A NEW PROOF AND AN EXTENSION OF A THEOREM OF MILLINGTON ON THE MODULAR GROUP

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A New Proof and an Extension of a Theorem of Millington on the Modular Group

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§1. Discussion and statement of the result

Let $\Gamma = PSL_2(\mathbb{Z})$ be the modular group acting canonically on the upper half plane $\mathcal{H}$. Let Φ be a subgroup of Γ of finite index d. The geometric invariants of Φ include g (resp. t) the genus (resp. the number of cusps) of $\Phi\backslash\mathcal{H}$, and e_2 (resp. e_3) the number of elliptic branch points with branching index 2 (resp. 3) on $\Phi\backslash\mathcal{H}$. The well-known relationship among these numbers is the Riemann-Hurwitz formula:

$$(1.1) \qquad d = 3e_2 + 4e_3 + 12g + 6t - 12.$$

A remarkable theorem of Millington, cf. [5] asserts that

(1.2) Theorem. Given positive integers d,t and non-negative integers g, e_2,e_3 satisfying (1.1) there exists a subgroup of the modular group of index d having the invariants g, t, e_2, e_3 with their meanings as attached above.

This theorem extends the previous work and answers a series of questions posed by H. Petersson cf. [4],[6],[7] and the references there. The method consists in relating the existence of a subgroup of index d to the existence of certain

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+ Partially supported by the Max Planck Institut für Mathematik, Bonn and the NSF grant MCS 83-01614.

permutations in the symmetric group Σ_d, and finding these permutations explicitly, cf. [5],[8].

Evidently the problem may be generalized and formulated in purely topological terms as follows. Let Γ be a finitely generated properly discontinuous group of orientation-preserving homeomorphisms of $\mathbb{R}^2$. Then $\Gamma\backslash\mathbb{R}^2$ is an orientable surface. Let g (resp. t) be the genus (resp. the number of ends) of $\Gamma\backslash\mathbb{R}^2$, and let $m_1, m_2, \ldots, m_k$ be the branching indices of the branch points on $\Gamma\backslash\mathbb{R}^2$. The <u>signature</u> of Γ is $\{g,t;\, m_1,\ldots,m_k\}$, cf. [2],[9]. Since Γ is finitely generated, g,t,k are finite. Let Φ be a subgroup of Γ of finite index d. Then Φ has its own signature $\{h,s;\, n_1,\ldots,n_\ell\}$. The well-known relations among the two signatures are

(1.3)

i) each n_i divides some m_j.

ii) (the Riemann Hurwitz relation)

$$2-2h-s-\sum_{j=1}^{\ell}\left(1-\frac{1}{n_j}\right) = d\left\{2-2g-t-\sum_{i=1}^{k}\left(1-\frac{1}{m_i}\right)\right\}$$

iii) $g \le h, \quad t \le s \le dt$

Henceforth assume that $t \geq 1$ i.e. $\Gamma\backslash\mathbb{R}^2$ is noncompact. Then, <u>as an abstract group</u>,

$$\Gamma \cong F_{2g+t-1} * \mathop{*}_{i=1}^{k} \mathbb{Z}_{m_i}, \qquad m_i > 0$$

where henceforth F_u will denote the free group of rank u, $\mathbb{Z}_a$ will denote a finite cyclic group of order a, and $*$ will stand for the free product. Then iii) implies

$$(1.4) \qquad \text{iii)}' \quad 2g+t-1 \leq 2h+s-1.$$

In [3] we studied, Γ, Φ with prescribed signatures as above on the level of abstract groups, the problem of embedding Φ in Γ as a subgroup of finite index, and found that the obvious necessary conditions i), ii), iii)' are not sufficient. We found a further diophantine condition, cf. [3], theorem 2, and showed that i), ii), iii)' and the diophantine condition are necessary and sufficient for finding a subgroup of Γ of index d which is isomorphic to Φ. This still leaves the problem of finding effective criteria for the existence of a subgroup $\approx\Phi$ with prescribed genus and the number of ends consistent with iii). A solution of this problem would be a proper generalization of Millington's theorem. This problem however is very difficult. It is closely related to the famous Hurwitz problem on realizability of branched coverings. In fact in case $k \geq 3$ and $s = dt$ it is equivalent to the Hurwitz problem. Recently the Hurwitz problem was studied in [1], and the results therein have some implication for the problem at hand. For example, if a) $g > 0$, or b) $s \leq (t-1)d+1$ the conditions (1.3) together with the diophantine condition mentioned above are sufficient for finding a subgroup of

Γ of index d, genus h, the number of ends = s, and $\neq \phi$. Although many other cases may be settled by using the results from [1], it is usually difficult to find succinct answers covering a large class of interesting cases.

The purpose of this note is to consider one general case which includes (1.2) and which is not covered by the results in [1].

(1.5) Theorem. Let $\Gamma = \mathbb{Z}_{n_1} * \mathbb{Z}_{n_2}$ ● and $\Phi = F_r * \prod_{u=1}^{*L} \mathbb{Z}_{m_u}$ ●●,

$r = 2h + s-1$ where $h \geq 0$ and $s \geq 1$ are integers. Then Φ can be realized as a subgroup of finite index d, genus h and the number of ends s iff the following conditions hold

(α) each m_u divides n_1 or n_2,

(β) $\sum_{u=1}^{L} \frac{1}{m_u} - L - r+1 = d\{\frac{1}{n_1} + \frac{1}{n_2} - 1\}$,

(γ) let $m_0 = 1$, and $m_1,\ldots,m_\ell$ the maximal set of distinct m_u's, and each m_q, $1 \leq q \leq \ell$ occur b_q times. Set

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● So Γ has signature $\{0,1; n_1,n_2\}$, and upto a topological equivalence, it can be realized as a properly discontinuous group of orientation-preserving homeomorphisms of $\mathbb{R}^2$ in a unique way.

●● The case $\Phi = F_r$ is covered unter $\Phi = F_r * \mathbb{Z}_{m_1}$ with $m_1 = 1$.

$$\delta_{iq} = \begin{cases} \dfrac{n_i}{m_q} & \text{if } m_q | n_i \\ 0 & \text{otherwise}, \quad 1 \le i \le 2,\ 0 \le q \le \ell \end{cases}$$

Then the system

$$x_{1q} + x_{2q} = b_q , \qquad q = 1,2 \ldots \ell$$

$$\sum_{q=0}^{\ell} \delta_{iq}\, x_{iq} = d, \qquad i = 1,2$$

has a solution for x_{iq}'s in nonnegative integers, with $x_{iq} = 0$ if $m_q \nmid n_i$.

An interesting special case is when n_1, n_2 are distinct primes. Then γ) is a consequence of α) and β), cf. [3], (5.1). This explains why the condition (1.1) sufficed in Millington's theorem which deals with the case $n_1 = 2$, $n_2 = 3$.

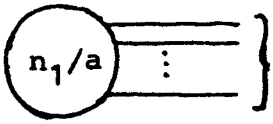
The proof of (1.5) actually is quite elementary. It replaces constructing permutations by the diagrams introduced in [3] and uses some elementary surface topology.

§2. Proof of (1.5)

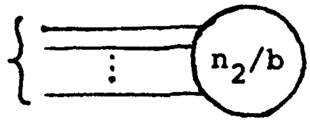
(2.1) From the discussion of (1.3) in §1 and [3], theorem 2, the conditions α), β), γ) are seen to be necessary. So we proceed to show their sufficiency. The diagrams for Γ and its thickening $\mathbb{X}_\Gamma$, cf. [3] §4 and (A.1.1) are

(2.1.1) 

respectively. A diagram corresponding to a subgroup Ψ of Γ is obtained from the building blocks

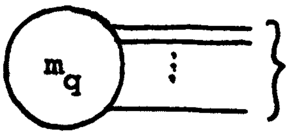
(2.1.2) 

a edges, and

(2.1.3) b edges 

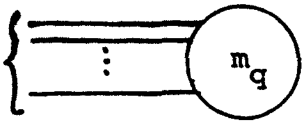
for suitable divisors a resp. b of n_1 resp. n_2 by identifying suitably the endpoints of the edges. Moreoerver if $\mathbb{X}_\psi$ is its thickening then the canonical projection int $\mathbb{X}_\psi \to$ int $\mathbb{X}_\Gamma$ provides the picture of the actual corresponding branched covering.

(2.2) Let $\{x^o_{iq}\}$ be a nonnegative integral solution of the diophantine system in γ), $1 \leq i \leq 2$, $0 \leq q \leq \ell$, and $x^o_{iq} = 0$ if $m_q \nmid n_i$. The significance of γ) is precisely that from the x^o_{1q} copies of

(2.2.1) 

n_1/m_q edges

and x_{2q}^{o} copies of

(2.2.2) n_2/m_q edges 

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$0 \le q \le \ell$, it is possible to construct a <u>connected</u> diagram which would correspond to a subgroup $\approx \Phi$.

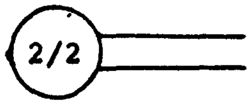
(2.3) Now we show how to obtain a subgroup $\approx \Phi$ of genus 0. Notice that $\mathbb{X}_\Gamma \approx$ a closed disk. Orient $\mathbb{X}_\Gamma$ and $\partial\mathbb{X}_\Gamma$ in some way. This induces orientations on the diagrams in (2.2.1) and (2.2.2), and on their thickenings. For simplicity we shall call the thickening of an edge in (2.2.1) or (2.2.2) as an <u>arm</u>. Notice that there is a well-defined <u>cyclic</u> <u>order</u> on the edges of each of the diagrams in (2.2.1) and (2.2.2) and also on the arms of their thickenings. Now embed the union of these thickened diagrams corresponding to (2.2.1) and (2.2.2) in $\mathbb{R}^2$ preserving orientation. We thus get a certain complex in $\mathbb{R}^2$ with, say, r_1 (resp r_2) components corresponding the diagrams in (2.2.1) (resp. (2.2.2)). For simplifying expression let us call these r_1 resp. r_2 components as the <u>first r_1 components</u> resp. the <u>last r_2 components</u> of this complex. Now we start connecting the arms of the last r_2 components with those of the first r_1 components <u>in pairs</u> so that we do not introduce any intersections (except where the arms meet) and we go on reducing the number r_1 of components as quickly as feasible until we are left with exactly one component, cf. the proof of theorem 1 in [3]. Let us call the new complex obtained this way as D_0. Evidently $D_0 \approx$ a closed disk.

Also certain of the arms of the first r_1 components (now combined into D_0) still may remain to be connected with some arms of some of the last r_2 components. But notice that the unconnected arms of D_0 have a well-defined cyclic order (w.r.t. D_0), and so also the arms of each of the last r_2 components which remain to be connected. Using these cyclic orders it <u>is</u> possible to connect the remaining arms of D_0 with those of the last r_2 components <u>in $\mathbb{R}^2$</u>, so that in this process no intersections (except where the arms meet) are introduced. In other words we obtain a thickened diagram of a subgroup $\cong \Phi$ of Γ of index d as a subset of $\mathbb{R}^2$ —— so this subgroup has genus 0.

(To motivate the reader, we interrupt the proof to illustrate the process described above by an example. Let $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_3$ and $\Phi = F_2 * \mathbb{Z}_3 * \mathbb{Z}_3$. Then the possible finite value of the index d is given by β):

$$d = \frac{\frac{1}{3} + \frac{1}{3} - 3}{\frac{1}{2} + \frac{1}{3} - 1} = 14 .$$

The diagram for Φ is constructed out of $\frac{14}{2} = 7$ copies of



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2 copies of

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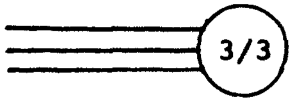
and $\frac{14 - 2}{3} = 4$ copies of



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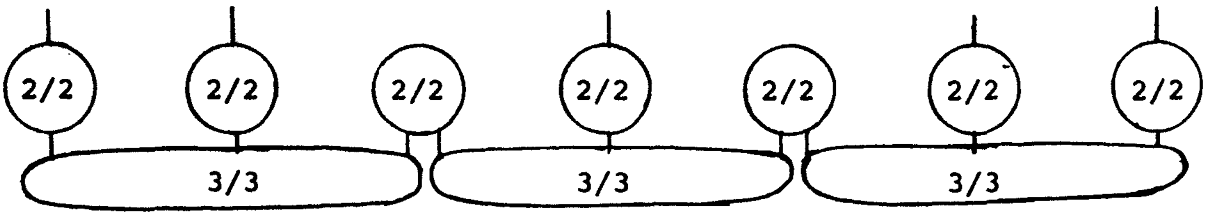
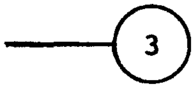
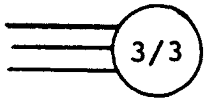
A choice of a disk D_0 is the thickened version of



There are 5 arms of D_0 which remains to be connected to the arms of the thickenings of



,


and


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Evidently these arms can be joined <u>in $\mathbb{R}^2$</u> without intersections except where they meet).

(2.4) It remains to obtain a subgroup Φ of orbitrary genus $h \geq 0$ satisfying $r = 2h+s-1$ with $s \geq 1$. Notice that the number s of ends of a subgroup Ψ of Γ is just the number of components of $\partial\mathbb{X}_\Psi$.

Let us denote a subgroup obtained by the process in (2.3) as Φ_0, so $\mathbb{X}_{\Phi_0}$ is a compact surface of genus 0 with $s = r+1$ boundary components. This $\mathbb{X}_{\Phi_0}$ is obtained from the disk D_0 (which has only one boundary component) by attaching some complexes each with a certain number of arms, and this process produces r extra boundary components. If $r = 0$ or 1 there is nothing to prove. If $r \geq 2$, then $s \geq 3$. It is an elementary fact from the topology of surfaces that as long as $s \geq 3$ we can interchange two connections of an appropriate pair of arms which reduces s by 2 and increases the genus by 1. In this way we can produce the thickened diagrams of subgroups $*\Phi$ of Γ of any prescribed genus $h \geq 0$ and number of ends $s \geq 1$ satisfying $r = 2h+s-1$.

This finishes the proof.

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