. $G_{M,V}$ and $\mathfrak{R}_{M,V}$. And let $\mathcal{G}_{M,V} \overset{\mathfrak{p}_1}{\underset{\mathfrak{p}_2}{\rightrightarrows}} \mathcal{R}_{M,V}$ be morphisms corresponding to the projections $\mathfrak{R}_{M,V} \overset{p_1}{\underset{p_2}{\rightrightarrows}} G_{M,V}$. If τ is subcanonical, then the exact diagram (4) induces an exact diagram of τ-spaces

$$\mathfrak{R}_{M,V} \rightrightarrows G_{M,V} \longrightarrow Gr^\tau_{M,V}. \tag{5}$$

If the projections $\mathfrak{R}_{M,V} \rightrightarrows G_{M,V}$ are τ-covers, then $Gr^\tau_{M,V}$ is a locally affine τ-space in terms of [KR2, 8.3].

3.1.5. Grassmannians and projective spaces. Fix two objects, $\mathcal{M}$ and $\mathcal{V}$, of the category $\mathcal{A}$. Specializing the constructions of 3.1 in the case of the standard right action of the monoidal category $\mathcal{A}^\sim$, we obtain the functors $Gr_{\mathcal{M},\mathcal{V}}$, $Gr^\tau_{\mathcal{M},\mathcal{V}}$, $G_{\mathcal{M},\mathcal{V}}$, and $\mathfrak{R}_{\mathcal{M},\mathcal{V}}$ from $Alg\mathcal{A}^\sim$ to $Sets$ together with the exact diagram

$$\mathfrak{R}_{\mathcal{M},\mathcal{V}} \rightrightarrows G_{\mathcal{M},\mathcal{V}} \longrightarrow Gr_{\mathcal{M},\mathcal{V}}$$

and a canonical morphism $Gr_{\mathcal{M},\mathcal{V}} \longrightarrow Gr^\tau_{\mathcal{M},\mathcal{V}}$. If τ is subcanonical, then the corresponding diagram of τ-spaces,

$$\mathfrak{R}_{\mathcal{M},\mathcal{V}} \rightrightarrows G_{\mathcal{M},\mathcal{V}} \longrightarrow Gr^\tau_{\mathcal{M},\mathcal{V}},$$

is exact. We denote by $\mathbf{P}^\tau_{\mathcal{M}}$ the τ-space $Gr^\tau_{\mathcal{M},\mathbf{1}}$ and call it the *projective τ-space of* $\mathcal{M}$.

3.2. Generic Grassmannians. Fix an object E of $\mathcal{C}$. For any algebra S in $\mathcal{A}^\sim$, denote by $Gr_E(S)$ the set of isomorphism classes of split epimorphisms $S(E) \longrightarrow L'$. The map $S \longmapsto Gr_E(S)$ extends naturally to a functor $Gr_E : Alg\mathcal{A}^\sim \longrightarrow Sets$.

For any $L \in Ob\mathcal{A}$, there is a natural functor morphism $G_{E,L} \xrightarrow{\rho_L} Gr_E$.

3.2.1. The functor $\mathfrak{Pr}_E$. Denote by $\mathfrak{Pr}_E$ the functor $Alg\mathcal{A}^\sim \longrightarrow Sets$ which assigns to any algebra S the set of projectors of $S(E)$, i.e. morphisms $S(E) \xrightarrow{p} S(E)$ such that $p^2 = p$. We have a natural functor epimorphism

$$\mathfrak{Pr}_E \longrightarrow Gr_E \tag{1}$$

3.2.2. Relations. Two projectors, $S(R) \overset{p_1}{\underset{p_2}{\rightrightarrows}} S(R)$ are equivalent if their images by (1) are isomorphic. The latter can be expressed by the equalities

$$p_1 p_2 p_1 = p_1 \quad \text{and} \quad p_2 p_1 p_2 = p_2. \tag{2}$$

Thus the functor of relations $\mathfrak{R}_E = \mathfrak{Pr}_E \times_{Gr_E} \mathfrak{Pr}_E$ of the morphism (1) assigns to each algebra S the subset of all $(p_1, p_2) \in \mathfrak{Pr}_E \times \mathfrak{Pr}_E$ satisfying the relations (2) above (in addition to the relations $p_i^2 = id$, $i = 1, 2$).

We have an exact diagram

$$\mathfrak{R}_E \rightrightarrows \mathfrak{Pr}_E \xrightarrow{\ \pi\ } Gr_E \tag{3}$$

3.2.2.1. It follows from 2.4.2 that if (E, E) is admissible, the functors $\mathfrak{R}_E$ and $\mathfrak{Pr}_E$ are corepresentable.

3.2.3. Generic τ-Grassmannians. Let τ be a topology on the category $\mathbf{Aff}_{\mathcal{A}^\sim}$ of affine schemes in the monoidal category $\mathcal{A}^\sim$. We denote by Gr_E^τ the τ-space associated with Gr_E. We call the functor Gr_E^τ a *generic τ-Grassmannian of the type E.*

Suppose the pair (E, E) is admissible. Let $\mathcal{P}r_E$ and $\mathcal{R}_E$ be algebras corepresenting the functors resp. $\mathfrak{Pr}_E$ and $\mathfrak{R}_E$. And let $\mathfrak{p}_i$, $i = 1, 2$, be the morphisms $\mathcal{P}r_E \rightrightarrows \mathcal{R}_E$ corresponding to the projections $\mathfrak{R}_E \rightrightarrows \mathfrak{Pr}_E$.

If τ is subcanonical, then the exact diagram (3) induces an exact diagram of τ-spaces

$$\mathfrak{R}_E \overset{p_1}{\underset{p_2}{\rightrightarrows}} \mathcal{P}r_E \longrightarrow Gr_E^\tau. \tag{4}$$

If the canonical morphisms p_i, $i = 1, 2$, in (4) are covers, then Gr_E^τ is a locally affine τ-space.

3.3. Some properties of Grassmannians.

3.3.1. Functoriality. One can see that the map $E \longmapsto Gr_E$ is functorial for split epimorphisms. Moreover, we have the following

3.3.1.1. Proposition. *For any locally split epimorphism $E' \longrightarrow E$, the corresponding morphism $Gr_E^\tau \longrightarrow Gr_{E'}^\tau$ is a closed immersion.*

Proof is left to the reader. ∎

There is a similar assertion for non-generic Grassmannian $Gr_{E,L}^\tau$:

3.3.1.2. Proposition. *The map $E \longmapsto Gr_E^\tau$ is functorial for locally split epimorphisms. For any locally split epimorphism $E' \longrightarrow E$, the corresponding morphism $Gr_{E,L}^\tau \longrightarrow Gr_{E',L}^\tau$ is a closed immersion.*

Proof. The argument is left to the reader. ∎

3.3.1.3. The canonical morphism $\rho_E : Gr^{\tau}_{E,L} \longrightarrow Gr^{\tau}_E$ is functorial in E. That is for any locally split epimorphism $E' \longrightarrow E$, the diagram

$$\begin{array}{ccc} Gr^{\tau}_{E,L} & \xrightarrow{\rho_E} & Gr^{\tau}_E \\ \downarrow & & \downarrow \\ Gr^{\tau}_{E',L} & \xrightarrow{\rho_{E'}} & Gr^{\tau}_{E'} \end{array} \qquad (1)$$

commutes.

3.3.2. Proposition. *Grassmannians are separated.*

Proof. Let $V = \mathfrak{Sp}(R)$ be an arbitrary affine scheme, and let $h_V \underset{u_2}{\overset{u_1}{\rightrightarrows}} Gr^{\tau}_{E,L}$ be a pair of morphisms over R. The claim is that the kernel of the pair (u_1, u_2) is representable by a morphism of affine schemes (cf. GA.3.1).

Let $\xi_i : R(E) \longrightarrow L_i$ be a locally split morphism corresponding to u_i, $i = 1, 2$. Let (v_1^i, v_2^i) be a pair of arrows $M_i \rightrightarrows R(E)$ such that u_i is a cokernel of (v_1^i, v_2^i), $i = 1, 2$. Consider the compositions

$$(\xi_1 v_1^2, \xi_1 v_2^2) : M_2 \rightrightarrows L_1 \quad \text{and} \quad (\xi_2 v_1^1, \xi_2 v_2^1) : M_1 \rightrightarrows L_2. \qquad (1)$$

By 2.4.2, there exists a universal affine scheme morphism $\psi : W \longrightarrow V$ such that the image of each of the pairs (1) by ψ^* belongs to the diagonal. And this morphism ψ is a closed immersion. ∎

3.3.3. Proposition. *Grassmannians are proper.*

Proof. Since Grassmannians are separated, it suffices to show that the canonical morphism $Gr_{E,L} \xrightarrow{\pi} S$ is a cover in τ_{can}. For an arbitrary morphism $\mathfrak{Sp}(A) \xrightarrow{f} S$, the induced projection is isomorphic to the canonical morphism $Gr_{f^*(E),f^*(L)} \longrightarrow \mathfrak{Sp}(A)$. The morphism $Gr_{E,L} \xrightarrow{\pi} S$ is a cover in the canonical topology, because

$$G_{f^*(E),f^*(L)} \longrightarrow \mathfrak{Sp}(A)$$

is represented by a strict epimorphism. ∎

4. Complements on Grassmannians.

4.1. Tautological morphism. Let $Gr_{E,L} \xrightarrow{p} X$ be the canonical projection. The *tautological* or *universal* morphism $p^*(E) \longrightarrow L(1)$ is defined uniquely up to isomorphism by the following universal property: for any morphism $Y \xrightarrow{f} X$ and any locally split epimorphism $f^*(E) \longrightarrow L'$ with L' locally isomorphic to $f^*(L)$, there exists a unique morphism

$$\begin{array}{ccc} Y & \xrightarrow{g} & Gr_{E,L} \\ f \searrow & & \swarrow p \\ & X & \end{array}$$

and a unique isomorphism $g^*L(1) \xrightarrow{\sim} L'$ such that the diagram

$$\begin{array}{ccc} g^*f^*(E) & \longrightarrow & g^*L(1) \\ \wr\big\downarrow & & \big\downarrow\wr \\ f^*(E) & \longrightarrow & L' \end{array} \tag{1}$$

commutes. In particular, any locally split epimorphism $\xi : E \longrightarrow L'$, where L' is locally isomorphic to L, determines a section $s_\xi : X \longrightarrow Gr_{E,L}$, i.e. $p \circ s_\xi = id_X$. On the other hand, the epimorphism ξ defines a closed embedding

$$\begin{array}{ccc} Gr_{L',L} & \xrightarrow{j_\xi} & Gr_{E,L} \\ p_{_{L'}} \searrow & & \swarrow\ p \\ & X & \end{array} \tag{2}$$

In the commutative case, the projection $p_{_{L'}}$ is an isomorphism. This is far from being the case in the noncommutative setting. Thus we have the following diagram

$$\begin{array}{ccc} Gr_{L',L} & \xrightarrow{j_\xi} & Gr_{E,L} \\ p_{_{L'}} \searrow\nwarrow s' & & s_\xi \nearrow\swarrow\ p \\ & X & \end{array} \tag{3}$$

with arrows satisfying the identities:

$$p_{_{L'}} = p \circ j_\xi, \quad s_\xi = j_\xi \circ s', \quad p_{_{L'}} \circ s' = id_X = p \circ s_\xi. \tag{4}$$

4.1.1. The universal hyperplane sheaf. Suppose the category $\mathcal{C}$ has zero (i.e. a final object which is initial too). Then for any two objects, M and M', there is a zero morphism from M to M', hence a kernel of any morphism $M \xrightarrow{f} M'$ which is by definition the kernel of the pair $M \underset{0}{\overset{f}{\rightrightarrows}} M'$. In particular, we have the *tautological* exact sequence

$$0 \longrightarrow H \longrightarrow p^*(E) \longrightarrow L(1) \longrightarrow 0 \tag{1}$$

The kernel $H = H_E$ of the universal morphism is called, as in the commutative case, the *universal hyperplane sheaf.*

4.2. Zero section and the hyperplane at infinity. Let $E' = E \prod L$, and let $L \xleftarrow{p_L} E \prod L \xrightarrow{p_E} E$ be canonical projections. The projection $p_{_L}$ determines a canonical section $X \longrightarrow Gr_{E',L}$ which (following the commutative tradition) will be called the *zero section.* The projection $E' \xrightarrow{p_E} E$ induces a closed embedding $Gr_{E,L} \longrightarrow Gr_{E',L}$ called the *hyperplane at infinity.*

4.3. Vector bundles and Grassmannians. Fix a morphism $L \xrightarrow{\phi} E$. For any algebra S, consider the set $F_{\phi;E,L}(S)$ of all morphisms $S(E) \xrightarrow{v} L'$ such that $v \circ S(\phi)$ is an isomorphism.

4.3.1. Proposition. *(a) The map* $S \longmapsto F_{\phi;E,L}(S)$ *is naturally extended to a subfunctor* $F_{\phi;E,L} : Alg\mathcal{A}^{\sim} \longrightarrow Sets$ *of* $Gr_{E,L}$.

(b) The functor $F_{\phi;E,L}$ *is representable by an affine scheme.*

(c) The canonical morphism $F_{\phi;E,L} \longrightarrow Gr_{E,L}$ *is an affine localization.*

Proof. (a) (i) In fact, if $v : S(E) \longrightarrow L'$ belongs to $F_{\phi;E,L}(S)$, i.e. $v \circ S(\phi)$ is an isomorphism, then for any morphism $h : S \longrightarrow T$, the composition $h^*(v) \circ h^*S(\phi)$ is an isomorphism. There is a natural morphism $F_{\phi;E,L} \longrightarrow Gr_{E,L}$.

(ii) Note that one can identify $F_{\phi;E,L}(S)$ with the set of epimorphisms $S(E) \xrightarrow{v} S(L)$ such that $v \circ S(\phi) = id_{S(L)}$. In fact, if $v' : S(E) \longrightarrow L'$ is such that

$$w = v' \circ S(\phi) : S(L) \longrightarrow L'$$

is an isomorphism, then $v = w^{-1} \circ v'$ has the required property.

(iii) One of the consequences of the observation (ii) is that the canonical morphism $F_{\phi;E,L} \longrightarrow Gr_{E,L}$ is a monomorphism.

(b) There are two maps,

$$Hom_S(S(E), S(L)) \underset{\beta_S}{\overset{\alpha_S}{\rightrightarrows}} Hom_S(S(L), S(L)),$$

defined by $\alpha_S : v \longmapsto v \circ S(\phi)$, $\quad \beta_S : v \longmapsto id_{S(L)}$. The maps α_S and β_S are functorial in S, hence they define morphisms, resp. α and β, from the functor

$$(S) \longmapsto Hom_S(S(E), S(L)) \simeq \mathcal{A}(E, S(L))$$

to the functor

$$(S) \longmapsto Hom_S(S(L), S(L)) \simeq \mathcal{A}(L, S(L)).$$

The first functor is representable by $\mathbf{V}(L^\vee E)$, the second one is representable by $\mathbf{V}(L^\vee L)$. Let α' and β' be morphisms from $\mathbf{V}(L^\vee E)$ to $\mathbf{V}(L^\vee L)$ corresponding to resp. α and β. The functor $F_{\phi;E,L} : X \longmapsto F_{\phi;E,L}(X)$ is the kernel of the pair (α, β), hence it is representable by the kernel, $\mathbf{F}_{\phi;E,L}$, of the pair (α', β').

(c) *The functor morphism* $F_{\phi;E,L} \longrightarrow Gr_{E,L}$ *is representable by an affine morphism*; i.e. for any affine $\mathcal{A}^{\sim}$-scheme $Y = \mathfrak{Sp}(S)$ and any morphism $h_Y \longrightarrow Gr_{E,L}$, the functor

$$\mathbf{Aff}_{\mathcal{A}^{\sim}} \longrightarrow Sets, \quad Z \longmapsto F_{\phi;E,L}(Z) \times_{Gr_{E,L}(Z)} h_Y(Z)$$

is representable by an affine subscheme of Y.

In fact, any morphism $h_Y \longrightarrow Gr_{E,L}$ is uniquely determined by an element of $Gr_{E,L}(Y)$, i.e. by the equivalence class, $[v]$, of a locally split epimorphism $v : S(E) \longrightarrow L'$. The corresponding map $h_Y(Z) \longrightarrow Gr_{E,L}(Z)$ sends any morphism $t : Z \longrightarrow Y$ into $[t^*(v)]$. Let $Z = \mathfrak{Sp}(T)$. The fiber product $F_{\phi;E,L}(Z) \times_{Gr_{E,L}(Z)} h_Y(Z)$ consists of all pairs (w, t), where $t \in h_Y(Z)$ and $[w : T(E) \longrightarrow T(L)]$ are such that $w \circ T(\phi) = id_{T(L)}$ and $w = t^*(v)$. Since v and ϕ here are fixed, the fiber product $F_{\phi;E,L}(Z) \times_{Gr_{E,L}(Z)} h_Y(Z)$ can be identified with the set of all morphisms $t : Z \longrightarrow Y$ such that $t^*(v \circ T(\phi)) = id_{T(L)}$. In other words, the fiber product $F_{\phi;E,L}Z \times_{Gr_{E,L}(Z)} h_Y(Z)$ is identified with the kernel of the pair of morphisms $\beta_Z, \alpha_Z : h_Y(Z) \longrightarrow Hom_T(T(L), T(L))$, defined by

$$\beta_Z : t \longmapsto id_{T(L)}, \ \ \alpha_Z : t \longmapsto t^*(v \circ T(\phi)).$$

The morphisms $\beta_Z, \ \alpha_Z$ are functorial in Z, and $Hom_T(T(L), T(L)) \simeq h_{\mathbf{V}(L^\vee L)}(Z)$. Hence the morphisms $\beta = (\beta_Z), \ \alpha = (\alpha_Z)$ define a pair of morphisms $Y \underset{\beta'}{\overset{\alpha'}{\rightrightarrows}} \mathbf{V}(L^\vee L)$, and the functor $Z \longmapsto F_{\phi;E,L}Z \times_{Gr_{E,L}Z} h_Y(Z)$ is representable by the kernel of the pair (α', β'). ■

4.3.2. Projective completion of a vector bundle. Let $E'' = E \coprod L$, and let $L \xrightarrow{j_L} E''$ be the canonical morphism. The functor $F_{j_L;E'',L}$ is isomorphic to the functor which assigns to an affine $\mathcal{A}^\sim$-scheme $Y = \mathfrak{Sp}(S)$ the set $Hom_S(S(E), S(L)$ (cf. (ii) and (b) in the argument of 4.3.1). The latter functor is representable by the vector bundle $\mathbf{V}(L^\vee E)$. By 4.3.1 we have an affine embedding (an open immersion) $\mathbf{V}(L^\vee E) \longrightarrow Gr_{E,L}$. In particular, taking $L = \mathcal{O}_X$, we obtain a canonical immersion $\mathbf{V}(E) \longrightarrow \mathbf{P}^\tau_E$. The projective space $\mathbf{P}^\tau_{E \sqcup \mathcal{O}_X}$ can be regarded (as in the commutative case) as the *projective completion* of the vector bundle $\mathbf{V}(E)$.

4.3.3. Suppose the category $\mathcal{C}$ has zero. Then we have canonical projections

$$L \xleftarrow{q_L} E \coprod L \xrightarrow{q_E} E$$

such that $q_L \circ j_L = id_L$. As in 4.2, the projection q_L determines the *zero section*

$$s_E : \mathbf{Spec1} = * \longrightarrow Gr^\tau_{E \sqcup L, L} \tag{1}$$

and the *hyperplane at infinity* $Gr^\tau_{E,L} \longrightarrow Gr^\tau_{E \sqcup L, L}$. Combining with 4.3.2, we obtain an open and a closed embeddings of resp. an affine vector bundle and a Grassmannian:

$$\mathbf{V}(L^\vee E) \longrightarrow Gr_{E \sqcup L, L} \longleftarrow Gr_{E,L}. \tag{2}$$

Applying the functor s_E^* to the tautological exact sequence

$$0 \longrightarrow H \longrightarrow p^*(E \coprod L) \longrightarrow L(1) \longrightarrow 0 \tag{3}$$

and using 4.1(1), we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & s_E^*(H) & \longrightarrow & s_E^* S(E) & \longrightarrow & s_E^* L(1) & \longrightarrow & 0 \\ & & \downarrow & & \wr\downarrow & & \downarrow\wr & & \\ 0 & \longrightarrow & E & \longrightarrow & E \coprod L & \longrightarrow & L & \longrightarrow & 0 \end{array} \tag{4}$$

Since E is the kernel of the projection $E \coprod L \longrightarrow L$, the left vertical arrow in the diagram (4), $s_E^*(H) \longrightarrow E$, is an isomorphism too.

4.3.4. The split epimorphisms $E \longleftarrow E \sqcup F \longrightarrow F$ correspond to closed immersions

$$Gr_{E,L} \longrightarrow Gr_{E \sqcup F,L} \longleftarrow Gr_{F,L} \tag{1}$$

which induce the canonical morphism

$$Gr_{E,L} \coprod Gr_{F,L} \longrightarrow Gr_{E \sqcup F,L} \tag{2}$$

4.3.4.1. Proposition. *The canonical morphism (2) is a closed immersion.*

Proof. Let $Y = \mathfrak{Sp}(S)$, and let

$$\begin{array}{ccc} Y & \xrightarrow{\ \phi\ } & Gr_{E,L} \\ \psi \big\downarrow & & \big\downarrow \\ Gr_{F,L} & \longrightarrow & Gr_{E \sqcup F,L} \end{array} \tag{3}$$

be a commutative square. The morphism ϕ corresponds to a locally split epimorphism $S(E) \longrightarrow L'$ and the morphism ψ corresponds to a locally split epimorphism $S(F) \longrightarrow L''$. The commutativity of the diagram (3) means that there exists an isomorphism $L' \xrightarrow{\sim} L''$ such that the diagram

$$\begin{array}{ccccc} SE \sqcup SF & & \longrightarrow & & SF \\ & \searrow & & & \big\downarrow \\ & & SE & & L'' \\ & & & \searrow \quad \nearrow & \\ & & & L' & \end{array} \tag{4}$$

in which arrows $SE \leftarrow SE \sqcup SF \longrightarrow SF$ are canonical projections, commutes. This implies that the morphism $SE \sqcup SF \longrightarrow L'$ is zero which contradicts to the assumptions. Therefore the intersection of the subfunctors $Gr_{E,L} \longrightarrow Gr_{E \sqcup F,L} \leftarrow Gr_{F,L}$ is empty, hence the assertion. ■

4.3.4.2. Remark. Restricting Grassmannians to the commutative rings, we obtain a well known statement (see [GrD2], 4.3.6). Note that the argument in [GrD2] uses the reduction to the local case and spectral considerations.

5. Generic flags. Let I be a preordered set with initial object $*$. Fix an object, E, of the category $\mathcal{C}$. For any algebra S, denote by $Fl_{E|I}$ set of isomorphism classes of functors $I \longrightarrow \mathcal{C}$ which map all arrows $* \longrightarrow i$ to split epimorphisms $S(E) \longrightarrow L_i$.

Denote by $\mathfrak{F}_E^I$ the functor $Alg\mathcal{A}^{\sim} \longrightarrow Sets$ which assigns to any algebra S the set of projectors $S(E) \xrightarrow{p_i} S(E)$ such that $p_i p_j = p_i$ if $i \leq j$. By 2.4.2, the functor $\mathfrak{F}_E^I$ is representable by an affine scheme provided E is a finite object.

We have a natural functor morphism

$$\mathfrak{F}_E^I \longrightarrow Fl_{E|I}^{\tau} \tag{1}$$

The functor of relations $\mathfrak{R}^I_E = \mathfrak{F}^I_E \times_{Fl^\tau_{E|I}} \mathfrak{F}^I_E$ consists of all $(\{p_i|i \in I\}, \{p'_i|i \in I\}) \in \mathfrak{F}^I_E \times_{Fl^\tau_{E|I}} \mathfrak{F}^I_E$ satisfying the relations

$$p_i p'_i p_i = p_i \quad \text{and} \quad p'_i p_i p'_i = p'_i \tag{2}$$

(see the argument of 3.2.1). By 2.4.2, the functor $\mathfrak{R}_E$ is representable by an affine scheme if E is finite. If $I = \{0, 1, 2, ..., r\}$ with the natural order, we shall write $Fl^\tau_{E|r}$ instead of $Fl^\tau_{E|I}$ and $\mathfrak{F}^r_E$ instead of $\mathfrak{F}^I_E$.

5.1.1. The functor $Fl^\tau_{E|I}$. Fix a topology τ on $\mathbf{Aff}_{\mathcal{A}^\sim}$. We denote by $Fl^\tau_{E|I}$ the τ-space associated with $Fl_{E|I}$. For any algebra S, $Fl^\tau_{E|I}(S)$ can be regarded as the set of isomorphism classes of functors $I \longrightarrow \mathcal{C}$ which map all arrows $* \longrightarrow i$ to*locally* split epimorphisms $S(E) \longrightarrow L_i$.

5.1.2. Proposition. *Suppose τ is a subcanonical pretopology whose covers consists of one arrow (which we call deflation). Let the object E be finite, and the projections $\mathfrak{R}_{E|I} \rightrightarrows \mathfrak{F}_{E|I}$ are represented by deflations. Then the functor $Fl^\tau_{E|I}$ is representable by a locally affine space $\mathbf{Fl}^\tau_E$.*

Proof. The argument is left to the reader (cf. 3.1.5). ∎

5.2. Proposition. *Assume that I is finite. Then $\prod_{i \in I} Gr^\tau_E$ is a locally affine space and the natural embedding*

$$Fl^\tau_{E|I} \longrightarrow \prod_{i \in I} Gr^\tau_E \tag{3}$$

is a closed immersion.

Proof. The argument is left to the reader. ∎

5.3. An action of GL_E on generic flag varieties. The functor GL_E acts naturally on the functor $\mathfrak{F}^I_E$ and on $Fl^\tau_{E|I}$, and the canonical morphism 5.1(1) is compatible with these actions. In particular, the induced action of GL_E on $\mathfrak{F}^I_E \times \mathfrak{F}^I_E$ preserves the subfunctor of relations $\mathfrak{R}^I_E$.

5.4. Stiefel schemes and flag varieties.

5.4.1. Universal Stiefel schemes. Let E be a finite right R-module. Fix a number r and consider the universal direct sum decomposition $E = \coprod_{1 \le n \le r} P_i$ by adjoining to R universal idempotent morphisms e_i, $1 \le i \le r$ such that $e_i e_j = \delta_{ij} e_i$. We denote the resulting algebra by $R\langle E; r, e_i e_j = \delta_{ij} e_i\rangle$. The corresponding affine scheme $\mathfrak{Sp}(R)\langle E; r, e_i e_j = \delta_{ij} e_i\rangle \longrightarrow \mathfrak{Sp}(R)$ over R will be denoted by $\mathbf{UStief}_{r+1}(E)$ and called the *universal Stiefel scheme of rank $r + 1$*.

There is a natural action of the affine group scheme $\mathbf{GL}_E$ on the Stiefel scheme $\mathbf{UStief}_{r+1}(E)$.

5.4.2. There is a natural functorial morphism

$$Stief_{r+1}(E)(S) \longrightarrow \mathfrak{F}^r_E(S) \tag{3}$$

which maps the set of projectors $(e_1, ..., e_r)$ to the set of projectors $(p_1, ..., p_r)$, where $p_i = e_1 + ... + e_i$, $1 \leq i \leq r$.

5.4.3. Proposition. *The canonical morphism (3) is an isomorphism. In particular, the composition*

$$Stief_{r+1}(E) \longrightarrow Fl^{\tau}_{E|I} \tag{4}$$

of the morphism (3) with the morphism $\mathfrak{F}^r_E \longrightarrow Fl^{\tau}_{E|r}$ *is a surjection of spaces.*

Proof is left to the reader. ■

5.4.4. Note that the morphism (3) is compatible with the action of GL_E, hence the composition of the morphism (3) with the morphism $\mathfrak{F}^r_E \longrightarrow Fl^{\tau}_{E|r}$ is compatible with the action of GL_E.

6. General grassmannian type spaces.

6.1. The functor $\mathfrak{F}_{\mathfrak{E}}$. Let B, D_1, D_2 be small categories and $B \xrightarrow{G_1} D_1$ and $D_1 \xrightarrow{G_2} D_2$ functors. Fix a functor $B \xrightarrow{E} \mathcal{C}$. Denote this data by $\mathfrak{E}$. For any algebra S, denote by $\mathfrak{F}^{\sim}_{\mathfrak{E}}(S)$ the category whose objects are functors $D_1 \xrightarrow{L} S - mod_{\mathcal{C}}$ such that $S \circ E$ and $L \circ G_1$ are isomorphic and the functor L extends to D_2. The correspondence $S \longmapsto \mathfrak{F}^{\sim}_{\mathfrak{E}}(S)$ is functorial in S. Hence the map $\mathfrak{F}_{\mathfrak{E}}$ which assigns to S the set $\mathfrak{F}_{\mathfrak{E}}(S)$ of isomorphism classes of objects of $\mathfrak{F}^{\sim}_{\mathfrak{E}}(S)$ extends to a functor $\mathfrak{F}_{\mathfrak{E}} : Alg\mathcal{A}^{\sim} \longrightarrow Sets$.

6.1.1. Functoriality. A morphism from the data $\mathfrak{E} = (D_2 \xleftarrow{G_2} D_1 \xleftarrow{G_1} B \xrightarrow{E} \mathcal{C})$ to the data $\mathfrak{E}' = (D'_2 \xleftarrow{G'_2} D'_1 \xleftarrow{G'_1} B' \xrightarrow{E'} \mathcal{C})$ is a commutative diagram of functors

$$\begin{tikzcd}
D_2 \arrow[d] & D_1 \arrow[l, "G_2"'] \arrow[d] & B \arrow[l, "G_1"'] \arrow[r, "E"] \arrow[d] & \mathcal{C} \arrow[d, "Id"] \\
D'_2 & D'_1 \arrow[l, "G'_2"'] & B' \arrow[l, "G'_1"'] \arrow[r, "E'"] & \mathcal{C}
\end{tikzcd} \tag{1}$$

The composition is defined in a natural way. It follows from the definitions that the presheaf $\mathfrak{F}_{\mathfrak{E}}$ depends functorially on the data $\mathfrak{E}$: to each morphism $\mathfrak{E} \longrightarrow \mathfrak{E}'$, there corresponds a presheaf morphism $\mathfrak{F}_{\mathfrak{E}'} \longrightarrow \mathfrak{F}_{\mathfrak{E}}$.

6.1.2. Functors $\mathfrak{L}_{E,G}$ and $\mathfrak{R}_{E,G}$. Set $G = G_2 \circ G_1 : B \longrightarrow D_2$. Consider the pseudo-functor $\mathfrak{L}^{\sim}_{E,G} : Alg\mathcal{A}^{\sim} \longrightarrow Cat$ which assigns to each algebra S the category of all functors $H : D_2 \longrightarrow S - mod_{\mathcal{C}}$ such that $H \circ G = S \circ E$ and the similar pseudo-functor $\mathfrak{L}^{\sim}_{E,G_1}$. There are natural pseudo-functor morphisms

$$\pi^{\sim}_1 : \mathfrak{L}^{\sim}_{E,G} \longrightarrow \mathfrak{L}^{\sim}_{E,G_1}, \quad H \longmapsto H \circ G_2,$$

and

$$\pi^{\sim}_0 : \mathfrak{L}^{\sim}_{E,G} \longrightarrow \mathfrak{F}^{\sim}_{E,G_1}.$$

Let $\mathfrak{L}_{E,G}$, $\mathfrak{L}_{E,G}$ denote the functors $Alg\mathcal{A}^{\sim} \longrightarrow Sets$ associated resp. with $\mathfrak{L}^{\sim}_{E,G}$ and $\mathfrak{L}^{\sim}_{E,G}$. Let $\pi_0 : \mathfrak{L}_{E,G} \longrightarrow \mathfrak{L}_{E,G_1}$ and $\pi_1 : \mathfrak{L}_{E,G} \longrightarrow \mathfrak{F}^{\tau}_{\mathfrak{E}}$ be the functor morphisms corresponding resp. to $\pi^{\sim}_0$ and $\pi^{\sim}_1$. Set $\pi = \pi_0 \circ \pi_1$.

The relations functor $\mathfrak{R}_{E,G} = \mathfrak{L}_{E,G} \times_{\mathfrak{F}^{\tau}_{\mathfrak{C}}} \mathfrak{L}_{E,G}$ is described as follows. For any algebra S, the set $\mathfrak{R}_{E,G}(S)$ consists of all pairs of functors $H, H' : D_2 \longrightarrow S - mod_{\mathcal{C}}$ such that $H \circ G = S \circ E = H' \circ G$ and there exists a functor isomorphism $\lambda : H \circ G_2 \longrightarrow H' \circ G_2$ such that $\lambda G_1 : \phi^* E \longrightarrow S \circ E$ is the identity. Since G_1 is a bijection on objects, the latter means that $\lambda = id$. Thus $\mathfrak{R}_{E,G}(S)$ consists of pairs of functors $H, H' : D_2 \longrightarrow S - mod_{\mathcal{C}}$ such that

$$H \circ G_2 = H' \circ G_2 \quad \text{and} \quad H \circ G_2 \circ G_1 = S \circ E, \tag{1}$$

or, equivalently, $\mathfrak{R}_{E,G} = \mathfrak{L}_{E,G} \times_{\mathfrak{L}_{E,G_1}} \mathfrak{L}_{E,G}$. Since both $\mathfrak{L}_{E,G}$ and $\mathfrak{L}_{E,G_1}$ are representable by affine schemes, the relations functor has the same property. It follows that the diagram

$$\mathfrak{R}_{E,G} \rightrightarrows \mathfrak{L}_{E,G} \longrightarrow \mathfrak{F}_{\mathfrak{C}} \tag{2}$$

is exact.

6.2. Proposition. *Suppose the functors $B \xrightarrow{G_1} D_1$ and $D_1 \xrightarrow{G_2} D_2$ are bijective on objects. Let $\mathcal{A}$ have small colimits and the pair of functors $(E, G_2 \circ G_1)$ satisfy the condition (†) of 2.6. Then the functors $\mathfrak{R}_{E,G}$ and $\mathfrak{L}_{E,G}$ are (representable by) affine schemes.*

Proof. By 2.6, the functors $\mathfrak{L}_{E,G}$ and $\mathfrak{L}_{E,G_1}$ are representable by affine schemes, resp. $\mathfrak{Sp}(R)\langle E, G\rangle$ and $\mathfrak{Sp}(R)\langle E, G_1\rangle$. ■

6.2.1. Note. Thanks to 2.6.1, the condition "G_1, G_2 are bijective on objects" can be replaced by the condition that the functors G_i are injective on objects and each object of the category D_i is isomorphic to an object of the image of G_i, $i = 1, 2$.

6.3. The space $\mathfrak{F}^{\tau}_{\mathfrak{C}}$. Suppose that τ is a topology on $\mathbf{Aff}_{\mathcal{A}^{\sim}}$. Denote by $\mathfrak{F}^{\tau}_{\mathfrak{C}}$ the τ-space associated with the functor $\mathfrak{F}_{\mathfrak{C}}$. If τ is subcanonical, then 6.1.2(2) induces an exact sequence of τ-spaces

$$\mathfrak{R}_{E,G} \rightrightarrows \mathfrak{L}_{E,G} \longrightarrow \mathfrak{F}^{\tau}_{\mathfrak{C}}$$

If, in addition, the conditions of 6.2 hold and the projections $\mathfrak{R}_{E,G} \rightrightarrows \mathfrak{L}_{E,G}$ are covers, then the space $\mathfrak{F}^{\tau}_{\mathfrak{C}}$ is τ-locally affine.

For any algebra R in $\mathcal{A}^{\sim}$, denote by $R^*(\mathfrak{C})$ the data obtained from $\mathfrak{C}$ via the base change: $R^*(\mathfrak{C}) = (D_2 \xleftarrow{G_2} D_1 \xleftarrow{G_1} B \xrightarrow{R \circ E} R - mod_{\mathcal{C}})$. We have the following

6.3.1. Proposition. *For any affine scheme $\mathfrak{Sp}(R)$, there is a canonical isomorphism $\mathfrak{F}^{\tau}_{\mathfrak{C}} \times \mathfrak{Sp}(R') \xrightarrow{\sim} \mathfrak{F}^{\tau}_{R^*(\mathfrak{C})}$.*

Proof. The fact is a consequence of 2.4.5. ■

6.3.2. Proposition. *Fix a data $\mathfrak{C} = (D_2 \xleftarrow{G_2} D_1 \xleftarrow{G_1} B \xrightarrow{E} \mathcal{C})$ as in 6.1. Suppose the category $\mathcal{A}$ has coproducts of $|ObB|$ objects. Then the space $\mathfrak{F}^{\tau}_{\mathfrak{C}}$ is proper. In particular, it is separated.*

Proof. The argument proving separatedness is based on the same idea as that of 3.1.2, and it uses the second half of 2.4.2. We prove that the morphism $\mathfrak{F}^{\tau}_{\mathfrak{C}} \xrightarrow{\pi} S$ is a cover in τ_{can} repeating the argument of 3.3.3. Details are left to the reader. ■

6.4. Examples: Grassmannians and flag varieties.

6.4.1. Grassmannians. Let D_2 be a category with two objects, $x_0,\ x_1$, and three non-identical morphisms: $x_0 \xrightarrow{e} x_1 \xrightarrow{m} x_0$, and $m \circ e$ such that $e \circ m = id_{x_1}$. Let B be the discrete subcategory of D_2 with objects $x_0,\ x_1$ and D_1 the subcategory $x_1 \xrightarrow{e} x_0$. The functors $B \xrightarrow{G_1} D_1 \xrightarrow{G_2} D_2$ are natural embeddings. Fix a functor

$$B \xrightarrow{E} \mathcal{C} \quad x_i \longmapsto E_i,\ i = 0, 1.$$

The locally affine space $\mathfrak{F}^\tau_{\mathfrak{E}}$ corresponding to the data $\mathfrak{E} = (E, G_1, G_2)$ coincides with the Grassmannian $Gr^\tau_{E_0,E_1}$.

6.4.2. Flag varieties. Let $\mathcal{I} = (I, \leq)$ be an ordered set regarded as a category. Let B be the discrete category with the set of objects I. Let D_1 coincide with $(I, \leq)$. Finally, D_2 is the category with $ObD_2 = I$ and the set of morphisms generated by morphisms $y \xrightarrow{e_{xy}} x$ and $x \xrightarrow{m_{yx}} y$ defined for all $x, y \in I$ such that $x \leq y$, which satisfy the relations: $e_{xy} m_{yx} = id_x$, and for any $x \leq y \leq z$, $e_{xy} e_{yz} = e_{xz},\ m_{zy} m_{yx} = m_{zx}$. In particular, there are projections $m_{yx} e_{xy} : y \longrightarrow y$. The functors $B \xrightarrow{G_1} D_1$ and $D_1 \xrightarrow{G_2} D_2$ are natural embeddings. Fix a functor $B \xrightarrow{E} \mathcal{C},\ x \longmapsto E_x$. The locally affine space corresponding to the data $\mathfrak{E} = (E, G_1, G_2)$ will be denoted by $\mathbf{Fl}^\tau_{E,\mathcal{I}}$ and called the *flag variety* corresponding to the ordered set $\mathcal{I} = (I, \leq)$ and the map $E : I \longrightarrow Ob\mathcal{C}$.

Taking $\mathcal{I} = (x_1 \leq x_0)$, we recover back Grassmannians.

6.5. Cutting off objects. Return to the general setting: the data $\mathfrak{E}$ consisting of three small categories, $B,\ D_1$ and D_2, and three functors,

$$D_2 \xleftarrow{G_2} D_1 \xleftarrow{G_1} B \xrightarrow{E} \mathcal{C}. \tag{1}$$

and a topology τ on $\mathbf{Aff}_{\mathcal{A}^\sim}$.

6.5.1. Assume that the functors $G_0,\ G_1$ are injective on objects. Let D'_1 be the full subcategory of the category D_1 whose objects are all objects of D_1 which are isomorphic to some objects of $G_1(B)$. Similarly, we denote by D'_2 the full subcategory the category D_2 whose objects are all objects of D_2 which are isomorphic to some objects of $G_2(D'_1)$. The functors $G_1,\ G_2$ induce functors respectively $B \xrightarrow{G'_1} D'_1$ and $D'_1 \xrightarrow{G'_2} D'_2$ and we have a natural morphism, γ, from the data $\mathfrak{E}' = (D'_2 \xleftarrow{G'_2} D'_1 \xleftarrow{G'_1} B \xrightarrow{E} \mathcal{C})$ to the data $\mathfrak{E} = (D_2 \xleftarrow{G_2} D_1 \xleftarrow{G_1} B \xrightarrow{E} \mathcal{C})$ (cf. 6.1.1). By functoriality, to this morphism there corresponds a canonical presheaf morphism

$$\mathfrak{F}_\gamma : \mathfrak{F}^\tau_{\mathfrak{E}} \longrightarrow \mathfrak{F}^\tau_{\mathfrak{E}'}. \tag{2}$$

Note that the data $\mathfrak{E}'$ satisfies the assumptions of 6.2, hence the presheaf $\mathfrak{F}^\tau_{\mathfrak{E}'}$ is a locally affine space. The following example shows that in some cases the morphism (2) is an isomorphism.

6.5.2. Example: from flag varieties to varieties of generic flags. Let $\mathcal{I} = (I, \le)$ be an ordered set with the initial element $*$. Let B be the discrete category with one object $*$. Let D_1 coincide with $(I, \le)$. Finally, D_2 is the category with $ObD_2 = I$ and the set of morphisms generated by morphisms $y \xrightarrow{e_{xy}} x$ and $x \xrightarrow{m_{yx}} y$ defined for all $x, y \in I$ such that $x \le y$, which satisfy the relations: $e_{xy}m_{yx} = id_x$, and for any $x \le y \le z$, $e_{xy}e_{yz} = e_{xz}$, $m_{zy}m_{yx} = m_{zx}$. The functors $B \xrightarrow{G_1} D_1$ and $D_1 \xrightarrow{G_2} D_2$ are natural embeddings. The functor $B \xrightarrow{E} \mathcal{C}$ maps the object $*$ to an object E of the category $\mathcal{C}$. Applying the precedure of 6.5.1, we obtain the data $\mathfrak{E}' = (D_2' \leftarrow D_1' \leftarrow B \longrightarrow \mathcal{C})$, where $D_1' = B = (*)$ and D_2' is the category with one object $*$ and the set of morphisms $D_2'(*, *)$ generated by arrows $\{p_x |\ x \in I\}$ satisfying the conditions: $p_x p_y = p_x$ if $x \le y$. In particular, all p_x are projectors: $p_x^2 = p_x$. The corresponding locally affine space $\mathfrak{F}^\tau_{\mathfrak{E}'}$ is the generic flag variety Fl^τ_E of E. We recover generic Grassmannians taking $\mathcal{I} = \{0, 1\}$.

6.5.2.1. Proposition. *The canonical morphism* $\mathfrak{F}^\tau_{\mathfrak{E}} \longrightarrow \mathfrak{F}^\tau_{\mathfrak{E}'} = Fl^\tau_E$ *is an isomorphism.*

Proof. A proof is implicitly contained in the argument of 5.1. Details are left to the reader. ∎

6.5.3. Partly generic flags. Let $\mathcal{I} = (I, \le)$ be an ordered set, and let I_0 is a subset of I such that for any $x \in I$, there exists an element y of I_0 such that $y \le x$. Let B is the discrete category with the set of objects I_0, and the categories D_1, D_2 are as in 6.5.2. The functors $G_1 : B \longrightarrow D_1$ and $G_2 : D_1 \longrightarrow D_2$ are natural embeddings. Fix a functor $E : B \longrightarrow \mathcal{C}$, $x \longmapsto E_x$, $x \in I_0$. Applying the precedure of 6.5.1, we obtain the data $\mathfrak{E}' = (D_2' \leftarrow D_1' \leftarrow B \longrightarrow \mathcal{C})$, where D_i' is the full subcategory of D_i, $i = 1, 2$, such that $ObD_1 = ObD_2 = I_0$. By 6.2, $\mathfrak{F}^\tau_{\mathfrak{E}'}$ is a locally affine space. Clearly the flag variety of 6.4.2 and the generic flag variety of 6.5.2 are particular extreme cases of this example. By an obvious reason, we call $\mathfrak{F}^\tau_{\mathfrak{E}'}$ *variety of partly generic flags.* We denote it by $Fl^\tau_{E,I_0,\mathcal{I}}$. As in the particular case 6.5.2, the canonoical morphism $\mathfrak{F}^\tau_{\mathfrak{E}'} \longrightarrow Fl^\tau_{E,I_0,\mathcal{I}}$ is an isomorphism.

6.6. Example. Consider the setting of 6.5 with the category D_2 consisting of three objects, x, y, z, and generating arrows

$$\begin{array}{ccc} x & \xrightarrow{\ a\ } & y \\ i \nwarrow \searrow b & & \nearrow c \\ & z & \end{array}$$

subject to the relations $b \circ i = id_z$, $c \circ b = a$ which imply that $\mathcal{C} = a \circ i$ and $e = i \circ b$ is an idempotent. Let D_1 be the subcategory

$$\begin{array}{ccc} x & \xrightarrow{\ a\ } & y \\ b \searrow & & \nearrow c \\ & z & \end{array}$$

and let B be the subcategory $x \xrightarrow{a} y$. The functors $B \xrightarrow{G_1} D_1 \xrightarrow{G_2} D_2$ are natural embeddings. Fix a functor $B \xrightarrow{E} \mathcal{C}$. Applying the procedure of 6.5.1, we obtain the functors

$$D_2' \xleftarrow{G_2'} D_1' = B \xleftarrow{G_1'} B \xrightarrow{E} \mathcal{C},$$

where G_1' is the identical functor and D_2' is the category generated by $x \xrightarrow{e} x \xrightarrow{a} y$ subject to the relations $e^2 = e,\ a \circ e = a$.

6.6.1. Proposition. *The canonical morphism $\mathfrak{F}^{\tau}_{\mathfrak{E}} \longrightarrow \mathfrak{F}^{\tau}_{\mathfrak{E}'} = \mathfrak{F}^{\tau}_{\mathfrak{E}}$ is an isomorphism.*

Proof is left to the reader. ∎

6.6.2. Corollary. *The presheaf $\mathfrak{F}^{\tau}_{\mathfrak{E}}$ is a locally affine space.*

More explicitly, we have a canonical exact diagram

$$\mathfrak{R}_{\mathfrak{E}} \rightrightarrows \mathfrak{L}_{\mathfrak{E}} \longrightarrow \mathfrak{F}^{\tau}_{\mathfrak{E}}, \tag{1}$$

where the affine cover $\mathfrak{L}_{\mathfrak{E}}$ is given by the relations

$$s \circ e = s, \quad e^2 = e \tag{2}$$

and the affine scheme (of relations) $\mathfrak{R}_{\mathfrak{E}}$ is defined by

$$s \circ e_i = s, \quad e_i e_j e_i = e_i, \quad e_i^2 = e_i, \quad i = 1, 2. \tag{3}$$

Here $s = E(a)$ and $e = E(i \circ b)$.

If the morphism s is trivial (either identical or zero), we recover the generic Grassmannian. In general, due to the presence of a fixed non-canonical morphism s in the relations (2) and (3), the locally affine space has a little chance to have good properties (say, to be formally smooth) unless s is 'good'.

6.7. Base change. Fix a data $\mathfrak{E} = \big(D_2 \xleftarrow{G_2} D_1 \xleftarrow{G_1} B \xrightarrow{E} \mathcal{C}\big)$. Due to the universality of our constructions, the diagram $\mathfrak{R}_{\mathfrak{E}} \rightrightarrows \mathfrak{L}_{\mathfrak{E}}$, is compatible with the base change. That is for any affine scheme $\mathfrak{Sp}(S)$, we have a canonical commutative diagram with isomorphic horizontal arrows

$$\begin{array}{ccc} \mathfrak{Sp}(S) \times \mathfrak{R}_{\mathfrak{E}} & \xrightarrow{\sim} & \mathfrak{R}_{S\circ\mathfrak{E}} \\ \downarrow\downarrow & & \downarrow\downarrow \\ \mathfrak{Sp}(S) \times \mathfrak{L}_{\mathfrak{E}} & \xrightarrow{\sim} & \mathfrak{L}_{S\circ\mathfrak{E}} \end{array} \tag{1}$$

Here $S \circ \mathfrak{E} = \big(D_2 \xleftarrow{G_2} D_1 \xleftarrow{G_1} B \xrightarrow{S\circ E} S - mod_{\mathcal{C}}\big)$.

This implies that the diagram $\mathfrak{R}_{\mathfrak{E}} \rightrightarrows \mathfrak{L}_{\mathfrak{E}} \to \mathfrak{F}^{\tau}_{\mathfrak{E}}$ is compatible with the base change. In particular, we have a unique isomorphism $\mathfrak{Sp}(S) \times \mathfrak{F}_{\mathfrak{E}} \longrightarrow \mathfrak{F}_{S\circ\mathfrak{E}}$ which makes the diagram

$$\begin{array}{ccc} \mathfrak{Sp}(S) \times \mathfrak{R}_{\mathfrak{E}} & \xrightarrow{\sim} & \mathfrak{R}_{S\circ\mathfrak{E}} \\ \downarrow\downarrow & & \downarrow\downarrow \\ \mathfrak{Sp}(S) \times \mathfrak{L}_{\mathfrak{E}} & \xrightarrow{\sim} & \mathfrak{L}_{S\circ\mathfrak{E}} \\ \downarrow & & \downarrow \\ \mathfrak{Sp}(S) \times \mathfrak{F}^{\tau}_{\mathfrak{E}} & \xrightarrow{\sim} & \mathfrak{F}^{\tau}_{S\circ\mathfrak{E}} \end{array} \tag{2}$$

commute.

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