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Abstract

We examine a setting appropriate for the analysis of many variational problems. We work on the closure of the space of measures induced by embeddings of submanifolds. We prove that this space coincides with the space of measures that vanish on exact forms. We characterize the space of derivatives of variations for these objects. We use this characterization to deduce some results for the critical points of the action of very general Lagrangians.

An intermediate result of independent interest is the characterization of the distributions that can appear as derivatives of families of Borel probabilities and signed measures on manifolds.

Contents

1 Introduction

In this paper we consider the space of holonomic measures, which are roughly all measures that can be approximated by closed submanifolds. These are defined carefully in Section 2.

To motivate the definition, we mention and prove an important equivalence of two classes measures: those that can be approximated by the measures induced by embeddings of closed submanifolds, and those for which teh integrals of exact forms vanish. This is Theorem 1 in Section 2.2, and the proof is in Appendix A.

We also prove a general result on families of measures. Consider a family of measures μ_s indexed by a real parameter s with values in an interval that contains 0, and such that $\mu = \mu_0$. We say that μ_s is differentiable at 0 if for every $f \in C_c^\infty(P)$ the function $s \mapsto \int f \, d\mu_s$ is differentiable at 0 and if the derivative induces a distribution. For example, if the family is a moving Dirac delta $\mu_s = \delta_s$ on $\mathbb{R}$, then the derivative at 0 is the distribution $-\partial\delta_0$ given by $\langle -\partial\delta_0, f\rangle = f'(0)$ for all $f \in C_c^\infty(\mathbb{R})$.

We address the question of characterizing the distributions that arise in this way. In other words, we characterize the velocity vectors for curves in the space of measures that pass through μ. This result is stated and proved in Section 3.

Then we apply that to study the ways in which holonomic measures can be deformed, thus characterizing the velocity vectors of all curves in the space of holonomic measures that are differentiable. We are thus able to give a good description of what would be the tangent space to the space of holonomic measures. We do this in Section 4.

This study is fruitful, as is shown by an initial set of applications presented in Section 5. Among other things, we are able to show that the conditions we obtain for criticality are effectively more general than the classical Euler-Lagrange equations. Other applications include the formulation of a sort of n-dimensional energy-conservation principle, a version of the Hamilton-Jacobi equation, and the statement that the Lagrangian must look like an exact form in the support of the minimizers.

Related literature. Geometric measure theory and variational analysis are vast subjects, so a discussion about how this research fits in those contexts is in place. However, since it seems impossible to give an exhaustive discussion, we choose to instead give just a brief one and hence minimize the number of mistakes we make in the process due to our lack of expertise in all these fields. Also, we will not define all the objects involved, but rather we will just mention them in the hope that readers familiar with these concepts will find the information they are looking for, while readers not familiar with them will be happy to ignore the discussion.

Holonomic measures appeared in the $n = 1$ case in Mather's [15] version of Mather-Aubry theory for minimizers of the action of Lagrangians on the torus. The theory of holonomic measures was extended by others; for example by Mañé [5, 13], Bangert [3], Bernard [4]. A certain case of codimension one of Mather-Aubry theory was considered by Moser [17–19].

In the more general context we treat here, in which n can be arbitrary, a similar theory should exist for a large class of Lagrangians. Under rather mild conditions in the Lagrangian (such as convexity, superlinearity, tightness) minimizers exist in all holonomy classes with coefficients in the real numbers $\mathbb{R}$. However, Mather's α and β functions are probably only defined for a very restricted set of Lagrangians.

Holonomic measures induce superpositions of *currents* (cf. [10, 16]) on a manifold M in an obvious way. However, they carry more information than currents because they take into account the parameterization of the minimizers, and hence allow for the study of anisotropic Lagrangians.

Holonomic measures also induce *varifolds* (cf. [1, 2, 21]). Again, they carry more information because they record not only the tangent planes, but also the velocity vectors of a 'parameterization.'

With holonomic measures the issue of rectifiability is not a concern since rectifiability is built into them. Whether or not one can find their volume (or the action of a Lagrangian) depends on the question of whether this function is integrable with respect to them. They have empty perimeter, so finiteness of perimeter is also not a concern.

Holonomic measures are suitable for the treatment of many problems that could be approached parametrically using functions for example in Sobolev or Lipschitz spaces (cf. [7, 8, 12]).

Superpositions of Young measures (cf. [4, 23]) are a special case of holonomic measures.

In Section 5.2 we deduce a sort of general Hamilton-Jacobi equation, a case of which has been studied to great depth (see for example [6, 9]).

The definition of differentiability of families of measures (i.e., of variations) that we use is only one possibility of many; see for example [22] for an exploration of other possibilities.

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2 Preliminaries

2.1 Setting

Phase space. Let M be a compact, oriented C^∞ manifold of dimension $m \geq 1$, possibly with boundary ∂M. Denote by TM its tangent bundle and, for $n \geq 1$, denote by T^nM the direct sum bundle

$$T^nM = \underbrace{TM \oplus \cdots \oplus TM}_{n}$$

of n copies of TM. The dimension of T^nM is $m(n+1)$. An element in T^nM can be denoted $(x, v_1, v_2 \ldots, v_n)$, where x is a point in M and $v_1, v_2, \ldots, v_n \in T_xM$ are vectors tangent to x. When taking local coordinates, we will write

$$x = (x_1, x_2, \ldots, x_m) \quad \text{and} \quad v_i = (v_{i1}, v_{i2}, \ldots, v_{im}).$$

Sometimes for brevity we will write (x, v) instead of $(x, v_1, v_2, \ldots, v_n)$.

The projection $\pi : T^nM \to M$ is given by $\pi(x, v_1, \ldots, v_n) = x$. We denote by $\Omega^n(M)$ the space of smooth differential n-forms on M. We will often consider these forms as smooth functions on T^nM.

Throughout, when referring to functions on these objects, we will use the term *smooth* to mean C^∞. We will denote by $C^\infty(X, Y)$ the space of all smooth functions $X \to Y$. If Y is the real line $\mathbb{R}$, we will sometimes omit it in our notation. We will denote by $C_c^\infty(X)$ the set of all real-valued, compactly-supported, smooth functions on the set X.

Riemannian structure. We fix, once and for all, a Riemannian metric $g \in C^\infty(T^2M)$ on M and its corresponding Levi-Civita connection ∇. We denote the operation of covariant differentiation in the direction of a vector field F by ∇_F.

We will denote $|v| = \sqrt{g(v, v)}$ for $v \in T_xM$, and we extend this norm to T_x^nM by letting

$$|(v_1, v_2, \ldots, v_n)| = \sqrt{|v_1|^2 + |v_2|^2 + \cdots + |v_n|^2}.$$

Forms. We will denote by $\Omega^k(M)$ the space of smooth differential k-forms on M. On this space we define a norm $\|\cdot\|$ by letting, for $\omega \in \Omega^k(M)$,

$$\|\omega\| = \sup\{\omega_x(v_1, \ldots, v_k) : (x, v_1, \ldots, v_k) \in T^nM, |v_i| \leq 1\}.$$

2.2 Definition of holonomic measures and their topology

Subpower functions. We let $\mathscr{P}_n$ be the space of *subpower functions*, that is, the space of real-valued continuous functions $f \in C^0(T^nM)$ such that

$$\sup_{(x,v)\in T^nM} \frac{|f(x, v)|}{1 + |v|^n} < +\infty.$$

Note that all differential n-forms on M belong to $\mathscr{P}_n$ when regarded as functions on T^nM. We endow $\mathscr{P}_n$ with the supremum norm and its induced topology.

Mild measures. A signed measure μ on T^nM is *mild* if

$$\int_{T^nM} 1 + |(v_1, \dots, v_n)|^n d|\mu| < +\infty,$$

where $|\mu| = \mu^+ + \mu^-$ is the absolute value of the measure with Hahn decomposition $\mu = \mu^+ - \mu^-$, for positive measures μ^+ and μ^-. We denote the space of mild measures by $\mathscr{M}_n$. We define the *mass* $\mathbf{M}(\mu)$ of $\mu \in \mathscr{M}_n$ to be

$$\begin{aligned}\mathbf{M}(\mu) = \int_{T^nM} \left[\sup_{\omega \in \Omega^n(M), \|\omega\| \leq 1} \omega_x(v_1, \dots, v_n) \right] d|\mu|(x, v_1, \dots, v_n) \\ = \int_{T^nM} \mathrm{vol}_n(v_1, v_2, \dots, v_n)\, d|\mu|(x, v_1, \dots, v_n).\end{aligned}$$

This is always a nonnegative number.

The space $\mathscr{M}_n$ is natually embedded in the dual space ${\mathscr{P}_n}^*$ and we endow it with the topology induced by the weak* topology on ${\mathscr{P}_n}^*$. Although the topology on ${\mathscr{P}_n}^*$ is not metrizable, the topology on $\mathscr{M}_n$ is. We can give a metric in $\mathscr{M}_n$ by picking a sequence of functions $\{f_i\}_{i\in\mathbb{N}} \subset C_c^\infty(T^nM)$ that are dense in $\mathscr{P}_n$, and then letting

$$\mathrm{dist}_{\mathscr{M}_n}(\mu_1, \mu_2) = \mathbf{M}(\mu_1 - \mu_2) + \sum_{k=1}^{\infty} \frac{1}{2^k \sup |f_k|} \left| \int |f_k| d\mu_1 - \int |f_k| d\mu_2 \right|. \quad (1)$$

Cellular complexes. An *n-dimensional cell* (or *n-cell*) γ is a smooth map

$$\gamma : D \subseteq \mathbb{R}^n \to M,$$

where D is a subset of $\mathbb{R}^n$ homeomorphic to a closed ball, *together* with a choice of coordinates $t = (t_1, t_2, \dots, t_n)$ on D. A *chain* of n-cells is a formal linear combination of the form

$$a_1\gamma_1 + a_2\gamma_2 + \cdots + a_k\gamma_k$$

for real numbers $a_1, a_2, \dots, a_k$ and n-cells $\gamma_1, \gamma_2, \dots, \gamma_k$. We will say that a chain is *positive* if $a_i > 0$.

Let $\gamma : D \subseteq \mathbb{R}^n \to M$ be an n-cell. Denote by $d\gamma$ the differential map associating, to each element in D, an element in $T^n M$. Explicitly, if we have coordinates $t = (t_1, t_2, \dots, t_n)$ on D, then

$$d\gamma(t) = \left(\gamma(t), \frac{\partial\gamma}{\partial t_1}(t), \frac{\partial\gamma}{\partial t_2}(t), \cdots, \frac{\partial\gamma}{\partial t_n}(t)\right).$$

This map *does* depend on our choice of coordinates t.

To an n-cell γ, we associate a measure μ_γ on $T^n M$ defined by

$$\int_{T^n M} f \, d\mu_\gamma = \int_D f(d\gamma(t)) \, dt,$$

where $dt = dt_1 \wedge \cdots \wedge dt_n$. Similarly, to a chain of n-cells $\alpha = \sum_{i=1}^k a_i \gamma_i$, we associate the measure μ_α given by

$$\mu_\alpha = \sum_{i=1}^k a_i \mu_{\gamma_i}.$$

The measure μ_α is an element of $\mathscr{M}_n$. We will say that the chain α is a *cycle* if for all forms $\omega \in \Omega^{n-1}(M)$,

$$\int_{T^n M} d\omega \, d\mu_\alpha = 0.$$

Holonomic measures. The proof of the following theorem can be found in Appendix A:

Theorem 1. *Assume that $1 \le n \le d$. Let $\mu \in \mathscr{M}_n$ be a probability measure on $T^n M$. Then the following conditions are equivalent:*

(Hol) The measure μ satisfies

$$\int d\omega \, d\mu = 0$$

for all $\omega \in \Omega^{n-1}(M)$.

(Cyc) There exists a sequence $\{\alpha_k\}_{k\in\mathbb{N}}$ of cycles such that $\mu_{\alpha_k} \to \mu$ as $k \to \infty$ in the topology induced by the distance (1)*.*

A mild measure $\mu \in \mathscr{M}_n$ is *holonomic* if it is a probability (that is, a positive measure such that $\mu(T^n M) = 1$) satisfies the Conditions (Hol) and (Cyc). The space $\mathscr{H}$ of holonomic measures is convex. By the Banach-Alaoglu theorem, it is also compact, since it is a closed subset of the unit ball of ${\mathscr{P}_n}^*$ (it is cut out by the closed condition (Hol)).

2.3 Relative holonomic measures

Since our proof of Theorem 1 relies on triangulations, it is easy to modify it in order to prove

Theorem 2. *Assume that* $1 \leq n \leq d$. *Let* $\mu \in \mathscr{M}_n$ *and* $U \subset M$ *be a closed set diffeomorphic to a union of simplices of a smooth triangulation of* M. *Then the following conditions are equivalent:*

1. *For all forms* $\omega \in \Omega^{n-1}(M)$ *such that* $\omega|_U = 0$,
$$\int_{T^n M} d\omega \, d\mu = 0.$$
2. *There exists a sequence* $\{\alpha_k\}_{k\in\mathbb{N}}$ *of chains such that the boundaries* $\partial\alpha_k$ *are contained in* U, *and such that* $\wr\alpha_k\wr \to \mu$ *as* $k \to \infty$ *in the topology induced by the distance* (1).

A probability measure $\mu \in \mathscr{M}_n$ that satisfies the conditions in Theorem 2 is said to be *holonomic relative to* U. The space of all these measures is again compact and convex.

3 Distributions that arise as derivatives of families of measures

Let P be a Riemannian manifold without boundary. In Section 4.1, we will use the results in this section for $P = T^n M$.

In this section, we characterize the distributions that arise as derivatives of families of probabilities and of positive and signed Borel measures on smooth manifolds.

3.1 Distributions and measures

3.1.1 Convolutions

A *mollifier* $\psi \in C_c^\infty(\mathbb{R})$ is a function such that $\psi(x) = \psi(-x)$, $\int \psi = 1$, and $\psi \geq 0$.

We will say that a tuple of vector fields $F = (F_1, \dots, F_\ell)$ on P is *generating* if at every point $p \in P$ the vectors $F_1(p), \dots, F_\ell(p)$ span all of $T_p P$.1

Fix a generating tuple of vector fields $F = (F_1, \dots, F_\ell)$. Denote by $\phi^i : P \times \mathbb{R} \to P$ the *flow* of F_i:

$$\phi_0^i(x) = 0, \quad \frac{d\phi_s^i(x)}{ds} = F_i(\phi_s^i(x)), \quad s \in \mathbb{R}.$$

For $f \in C_c^\infty(P)$, we will denote by $P_i(f)$ the function given by

$$P_i(f)(x) = \int_{\mathbb{R}} f \circ \phi_s^i(x)\, \psi(s)\, ds.$$

This is a *convolution* in the direction F_i.

For $f \in C_c^\infty(P)$, we will denote

$$\psi *_F f := P_1 P_2 \cdots P_\ell(f).$$

3.1.2 Definition and smoothing of distributions

A *distribution* on the open set $U \subseteq \mathbb{R}^m$ is a linear functional $\eta : C_c^\infty(U) \to \mathbb{R}$ such that for each compact set $K \subset U$ there are some constants $N > 0$ and $C > 0$ (depending only on K and η) such that

$$|\langle \eta, f\rangle| \leq C \sum_{|I| \leq N} \sup_{p \in U} |\partial^I f(p)|$$

for all $f \in C_c^\infty(U)$. Here, the sum is taken over all multi-indices I with m nonnegative entries adding up to at most N, and ∂^I denotes the iterated partial derivatives in the corresponding directions in $\mathbb{R}^m$.

We fix, once and for all, an n-dimensional C^∞ manifold P without boundary, and with a Riemannian metric that induces the distance dist_P between points of P.

Let $\eta : C_c^\infty(P) \to \mathbb{R}$ be a linear functional. For a chart $\varepsilon : U \to W$ from the open set $U \subseteq P$ to the open set $W \subseteq \mathbb{R}^n$, the *pushforward* $\varepsilon_* \eta$ is defined by

$$\langle \varepsilon_* \eta, f\rangle = \langle \eta, f \circ \varepsilon^{-1}\rangle$$

for f in $C_c^\infty(W)$.

The functional η is a *distribution* if for each chart ε as above, $\varepsilon_*\eta$ is a distribution on W. We will denote by $\mathscr{D}'(P)$ the space of distributions on P. The topology on $\mathscr{D}'(P)$ is induced by the seminorms

$$\eta \mapsto |\langle \eta, f\rangle|$$

for $f \in C_c^\infty(P)$. In other words, we have $\eta_i \to \eta$ if, and only if, $\langle \eta_i, f\rangle \to \langle \eta, f\rangle$ for all $f \in C_c^\infty(P)$. We remark that any measure on P determines a distribution, but that not all distributions arise in this way.

For a distribution $\eta \in \mathscr{D}'(P)$, we define the convolution by duality:

$$\langle \psi *_F \eta, f\rangle = \langle \eta, \psi *_F f\rangle.$$

Lemma 3. *If η is a distribution in $\mathscr{D}'(P)$, F is a generating tuple of vector fields, and ψ is a mollifier, then $\psi *_F \eta$ is a smooth signed Borel measure.*

For a proof see for example [11, §5.2].

3.1.3 Structure

We fix a generating tuple $F = (F_1, \ldots, F_\ell)$ of vector fields. As before, we denote by I a multi-index $I = (i_1, \ldots, i_\ell)$ with ℓ nonnegative entries, and by ∂^I the operator that iteratively takes i_j covariant derivatives in the direction F_j, $j = 1, \ldots, \ell$.

Lemma 4 (Structural representation in terms of measures)**.** *A distribution $\eta \in \mathscr{D}'(P)$ can be written as a sum*

$$\eta = \sum_I \partial^I \nu_I \tag{2}$$

where I ranges over all multi-indices as above; for each I, ν_I is a signed measure. For a compact set $K \subseteq P$,

$$K \cap \operatorname{supp} \nu_I = \varnothing$$

for all but finitely many multi-indices I.

Proof. Take a partition of unity $\{\xi_j\}_{j\in\mathbb{N}} \subseteq C_c^\infty(P)$ of P, that is, a countable set of smooth functions p_i with compact support such that $\sum_j \xi_j(p) = 1$ and $\xi_j(p) \geq 0$ for all $p \in P$, and $\xi_j(p) = 0$ for all but finitely $j \in \mathbb{N}$ at any point $p \in P$. We make the further assumption that the support of each of the functions ξ_j is contained in an open set $U_j \subseteq P$ that is diffeomorphic to a cube $[0,1]^n$, and we let $\phi_j : U_j \to [0,1]^n$ be the corresponding diffeomorphism.

We let $\tilde{\eta}_j$ be the distribution on $\mathbb{R}^n$ that results from pushing $\xi_j\eta$ forward to the cube $[0,1]^n$ and extending periodically. In other words, for all rapidly-decreasing (Schwartz) functions $f \in C^\infty(\mathbb{R}^n)$, we let $\tau_z f(x) = f(x-z)$ and

$$\langle \tilde{\eta}_j, f \rangle = \sum_{z\in\mathbb{Z}^n} \langle \eta, p_j \cdot (\tau_z f) \circ \phi_j^{-1} \rangle.$$

Like all periodic distributions, $\tilde{\eta}_j$ is a tempered distribution. We have

Lemma 5. *Every tempered distribution is a derivative of finite order of some continuous function of polynomial growth.*

For a proof, see for example [11, Theorem 3.8.1].

Let ζ_j be the continuous function of polynomial growth corresponding to $\hat{\eta}_j$ and let α_j be the multi-index corresponding to the derivative in the lemma, so that

$$\hat{\eta}_j = \partial^{\alpha_j} \zeta_j.$$

Let D_j be the (smooth) differential operator on P such that $\phi_j^* \partial^{\alpha_j} = D_j \phi_j^*$, and write

$$\xi_j \eta = D_j \phi_j^* \zeta_j.$$

Since ζ_j is a continuous function, so is $\phi_j^* \zeta_j$, and hence it induces a measure on U_j. Then we can write

$$\eta = \sum_j p_j \eta = \sum_j D_j \phi_j^* \zeta_j,$$

and since each of the summands on the right can be expressed as a finite sum of derivatives of a continuous function, this proves the lemma. □

3.2 Variations

Let μ_s be a family of Borel measures on the manifold P parameterized by a real parameter s with values in an open interval $J \subseteq \mathbb{R}$ that contains 0. We say that the family μ_s is *differentiable* at 0 if there is a distribution $\eta \in \mathscr{D}'(P)$ such that, for every function $f \in C_c^\infty(P)$,

$$\left.\frac{d}{ds}\right|_{s=0} \int f \, d\mu_s = \langle \eta, f \rangle.$$

The distribution η is the *derivative* $d\mu_s/ds|_{s=0}$ of μ_s at 0.

Remark 6. This is just one way to define differentiability of families of distributions; other ways have been explored for example in [22].

Proposition 7. *For every Borel measure μ and every distribution η, there exists a family of signed measures with*

$$\mu_0 = \mu \quad \textit{and} \quad \left.\frac{d\mu_s}{ds}\right|_{s=0} = \eta.$$

Proof. Take a mollifier ψ and a tuple F of generating vector fields. Then, as follows from Lemma 3, the family $\mu_s = \psi *_{sF} (\mu_0 + s\eta)$ has the required properties. □

For families of *positive* measures, the situation is different.

Theorem 8. *Let μ be a positive Borel measure and let η be a distribution. Denote by $\mathscr{F}_\mu$ the space of nonnegative functions $f \in C_c^\infty(P)$ that vanish identically on* $\operatorname{supp}\mu$. *Then there exists a family μ_s of positive measures with $\mu_0 = \mu$ and derivative $d\mu_s/ds|_{s=0} = \eta$ if, and only if, η satisfies the following condition:*

$$\langle \eta, f\rangle = 0 \text{ for every } f \in \mathscr{F}_\mu. \tag{(Pos)}$$

If μ is a probability measure and η additionally satisfies that $\langle \eta, 1\rangle = 0$, then μ_s can be realized as a family of probability measures.

Remark 9. Condition (Pos) implies that $\operatorname{supp}\eta \subseteq \operatorname{supp}\mu$. Apart from this, Condition (Pos) is relevant only when $\operatorname{supp}\mu$ has parts that are very thin — only one point thick.

For example, if $P = \mathbb{R}$, μ is the Dirac delta δ_0, and $\rho : \mathbb{R} \to \mathbb{R} \in C_c^\infty(\mathbb{R})$ is a cutoff function ($\rho \geq 0$, $\rho \equiv 1$ in a neighborhood of 0 and $\rho \equiv 0$ outside a slightly larger neighborhood), then taking $f(x) = \rho(x)\sum_{i\geq 2} c_i x^i$ (with c_2 large enough to ensure that $f \geq 0$) we see that η must be of the form $A\delta_0 + B\partial\delta_0$, for involving any higher-degree derivatives would contradict the condition.

On the other hand, if we again had $P = \mathbb{R}$, but now $\mu = \chi_{[0,1]}$ the characteristic function on the unit interval, then as long as $\operatorname{supp}\eta \subseteq \operatorname{supp}\mu$, η can be any distribution and still comply with Condition (Pos).

Remark 10. If μ_s is any family of measures that is differentiable at $s = 0$ and if ψ is a mollifier and F is a generating tuple, then the measure $\tilde{\mu}_s = \psi *_{sF} \mu_s$ has the same derivative at 0 and the same mass as μ_s, and $\tilde{\mu}_s$ is a positive measure if $\tilde{\mu}_s$ is. By Lemma 3, the measure $\tilde{\mu}_s$ is a smooth density for all $s \neq 0$. Hence, the family μ_s can always be realized as a family of smooth measures (except maybe at $s = 0$).

Lemma 11. *Fix a point $p \in \operatorname{supp}\mu \subseteq P$. Let η_p be a distribution supported on p that satisfies Condition (Pos). Then there is a family of positive measures μ_s^p such that $\mu_0^p = \mu$ and*

$$\left.\frac{d\mu_s^p}{ds}\right|_{s=0} = \eta_p.$$

Moreover, the dependence of μ_s^p on p is measurable.

If μ is a probability measure and $\langle \eta_p, 1\rangle = 0$, then μ_s^p is a family of probability measures too, when it exists.

For the proof of the lemma we will need a metric defined on the space of distributions involving up to k^{th} derivatives, $k \geq 1$, and given by

$$\operatorname{dist}_k(\theta_1, \theta_2) = \sum_{j=1}^{\infty} \frac{1}{2^j \|f_j\|_k} |\langle \theta_1, f_j \rangle - \langle \theta_2, f_j \rangle|$$

for two distributions θ_1 and θ_2, and with $\{f_j\}_j \subset C_c^\infty(P)$ a sequence of functions that is dense with respect to the norm

$$\|f\|_k = \sum_{|I| \leq k} \sup_{q \in P} |\partial^I f(q)|.$$

Proof of Lemma 11. Let $V \subseteq T_pP$ be the subspace that is null for the Hessians at p of all the functions in $\mathscr{F}_\mu$:

$$V = \{v \in T_pP : \operatorname{Hess}_p f(v,v) = 0 \text{ for all } f \in \mathscr{F}_\mu\}.$$

Let $m = \dim V \leq n = \dim P$. Take coordinates $(x_1, x_2, \ldots, x_n)$ around p such that the vectors

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m} \in T_pP$$

form a basis of V and $\partial/\partial x_1, \ldots, \partial/\partial x_n$ is an orthonormal basis of T_pP. Assume that p corresponds to the origin in these coordinates. Then by Lemma 4 we know that η must be a finite linear combination of distributions of the form

$$\left(\frac{\partial}{\partial x_u}\right)^{e_0} \left(\frac{\partial}{\partial x_1}\right)^{e_1} \left(\frac{\partial}{\partial x_2}\right)^{e_2} \cdots \left(\frac{\partial}{\partial x_m}\right)^{e_m} \delta_p$$

where $e_0 \in \{0,1\}$, $u > m$, and the integers $e_1, \ldots, e_m$ are nonnegative. For reasons analogous to those explained in Remark 9, Condition (Pos) makes it impossible to have higher derivatives in the directions outside V.

Note that if ν_s is a family of positive measures such that $\nu_0 = \mu$ and

$$\left.\frac{d\nu_s}{ds}\right|_{s=0} = \left(\frac{\partial}{\partial x_1}\right)^{e_1} \left(\frac{\partial}{\partial x_2}\right)^{e_2} \cdots \left(\frac{\partial}{\partial x_m}\right)^{e_m} \delta_p, \tag{3}$$

and if ϕ is the flow of the vector field $\partial/\partial x_u$, then

$$\left.\frac{d}{ds}\phi_s^* \nu_s\right|_{s=0} = \frac{\partial}{\partial x_u} \left(\frac{\partial}{\partial x_1}\right)^{e_1} \left(\frac{\partial}{\partial x_2}\right)^{e_2} \cdots \left(\frac{\partial}{\partial x_m}\right)^{e_m} \delta_p.$$

So we will focus on finding such a family ν_s. In particular, we will assume that η_p is of the form given in (3). In other words, we will assume that it only involves derivatives in the directions of V.

Lemma 12. *For η_p as in equation* (3) *and for $k = \sum_{i=1}^m e_i$, we have*

$$\inf_g \operatorname{dist}_k(g\mu, \eta_p) = 0,$$

where the infimum is taken over all measurable functions $g : P \to \mathbb{R}$.

The reader will find the proof of Lemma 12 below.

Take a strictly-decreasing sequence of positive numbers $\{v_j\}_{j=-\infty}^0$ such that $\sum_j 1/v_j = 1$. Let k be as in Lemma 12. For each $j = 0, -1, -2, \ldots$, take a measurable function g_j such that $\sup_{q\in P} |g_j| \le 1$ and

$$\operatorname{dist}_k(v_j g_j \mu, \eta) < 2^j + \inf_{|g|\le 1} \operatorname{dist}_k(v_j g \mu, \eta).$$

With this definition, Lemma 12 implies that if we let $j \to -\infty$, we get $v_j g_j \mu \to \eta$.

We let, for $\sum_{j=-\infty}^{i} v_j^{-2} \le |s| < \sum_{j=-\infty}^{i+1} v_j^{-2}$,

$$\nu_s = \left(1 + \left(\left(s - \sum_{j=-\infty}^{i} \frac{\operatorname{sgn} s}{v_j^2}\right) v_{i+1} g_{i+1} + \sum_{j=-\infty}^{i} \frac{\operatorname{sgn} s}{v_j} g_j\right)\right) \mu.$$

By construction, ν_s is a family of positive measures and its derivative at $s = 0$ is η.

To ensure the measurability of the dependence of this construction in p, we further specify the construction as follows. For each $j \in \mathbb{Z}_-$, we take a covering of P by measurable sets A_j of diameter at most $-1/j$. For all $p \in A_i$, we take the same function g_j. This ensures that these choices are made in a 'measurable' way. The rest of the construction does not depend on arbitrary choices, so the dependence becomes measurable immediately.

The last statement of the lemma follows from the fact that if η_p satisfies $\langle \eta, 1\rangle = 0$, then either g_j can be chosen so that $g_j\mu$ satisfies this too, or else $e_1 = e_2 = \cdots = e_n = 0$, and in both cases the coordinates can be picked so that the mass is preserved by the flow ϕ_s for small-enough $|s|$. □

Proof of Lemma 12. Let V and $x_1, x_2, \ldots, x_n$ be as in the proof of Lemma 11. Let U be a small neighborhood of p on which the exponential map $\exp_p : T_pP \to P$ is injective.

The distribution η_p induces a distribution on V, $\bar\eta_p$, defined by

$$\langle \bar\eta_p, f\rangle = \langle \eta_p, \xi \cdot (f \circ \operatorname{proj}_V \circ \exp_p^{-1})\rangle, \quad f \in C_c^\infty(V),$$

where ξ is any compactly-supported step-function with $\xi \equiv 1$ in a small neighborhood of p and $\xi \equiv 0$ outside U. Clearly, if we can find a sequence of functions $g_i : T_pP \to \mathbb{R}$ such that $\operatorname{dist}_k(g_i \exp_p^* \mu, \bar{\eta}_p) \to 0$ as $i \to \infty$, then the sequence $\{\operatorname{dist}_k((g_i \circ \exp_p^{-1})\mu, \eta_p)\}_i$ will also approach 0 and the lemma will be proved. We may thus assume that P is a Euclidean space $\mathbb{R}^n$, that μ and η_p are defined on $\mathbb{R}^n$, and that p is at the origin of $\mathbb{R}^n$.

Condition (Pos) implies that for any open set $A \subseteq U$ that contains p, the set $\operatorname{proj}_V(\operatorname{supp}\mu \cap A)$ contains infinitely many vectors, and that these vectors span V as a vector space.

For each $j = 1, 2, \ldots$, let $\{x_i^j\}_{i=1}^\infty \subseteq \operatorname{supp}\mu \subset \mathbb{R}^n$ be a sequence of points contained within distance $1/j$ of p and within distance $1/j^2$ of V. We also assume that $\{\operatorname{proj}_V x_i^j\}_i$ span all of V. For a large-enough finite subset I_j of $\mathbb{N}$, there is always a solution to the problem of finding real numbers c_{ij} such that

$$\langle \eta_p, f\rangle = \lim_{h\to 0} \frac{1}{h^k} \sum_{i \in I_j} c_{ij} f(h \operatorname{proj}_V x_i^j) \tag{4}$$

for all $f \in C_c^\infty(P)$. To see this, note that expanding the right-hand-side as Taylor series in h and comparing coefficients, one obtains a linear system in the variables c_{ij}, and that this system has solutions if sufficiently many points x_i^j are available.

For each $j = 1, 2, \ldots$, let $\varepsilon_j > 0$ be small enough that the balls $B_{\varepsilon_j}(x_i^j)$ are disjoint. For $q \in B_{\varepsilon_j}(x_i^j) \cap \operatorname{supp}\mu$ for some $i \in I_j$, let

$$g_j(q) = \frac{c_{ij}}{\mu(B_{\varepsilon_j}(x_i^j))},$$

and let $g_j(q) = 0$ for all other $q \in P$. Then $g_j\mu \to \eta$ as $j \to \infty$ because for each $y \in C_c^\infty(P)$ the set

$$A(y, R) = \Big\{ \sum_{i=1}^N c_i y(x_i) : c_j \in \mathbb{R},\ x_i \in B_R(p) \subset P,\ N \in \mathbb{N}, \text{ and} \\ \langle \eta_p, f\rangle = \lim_{h\to 0} \frac{1}{h^k} \sum_{i=1}^N c_i f(h \operatorname{proj}_V x_i) \text{ for all } f \in C_c^\infty(P) \Big\}$$

contains the value of $\sum_i c_{ij} y(x_i^j) \approx \langle g_j\mu, y\rangle$ for $R > 1/j$, and the diameter of $A(y, R)$ tends to 0 as $R \to 0$. This proves the lemma. □

Proof of Theorem 8. Assume first that the family μ_s exists. To prove that Condition (Pos) must hold, let $f \in C_c^\infty(P)$ be a nonnegative function $f \in \mathscr{F}_\mu$, and consider the function

$$g(s) = \int f \, d\mu_s.$$

Since f is nonnegative and μ_s is a positive measure for all s, g must be nonnegative as well. Since $g(0) = 0$, it must also be true that $g'(0) = 0$, and this is equivalent to Condition (Pos).

Now assume that we have a measure μ and a distribution η such that Condition (Pos) holds, and let us construct a family μ_s as in the statement of the theorem. Let ν_I be the measures as in Lemma 4. These induce a measure γ on P and a family of distributions η_p supported at $p \in P$ such that for all $f \in C_c^\infty(P)$

$$\eta = \int \eta_p d\gamma(p).$$

For γ-almost all p, the distributions η_p also satisfy Condition (Pos). From Lemma 11, we get families μ_s^p of measures whose derivatives at 0 are precisely the distributions η_p. Thus,

$$\mu_s = \int \mu_s^p d\gamma(p)$$

is a family as in the statement of the theorem.

If μ is a probability, since each μ_s^p preserves the probability, so does μ_s. □

4 The tangent space

4.1 Mild distributions

Recall that distributions on a manifold P were defined in Section 3.1. Now take the case in which $P = T^n M$. A *partition of unity* in $T^n M$ is a set of nonnegative functions $\{\psi_i\}_i \subset C_c^\infty(T^n M)$ such that for all $x \in T^n M$

$$\sum_i \psi_i(x) = 1.$$

Given a distribution $\eta \in \mathscr{D}'(T^n M)$, we want to make sense of its value at a form $\omega \in \Omega^n(M)$. We let

$$\langle \eta, \omega \rangle = \sum_i \langle \eta, \psi_i \omega \rangle,$$

We denote by $\mathscr{D}'_n \subset \mathscr{D}'(T^n M)$ the set of distributions for which the series in the right-hand-side converges absolutely for all $\omega \in \Omega^n(M)$. This is independent of our choice of partition of unity $\{\psi_i\}_i$. Also, the spaces of mild measures $\mathscr{M}_n$ and of holonomic measures $\mathscr{H}$ are subsets of $\mathscr{D}'_n$.

A family of measures $\mu_t \in \mathscr{M}_n$ is *differentiable* at 0 if there is a distribution $\eta \in \mathscr{D}'_n$ such that for all $f \in C_c^\infty(T^n M)$

$$\left.\frac{d}{dt}\right|_{t=0} \int f \, d\mu_t = \langle \eta, f \rangle.$$

4.2 Variations of holonomic measures

Theorem 13. *Let μ be a holonomic measure in $T^n M$ and let $\eta \in \mathscr{D}'_n$ be a distribution on $T^n M$. Then there exists a family of holonomic measures $\mu_t \in \mathscr{M}_n$, $t \in \mathbb{R}$, such that $\mu_0 = \mu$ and*

$$\left.\frac{d}{dt}\right|_{t=0} \int f \, d\mu_t = \langle \eta, f \rangle \tag{5}$$

for all $f \in C_c^\infty(T^n M)$ if, and only if, the following two conditions are satisfied:

(Pos) *For all nonnegative $f \in C_c^\infty(T^n M)$ that vanish on $\operatorname{supp} \mu$, $\langle \eta, f \rangle = 0$.*

(Hol) *For all differential forms $\omega \in \Omega^{n-1}(M)$, $\langle \eta, d\omega \rangle = 0$.*

Remark 14. In other words, the tangent space to the space of holonomic measures at the point μ is characterized by Conditions (Pos) and (Hol).

Proof. By Theorem 8, Condition (Pos) is necessary. If μ_t exists, then we have

$$0 = \left.\frac{d}{dt}\right|_{t=0} \int d\omega \, d\mu_t = \langle \eta, d\omega \rangle$$

for all $\omega \in \Omega^{n-1}(M)$. Hence, Condition (Hol) is also necessary.

To prove that Conditions (Pos) and (Hol) are sufficient, assume that they are satisfied. Then by Theorem 8 we have a family of probability measures θ_t for t in some interval that contains 0, with $\theta_0 = \mu$ and with (5). Now we need to modify θ_t so that it is also a family of holonomic measures. Moreover, the proofs of Theorem 8 and Lemma reflem:variationpoint show that θ_t can be assumed to be in $\mathscr{M}_n$ for all t.

There exists a family of measures ν_t such that for all $\omega \in \Omega^{n-1}(M)$ and all t

$$\int d\omega \, d\theta_t + \int d\omega \, d\nu_t = 0.$$

The measure ν_t can for example be obtained from θ_t as follows. For each $x \in M$, let $r_x : T_x^n M \to T_x^n M$ be some reflection such that the multivector

$r_x(v)$ has the opposite orientation as the multivector $v \in T^n M$. These reflections can be chosen in a piecewise-continuous (and hence measurable) way with respect to the variable x. Then one can take the family of measures determined by $\nu_t|_{T^n_x M} = r_x^*(\theta_t|_{T^n_x M})$.

For $a > 0$, let $\lambda_a : T^n M \to T^n M$ be the map given by

$$\lambda_a(x, v_1, v_2, \ldots, v_n) = (x, av_1, v_2, v_3, \ldots, v_n).$$

The measure $\nu_t^a = \lambda_a^* \nu_t / a$ satisfies

$$\int d\omega \, d\nu_t^a = \frac{1}{a} \int d\omega(x, av_1, \ldots, v_n) \, d\nu_t = \int d\omega \, d\nu_t$$

for all $\omega \in \Omega^{n-1}(M)$. However, as $a \to \infty$, the mass $\int d\nu_t^a$ of ν_t^a tends to 0. It is hence possible to find a function $b : \mathbb{R} - \{0\} \to \mathbb{R}_+$ such that $\nu_t^{b(t)}$ is a family of measures with

$$\left. \frac{d\nu_t^{b(t)}}{dt} \right|_{t=0} = 0 \qquad \text{and} \qquad \lim_{t \to 0} \frac{1}{t^2} \int d\nu_t^{b(t)} = 0.$$

We let

$$\mu_t = \frac{\theta_t + \nu_t^{b(t)}}{1 + \int d\nu_t^{b(t)}}$$

for $t \neq 0$ and $\mu_0 = \mu$. This is a family of measures as in the statement of the theorem. $\square$

5 Examples

Results in this section are valid for measures that are critical with respect to the action of a general smooth Lagrangian $L \in C^\infty(T^n M)$. Unless explicitly stated, we do not require, for example, that L be convex.

A *variation* of a holonomic measure $\mu \in \mathscr{H}$ is a family μ_t of holonomic measures that is defined for t in an interval $I \subseteq \mathbb{R}$ containing 0 and is differentiable at 0.

We denote by A_L the action of the Lagrangian L,

$$A_L(\mu) = \int_{T^n M} L \, d\mu.$$

We say that $\mu \in \mathscr{H}$ is *critical* for A_L if for every variation μ_t with $\mu_0 = \mu$

$$\left. \frac{d}{dt} \right|_{t=0} A_L(\mu_t) = 0. \tag{6}$$

By Theorem 13, μ is critical if, and only if, for all distributions $\eta \in \mathscr{D}'_n$ that satisfy Conditions (Pos) and (Hol), we have

(Crit) $\langle \eta, L\rangle = 0.$

Homology. A holonomic measure $\mu \in \mathscr{H}$ is assigned its *homology class* $\rho(\mu) \in H^n(M;\mathbb{R})$ by requiring

$$\langle \rho(\mu), \omega\rangle = \int \omega \, d\mu$$

for all closed forms $\omega \in \Omega^n(M)$, $d\omega = 0$. If for each t the measure μ_t has the same associated homology class as μ_0, $\rho(\mu_t) = \rho(\mu_0)$, then we say that the variation μ_t is *homology preserving.* Clearly, for this to happen the following condition is necessary on the derivative $\eta = d\mu_t/dt|_{t=0}$:

(Hom) $\langle \eta, \omega\rangle = 0$ for all $\omega \in \Omega^n(M)$ with $d\omega = 0$.

Conjecture 15. Condition (Hom) is sufficient for the existence of a homology preserving variation μ_t.

We will say that $\mu \in \mathscr{H}$ is *critical for A_L within its homology class* if equation (6) holds for every homology variation μ_t of μ. In particular, if μ is critical for A_L, then it is also critical within its homology class.

5.1 Horizontal variations

Let $X : M \to TM$ be a smooth vector field on M. For $f \in C_c^\infty(T^nM)$, denote by Xf the Lie derivative in the (horizontal) direction X. This is given by $Xf = d_xf(X)$, and is independent of the Riemannian metric on M. For a differential form $\omega \in \Omega^n(M)$, the action of X on ω is also defined, and it is equal to the Lie derivative $\mathcal{L}_X\omega = i_Xd\omega + di_X\omega$. Here, i_X denotes the contraction.

Let μ be a holonomic measure on T^nM. The distribution η given by

$$\langle \eta, f\rangle = \int_{T^nM} Xf \, d\mu \tag{7}$$

for $f \in C_c^\infty(T^nM)$ clearly satisfies Condition (Pos). It also satisfies Condition (Hol) because for all $\omega \in \Omega^{n-1}(M)$,

$$\langle \eta, d\omega\rangle = \int \mathcal{L}_X d\omega \, d\mu = \int i_Xd^2\omega + di_Xd\omega \, d\mu = 0.$$

Therefore, η is in the tangent space to μ.

It also satisfies Condition (Hom) because, if ω is a closed n-form,

$$\frac{d}{ds}\int \omega \, d\mu_s = \int \mathcal{L}_X \omega \, d\mu_s = \int i_X d\omega + d i_X \omega \, d\mu_s = 0.$$

The last equality is true since $d\omega = 0$ because ω is closed, and $\int d i_X \omega \, d\mu_s = 0$ because μ_s is holonomic.

In fact, it is easy to explicitly construct a family μ_t with derivative η and $\mu_0 = \mu$. To do this, take the flow $\phi_t : \mathbb{R} \times M \to M$ of X on M, determined by

$$\phi_0(x) = x, \quad \frac{d}{dt}\phi_t(x) = X(x), \quad \text{for } x \in M, t \in \mathbb{R}.$$

Extend this to an isotopy $r : \mathbb{R} \times T^n M \to T^n M$ by

$$r_t(x, v_1, \ldots, v_n) = (\phi_t(x), d\phi_t(v_1), \ldots, d\phi_t(v_n)),$$

where $d\phi_t : T_x M \to T_{\phi_t(x)} M$ denotes the derivative of ϕ_t at x. Then we can simply let $\mu_t = r_t^* \mu$. From this construction and Proposition 1, it is clear that μ_t is homology preserving. We thus have

Proposition 16. *If μ is critical for A_L within its homology class, then Condition (Crit) must hold for all distributions η of the form given in equation (7).*

Euler-Lagrange equations. Assume that the holonomic measure μ is induced by a cycle α, that is,

$$\mu = \mu_\alpha.$$

We will now recover the traditional Euler-Lagrange equations in this special case.

For $t \in \mathbb{R}$, $r_t \circ \alpha$ denote the cycle that results from the operation of composing each of the n-cells γ_i that appear in α with the isotopy r:

$$\text{if } \alpha = \sum_i c_i \gamma_i, \; c_i \in \mathbb{R}, \text{ then } r_t \circ \alpha = \sum_i c_i \, r_t \circ \gamma_i.$$

The variation $\mu_t = r_t^* \mu_\alpha$ constructed above is precisely the same as $\mu_{r_t \circ \alpha}$.

We want to examine what happens when the measure μ is critical for A_L with respect to all such variations μ_t for all vector fields X. For clarity, we use the time variable s instead of t, and we use the variables t_j on the domain of γ_i. Also, we write $dt = dt_1 \cdots dt_n$. We denote the partial derivatives of L by L_x and L_{v_i}. For each such variation have:

$$
\begin{aligned}
0 = \frac{d}{ds}\bigg|_{s=0} \int L\, d\mu_s &= \frac{d}{ds}\bigg|_{s=0} \sum_i c_i \int L(d(r_s \circ \gamma_i))\, dt \\
&= \sum_i c_i \int \left[L_x(d\gamma_i)\, \frac{\partial r_s \circ \gamma_i}{\partial s}\bigg|_{s=0} + \sum_j L_{v_j}(d\gamma_i)\, \frac{\partial^2 (r_s \circ \gamma_i)}{\partial s\, \partial t_j}\bigg|_{s=0} \right] dt \\
&= \sum_i c_i \int \left[L_x(d\gamma_i) - \sum_j \frac{\partial L_{v_j}(d\gamma_i)}{\partial t_j} \right] \frac{\partial (r_s \circ \gamma_i)}{\partial s}\bigg|_{s=0} dt \\
&= \sum_i c_i \int \text{(E-L)}\, \frac{\partial r_s}{\partial s}\bigg|_{s=0} dt
\end{aligned}
$$

where

$$
\text{(E-L)} := \frac{\partial L}{\partial x} - \sum_{i=1}^{n} \left(\frac{\partial^2 L}{\partial x \partial v_i} v_i + \sum_{j=1}^{n} \frac{\partial^2 L}{\partial v_i \partial v_j} \mathsf{X}_{ij} \right),
$$

and a point in the vector space $T_{(v_1,\dots,v_n)}(T^n_x M)$ has coordinates X_{ij}, $1 \leq i, j \leq n$. Since the above is true for all smooth vectorfields $X = \partial r_s / \partial s|_{s=0}$, we conclude that (E-L) must vanish identically throughout the support of $\mu = \mu_\alpha$.

In other words, Condition (Crit) for measures μ_α and for distributions of the form (7) is equivalent to the Euler-Lagrange equations.

Remark 17. In the case of an arbitrary holonomic measure (not necessarily induced by a cycle) we have no information about the ‘second derivatives,’ so we find no clear way to give this deduction in that general case. While it can be ascertained that these equations must be respected in a ‘weak’ way (if $\mu = \lim_i \mu_{\alpha_i}$, the measures μ_{α_i} will asymptotically satisfy Euler-Lagrange in the sense of distributions, so (E-L) must vanish μ-almost everywhere), it is not clear to us how this can be useful.

5.2 Vertical variations

Let μ be a holonomic measure in $T^n M$.

We introduce the Hilbert space $\mathcal{H}$ of all functions

$$
u : \operatorname{supp} \mu \subseteq T^n M \to T^n M
$$

such that $u(x,v) \in T^n_x M$ for all $(x,v) \in T^n M$, and $\int g(u,u)\, d\mu < +\infty$, where g is the Riemannian metric on M. The inner product in $\mathcal{H}$ is defined by

$$(u_1, u_2) = \int g(u_1, u_2)\, d\mu.$$

The set of gradients $\nabla_v d\omega$ of exact differential forms (viewed as functions on $T^n M$) is a subspace F of $\mathcal{H}$.

Each function u in $\mathcal{H}$ induces a distribution η^u of the form

$$\langle \eta^u, f\rangle = \int_{T^n M} g(u, \nabla_v f)\, d\mu.$$

for $f \in C_c^\infty(T^n M)$. This distribution clearly satisfies Condition (Pos). The set of all functions u in $\mathcal{H}$ such that η^u satisfies Condition (Hol) as well are exactly the orthogonal complement $F^\perp$ to F in $\mathcal{H}$ because Condition (Hol) is

$$0 = \langle \eta^u, d\omega\rangle = \int g(u, \nabla_v d\omega)\, d\mu = (u, d\omega).$$

for $\omega \in \Omega^{n-1}(M)$.

It follows that, if Condition (Crit) is satisfied for all η^u satisfying Conditions (Pos) and (Hol), then $\nabla_v L$ must be contained in the space $F^{\perp\perp}$, which coincides with the topological closure $\overline{F}$. We have proved

Proposition 18 ("$L_v = d\omega$")**.** *If μ is a holonomic measure that is critical for A_L, then there exist a sequence $\{\omega^i\}_i \subset \Omega^{n-1}(M)$ such that*

$$\nabla_v L|_{\operatorname{supp}\mu} = \lim_{i\to\infty} \nabla_v d\omega^i.$$

The limit is taken in $\mathcal{H}$.

It is possible to produce an explicit variation μ^u_t of μ with derivative η^u by letting

$$\int_{T^n M} f\, d\mu^u_s = \int_{T^n M} f(x, v + s u(x,v))\, d\mu(x,v)$$

for all $f \in C_c^\infty(T^n M)$ and $s \in \mathbb{R}$. It follows from the construction that this variation preserves homology whenever Condition (Hom) holds. That is, whenever u is such that

$$0 = \langle \eta^u, \omega\rangle = (u, \omega)$$

for all closed forms $\omega \in \Omega^n(M)$. Hence, the same argument as before yields

Proposition 19. *If μ is a holonomic measure that is critical for A_L within its homology class, then there exists a sequence of closed n-forms $\{\omega^i\}_i \subset \Omega^n(M)$, $d\omega^i = 0$, such that*

$$\nabla_v L|_{\operatorname{supp}\mu} = \lim_{i\to\infty} \nabla_v \omega^i.$$

The limit is taken in $\mathcal{H}$.

5.3 Transpositional variations

Let $\mu \in \mathscr{H}$ again be a holonomic measure, and let L be a Lagrangian.

Let $\sigma \in C_c^\infty(T^n M)$ and fix some $1 \leq i \leq n$. We consider the distribution on $T^n M$ given by

$$\langle \eta, f\rangle = \int \sigma f \, d\mu - \int \sigma \frac{\partial f}{\partial v_i} \cdot v_i \, d\mu - \int f \, d\mu \int \sigma d\mu$$

for $f \in C_c^\infty(T^n M)$. The distribution η clearly satisfies Condition (Pos). To see that it also satisfies Condition (Hol), we compute, for $\omega \in \Omega^{n-1}(M)$,

$$\langle \eta, d\omega\rangle = \int \sigma \, d\omega \, d\mu - \int \sigma \, d\omega \, d\mu - \int d\omega \, d\mu \int \sigma \, d\mu = 0.$$

Here, we used that $\partial d\omega/\partial v_i = d\omega$ by linearity, and we also used the fact that μ is holonomic.

If μ is critical for A_L, it must satisfy Condition (Crit) for all variations arising in this way from any $\sigma \in C_c^\infty(T^n M)$. This translates to

$$0 = -\left.\frac{d}{ds}\right|_{s=0} A_L(\mu_s^\sigma) = \int \sigma L \, d\mu - \int \sigma L_{v_i} \cdot v_i \, d\mu - \int L \, d\mu \int \sigma \, d\mu.$$

If the domain of σ is very small around a point $(x, v) \in T^n M$, this can be very well approximated by

$$0 \approx \int \sigma \, d\mu \left(v_i \cdot L_{v_i}(x, v) - L(x, v) - \int L \, d\mu \right).$$

This is how we deduce

Proposition 20 (Energy conservation)**.** *If a holonomic measure is critical with respect to all transpositional variations, then its support is a subset of the set where*

$$v_i \cdot L_{v_i} - L = A_L(\mu).$$

Remark 21. In the cases in which we can define the change of variables $p_i = L_{v_i}$ (for example, in the case of convex, superlinear Lagrangians), we can also define the Hamiltonians

$$H_i(x, p_i) = H_i(x, p_i; v_1, \ldots, \widehat{v}_i, \ldots, v_n) = \max_{v_i \in T_x^n M} p_i v_i - L(x, v_1, \ldots, v_n),$$

and what we have here is just a higher-dimensional version of the usual energy conservation principle.

Remark 22 (Hamilton-Jacobi equation). We can form the full Hamiltonian $H = \sum_i H_i$, which in the case of a convex, superlinear Lagrangian L is the convex dual to the Lagrangian nL. Then it follows from Proposition 18 that there is a sequence of $(n-1)$-forms ω^k such that, abusing the notation a little,

$$\lim_{k \to \infty} H(x, d\omega^k) = nA_L(\mu)$$

on $\operatorname{supp} \mu$. This is a generalized form of the Hamilton-Jacobi equation.

The distribution η is in fact the derivative of the variation μ_t^σ given by

$$\int f \, d\mu_t^\sigma = \frac{\int (1 - t\sigma) f(x, v_1, \ldots, v_{i-1}, (1 - t\sigma)^{-1} v_i, v_{i+1}, \ldots, v_n) \, d\mu}{\int (1 - t\sigma) \, d\mu}$$

for $f \in C_c^\infty(T^n M)$ and for t in an open interval that contains 0.

If we require the variation μ_t^σ to preserve homology, then we find that we must require $\int \sigma \, d\mu = 0$ because

$$\int \omega \, d\mu_t^\sigma = \frac{\langle \rho(\mu), \omega \rangle}{\int (1 - t\sigma) \, d\mu}$$

must be constant for each closed form $\omega \in \Omega^n(M)$, $d\omega = 0$. It follows that if μ is critical for A_L within its homology class then it must satisfy

$$\int \sigma (L - L_{v_i} \cdot v_i) \, d\mu = 0$$

for all σ with $\int \sigma \, d\mu = 0$. We can use $\sigma d\mu$ to approximate the derivative at any point in $\operatorname{supp} \mu$ arbitrarily well. Hence, we get

Proposition 23 (Energy conservation for homological minimizers)**.** *If a holonomic measure is critical for A_L within its homology class, then there are some $c_1, \ldots, c_n \in \mathbb{R}$ such that the support of μ contained in the set where*

$$L - v_i \cdot L_{v_i} = c_i, \quad i = 1, 2, \ldots, n.$$

A Proof of Theorem 1

This section is devoted to the proof of Theorem 1, which will be given in Section A.6.

The $n = 1$ case of this result was proved by Bangert [3] and Bernard [4]. The author saw a letter by Mather [14] in which an idea similar to Bangert's was sketched. Our proof if that case is different to theirs.

The idea of the proof is the following. The fact that Condition (Cyc) implies Condition (Hol) is an easy consequence of Stokes's theorem, so we concentrate in the other implication.

We start with a positive measure μ that satisfies Condition (Hol). We prove in Section A.1 that we may assume that the measure μ is a smooth density. In Section A.2 we specify a family of triangulations T_k on M for $k \in \mathbb{N}$. Then in Section A.3.1 we construct 'base measures' $\bar{\mu}_k$, which are approximations to our smooth density that are (in a sense) constant on each simplex of T_k; this is analogous to approximating a smooth function on $\mathbb{R}$ with simple functions. In Section A.3.2 we construct an n-chain β_k that is again (in a sense) constant on each simplex of T_k.

In Section A.4 we derive a condition on the $(d-n)$-dimensional skeleton of T_k that in Section A.5.1 allows us to construct cycles that contain the chains β_k, and whose mass $\mathbf{M}$ can be estimated. We work on the estimates for the mass in Sections A.5.2 and A.5.3. Finally, we put everything together in Section A.6.

A.1 Smoothing revisited

Lemma 24. *Any measure μ in $\mathscr{M}_n$ can be approximated arbitrarily well using a smooth density on T^nM. If μ is a probability measure that satisfies Condition (Hol) then it can be approximated by smooth probability densities that also satisfies Condition (Hol).*

Proof. Denote the exponential map by $\exp_x : T_xM \to M$.

A *mollifier* $\psi \in C_c^\infty(\mathbb{R})$ is a function such that $\psi(x) = \psi(-x)$, $\int \psi = 1$, and $\psi \geq 0$.

Recall that the operators P_i were defined in Section 3.1.1. These are a convolutions in the horizontal directions defined by vector fields F_i, which are taken to form a generating tuple.

Also, for $f \in C_c^\infty(T^nM)$ we let $V(f)$ be the convolution in the vertical direction,

$$V(f)(x,v_1,\dots,v_n) = \int_{T_xM} dw_1\psi(|w_1-v_1|)\int_{T_xM} dw_2\psi(|w_2-v_2|)$$
$$\cdots\int_{T_xM} dw_n\psi(|w_n-v_n|)f(x,w_1,w_2,\dots,w_n).$$

For $f\in C_c^\infty(T^nM)$, we will denote

$$\psi * f = P_1P_2\cdots P_\ell V(f).$$

Note that $\psi * f$ is a C^∞ function even if f is only measureable. Moreover, if the diameter of the support of ψ is sufficiently small, and if f is an exact form, $f = d\omega$, then $\psi * d\omega$ is the exact form $d(\psi*\omega)$. To see this, note first that by linearity of ω on each entry $V(d\omega)=d\omega$. Also, for s small enough, ϕ_s^* is a diffeomorphism and hence

$$P_i(d\omega) = \int \psi(s)\phi_s^{i*}d\omega\,ds = d\left[\int \psi(s)\phi_s^{i*}\omega\,ds\right] = d(P_i\omega).$$

Now let μ be a probability measure on T^nM. Recall that we define the convolution $\psi*\mu$ by duality, setting

$$\int_{T^nM} f\,d(\psi*\mu) = \int_{T^nM}(\psi*f)\,d\mu.$$

Then $\psi*\mu$ is a smooth density (see for example [11, §5.2]), and in the topology of $\mathscr{M}_n$,

$$\psi*\mu\to\mu \quad\text{as}\quad \operatorname{diam}\operatorname{supp}\psi\to 0.$$

Also, if μ satisfies Condition (Hol), then

$$\int_{T^nM} d\omega\,d(\psi*\mu) = \int_{T^nM} d(\psi*\omega)d\mu = 0,$$

so $\psi*\mu$ also satisfies Condition (Hol). $\square$

A.2 Triangulations

A triangulation $T=(K,h)$ of M is a simplician complex K homeomorphic to M together with a homeomorphism $h: K\to M$. When talking about such a triangulation T, we will speak indistinctly of a simplex $U\subseteq K$ and of its image $h(U)\subseteq M$. In other words, we will ignore K as a topological space, and we will instead think of the triangulation as being ‘drawn’ directly on M.

On $\mathbb{R}^d$, we will use the standard inner product. For a subspace $W \subset \mathbb{R}^d$ passing through the origin, let $\operatorname{proj}_W : \mathbb{R} \to W$ denote the orthogonal projection onto W. Also, for a subspace W, we will denote the subspace perpendicular to W by $W^\perp \subseteq \mathbb{R}^d$.

We fix a sequence of triangulations $\{T_k\}_{k\in\mathbb{N}}$ on M such that:

T1. (Successive refinements) For $k > 1$, T_k is a refinement of T_{k-1}.

For each simplex V in T_k, $k \geq 1$, we denote by $U(V)$ the simplex of dimension d of T_1 in which V is contained. (This is ambiguous for the simplices of dimension less than d, but any choice will work, so we assume that this choice has been made for each V once and for all.)

T2. (Finite) T_k has finitely many simplices.

T3. (Charted) For each simplex U of dimension d of T_1, there is a chart $\varphi_U : M \to \mathbb{R}^d$ such that the image $\varphi_U(U)$ is the standard simplex with vertices at the origin and at the vectors of the standard basis of $\mathbb{R}^d$.

For brevity, we will denote $\varphi_{U(V)}$ by φ_V for all simplices V in the triangualtions T_k, $k \geq 1$.

T4. (Affine) For every simplex V in T_k, $\varphi_V(V)$ is contained in a translate of a vector space $Y(V) \subset \mathbb{R}^d$ of dimension $\dim V$.

T5. (Nondegeneracy) All simplices of T_k are non-degenerate. In other words, if a simplex V has dimension m, then also

$$\operatorname{vol}_m V > 0.$$

T6. (Vanishing diameter)

$$\lim_{k\to\infty} \operatorname{diam} T_k = 0.$$

Existence of triangulations on manifolds is discussed in great detail for example in [20]. A triangulation T_1 satisfying T2–T5 always exists. To obtain all other refinements T_k of T_1, one successively refines the standard simplex $\varphi_U(U)$ (for U a simplex in T_1) making sure that the rules T2–T5 are respected every time. It can be seen by induction on k that this is possible. It is quite obvious how to take a refinement that respects T2–T5. Ensuring overall compliance with T6 is easy . Then one pulls the resulting triangulation onto M using the maps φ_U.

We will denote by E_m^k the m-dimensional skeleton of the triangulation T_k.

A.3 The base measure and its approximation

A.3.1 Construction of the base measure

In Section A.2 we specified the triangulations T_k, $k \in \mathbb{N}$, and we introduced the notation φ_V.

Let μ be a smooth density in $\mathscr{M}_n$. We will define *base measures* $\bar{\mu}_k \leq \mu$ depending on the triangulations T_k such that $\bar{\mu}_k \to \mu$ as $k \to \infty$. Roughly speaking, the measure $\bar{\mu}_k$ is the largest density, constant on a constant section of $T^n M$ in the interior of each d-dimensional simplex U of T_k. Our goal here is *not* to produce measures that satisfy Condition (Hol).

For a simplex V of dimension d in the triangulation T_k, we take the chart φ_V and extend it to a trivialization of $T^n M$, $d\varphi_V : T^n M \to \mathbb{R}^{d(n+1)}$, by setting

$$d\varphi_V(x, v_1, v_2, \ldots, v_n) = (\varphi_V(x), d\varphi_V(v_1), \ldots, d\varphi_V(v_n)) .$$

Let m denote Lebesgue measure on $\mathbb{R}^{d(n+1)}$ and let ρ be the Radon-Nikodym derivative of the pushforward measure $(\varphi_V)_*\mu = \rho\mathsf{m}$ on $\mathbb{R}^{d(n+1)}$.

For $(x, v) \in \mathbb{R}^{d(n+1)}$ with $x \in \varphi_V(V)$ for a simplex V of dimension d, we let

$$\bar{\rho}_k(x, v) = \inf_{y \in \varphi_V(V)} \rho(y, v).$$

Note that v is the same on both sides of the equation, and the dependence of the right-hand-side on x comes from the choice of V. Also, this is ambiguous when x is in a simplex of dimension $< d$. This ambiguity happens only on a set of m-measure zero, so we may just ignore it, as it will not affect the rest of our argument. We let

$$\bar{\mu}_k|_{T^n V} = \varphi_V^*(\bar{\rho}_k \mathsf{m}).$$

This completely determines $\bar{\mu}_k$ on the whole bundle $T^n M$. Also, $\rho_k \to \rho$ uniformly on compact sets, because ρ is smooth and $\operatorname{diam} T_k \to 0$ by T6. Similarly, $\mathbf{M}(\bar{\mu}_k - \mu) \to 0$. Hence $\operatorname{dist}_{\mathscr{M}_n}(\bar{\mu}_k, \mu) \to 0$.

A.3.2 Construction of the approximation

For each $k \in \mathbb{N}$, we will construct a chain β_k whose induced measure $\wr\beta_k\wr$ will approximate the base measure $\bar{\mu}_k$ very well. We do this in the following steps.

Step 1. On each d-dimensional simplex V of T_k, we sample the distribution $\bar{\rho}_k \mathsf{m}$ to get a finite sequence of points $p_1^V, \dots, p_{\ell_V}^V \in \mathbb{R}^{d(n+1)}$. We may assume that the following conditions are true for these points:

A1. Each point p_i^V is in the interior of $\varphi_V(V)$.

A2. Write p_i^V as $(x, v_1, \dots, v_n)$ in the standard basis of $(\mathbb{R}^d)^{n+1}$. Let Π be the plane

$$\Pi_i^V = \{x + t_1 v_1 + t_2 v_2 + \cdots + t_n v_n : t_i \in \mathbb{R}\}.$$

We assume that Π_i^V intersects all the simplices $W \subseteq \partial\varphi_V(V)$ of dimension $\dim W \geq d - n$ transversally.

A3. For a $(d-n)$-dimensional simplex $W \subset E_{d-n}^k$, let V_1 and V_2 be two d-dimensional simplices adjacent to W. Let A_i, $i = 1, 2$, be the set of points of the form $\Pi_i^{V_i} \cap W$. There is a finite partition of W by disjoint, convex sets $U_1, \dots, U_m$ with $\operatorname{diam} U_i < a(k)$ such that each of them contains at least one point in A_1, and

$$\left| \frac{\bar{\mu}_k(V_2)}{\bar{\mu}_k(V_1)} - \frac{\# U_i \cap A_2}{\# U_i \cap A_1} \right| < a(k), \tag{8}$$

where $a : \mathbb{N} \to \mathbb{R}_+$ is an asymptotically-vanishing function that will be specified at the end of Section A.5.2.

Note that the measure

$$\sum_{V \subset E_d^k} \frac{1}{\ell_V} \sum_i \varphi_V^* \delta_{p_i^V} \tag{9}$$

is a good approximation of $\bar{\mu}_k$. Compliance with item A3 can be achieved by increasing the number of points, thus making the sample more dense.

Step 2. Let V be a d-dimensional simplex in T_k. Let $\gamma_i^V : D_i^V \subseteq \mathbb{R}^n \to \mathbb{R}^d$ be the solution to the equations

$$\gamma_i^V(0, 0, \dots, 0) = x, \quad \frac{\partial \gamma_i^V}{\partial t_j} = v_j, \quad i = 1, \dots, n. \tag{10}$$

Assume that the domain D_i^V of γ_i^V is the largest closed subset of $\mathbb{R}^n$ such that γ_i^V remains within $\varphi_V(V)$. Note that $\operatorname{image} \gamma_i^V = \gamma_i^V(D_i^V) \subset \Pi_i^V$, so by A2 this image also intersects the simplices in $\partial\varphi_V(V)$ transversally.

We let

$$\beta_k = \sum_{V \subset E_d} \frac{1}{\ell_V} \sum_i \frac{1}{|D_i^V|} \varphi_V^* \gamma_i^V.$$

When we consider the measure $\wr\beta_k\wr$, this is like taking the measure in equation (9) and spreading the mass of each point along a simplex determined by its velocity vectors $v_1, \dots, v_n$. Since $\bar{\mu}_k|_V$ is 'constant' for each such set of velocity vectors, $\wr\beta_k\wr$ is in fact a very natural approximation to $\bar{\mu}_k$. Note that we divide by the n-dimensional lebesgue measure of the domain, $|D_i^V|$, in order to normalize and obtain the correct weights.

A.4 Conditions on the boundary

We say that a sequence of simplices $V_1, \dots V_\ell$ of a triangulation is *properly nested* if $V_i \subset \partial V_{i-1}$ and $\dim V_i = d - i$.

Let V be a simplex in a triangulation T of M. For x in V, let

$$u_V(x) = \operatorname{dist}(x, \partial V).$$

If the triangulation T is reasonably nice, u_V can then be extended to all of M in such a way that u_V will be smooth on the interiors of the simplices of ∂V. In our case, this can be done because the triangulation satisfies T3–T5. There is some ambiguity in the choice of the extension, but it is immaterial in our argument.

Let, for $\varepsilon > 0$,

$$u_V^\varepsilon(x) = \begin{cases} u_V(x)/\varepsilon, & \text{if } |u_V(x)| < \varepsilon. \\ -1, & \text{if } u_V(x) < -\varepsilon \\ 1, & \text{if } u_V(x) > \varepsilon \end{cases}$$

Finally, let $\bar{u}_V^\varepsilon$ be a smoothed version of u_V^ε, such that the amount of smoothing tends to 0 as $\varepsilon \to 0$. This can be obtained, for example, by convolving as in Section A.1 and ensuring that one uses mollifiers ψ such that $\operatorname{diam} \operatorname{supp} \psi < \varepsilon^2$.

Let $C = \{V_1 \supset \cdots \supset V_n\} \subseteq T_k$ be a set of n properly nested simplices. Observe that the form

$$\omega_\varepsilon = d\bar{u}_{V_1}^\varepsilon \wedge d\bar{u}_{V_2}^\varepsilon \wedge \cdots \wedge d\bar{u}_{V_n}^\varepsilon$$

is exact.

Let ν be a measure on $T^n M$. Let $C = \{V_1 \supset V_2 \supset \cdots \supset V_\ell\}$ be properly nested simplices in some triangulation of M. Let

$$B_\varepsilon(C) = \{x \in M : |u_{V_i}(x)| \le \varepsilon, i = 1, 2, \dots, \ell\}.$$

Define the measure ν^C by

$$\int f\, d\nu^C = \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon^\ell} \int_{B_\varepsilon(C)} f\, d\nu,$$

where $f \in C_c^\infty(T^n M)$.

Notice that

$$\lim_{\varepsilon\to 0+} \int \omega_\varepsilon\, d\mu = \int du_{V_1} \wedge du_{V_2} \wedge \cdots \wedge du_{V_n}\, d\mu^C.$$

Since the left-hand-side vanishes when μ satisfies Condition (Hol), we get

Lemma 25. *If the smooth density $\mu \in \mathscr{M}_n$ satisfies Condition (Hol), then for every $k \in \mathbb{N}$ and for every properly nested sequence of simplices $C = \{V_1 \supset V_2 \supset \cdots \supset V_n\}$ of the triangulation T_k, we have*

$$\int_{T^n M} du_{V_1} \wedge du_{V_2} \wedge \cdots \wedge du_{V_n}\, d\mu^C = 0. \tag{11}$$

A.5 Closing up the base measure

A.5.1 Inductive construction of cycles

The 0-dimensional chain. Recall that the chain β_k was constructed in Section A.3.2. It is a linear combination of n-cells $\varphi_V^* \gamma_i^V$, determined by the equations (10). For each $k > 0$, we let $\tilde{\beta}_k$ be the chain that results from extending the domain of the n-cell γ_i^V (still respecting (10)) to an open set very slightly larger than its original domain D_i^V, so that it now intersects the skeleton E_{d-1}^k of T^k. By property A2, this intersection is transversal. Then, for properly-nested simplices $C = \{V_1 \supset \cdots \supset V_\ell\}$ the measure $\wr\tilde{\beta}_k\wr^C$ reflects the way the boundary of β_k intersects ∂V_ℓ.

For a point p in $T^n M$ such that $\pi(p) \in V_\ell$, and for a set of n properly nested simplices $C = \{V_1 \supset \cdots \supset V_n\}$ let

$$W(p, C) = du_{V_1} \wedge du_{V_2} \wedge \cdots \wedge du_{V_n}(p),$$

where the functions u_{V_i} are as in Section A.4. Observe that if C and C' are two sets of n properly nested simplices that differ only in the ℓ^{th} simplex, $\ell < n$, and the corresponding simplices V_ℓ and V'_ℓ are adjacent, then

$$W(p, C) = -W(p, C') \tag{12}$$

because $du_{V_\ell} = -du_{V_\ell}$ at p.

For each k, we pick a finite set of points $\{p_i^k\}_i \subset T^n M$, and weights $r_i^k \in \mathbb{R}_+$ such that Conditions U1–U4 below are true. We will imagine that there is an n-chain whose (degenerate) cells are the points $\{\pi(p_i)\}_i \subseteq M$, so that η_k^0 is given by

$$\eta_k^0 = \sum_i r_i^k \pi(p_i^k),$$

and parameterized so that

$$\wr\eta_k^0\wr = \sum_i r_i^k \delta_{p_i^k}.$$

Strictly speaking, such chain η_k^0 does not exist, but the measure $\wr\eta_k^0\wr$ does, and this is the object we need. The conditions are:

U1. $\pi(p_i^k) \in E_{d-n}^k$ for all i.

U2. We require the points in the support of $\wr\tilde{\beta}_k\wr^C$ to be contained in $\{p_i^k\}_i$, and the corresponding weights r_i^k to be at least as large as the weights these points have in the measure $\wr\tilde{\beta}_k\wr^C$.

U3. For each set of n properly nested simplices $C = \{V_1 \supset \cdots \supset V_n\} \subseteq T^k$,

$$\sum_i W(p_i^k, C)\, r_i^k = 0,$$

where the sum is taken over all i such that $\pi(p_i)$ is in V_n.

U4. The measure $\wr\eta_k^0\wr$ approximates the restriction of μ to the skeleton E_{d-n}^k:

$$\operatorname{dist}_{\mathscr{M}_n}\left(\sum_C \mu^C, \sum_C \wr\eta_k^0\wr^C\right) \leq \frac{1}{k}$$

where the sums are taken over all sets of n properly nested simplices of T^k.

The idea is that $\{p_i^k\}_i \cap \pi^{-1}(V_n)$ should be a very good sample of the measure μ^C. The set of points and weights can be found as follows. Start with the points in the support of $\wr\tilde{\beta}_k\wr^C$, with the weights they inherit from β_k. Then by further sampling the measure μ^C, and invoking the fact that it satisfies the conclusion of Lemma 25, a solution for the condition in item U3 is guaranteed to exist. Note that the condition in item U3 is essentially a rephrasing of the conclusion of Lemma 25 adapted to $\wr\eta_k^0\wr^C$. Taking a sufficiently large sample of μ^C, one can also guarantee that item U4 will be satisfied.

The higher-dimensional chains. For every set of $n+1$ properly nested simplices $C = \{V_1 \supset \cdots \supset V_{n+1}\}$, we let η_k^C denote the 0-dimensional chain

$$\eta_k^C = \sum_i (\operatorname{sgn} W(p_i^k, C)) r_i^k \pi(p_i^k)$$

where the sum is taken over all indices i such that p_i^k is contained in V_{n+1}.

For every set $C = \{V_1 \supset \cdots \supset V_{n-j}\} \subseteq T_k$ of $n-j$ properly nested simplices, $1 \leq j < n$, $\tilde{\beta}_k$ induces an j-dimensional chain β_k^C on ∂V_{n-j} that satisfies, for all $\omega \in \Omega^j(M)$,

$$\int_{\beta_k^C} \omega = \int_{T^n M} \omega \wedge du_{V_1} \wedge du_{V_2} \wedge \cdots \wedge du_{V_{n-j}} \, d\wr\tilde{\beta}_k\wr$$

Observe that the chain β_k^C is in general not unique, but any choice will do for our purposes. We also let $\beta_k^{\emptyset} = \beta_k$.

For sets of properly nested simplices

$$C' = \{V_1 \supset \cdots \supset V_{n-j-1}\} \subset C = \{V_1 \supset \cdots \supset V_{n-j}\},$$

we refine the chain $\beta_k^{C'}$ so that each of its $(j+1)$-dimensional cells intersects only one of the $(d-n+j+1)$-dimensional simplices of the boundary ∂V_{n-j-1}. We then let $\bar{\beta}_k^C$ be the part of $\beta_k^{C'}$ that is contained in V_{n-j}. In other words,

$$\beta_k^{C'} = \sum_{V \subset \partial V_{n-j}} \bar{\beta}_k^{C' \cup \{V\}}.$$

We proceed to construct, inductively on $j = 0, 1, \ldots, n-1$, $(j+1)$-dimensional cycles η_k^C corresponding to each set of $n-j$ properly nested simplices $C = \{V_1 \supset \cdots \supset V_{n-j}\} \subseteq T_k$, such that:

E1. The cells of η_k^C are contained in $V_{n-j} \subseteq E_{d-n+j+1}^k \subseteq M$.

E2. We require that $\bar{\beta}_k^C$ be contained in η_k^C, in the sense that all the cells of $\bar{\beta}_k^C$ appear in η_k^C with coefficients of magnitud greater or equal to those they have in $\bar{\beta}_k^C$.

If $j = n-1$, $C = \{V_1\}$ and η_k^C contains precisely the cells of β_k that are contained in V_1, and with exactly the same parameterization for each cell.

E3. We have

$$\partial \eta_k^C = \sum_{V \subset \partial V_{n-j}} \eta_k^{C \cup \{V\}},$$

where the sum is taken over all simplices in the boundary of V_{n-j}.

E4. If C and C' are sets of $n-j$ properly nested simplices of T_k that only differ in the ℓ-th simplex, $1 \leq \ell < n-j$, and the corresponding simplices V_ℓ and V'_ℓ are adjacent, then

$$\eta_k^C = -\eta_k^{C'}.$$

This should hold in the sense that the induced functionals on $\Omega^{j+1}(M)$ (i.e., the induced currents) must be equal.

E5. If $C' = \{V_1 \supset \cdots \supset V_{n-j-1}\} \subseteq T_k$ is not empty,

$$\sum_{V \subset \partial V_{n-j-1}} \partial \eta_k^{C' \cup \{V\}} = 0,$$

where the sum is taken over all simplices in the boundary of V_{n-j-1}. If C' is empty, then the same equation should hold, but now taking the sum over all simplices V of dimension d in T_k.

E6. The cells of η_k^C that are not inherited from $\bar{\beta}_k^C$ are almost $\mathbf{M}$-mass minimizing, in a sense that will be specified at the end of Section A.5.3.

E7. If $j = n-1$, the cells of η_k^C that are not inherited from $\bar{\beta}_k^C$ are parameterized with very high speed (and thus the induced total measure $\wr\bar{\beta}_k^C\wr(T^n M)$ is very small), in a sense that will be specified in Section A.6.

First we show how to create the 1-chain η_k^C corresponding to the case in which C contains n properly nested simplices. We start with $\bar{\beta}_k^C$, which will provide for compliance with item E2. By U2, the boundary of $\bar{\beta}_k^C$ is also contained in $\sum_{V \subset \partial V_{n-1}} \eta_k^{C \cup \{V\}}$. So what we do, in order to comply with E1 and E3, is that we connect the remaining dots in $\sum_{V \subset \partial V_{n-1}} \eta_k^{C \cup \{V\}}$ with curves contained in V_{n-1} in the way prescribed by the weights of the dots; because of property U3, this is possible. By taking very short curves, we ensure compliace with E6. Because of identity (12), the construction of $\eta_k^{C \cup \{V\}}$ ($V \subset \partial V_{n-1}$) immediately implies E4. Property E5 also follows from the identity (12).

Now assume that we have η_k^C for $j = m-1$, and let us construct it for $j = m$, $m > 1$. Let $C = \{V_1 \supset \cdots \supset V_{n-m}\} \subseteq T_k$. For each simplex $V \subset \partial V_{n-m}$, we are assuming that there exists $\eta_k^{C \cup V}$ that satisfies E1–E6. To close these up, we again start with $\bar{\beta}_C$ (whence complying with E2) and we add cells of dimension $m+1$ contained in V_{n-m} (complying with E1) so that property E3 will hold; this is possible because V_{n-m} has trivial homology and because $\sum_{V\subset\partial V_{n-1}} \eta_k^{C\cup\{V\}}$ is a cycle as it satisfies E5. Properties E4 and E5 for $j=m$ follow from property E4 for $j=m-1$. Compliance with property E6 can be attained by choosing an almost mass-minimizing set of $(m+1)$-cells. Property E7 can be achieved by adjusting the parameterization of the cells involved.

Write $\eta_k = \eta_k^{\emptyset}$. We have proved:

Lemma 26. *There is a sequence of cycles η_k that contain β_k and such that*

$$\mathbf{M}(\wr\eta_k\wr) - \mathbf{M}(\wr\beta_k\wr) \tag{13}$$

is almost minimal while respecting

$$\operatorname{dist}_{\mathscr{M}_n}\left(\sum_C \mu^C, \sum_C \wr\eta_k\wr^C\right) \leq \frac{1}{k}, \tag{14}$$

where the sums are taken over all sets C of n properly nested simplices of T^k. Also, the part of $\wr\eta_k\wr^C$ that comes inherited from β_k satisfies A3.

By construction, equation (14) is exactly the same as the condition in U4.

A.5.2 Density lemma

For each set $C = \{V_1 \supset \cdots \supset V_{n-1}\} \subset T_k$ of properly nested simplices, in Section A.5.1 the 1-dimensional chain η_k^C was constructed. Our goal in this section is to estimate the asymptotic behavior of its mass $\mathbf{M}(\wr\eta_k^C\wr)$ as $k \to \infty$.

For a set $U \subset \mathbb{R}^m$, the *diameter of U within U* is defined to be

$$\operatorname{diam}_U U = \sup_{x,y\in U} \inf_{\gamma} \operatorname{arclength}(\gamma)$$

where the infimum is taken over all absolutely-continuous curves γ parameterized on any interval $[a,b] \subseteq \mathbb{R}$ and such that $\gamma(a) = x$ and $\gamma(b) = y$.

Lemma 27. *Let U be a path-connected, bounded open set in $\mathbb{R}^m$, $m \geq 1$. There is a number $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$ and A and B are two finite subsets of U of equal cardinality, then the following is true. Assume that there is a finite partition of U by disjoint, path-connected sets $U_1, \ldots, U_m$*

with $\operatorname{diam}_{U_i} U_i < \varepsilon$ *such that each of them contains at least one point of* A*, and*

$$\left|1 - \frac{\# U_i \cap B}{\# U_i \cap A}\right| < \varepsilon^2. \tag{15}$$

Then there is a 1-dimensional chain θ *such that* $\wr\theta\wr(U) = 1$*,* $\mathbf{M}(\wr\theta\wr) < 2\varepsilon$*, and*

$$\partial\wr\theta\wr = \frac{1}{\# A}\left(\sum_{x\in A}\delta_x - \sum_{y\in B}\delta_y\right). \tag{16}$$

Proof. Let

$$\varepsilon_0 = \frac{1}{2\operatorname{diam}_U U}.$$

Condition (15) implies that at least $\lfloor(1-\varepsilon^2)\# U_i \cap A\rfloor$ points of A can be joined to points of B within U_i. Since $\operatorname{diam}_{U_i} U_i < \varepsilon$, this can be done using curves γ of length smaller than ε. Let λ_1 be the chain formed by all those curves γ, each parameterized at the right speed that its induced measure will be a probability, $\wr\gamma\wr(TU) = 1$. The remaining $\sim \varepsilon^2 \# U_i \cap A$ points of A (and a similar amount of points of B) need to be paired with points outside U_i. Since $\# A = \# B$, this is always possible, and it can be done using curves of length $\leq \operatorname{diam}_U U$. Let λ_2 be the chain corresponding to these longer curves, again parameterized at a speed that will make the induced measure a probability.

We let $\theta = (\lambda_1 + \lambda_2)/\# A$. It is clear then that $\wr\theta\wr$ is a probability, and that (16) holds. We estimate

$$\begin{aligned}\mathbf{M}(\wr\theta\wr) &= \frac{\operatorname{arclength}(\lambda_1) + \operatorname{arclength}(\lambda_2)}{\# A} \\ &\leq \frac{\varepsilon(1-\varepsilon^2)\# A + (\operatorname{diam}_U U)\varepsilon^2 \# A}{\# A} \leq 2\varepsilon. \quad \square\end{aligned}$$

Let $k \geq 1$ and let C be a set of properly nested simplices in T_k. Decompose the chain η_k^C into the part of it that comes from $\bar\beta_k^C$ and a remainder ζ_k^C,

$$\eta_k^C = \bar\beta_k^C + \zeta_k^C.$$

Fix $k \in \mathbb{N}$ and a set C of n properly nested simplices. From the construction of η_k^C, it follows that ζ_k^C is formed from two types of components:

- Curves joining two points in the 0-chains β_k^C; call the corresponding chain ζ_{short}.

- Curves joining points in various 0-chains $\eta_k^{C'}$ ($C' \supset C$) that are not both already in β_k^C; call the corresponding chain ζ_{long}.

Observe that as $k \to \infty$, the first quotient in (8) behaves as

$$\frac{\bar{\mu}_k(V_2)}{\bar{\mu}_k(V_1)} \to 1$$

since the triangulations T_k satisfy T6 and μ is assumed to be a smooth density. So (8) tends to look like (15). It follows that if k is large, we can apply Lemma 27 to a large subset of the points of $\partial\zeta_{short}$, with the conclusion that the part of ζ_{short} joining them has very small mass $\mathbf{M}$. What remains of $\partial\zeta_{short}$ tends to have 0 weight, so the mass of the corresponding part of ζ_{short} also vanishes asymptotically.

Similarly, since $\bar{\mu}_k \to \mu$ as $k \to \infty$, and since the points $\{p_i^k\}_i$ are a sample of $\mu|_{E_{d-n}^k}$ (they satisfy U4), the weight of $\partial\zeta_{long}$ vanishes asymptotically, and hence so does the mass of ζ_{long}.

We let the function a in A3 decrease rapidly enough that the following lemma will hold as per the preceding argument.

Lemma 28. *As $k \to \infty$,*

$$\sum_C \mathbf{M}(\wr\zeta_k^C\wr) \to 0 \qquad \textit{and} \qquad \frac{\sum_C \mathbf{M}(\wr\eta_k^C\wr)}{\sum_C \mathbf{M}(\wr\bar{\beta}_k^C\wr)} \to 1,$$

where the sums are taken over all sets C of $n-1$ properly nested simplices in T_k.

A.5.3 Isoperimetric inequality

In this section we want to find an upper bound for the mass difference (13).

Recall the isoperimetric inequality:

Proposition 29 (Federer [10, §4.2.10], [16, §5.3])**.** *There is a constant $C_4 > 1$ such that if θ is an m-chain with $\partial\theta = 0$ and contained in a simplex V of some triangulation T_k and of diameter $\operatorname{diam}_V V < 1$, then there exists an $(m+1)$-chain σ with $\partial\sigma = \theta$ contained in V and with mass bounded by*

$$\mathbf{M}(\wr\sigma\wr) \leq C_4 \mathbf{M}(\wr\theta\wr)^{\frac{k+1}{k}}.$$

The original proposition is valid for chains θ in $\mathbb{R}^d$. It is true as stated because when we pullback a chain from $\mathbb{R}^d$ to M via any of the functions

φ_V, the modulus of these mappings is globally bounded. This in turn is true because there are only finitely many of them, and they have compact domains.

Let $k \geq 1$ and let V_1 be a d-dimensional simplex in T_k. Recall that the chains ζ_k^C were defined in Section A.5.2. It follows from Lemma 28 and Proposition 29 that we can take the cells in ζ_k^C to be such that, as $k \to \infty$

$$\begin{aligned}
\mathbf{M}(\zeta_k^{\{V_1\}}) &\leq C_4 \sum_{V_2 \subset \partial V_1} \mathbf{M}(\zeta_k^{\{V_1,V_2\}})^2 + \varepsilon_2^k \\
&\leq C_4{}^{1+\frac{3}{2}} \sum_{V_3 \subset \partial V_2} \sum_{V_2 \subset \partial V_1} \mathbf{M}(\zeta_k^{\{V_1,V_2,V_3\}})^3 + \varepsilon_3^k \\
&\leq \cdots \\
&\leq C_4{}^{p_n} \sum_{V_{n-1} \subset \partial V_{n-2}} \cdots \sum_{V_2 \subset \partial V_1} \mathbf{M}(\zeta_k^{\{V_1,V_2,\ldots,V_{n-1}\}})^{n-1} + \varepsilon_{n-1}^k \to 0,
\end{aligned}$$

where $p_n > 1$ is some number depending only on n, ε_k^ℓ is arbitrarily small (it is the error we may get from not taking exactly the cell provided by Proposition 29, but one with slightly larger mass; we thus specify property E6 to mean that $\varepsilon_\ell^k \to 0$ as $k \to \infty$ for all ℓ), and the sums are taken over all simplices in the corresponding boundaries. We conclude

Lemma 30.

$$|\mathbf{M}(\wr\eta_k\wr) - \mathbf{M}(\wr\beta_k\wr)| \to 0 \quad \textit{as} \quad k \to \infty.$$

A.6 Conclusion

Proof of Theorem 1. Let $\mu \in \mathscr{M}_n$ be a positive measure. If μ satisfies Condition (Cyc), it follows from Stokes's theorem that it also satisfies Condition (Hol).

To prove the other direction, assume that μ satisfies Condition (Hol). By Lemma 24, we can assume that μ is smooth. We can thus construct for $k \geq 1$ triangulations T_k as in Section A.2, base measures $\bar{\mu}_k$ as in Section A.3.1, chains β_k approximating these as in Section A.3.2, and cycles η_k as in Section A.5.1 that contain β_k. We have

$$\operatorname{dist}_{\mathscr{M}_n}(\mu, \wr\eta_k\wr) \leq \operatorname{dist}_{\mathscr{M}_n}(\mu, \bar{\mu}_k) + \operatorname{dist}_{\mathscr{M}_n}(\bar{\mu}_k, \wr\beta_k\wr) + \operatorname{dist}_{\mathscr{M}_n}(\wr\beta_k\wr, \wr\eta_k\wr).$$

The first two summands on the right-hand-side vanish asymptotically by construction. The last term, as per the definition of $\operatorname{dist}_{\mathscr{M}_n}$ in equation (1), has two parts: the mass difference, which tends to zero by Lemma 30, and the one involving the functions f_i. The second one can be arranged to tend to zero by having the cells of η_k not present in β_k be parameterized at very high speeds, thus specifying property E7. We conclude that the measures induced by the cycles η_k indeed approximate μ, so μ satisfies Condition (Cyc). $\square$

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