A new example of $N = 2$ supersymmetric Landau-Ginzburg theories: the two-ring case

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Abstract

The new example of $N = 2$ supersymmetric Landau-Ginzburg theories is considered when the critical values of the superpotential $w(x)$ form the regular two-ring configuration. It is shown that at the deformation, which does not change the form of this configuration, the vacuum state metric satisfies the equation of nonabelian 2×2 Toda system.

In the very interesting paper by Cecotti and Vafa [1] they have considered N=2 supersymmetric Landau-Ginzburg theories and have showed that in many cases the metric for supersymmetric ground states for special deformations of this metric satisfies the certain system of PDE's, such for example as Toda equations. Further development related to the theory see [2, 3] .

In the note [4] two new examples of such theories were considered. The purpose of the present paper is to give ones more additional example of such theories. Namely we will show here that the two-ring case gives instead the standard Toda system, so-called the nonabelian 2×2 Toda system.

1. Let us remind first at all some basic facts from N=2 supersymmetric Landau-Ginzburg theory (for more details see [1]). The basic quantities here are the chiral fields ϕ_i, the vacuum state $|0>$ and the states

$$|j>=\phi_j|0> \tag{1}$$

The action of ϕ_j on this state gives

$$\phi_i|j>=\phi_i\phi_j|0>=C^k_{ij}\phi_k|0>=C^k_{ij}|k>. \tag{2}$$

So the action of the chiral field ϕ_i in the subsector of vacuum states is given by the matrix $(C_i)^k_j = C^k_{ij}$. Analogously, we have anti-chiral fields $\phi_{\bar{i}}$ and the states $|\bar{j}>$. So we may define two metric tensors

$$\eta_{ij}=<j|i> \tag{3}$$

and

$$g_{i\bar{j}}=<\bar{j}|i>, \tag{4}$$

which should satisfy the condition

$$\eta^{-1}g(\eta^{-1}g)^*=1. \tag{5}$$

The theory is determined by the superpotential $w(x_j)$ which is a holomorphic function of complex variables x_i. The superpotential completely determines the chiral ring

$$\mathcal{R}=\mathbf{C}[x_i]/\partial_i w \tag{6}$$

and we may also determine the metric η_{ij} by the formula

$$\eta_{ij}=<i|j>=Res_w[\phi_i\phi_j], \tag{7}$$

where

$$Res_w[\phi]=\sum_{dw=0}\phi(x)H^{-1}(x); \quad H=det(\partial_i\partial_j w). \tag{8}$$

As for the metric $g_{i\bar{j}}$, then, as was shown in [1], it should satisfy the zero-curvature conditions

$$\bar{\partial}_i(g\partial_j g^{-1})-[C_j,g(C_i)^+g^{-1}]=0, \tag{9}$$

$$\partial_i C_j - \partial_j C_i + [g(\partial_i g^{-1}), C_j] - [g(\partial_j g^{-1}), C_i] = 0. \tag{10}$$

Note also that the metric g_{ij} should satisfy the " reality constraint "

$$\eta^{-1} g (\eta^{-1} g)^* = 1. \tag{11}$$

2. Let us begin to describe the system. As for the superpotential $w(x)$ we take

$$w(x) = t(\frac{x^{2n+1}}{2n+1} + 2c\frac{x^{n+1}}{n+1} - x). \tag{12}$$

Then

$$w'(x) = t(x^{2n} + 2cx^n - 1). \tag{13}$$

So the zeros of $w'(x)$ are located on two rings

$$w'(x) = \prod_{j=0}^{n-1}(x - a_j)\prod_{j=0}^{n-1}(x - b_j), \tag{14}$$

where

$$a_j = a\omega^j, \quad b_j = b\omega^j\epsilon, \tag{15}$$

$$\omega = \exp\frac{2i\pi}{n}, \epsilon = \exp\frac{i\pi}{n}, a < b, \quad ab = 1, \quad b^n - a^n = 2c. \tag{16}$$

So the chiral ring take the form

$$\mathcal{R} = \mathbf{C}[x]/w'(x) = \mathbf{C}[x]/(x^{2n} + 2cx^n - 1). \tag{17}$$

Then

$$w''(x) = nx^{n-1}[(x^n + b^n) + (x^n - a^n)] \tag{18}$$

and

$$w''(a_j) = \alpha a_j^{-1}, \quad w''(b_j) = \beta b_j^{-1} \tag{19}$$

with

$$\alpha = na^n(a^n + b^n), \quad \beta = nb^n(a^n + b^n). \tag{20}$$

3. Now using (8) we can calculate the metric tensor

$$\eta_{ij} =< i|j > . \tag{21}$$

We have

$$Res_w(x^k) = \sum_{dw=0}\frac{x^k}{w''(x)} = \sum_{j=0}^{n-1}(\frac{a_j^{k+1}}{\alpha} + \frac{b_j^{k+1}}{\beta}) = (\frac{a^{k+1}}{\alpha} + (-1)^{\frac{k+1}{n}}\frac{b^{k+1}}{\beta})\sum_{j=0}^{n-1}(\omega^j)^{k+1} =$$

$$= \begin{cases} 0, & \text{if } n \nmid (k+1); \\ \frac{a^{k+1-n}+(-1)^{(k+1)/n}b^{k+1-n}}{t(a^n+b^n)}, & \text{if } n \mid k+1. \end{cases} \tag{22}$$

So for $0 \le i \le 2n-1, \quad 0 \le j \le 2n-1$ we have

$$Res_w(x^{i+j}) = \begin{cases} 0, & \text{if } n \nmid (i+j+1), \\ 0, & \text{if } i+j+1 = n, \\ 1/t, & \text{if } i+j+1 = 2n, \\ -(2c)/t, & \text{if } n+j+1 = 3n. \end{cases} \tag{23}$$

and the matrix η_{ij} in the basis $\sqrt{t}\{1, x, \dots, x^{2n-1}\}$ takes the form

$$\begin{pmatrix} 0 & J \\ J & -2cJ \end{pmatrix}, \tag{24}$$

where J is the $n \times n$ matrix

$$J = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix} \tag{25}$$

4. Structure of chiral algebra.

Note first at all that since

$$x^{2n} + 2cx^n - 1 = \prod_{j=0}^{n-1}(x-a_j)(x-b_j), \tag{26}$$

where $a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1}$ are distinct, we have the $\mathbf{C}$-algebra isomorphism

$$\mathcal{R} = \mathbf{C}[x]/(x^{2n}+2cx^n-1) \simeq \bigoplus_{j=0}^{n-1} \mathbf{C}[x]/(x-a_j) \oplus \bigoplus_{j=0}^{n-1} \mathbf{C}[x]/(x-b_j) \tag{27}$$

(by the Chinese Reminder Theorem).

Let $\delta_j, \delta_j' \quad (j = 0, \dots, n-1)$ be the elements in $\mathcal{R}$

$$\delta_j \Longleftrightarrow (0, \dots, \frac{\bar{x}}{a_j}, 0, \dots, 0), \frac{\bar{x}}{a_j} \in \mathbf{C}[x]/(x-a_j),$$

$$\delta_j' \Longleftrightarrow (0, \dots, 0, \dots, \frac{\bar{x}}{b_j}, \dots, 0), \frac{\bar{x}}{b_j} \in \mathbf{C}[x]/(x-b_j). \tag{28}$$

Then $\{\delta_0, \ldots, \delta_{n-1}, \delta'_0, \ldots, \delta'_{n-1}\}$ is a basis of $\mathcal{R}$, and its relation with the monomial basis $\{1, x, \ldots, x^{2n-1}\}$ is given by the formula

$$\begin{pmatrix} 1 \\ x \\ \vdots \\ x^{2n-1} \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 \\ a & a\omega & \ldots & & b\epsilon & b\epsilon\omega & \ldots & b\epsilon\omega^{n-1} \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ a^{2n-1} & a^{2n-1}\omega^{2n-1} & \ldots & & b^{2n-1}\epsilon^{2n-1} & & \ldots & \end{pmatrix} \begin{pmatrix} \delta_0 \\ \delta_1 \\ \vdots \\ \delta_{n-1} \\ \delta'_0 \\ \vdots \\ \delta'_{n-1} \end{pmatrix}. \tag{29}$$

Therefore the topological-topological coupling with respect to basis $\{\sqrt{t}(\delta_0, \ldots, \delta_{n-1}, \delta'_0, \ldots, \delta'_{n-1})\}$ is given by formula

$$\frac{1}{n(a^n + b^n)} \begin{pmatrix} a^{-(n-1)} \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & \omega & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \omega^{n-1} \end{pmatrix} & 0 \\ 0 & b^{-(n-1)}\epsilon \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & \omega & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \omega^{n-1} \end{pmatrix} \end{pmatrix} \tag{30}$$

5. Lemma. The group of **C**-algebra automorphisms of $\mathcal{R}$ is generated by the transformation

$$\theta: x \to \omega x. \tag{31}$$

Hence

$$\theta: \begin{pmatrix} 1 \\ x \\ \vdots \\ x^{2n-1} \end{pmatrix} \to \begin{pmatrix} \begin{pmatrix} 1 & 0 & & 0 \\ 0 & \omega & & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \omega^{n-1} \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & 0 & & 0 \\ 0 & \omega & & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \omega^{n-1} \end{pmatrix} \end{pmatrix} \begin{pmatrix} 1 \\ x \\ \vdots \\ x^{2n-1} \end{pmatrix}, \tag{32}$$

$$
\theta \colon \begin{pmatrix} \delta_0 \\ \delta_1 \\ \vdots \\ \delta'_0 \\ \vdots \\ \delta'_{n-1} \end{pmatrix} \to \begin{pmatrix} \begin{pmatrix} 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \delta_0 \\ \delta_1 \\ \vdots \\ \delta'_0 \\ \vdots \\ \delta'_{n-1} \end{pmatrix}. \tag{33}
$$

Let us go now to the consideration of topological-antitopological couplings $(g_{i\bar{j}})$. As was shown in [1] this matrix should satisfy the equation

$$
\bar{\partial}(g\partial g^{-1}) - [C, gC^{+}g^{-1}] = 0. \tag{34}
$$

and the reality consraint

$$
\eta^{-1}g(\eta^{-1}g)^{*} = 1. \tag{35}
$$

Note first of all that in the ring $\mathcal{R}$ we have

$$
w = t\Big(\frac{x^{2n+1}}{2n+1} + 2c\frac{x^{2n+1}}{n+1} - x\Big) = \frac{-2nt}{2n+1}\Big(x - \frac{c}{n+1}x^{n+1}\Big). \tag{36}
$$

Then, to the multiplication to x in the basis $\{\sqrt{t}, \sqrt{t}x, \dots, \sqrt{t}x^{2n-1}\}$ there corresponds the matrix

$$
\begin{pmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ 1 & & & & (-2c) & \dots & 0 \end{pmatrix} \tag{37}
$$

and to the multiplication to x^{n+1} there corresponds the matrix

$$
\begin{pmatrix} & & & & 0 & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \\ 1 & & & & (-2c) & & & 0 \\ 0 & 1 & & & 0 & (-2c) & & \\ & \ddots & \ddots & & & & & \\ & & & 1 & & & & (-2c) \\ (-2c) & & & 0 & (1+4c^2) & & & 0 \end{pmatrix}. \tag{38}
$$

Hence the matrix C in this basis has the form

$$C = \begin{pmatrix} 0 & 1 & & & & & & & \\ & \ddots & \ddots & & & & & & \\ & & 0 & 1 & & & & & \\ & & & 0 & 1 & & & & \\ & & & & 0 & 1 & & & \\ & & & & & \ddots & \ddots & & \\ 1 & & & & (-2c) & & & & 0 \end{pmatrix} -$$

$$\frac{c}{n+1} \begin{pmatrix} & & & 0 & & 1 & \dots & 0 \\ & & & & & & \ddots & \\ 1 & & & -2c & & & & 1 \\ 0 & 1 & & & -2c & & & \\ & & \ddots & \vdots & & & \ddots & \\ & & 1 & 0 & & & & -2c \\ -2c & & 0 & 1+4c^2 & & & & 0 \end{pmatrix}. \tag{39}$$

As for the matrix $(g_{i\bar{j}})$ it has to be invariant relative to the automorphism θ of $\mathcal{R}$ and hence it should have the form

$$\begin{pmatrix} g_{0\bar{0}} & 0 & \dots & 0 & g_{0\bar{n}} & 0 & \dots & 0 \\ 0 & \ddots & \dots & \vdots & 0 & \ddots & \dots & \vdots \\ \vdots & & \ddots & 0 & & & \ddots & 0 \\ 0 & & & g_{n-1,\overline{n-1}} & & & & g_{n-1,\overline{2n-1}} \\ \bar{g}_{0\bar{n}} & 0 & \dots & 0 & \ddots & & & \\ 0 & \ddots & & & & \ddots & & \\ \vdots & & \ddots & & & & \ddots & \\ 0 & & & \bar{g}_{n-1,\overline{2n-1}} & & & & g_{2n-1,\overline{2n-1}} \end{pmatrix}. \tag{40}$$

Let us consider now the reality constraint. For this purposes it is convenient to use the basis ϕ_j in $\mathcal{R}$:

$$\sqrt{t}\{1, x, \dots . x^{n-1}, (x^n + c), \dots, (x^{2n-1} + cx^{n-1})\}, \tag{41}$$

so that in this basis the matrix η take very simple form

$$\eta \to \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}, \tag{42}$$

where the matrix J is given by the formula (25). From this we have

$$\eta = \eta^* = \eta^{-1} \tag{43}$$

and from the reality constraint and $g^* = g^t$, it follows

$$g \in SO(J, \mathbf{C^{2n}}) \tag{44}$$

$$g = \exp\{i \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix}\} \tag{45}$$

Let us consider now the behaviour at $c \to 0$. In the basis $\{1, x, \ldots, x^{2n-1}\}$ we have

$$(g_{i\bar{j}}) = (g^{(0)}_{i\bar{j}}) + c \begin{pmatrix} & & & h_1 & & \\ & 0 & & & \ddots & \\ & & & & & h_n \\ \bar{h}_1 & & & & & \\ & \ddots & & & & 0 \\ & & \bar{h}_n & & & \end{pmatrix}, \tag{46}$$

$$C = \begin{pmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix} + cC' \tag{47}$$

So that in the limit $c \to 0$ we have

$$w(x) \sim t(\frac{x^{2n+1}}{2n+1} - x) \tag{48}$$

and we get the Toda equations.

It is also possible to got the precise results for arbitrary value of it, if we use the theory of Tchebichef polynomials.

Let us remind that the Tchebichef polynomial $U_k(t)$ is defined by the recurrent relation

$$U_{k+1}(t) = 2tU_k(t) - U_{k-1}(t) \tag{49}$$

and by "initial conditions"

$$U_0(t) = 1, \quad U_1(t) = 2t. \tag{50}$$

So we have

$$U_k(\cos\theta) = \frac{\sin(k+1)\theta}{\sin\theta} \tag{51}$$

and the generating function for these polynomial

$$\frac{1}{1 - 2tz + z^2} = U_0(t) + U_1(t)z + \ldots + U_k(t)z^k + \ldots; \quad |z| < 1; \quad |t| < 1. \tag{52}$$

Substituting $z \to iz, t \to it$, we obtain

$$\frac{1}{1+2tz-z^2} = \tilde{U}_0(t) + \tilde{U}_1(t)z + \ldots + \tilde{U}_k(t)z^k + \ldots \tag{53}$$

where

$$\tilde{U}_k(t) = i^k U_k(it). \tag{54}$$

Then we have

$$\tilde{U}_{k+1}(t) = -2t\tilde{U}_k(t) + \tilde{U}_{k-1}(t), \tag{55}$$

$$\tilde{U}_0(t) = 1, \quad \tilde{U}_1(t) = -2t, \tag{56}$$

and

$$\tilde{U}_k(-i\cos\theta) = i^k \frac{\sin(k+1)\theta}{\sin\theta}. \tag{57}$$

Now it easy to prove the

Lemma. For $d \geq 0$

$$\frac{x^{d+2}}{x^2+2tx-1} = \tilde{U}_0(t)x^d + \tilde{U}_1(t)x^{d-1} + \ldots + \tilde{U}_d(t) + \frac{\tilde{U}_{d+1}(t)x + \tilde{U}_d(t)}{x^2+2tx-1} \tag{58}$$

Proof. Let us take the change $z \to x^{-1}$ in (52). We have

$$\frac{x^2}{x^2+2tx-1} = \tilde{U}_0(t) + \tilde{U}_1(t)\frac{1}{x} + \ldots \tilde{U}_n(t)\frac{1}{x^n} + \ldots, \quad |x| > 1. \tag{59}$$

Hence

$$\frac{x^{d+2}}{x^2+2tx-1} = \tilde{U}_0(t)x^d + \tilde{U}_1(t)x^{d-1} + \ldots + \tilde{U}_d(t) + \frac{\tilde{U}_{d+1}(t)}{x} + \frac{\tilde{U}_{d+2}(t)}{x^2} + \ldots \tag{60}$$

and we have

$$\frac{x^{d+2}}{x^2+2tx-1} = \tilde{U}_0(t)x^d + \tilde{U}_1(t)x^{d-1} + \ldots + \tilde{U}_d(t) + \frac{\tilde{U}_{d+1}(t)x + C(t)}{x^2+2tx-1} \tag{61}$$

If we put here $x = 0$ we obtain $C(t) = \tilde{U}_d(t)$.

To obtain explicit expression for matrix C, note that this matrix is determined by the multiplication to the element $(x - \frac{c}{n+1}x^{n+1})$ in the ring $\mathcal{R}$.

Hence

$$C^n(1) = (x - \frac{c}{n+1}x^{n+1})^n = (1 - \frac{c}{n+1}x^n)^n x^n = A_n(c) + B_n(c)x^n, \tag{63}$$

$$C^n(x^n) = B_n(c) + (A_n(c) - 2cB_n(c))(x^n).$$

So

$$\mathbf{C}\cdot 1+\mathbf{C}\cdot x^n \rightarrow^{C^n} \mathbf{C}\cdot 1+\mathbf{C}\cdot x^n, \tag{64}$$

$$\begin{pmatrix} C^n(1) \\ C^n(x^n) \end{pmatrix} = \begin{pmatrix} A & B \\ B & A-2cB \end{pmatrix} \begin{pmatrix} 1 \\ x^n \end{pmatrix}, \tag{65}$$

$$A = A_n, \quad B = B_n.$$

¿From here we obtain the characteristic equation

$$T^2 - 2(A-cB)T + (A^2 - 2cAB - B^2) = (T-\lambda)(T-\mu), \tag{66}$$

where

$$\lambda = A - cB + \sqrt{1+c^2}B = A - (c - \sqrt{1+c^2})B, \tag{67}$$

$$\mu = A - cB - \sqrt{1+c^2}B = A - (\sqrt{1+c^2}+c)B.$$

Let us define

$$\lambda_n(t) = A_n(t) - (t - \sqrt{1+t^2})B_n(t), \tag{68}$$

$$\mu_n(t) = A_n(t) - (t + \sqrt{1+t^2})B_n(t).$$

We have

$$B_n(c) \neq 0 \Leftrightarrow \lambda_n(c) \neq \mu_n(c). \tag{69}$$

In this situation

$$\mathbf{C}\cdot 1+\mathbf{C}\cdot x^n = \mathbf{C}\cdot\phi + \mathbf{C}\cdot\phi'$$

with

$$\phi = 1 + (\frac{\lambda_n(c)}{B_n(c)} - \frac{A_n(c)}{B_n(c)})x^n, \tag{70}$$

$$\phi' = 1 + (\frac{\mu_n(c)}{B_n(c)} - \frac{A_n(c)}{B_n(c)})x^n.$$

Let us define

$$\phi_0 = \phi, \quad \phi_1 = \frac{c\phi}{(\lambda_n(c))^{1/n}}, \ldots, \phi_j = \frac{c^j\phi}{(\lambda_n(c))^{j/n}}, \ldots, \phi_{n-1} = \frac{c^{n-1}\phi}{(\lambda_n(c))^{(n-1)/n}}, \tag{71}$$

$$\phi'_0 = \phi', \quad \phi'_1 = \frac{c\phi'}{\mu_n(c)^{1/n}}, \ldots, \phi'_j = \frac{c^j\phi'}{\mu_n(c)^{j/n}}, \ldots, \phi'_{n-1} = \frac{c^{n-1}\phi'}{(\mu_n(c))^{(n-1)/n}}. \tag{72}$$

Then $\{\phi_j, \phi'_j\}_{j=0}^{n-1}$ is the basis of $\mathcal{R}$ and we have

$$C \begin{pmatrix} \phi_0 \\ \vdots \\ \phi_{n-1} \\ \phi'_0 \\ \vdots \\ \phi'_{n-1} \end{pmatrix} =$$

$$
\begin{pmatrix}
(\lambda_n(c))^{1/n}\begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix} & 0 \\
0 & (\mu_n(c))^{1/n}\begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}
\end{pmatrix}
\begin{pmatrix} \phi_0 \\ \vdots \\ \phi_{n-1} \\ \phi'_0 \\ \vdots \\ \phi'_{n-1} \end{pmatrix}. \tag{73}
$$

It is more convenient to rewrite the equation (73) in another basis

$$
C\begin{pmatrix} \phi_0 \\ \phi'_0 \\ \phi_1 \\ \phi'_1 \\ \vdots \\ \phi_{n-1} \\ \phi'_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & D & & & \\ & 0 & D & & \\ & & \ddots & \ddots & \\ & & & \ddots & D \\ D & & & & 0 \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi'_0 \\ \phi_1 \\ \phi'_1 \\ \vdots \\ \phi_{n-1} \\ \phi'_{n-1} \end{pmatrix}, \tag{74}
$$

where

$$
D = \begin{pmatrix} (\lambda_n(c))^{1/n} & 0 \\ 0 & (\mu_n(c))^{1/n} \end{pmatrix}. \tag{75}
$$

In this basis the matrix of topological- antitopological couplings has the form

$$
\begin{pmatrix} G_0 & & & \\ & G_1 & & \\ & & \ddots & \\ & & & G_{n-1} \end{pmatrix} \tag{76}
$$

and the basic equation (9) takes the form

$$
\bar{\partial}(G_j \partial G_j^{-1}) - G_{j+1}G_j^{-1} + G_j G_{j-1}^{-1} = 0, \tag{77}
$$

$$
i = 0, \dots, n-1; \quad G_i = G_{i+n}.
$$

So this equation is the equation for the nonabelian Toda system, where quantities G_j are elements of the nonabelian group $SL(2, \mathbf{R})$.

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