CLUSTER ALGEBRAS AND POISSON GEOMETRY

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0. Introduction

The aim of the present paper is to study Poisson structures naturally related to cluster algebras recently introduced by Fomin and Zelevinsky in [FZ2].

Roughly speaking, a cluster algebra is defined by an n-regular tree whose vertices correspond to n-tuples of cluster variables and edges describe birational transformations between two n-tuples of variables; the cluster algebra itself is generated by the union of all cluster variables. Model examples of cluster algebras are coordinate rings of double Bruhat cells (see [FZ2]). Edge transformation rules imitate simplest (3-term) Plücker relations. Given the set of transformation rules for all the edges incident to one vertex of the tree, one can restore all the other transformation rules. The evolution of transformation rules provides the so-called Laurent phenomenon ([FZ3]), which means the following: fix one cluster with cluster variables $x_1,\dots,x_n$ and express any other cluster variable in terms of $x_1,\dots,x_n$; then the expression is a Laurent polynomial in $x_1,\dots,x_n$.

The first goal of this paper is to give less formal explanation for the evolution of edge transformation rules from the Poisson point of view. Namely, we introduce a Poisson structure compatible with the cluster algebra structure. Compatibility simply means that the Poisson structure is homogeneously quadratic in any set of cluster variables. Then edge transformations describe simple transvections with respect to the Poisson structure. In particular, the transition of transformation rules can be simply explained as the transformation of coefficients of the Poisson structure under a transvection.

The second goal of our paper is to extend our previous calculations of the number of connected components in double Bruhat cells (see [SSV1, SSV2, SSVZ, GSV]) to a more general setting of geometric cluster algebras and compatible Poisson structures. Namely, given a cluster algebra $\mathcal{A}$ over $\mathbb{R}$ we compute the number of connected components in the union of generic symplectic leaves of any compatible Poisson structure in a certain "large" nonsingular subset of $\mathrm{Spec}(\mathcal{A})$.

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Finally, we apply the general formula to a special case of Grassmanian coordinate ring.

The structure of the paper is as follows.

In the first chapter we recall a notion of (geometric) cluster algebra $\mathcal{A}$, introduce a notion of a Poisson bracket compatible with $\mathcal{A}$ and describe all Poisson brackets compatible with $\mathcal{A}$. Moreover, we also prove the following (partial) inverse result. Assume that a homogeneous quadratic Poisson bracket in the affine space $\mathbb{A}^n$ is given. We are looking for birational involutions preserving quadratic homogeneity and satisfying some locality and universality properties. Then there exists a cluster algebra compatible with the Poisson structure, such that these birational transformations are exactly edge transitions for this cluster algebra.

In the second chapter we introduce an $\mathbb{F}^*$-action compatible with the cluster algebra $\mathcal{A}$ (here $\mathbb{F}$ is a field of characteristic 0). Compatibility of the $\mathbb{F}^*$-action means that all edge transformations of $\mathcal{A}$ are preserved under this action. The union $\mathcal{X}^0$ of generic orbits with respect to this $\mathbb{F}^*$-action is "almost" the union of generic symplectic leaves of a compatible Poisson structure in $\mathrm{Spec}(\mathcal{A})$. We compute the number of connected components of $\mathcal{X}^0$ for a cluster algebra over $\mathbb{R}$.

Finally we apply the previous result to the case of refined open Bruhat cells in the Grassmanian $G(k,n)$. Namely, the famous Sklyanin Poisson-Lie bracket on $SL_n(\mathbb{R})$ induces a Poisson bracket on the open Bruhat cell in $G(k,n)$. This Poisson bracket is compatible with the structure of a special cluster algebra, one of whose clusters consists only of Plücker coordinates. The corresponding $\mathbb{R}^*$-action determines the union of generic orbits, which is simply described as a subset of the Grassmanian defined by inequalities $X_i \neq 0$, $i \in [1,n]$, where X_i is the (cyclically solid) minor containing the ith, $(i+1)$th, $\dots$, $(i+k)$th $(\mod n)$ columns. We call this subset a refined open Bruhat cell in the Grassmanian $G(k,n)$; indeed, this subset is an intersection of n open Bruhat cells in general position. In the last part we compute the number of connected components of a refined open Bruhat cell in $G(k,n)$ over $\mathbb{R}$. This number is equal to $3 \cdot 2^{n-1}$ if $k \geqslant 3$ and $n \geqslant 7$.

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1. Cluster algebras of geometric type and Poisson brackets

1.1. Cluster algebras of rational functions on $\mathbb{A}^n$. Let A be an arbitrary matrix, $I = \{i_1, \dots, i_m\}$, $J = \{j_1, \dots, j_n\}$ be two ordered multi-indices. We denote by $A(I;J)$ the $m \times n$ submatrix of A whose entries lie in the rows $i_1, \dots, i_m$ and columns $j_1, \dots, j_n$. Instead of $A([1,m];[1,n])$ we write just $A[m;n]$ (here and in what follows we use the notation $[i,j]$ for a contiguous index set $\{i, \dots, j\}$). Given a diagonal matrix D with positive integer diagonal entries $d_1, \dots, d_m$, let $\mathcal{Z}^D_{mn}$ be the set of all $m \times n$ integer matrices Z such that $m \leqslant n$ and $Z[m;m]$ is D-skew-symmetrizable (that is, $DZ[m;m]$ is skew-symmetric); clearly, $\mathcal{Z}^D_{mn} = \mathcal{Z}^{\lambda D}_{mn}$ for any positive integer λ. According to [FZ2], any $Z = (z_{ij}) \in \mathcal{Z}^D_{mn}$ defines a cluster algebra of geometric type in the following way. Let us fix a set of m cluster variables $f_1, \dots, f_m$, and a set of $n-m$ tropic variables $f_{m+1}, \dots, f_n$. For each $i \in [1,m]$ we introduce a transformation T_i of cluster variables by

$$T_i(f_j) = \bar f_j = \begin{cases} \dfrac{1}{f_i}\left(\prod\limits_{z_{ik}>0} f_k^{z_{ik}} + \prod\limits_{z_{ik}<0} f_k^{-z_{ik}}\right) & \text{for } j = i \\ f_j & \text{for } j \neq i, \end{cases} \tag{1.1}$$

and the corresponding matrix transformation $\bar Z = T_i(Z)$, called mutation, by

$$\bar z_{kl} = \begin{cases} -z_{kl} & \text{for } (k-i)(l-i) = 0 \\ z_{kl} + \dfrac{|z_{ki}|z_{il} + z_{ki}|z_{il}|}{2} & \text{for } (k-i)(l-i) \neq 0. \end{cases} \tag{1.2}$$

Observe that the tropic variables are not affected by T_i, and that $\bar Z$ belongs to $\mathcal{Z}_{mn}^D$. Thus, one can apply transformations T_i to the new set of cluster variables (using the new matrix), etc. The cluster algebra (of geometric type) is the subalgebra of the field of rational functions in cluster variables $f_1, \dots, f_m$ generated by the union of all clusters; its ground ring is the ring of integer polynomials over tropic variables. We denote this algebra by $\mathcal{A}(Z)$.

One can represent $\mathcal{A}(Z)$ with the help of an m-regular tree $\mathbb{T}_m$ whose edges are labeled by the numbers $1, \dots, m$ so that the m edges incident to each vertex receive different labels. To each vertex v of $\mathbb{T}_m$ we assign a set of m cluster variables $f_{v,1}, \dots, f_{v,m}$ and a set of $n-m$ tropic variables $f_{m+1}, \dots, f_n$. For an edge $(v, \bar v)$ of $\mathbb{T}_m$ that is labeled by $i \in [1, m]$, the variables $f = f_v$ and $\bar f = f_{\bar v}$ are related by the transformation T_i given by (1.1). The first monomial in the right hand side of (1.1) is sometimes denoted by $M^i = M_v^i$. Transformations (1.2) then guarantee that the second monomial is $\bar M^i = M_{\bar v}^i$.

Let us say that cluster and tropic variables together form an extended cluster. Assume that the entries of the initial extended cluster are coordinate functions on $\mathbb{A}^n$. We thus get a realization of a cluster algebra of geometric type as a cluster algebra of rational functions on $\mathbb{A}^n$. It is easy to see that in this situation entries of any extended cluster are functionally independent.

Remark. If all entries of Z belong to the set $\{0, \pm 1\}$, it is sometimes convenient to represent Z by a directed graph $\mathcal{E}$ with vertices corresponding to the variables (both cluster and tropic) and with edges $i \to j$ for every pair of vertices i, j, such that $Z_{ij} = 1$ (in particular, there are no edges between vertices corresponding to tropic variables). If we assume, in addition, that the resulting graph has no nonoriented 3-cycles, then the graph that corresponds to $\bar Z$ differs from the one that corresponds to Z as follows. All edges through i change directions. Furthermore, for every two vertices j, k such that edges $j \to i$ and $i \to k$ belong to the graph of Z, the graph of $\bar Z$ contains an edge $j \to k$ if and only if the graph of Z does not contain an edge $k \to j$.

1.2. τ-coordinates. For our further purposes it is convenient to consider, along with f, another n-tuple of rational functions on $\mathbb{A}^n$. This n-tuple is denoted $\tau = (\tau_1, \dots, \tau_n)$, and is related to the initial n-tuple f as follows. Let $\widehat D$ be an $n \times n$ diagonal matrix $\widehat D = \operatorname{diag}(d_1, \dots, d_m, 1, \dots, 1)$. Denote by $\mathcal{Z}_{nn}^{\widehat D}(Z)$ the set of all $n \times n$ integer matrices $Z' \in \mathcal{Z}_{nn}^{\widehat D}$ such that $Z'[m; n] = Z$. Fix an arbitrary matrix

$Z' \in \mathcal{Z}_{nn}^{\widehat{D}}(Z)$ and put

$$\tau_j = f_j^{\varkappa_j} \prod_{k=1}^{n} f_k^{z'_{jk}}, \tag{1.3}$$

where $\varkappa_j$ is an integer, $\varkappa_j = 0$ for $1 \leqslant j \leqslant m$. Given an extended cluster f, we say that the entries of the corresponding τ form a τ-cluster.

We say that the transformation $f \mapsto \tau$ is *nondegenerate* if

$$\det(Z' + K) \neq 0, \tag{1.4}$$

where $K = \operatorname{diag}(\varkappa_1, \dots, \varkappa_n)$. It is easy to see that if the transformation $f \mapsto \tau$ is nondegenerate and the entries of the extended cluster are functionally independent, then so are the entries of the τ-cluster.

Lemma 1.1. *A nondegenerate transformation $f \mapsto \tau$ exists if and only if* $\operatorname{rank} Z = m$.

Proof. The only if part is trivial. To prove the if part, assume that $\operatorname{rank} Z = m$ and $\operatorname{rank} Z[m;m] = k \leqslant m$. Then there exists a nonzero $m \times m$ minor of Z contained in the columns $j_1 < j_2 < \dots < j_m$ so that $j_k \leqslant m$, $j_{k+1} > m$ (here $j_0 = 0$, $j_{m+1} = n+1$). Without loss of generality assume that $j_i = i$ for $i \in [1,k]$. Define

$$\varkappa_j = \begin{cases} 0 & \text{for } j = 1, \dots, m, \\ 1 & \text{for } j = j_{k+1}, \dots, j_m, \\ \varkappa & \text{otherwise.} \end{cases}$$

Let us prove that there exists an integer $\varkappa$ such that $\det(Z' + K) \neq 0$. Indeed, the leading coefficient of this determinant (regarded as a polynomial in $\varkappa$) is equal to the $(2m-k) \times (2m-k)$ minor contained in the rows and columns $1, 2, \dots, m, j_{k+1}, \dots, j_m$. It is easy to see that using the same elementary row and column operations one can reduce the corresponding submatrix to the form

$$M = \begin{pmatrix} Z_1 & 0 & 0 \\ 0 & Z_2 & Z_3 \\ 0 & Z_4 & Z_5 \end{pmatrix},$$

where Z_1 is just $Z[k;k]$, Z_2 is an $(m-k) \times (m-k)$ matrix depending only on the entries of $Z[m;m]$, Z_3, Z_4, Z_5 are $(m-k) \times (m-k)$ matrices. Moreover, $Z_2 = 0$, since otherwise $\operatorname{rank} Z[m;m]$ would exceed k, and hence $\det M = \det Z_1 \det Z_3 \det Z_4$. On the other hand, condition $\operatorname{rank} Z = m$ implies $\det Z_1 \det Z_3 \neq 0$, while the skew-symmetrizability of Z' implies $\det Z_4 \neq 0$. Therefore, the leading coefficient of $\det(Z' + K)$ is distinct from zero, and we are done. □

Let us find explicit expressions for the transformations T_i in the new coordinates (τ, Z'). First, we extend the rules (1.2) to all the entries of Z'. Observe that for any Z' as above, one has $\bar{Z}'[m;n] = \bar{Z}$. Moreover, the coordinate change $\bar{f} \mapsto \bar{\tau}$ remains nondegenerate, due to Lemma 1.1 and the following proposition.

Lemma 1.2. *If* $\operatorname{rank} Z = m$, *then* $\operatorname{rank} \bar Z = m$.

Proof. Indeed, consider the following sequence of row and column operations with the matrix Z'. For any l such that $z'_{il} < 0$, subtract the ith column multiplied by z'_{il} from the lth column. For any k such that $z'_{ki} > 0$, add the ith row multiplied by z'_{ki} to the kth row. Finally, multiply the ith row and column by -1. It is easy to see that the result of these operations is exactly $\bar Z'$, and the lemma follows. □

Finally, coordinates τ are transformed as follows.

Lemma 1.3. *Let* $i \in [1,m]$ *and let* $\bar\tau_j = T_i(\tau_j)$ *for* $j \in [1,n]$. *Then* $\bar\tau_i = 1/\tau_i$ *and* $\bar\tau_j = \tau_j\psi_{ji}(\tau_i)$, *where*

$$\psi_{ji}(\xi) = \begin{cases} \left(\dfrac{1}{\xi}+1\right)^{-z'_{ji}} & \text{for } z'_{ji} > 0, \\ (\xi+1)^{-z'_{ji}} & \text{for } z'_{ji} < 0, \\ 1 & \text{for } z'_{ji} = 0 \text{ and } j \neq i. \end{cases}$$

Proof. Let us start from the case $j = i$. Since $i \in [1,m]$ and $z'_{ii} = 0$, we can write

$$\bar\tau_i = \prod_{k\neq i} \bar f_k^{\bar z'_{ik}} = \prod_{k\neq i} f_k^{-z'_{ik}} = \frac{1}{\tau_i},$$

as required.

Now, let $j \neq i$. Then

$$\begin{aligned} \bar\tau_j &= \bar f_j^{\varkappa_j} \bar f_i^{\bar z'_{ji}} \prod_{k\neq i} \bar f_k^{\bar z'_{jk}} \\ &= f_j^{\varkappa_j} \left(\prod_{z'_{ik}>0} f_k^{z'_{ik}} + \prod_{z'_{ik}<0} f_k^{-z'_{ik}} \right)^{-z'_{ji}} f_i^{z'_{ji}} \prod_{k\neq i} f_k^{z'_{jk}} \prod_{k\neq i} f_k^{(|z'_{ji}|z'_{ik}+z'_{ji}|z'_{ik}|)/2} \\ &= \tau_j \left(\prod_{z'_{ik}>0} f_k^{z'_{ik}} + \prod_{z'_{ik}<0} f_k^{-z'_{ik}} \right)^{-z'_{ji}} \prod_{k\neq i} f_k^{(|z'_{ji}|z'_{ik}+z'_{ji}|z'_{ik}|)/2}. \end{aligned}$$

If $z'_{ji} = 0$, then evidently $\bar\tau_j = \tau_j$.

Let $z'_{ji} > 0$, then

$$\begin{aligned} \bar\tau_j &= \tau_j \left(\prod_{z'_{ik}>0} f_k^{z'_{ik}} + \prod_{z'_{ik}<0} f_k^{-z'_{ik}} \right)^{-z'_{ji}} \prod_{z'_{ik}>0} f_k^{z'_{ji}z'_{ik}} \\ &= \tau_j \left(\prod_{z'_{ik}\neq 0} f_k^{-z'_{ik}} + 1 \right)^{-z'_{ji}} = \tau_j(1/\tau_i+1)^{-z'_{ji}}, \end{aligned}$$

as required.

Let $z'_{ji} < 0$, then

$$\bar\tau_j = \tau_j \left(\prod_{z'_{ik}>0} f_k^{z'_{ik}} + \prod_{z'_{ik}<0} f_k^{-z'_{ik}} \right)^{-z'_{ji}} \prod_{z'_{ik}<0} f_k^{-z'_{ji}z'_{ik}}$$
$$= \tau_j \left(\prod_{z'_{ik}\neq 0} f_k^{z'_{ik}} + 1 \right)^{-z'_{ji}} \prod_{z'_{ik}\neq 0} f_k^{-z'_{ij}z'_{ik}} = \tau_j(\tau_i+1)^{-z'_{ji}},$$

as required. □

1.3. Poisson brackets. Let ω be a Poisson bracket on an n-dimensional manifold M. We say that functions $g_1,\dots,g_n$ are *log-canonical* with respect to ω if $\omega(g_i,g_j) = \omega_{ij}g_ig_j$, where ω_{ij} are integer constants. The matrix $\Omega = (\omega_{ij})$ is called the *coefficient matrix* of ω (in the basis g); evidently, $\Omega \in so_n(\mathbb{Z})$.

Fix some $Z \in \mathcal{Z}^D_{mn}$ and consider the following question: are there any Poisson structures such that all clusters in the cluster algebra $\mathcal{A}(Z)$ are log-canonical with respect to them?

We say that a skew-symmetrizable matrix A is *reducible* if there exists a permutation matrix P such that PAP^T is a block-diagonal matrix, and *irreducible* otherwise; $r(A)$ is defined as the maximal number of diagonal blocks in PAP^T. The partition into blocks defines an obvious equivalence relation $\sim$ on the rows (or columns) of A.

Theorem 1.4. *Assume that $Z \in \mathcal{Z}^D_{mn}$ and $\operatorname{rank} Z = m$. Then the Poisson brackets for which all extended clusters in $\mathcal{A}(Z)$ are log-canonical form a vector space of dimension $r + \binom{n-m}{2}$, where $r = r(Z[m;m])$. Moreover, the coefficient matrices of these Poisson brackets in the basis τ are characterized by the equation $\Omega^\tau[m;n] = \Lambda Z\widehat{D}^{-1}$, where $\Lambda = \operatorname{diag}(\lambda_1,\dots,\lambda_m)$ with $\lambda_1\lambda_2\cdots\lambda_m \neq 0$ and $\lambda_i = \lambda_j$ whenever $i \sim j$. In particular, if $Z[m;m]$ is irreducible, then $\Omega^\tau[m;n] = \lambda Z\widehat{D}^{-1}$, $\lambda \neq 0$.*

Proof. Let us note first that τ-coordinates are expressed in a monomial way in terms of initial coordinates f, and that this transformation is invertible. Therefore, all extended clusters in $\mathcal{A}(Z)$ are log-canonical w.r.t. some bracket ω if and only if so are all the corresponding τ-clusters. Denote by Ω^f and by Ω^τ the matrices of ω in the bases f and τ, respectively. It is easy to see that $\Omega^\tau = (Z'+K)\Omega^f(Z'+K)^T$. Evidently, transformation T_i preserves the log-canonicity if and only if for any $j \neq i$, $\omega(\bar f_i, \bar f_j) = \bar\omega_{ij}\bar f_i \bar f_j$ provided $\omega(f_i,f_j) = \omega_{ij}f_if_j$. Using (1.1) we get

$$\omega(\bar f_i,\bar f_j) = \omega\left(\frac{1}{f_i}\left(\prod_{z_{ik}>0} f_k^{z_{ik}} + \prod_{z_{ik}<0} f_k^{-z_{ik}}\right), f_j\right)$$
$$= \frac{f_j}{f_i}\prod_{z_{ik}>0} f_k^{z_{ik}}\left(\sum_{z_{ik}>0} z_{ik}\omega_{kj} - \omega_{ij}\right) + \frac{f_j}{f_i}\prod_{z_{ik}<0} f_k^{z_{ik}}\left(-\sum_{z_{ik}<0} z_{ik}\omega_{kj} - \omega_{ij}\right),$$

and hence the above conditions are satisfied if and only if $\sum_{z_{ik}>0} z_{ik}\omega_{kj} - \omega_{ij} = -\sum_{z_{ik}<0} z_{ik}\omega_{kj} - \omega_{ij}$ for $j \neq i$. This means that

$$(Z'+K)\Omega^f[m;n] = Z\Omega^f = \begin{pmatrix} \Delta & 0 \end{pmatrix}, \tag{1.7}$$

where Δ is a diagonal matrix. Consequently, we get $\Omega^\tau[m;n] = \Delta(Z'[n;m])^T$, and hence $\Delta Z[m;m] = \Omega^\tau[m;m]$ is skew-symmetric. Therefore, $\Delta = -\Lambda D^{-1}$ where $\Lambda = \operatorname{diag}(\lambda_1,\dots,\lambda_m)$ with $\lambda_1\lambda_2\cdots\lambda_m \neq 0$ and $\lambda_i = \lambda_j$ whenever $i \sim j$. It remains to notice that $Z' = -\widehat{D}^{-1}Z'^T\widehat{D}$, and therefore $Z'[n;m] = -\widehat{D}^{-1}Z^TD$, and the equation $\Omega^\tau[m;n] = \Lambda Z\widehat{D}^{-1}$ follows. The entries ω^f_{ij} for $m+1 \leqslant i < j \leqslant n$ are free parameters. □

1.4. Recovering cluster algebra transformations. In this section we recover transformations (1.1), (1.2) (in their equivalent form presented in Lemma 1.3) as unique involutive transformations of log-canonical bases satisfying certain additional restrictions.

Local data F is a family of rational functions in one variable ψ_w, $w = 0, \pm1, \pm2, \dots$, and an additional function in one variable φ. For any Poisson bracket ω and any log-canonical (with respect to ω) basis $t = (t_1,\dots,t_n)$, the local data F gives rise to n transformations F_i^ω defined as follows:

(i) $F_i^\omega(t_i) = \bar t_i = \varphi(t_i)$;

(ii) let $\Omega = (\omega_{ij})$ be the coefficient matrix of ω in the basis t, then $F_i^\omega(t_j) = \bar t_j = t_j\psi_{\omega_{ij}}(t_i)$ for $j \neq i$.

We say that local data F is *canonical* if for any Poisson bracket ω, any log-canonical (with respect to ω) basis t, and any index i, the set $F_i^\omega(t)$ is a log-canonical basis of ω as well. Local data is called *involutive* if any F_i^ω is an involution, and is called *normalized* if $\lim_{z\to0}\psi_w(z) = \pm1$ for any integer $w \geqslant 0$.

We say that a polynomial P of degree p is *a-reciprocal* if $P(0) \neq 0$ and there exists a constant c such that $\xi^pP(a/\xi) = cP(\xi)$ for any $\xi \neq 0$.

The following result gives a complete description of normalized involutive canonical local data.

Lemma 1.5. *Any normalized involutive canonical local data has one of the following forms:*

(i) *$\varphi(\xi) = \xi$ and $\psi_w(\xi) = \pm1$ for any integer w (trivial local data);*

(ii) *$\varphi(\xi) = -\xi$ and $\psi_w(\xi) = \pm\left(\frac{P(\xi)}{P(-\xi)}\right)^w$, where P is a polynomial without symmetric roots;*

(iii) *$\varphi(\xi) = \frac{a}{\xi}$, $\psi_w(\xi) = a_w\xi^{c_w}\psi_1^w(\xi)$, and $\psi_1(\xi) = \frac{P(\xi)}{Q(\xi)}$, where P and Q are coprime a-reciprocal polynomials of degrees p and q, and the constants a_w, c_w, p, q satisfy relations $a_{-1}^2 = a^{-c_{-1}}$, $p - q = c_{-1}$, and*

$$a_w = \begin{cases} \pm1 & \text{for } w \geqslant 0,\\ a_{-w}^{-1}a_{-1}^{-w} & \text{for } w < 0,\end{cases}\qquad c_w = \begin{cases} 0 & \text{for } w \geqslant 0,\\ -wc_{-1} & \text{for } w < 0.\end{cases}$$

Proof. Let F be canonical local data; fix an arbitrary ω and pick any nonzero entry ω_{ij}. Applying F_i^ω gives

$$\omega(\bar t_i, \bar t_j) = \omega(\varphi(t_i), \psi_w(t_i)t_j) = \frac{\varphi'(t_i)t_i}{\varphi(t_i)}w\bar t_i\bar t_j,$$

where $w = \omega_{ij}$. The canonicity of F yields

$$\frac{\varphi'(\xi)\xi}{\varphi(\xi)} = c,$$

where cw is an integer, therefore $\varphi(\xi)=a\xi^c$. Since F is involutive, we get $a^{c+1}\xi^{c^2}=\xi$, hence either $c=1$ and $a=\pm 1$, or $c=-1$ and a is an arbitrary nonzero constant. Let $\bar\Omega=(\bar\omega_{ij})$ be the coefficient matrix of ω in the basis $\bar t$. It follows immediately from the above calculations that $\bar\omega_{ij}=\omega_{ij}$ if $c=1$ and $\bar\omega_{ij}=-\omega_{ij}$ if $c=-1$.

Assume that $c=1$ and $a=1$. Then the involutivity of F (applied to t_j) gives $\psi_w^2(\xi)\equiv 1$ for any w, and hence $\psi_w(\xi)=\pm 1$, and we obtained the first case described in the lemma.

To proceed further, consider two arbitrary indices $j,k\neq i$ and put $u=\omega_{ij}$, $v=\omega_{ik}$. Then we get

$$\omega(\bar t_j,\bar t_k)=\omega(t_j\psi_u(t_i),t_k\psi_v(t_i))=\left(\omega_{jk}+v\frac{\psi_u'(t_i)t_i}{\psi_u(t_i)}-u\frac{\psi_v'(t_i)t_i}{\psi_v(t_i)}\right)\bar t_j\bar t_k,$$

so the canonicity of F yields

$$v\frac{\psi_u'(\xi)\xi}{\psi_u(\xi)}-u\frac{\psi_v'(\xi)\xi}{\psi_v(\xi)}=c_{uv}$$

for some integer constant c_{uv}. Integrating both sides we obtain equation $\psi_u^v(\xi)=a_{uv}\xi^{c_{uv}}\psi_v^u(\xi)$, which is valid for any u and v. In particular, taking $v=1$ we get

$$\psi_u(\xi)=a_u\xi^{c_u}\psi_1^u(\xi), \tag{1.8}$$

where $a_u=a_{u1}$ and $c_u=c_{u1}$.

Let us return to the case $\varphi(\xi)=-\xi$. In this case the involutivity of F applied to t_j gives $\psi_w(\xi)\psi_w(-\xi)\equiv 1$ for any integer w. Using (1.8) we get

$$(-1)^{c_w}a_w^2\xi^{2c_w}\left(\psi_1(\xi)\psi_1(-\xi)\right)^w\equiv 1,$$

which immediately yields $c_w=0$ and $a_w=\pm 1$. Let us represent the rational function ψ_1 as the ratio of two coprime polynomials P and Q. Then the above involutivity condition gives $P(\xi)P(-\xi)=Q(\xi)Q(-\xi)$, which means that $Q(\xi)=\pm P(-\xi)$. Therefore,

$$\psi_w(\xi)=\pm\left(\frac{P(\xi)}{P(-\xi)}\right)^w,$$

and the coprimality condition translates to the nonexistence of symmetric roots of P.

Finally, consider the case $\varphi(\xi)=a/\xi$. The involutivity of F applied to t_j gives $\psi_w(\xi)\psi_{-w}(a/\xi)\equiv 1$ for any integer w. Using (1.8) we get

$$\left(\frac{\psi_1(a/\xi)}{\psi_1(\xi)}\right)^w=a^{c_{-w}}a_wa_{-w}\xi^{c_w-c_{-w}}; \tag{1.9}$$

in particular, for $w=-1$ this can be rewritten as $\psi_1(\xi)=a_{-1}\xi^{c_{-1}}\psi_1(a/\xi)$, and for $w=1$, as $\psi_1(a/\xi)=a_{-1}a^{c_{-1}}\xi^{-c_{-1}}\psi_1(\xi)$ (since by definition, $a_1=1$ and $c_1=0$). Comparing the two latter identities one immediately gets $a_{-1}^2=a^{-c_{-1}}$. Besides, plugging

$$\frac{\psi_1(a/\xi)}{\psi_1(\xi)}=a_{-1}a^{c_{-1}}x^{-c_{-1}}$$

into (1.9), one gets $c_w - c_{-w} = -wc_{-1}$ and $a_w a_{-w} = a_{-1}^w a^{wc_{-1}-c_{-w}} = a_{-1}^w a^{-c_w}$. Observe that the normalization condition applied to (1.7) immediately gives $c_w = 0$ and $a_w = \pm 1$ for $w \geqslant 0$. Therefore, we get $c_w = -wc_{-1}$ and $a_w = a_{-w}^{-1} a_{-1}^{-w}$ for $w < 0$.

Finally, let us represent the rational function ψ_1 as $\psi_1(\xi) = \frac{P(\xi)}{Q(\xi)}$, where P and Q are coprime polynomials; by the normalization condition, $P(0)/Q(0) = \pm 1$. Plugging this into (1.9) for $w = 1$ one gets

$$a_{-1}\xi^{c_{-1}} \frac{P(a/\xi)}{Q(a/\xi)} = \frac{P(\xi)}{Q(\xi)}.$$

Since P and Q are coprime, the above identity can only hold when they both are a-reciprocal. Equating the degrees on both sides of the identity gives $p - q = c_{-1}$. □

We say that local data F is *finite* if it has the following finiteness property: let $n = 2$, and let ω possess a log-canonical basis $t = (t_1, t_2)$ such that the corresponding coefficient matrix has the simplest form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; then the group generated by F_1^ω and F_2^ω has a finite order.

Theorem 1.6. *Any nontrivial finite involutive canonical local data has one of the following forms:*

(i) $\varphi(\xi) = a/\xi$ *with* $a \neq 0$, $\psi_w(\xi) = (\pm 1)^w a_w$, *where* $a_w = \pm 1$ *and* $a_w = a_{-w}$*;*

(ii) $\varphi(\xi) = b^2/\xi$ *with* $b \neq 0$,

$$\psi_w(\xi) = \begin{cases} (\pm 1)^w a_w \left(\dfrac{\xi + b}{b}\right)^w & \text{for } w \geqslant 0, \\[2ex] (\pm 1)^w a_{-w} \left(\dfrac{\xi + b}{\xi}\right)^w & \text{for } w < 0, \end{cases}$$

where $a_w = \pm 1$ *and* $a_w = a_{-w}$.

Proof. Assume first that local data is of type (ii) described in Lemma 1.5. Let us prove that the transformation of the plane (x, y) corresponding to the composition $T = F_2^\omega \circ F_1^\omega$ has an infinite order. Indeed, let as start from the pair $x = R_1(\xi)/S_1(\xi)$, $y = R_2(\xi)/S_2(\xi)$, where R_i and S_i are coprime, $i = 1, 2$, and

$$s_2 < r_2 < s_1 < r_1 \tag{1.10}$$

(here and in what follows a small letter denotes the degree of the polynomial denoted by the corresponding capital letter, e.g., $s_2 = \deg S_2$). Let $T(x, y) = (\widehat{x}, \widehat{y})$, then

$$\widehat{y} = \mp \frac{R_2 \widetilde{P}(R_1, S_1)}{S_2 \widetilde{P}(-R_1, S_1))},$$

where $\widetilde{P}(\xi, \zeta) = \zeta^p P(\xi/\zeta)$. Observe that if R and S are coprime and P does not have symmetric roots then $\widetilde{P}(R, S)$ and $\widetilde{P}(-R, S)$ are coprime as well. Therefore, if $\mp \widehat{R}_2/\widehat{S}_2$ is the representation of $\widehat{y}$ as the ratio of coprime polynomials, then $\widehat{r}_2 \geqslant pr_1 - s_2$. Besides, it is easy to see that $\widehat{r}_2 - \widehat{s}_2 = r_2 - s_2 > 0$. Next,

$$\widehat{x} = \mp \frac{R_1 \widetilde{P}(\mp \widehat{R}_2, \widehat{S}_2)}{S_1 \widetilde{P}(\pm \widehat{R}_2, \widehat{S}_2)},$$

so if $\mp\widehat{R}_1/\widehat{S}_1$ is the representation of $\widehat{x}$ as the ratio of coprime polynomials, then $\widehat{s}_1 \geqslant p\widehat{r}_2 - r_1$. Besides, $\widehat{r}_1 - \widehat{s}_1 = r_1 - s_1 > 0$. Finally, $\widehat{s}_1 - \widehat{r}_2 \geqslant p\widehat{r}_2 - r_1 - \widehat{r}_2 \geqslant (p-1)(pr_1 - s_2) - r_1 > (p-1)(pr_1 - r_1) - r_1 = p(p-2)r_1$. Therefore, relation (1.10) is preserved under the action of T, provided $p \geqslant 2$. It remains to notice that $\widehat{r}_2 - r_2 \geqslant pr_1 - s_2 - r_2$, and hence $p \geqslant 2$ implies $\widehat{r}_2 > r_2$. Therefore, r_2 grows monotonically with the iterations of T, and hence T has an infinite order.

It remains to consider the case $p = 1$. Without loss of generality we may assume that $P(\xi) = \xi + b$, where $b \neq 0$. The choice of the sign in the expressions for F_1^ω and F_2^ω leads to the following four possibilities:

$$T(x,y) = \left(x\frac{-b^2+bx+by+xy}{b^2-bx+by+xy}, -y\frac{b+x}{b-x}\right),$$
$$T(x,y) = \left(x\frac{b^2-bx-by-xy}{b^2-bx+by+xy}, -y\frac{b+x}{b-x}\right),$$
$$T(x,y) = \left(x\frac{b^2-bx+by+xy}{-b^2+bx+by+xy}, y\frac{b+x}{b-x}\right),$$
$$T(x,y) = \left(x\frac{b^2-bx+by+xy}{b^2-bx-by-xy}, y\frac{b+x}{b-x}\right).$$

In the first case,

$$T^{2k}(x,y) = \left(x - \frac{4k}{b^2}x^2y, y + \frac{4k}{b^2}xy^2\right) + o((x+y)^3),$$

in the second case,

$$T^{2k}(x,y) = \left(x, y - \frac{4k}{b}xy\right) + o((x+y)^2),$$

in the third case,

$$T^{2k}(x,y) = \left(x + \frac{4k}{b}xy, y\right) + o((x+y)^2),$$

and in the fourth case,

$$T^{k}(x,y) = \left(x + \frac{2k}{b}xy, y + \frac{2k}{b}xy\right) + o((x+y)^2).$$

Therefore, in all these cases T has an infinite order.

Assume now that local data is of type (iii) described in Lemma 1.5. Instead of looking at $T = F_2^\omega \circ F_1^\omega$ we are going to study $\widehat{T} = \sigma \circ F_1^\omega$, where $\sigma(x,y) = (y,x)$. It is easy to check that $T = \widehat{T}^2$, so T has a finite order if and only if $\widehat{T}$ has a finite order.

Consider a pair $x = R_1(\xi)/S_1(\xi)$, $y = R_2(\xi)/S_2(\xi)$, where R_i and S_i are coprime, $i = 1,2$, and denote $\widehat{T}(x,y) = (\widehat{x},\widehat{y})$. Then

$$(\widehat{x},\widehat{y}) = \left(\frac{R_2\widetilde{P}(R_1,S_1)}{S_2\widetilde{Q}(R_1,S_1)S_1^{p-q}}, \frac{aS_1}{R_1}\right),$$

where $\widetilde{P}$ and $\widetilde{Q}$ are defined as before. Let $\widehat{R}_1/\widehat{S}_1$ and $\widehat{R}_2/\widehat{S}_2$ be the representations of $\widehat{x}$ and $\widehat{y}$ as the ratios of coprime polynomials, then

$$\begin{aligned} \widehat{r}_1 &\geqslant p\max\{r_1, s_1\} - s_2, & \widehat{r}_2 &= s_1,\\ \widehat{s}_1 &\geqslant q\max\{r_1, s_1\} + s_1\max\{0, p-q\} - r_2, & \widehat{s}_2 &= r_1. \end{aligned}$$

Assume first that $p \geqslant 2$. Then we start from a pair (x, y) satisfying an additional condition $r_1 > s_2$. Observe that $\widehat{r}_1 - \widehat{s}_2 \geqslant pr_1 - s_2 - r_1 \geqslant (r_1 - s_2) + (p-2)r_1 > 0$, and hence the above additional condition is preserved under the action of $\widehat{T}$. Moreover, $\widehat{s}_2 = r_1 > s_2$, which means that s_2 grows monotonically with the iterations of $\widehat{T}$, and hence $\widehat{T}$ has an infinite order.

Assume now that $p < 2$ and $q \geqslant 2$. Then we start from a pair (x, y) satisfying an additional condition $s_1 > r_2$. Observe that $\widehat{s}_1 - \widehat{r}_2 \geqslant qs_1 - r_2 - s_1 \geqslant (s_1 - r_2) + (q-2)s_1 > 0$, and hence the above additional condition is preserved under the action of $\widehat{T}$. Moreover, $\widehat{r}_2 = s_1 > r_2$, which means that r_2 grows monotonically with the iterations of $\widehat{T}$, and hence $\widehat{T}$ has an infinite order.

It remains to consider the case $\max\{p, q\} < 2$, which amounts to the following four possibilities: $p = 1$, $q = 1$; $p = 1$, $q = 0$; $p = 0$, $q = 1$; $p = 0$, $q = 0$.

Consider the first possibility, when $p = 1$, $q = 1$. Recall that by Lemma 1.5, both P and Q are a-reciprocal. It is easy to check that a linear a-reciprocal polynomial can be represented as $c(\xi + b)$ with $b^2 = a$ and $c \neq 0$. Taking into account the normalization condition, we can write $\widehat{T}$ as follows: $\widehat{T}\colon (x, y) \mapsto (\pm y(x+b)/(x-b), b^2/x)$. Consider the transformation $\widetilde{T} = \widehat{T}^8$. In a neighborhood of the point $(0, 0)$ $\widetilde{T}$ can be expanded as

$$\widetilde{T}\colon (x, y) \mapsto \left(x - \frac{8}{b^2}x^2y, y + \frac{8}{b^2}xy^2\right) + o((x+y)^3).$$

Therefore, $\widetilde{T}^k\colon (x, y) \mapsto (x - 8kx^2y/b^2, y + 8kxy^2/b^2) + o((x+y)^3)$, and hence $\widetilde{T}$ has an infinite order.

Consider the second possibility, when $p = 1$, $q = 0$. In this case the transformation $\widehat{T}$ can be written as follows: $\widehat{T}\colon (x, y) \mapsto (\pm y(x+b)/b, a/x)$ with $b^2 = a$. It is easy to check that in this case indeed $\widehat{T}^5 = \mathrm{id}$. Besides, by Lemma 1.5, $c_{-1} = p - q = 1$ and $a_{-1}^2 = b^{-2}$. Therefore,

$$\psi_w(\xi) = \begin{cases} (\pm 1)^w a_w \left(\dfrac{\xi + b}{b}\right)^w & \text{for } w \geqslant 0,\\ \dfrac{(\pm 1)^w}{a_{-w}} \left(\dfrac{\xi + b}{\xi}\right)^w & \text{for } w < 0, \end{cases}$$

where $a_w = \pm 1$ for $w \geqslant 0$.

Consider the third possibility, when $p = 0$, $q = 1$. In this case the transformation $\widehat{T}$ can be written as follows: $\widehat{T}\colon (x, y) \mapsto (cy/(x+b), a/x)$ with $b^2 = a$ and $c = \pm b$. Consider the transformation $\widetilde{T} = \widehat{T}^7$. In a neighborhood of the point $(\infty, 0)$ $\widetilde{T}$ can be expanded in local coordinates $z = 1/x$, y as

$$\begin{aligned} \widetilde{T}\colon (z, y) \mapsto \Bigg(&\frac{cz}{b} - \frac{c-b}{b^2}zy - \frac{b^2 - 3bc + 3c^2}{b^2}z^2y - \frac{b-c}{b^3}zy^2,\\ &\frac{by}{c} + \frac{b(c-b)}{c}zy - \frac{b^2(c-b)}{c}z^2y - \frac{b-2c}{c}zy^2\Bigg) + o((z+y)^3). \end{aligned}$$

Therefore, $\widetilde{T}$ can have a finite order only if $c = b$, in which case we get $\widetilde{T}\colon (z,y) \mapsto (z - z^2y, y + zy^2) + o((z+y)^3)$. In its turn, this gives $\widetilde{T}^k\colon (z,y) \mapsto (z - kz^2y, y + kzy^2) + o((z+y)^3)$, and hence $\widetilde{T}$ has an infinite order.

Finally, consider the last possibility, when $p = q = 0$. In this case the transformation $\widehat{T}$ can be written as follows: $\widehat{T}\colon (x,y) \mapsto (\pm y, a/x)$. It is easy to check that in this case indeed $\widehat{T}^4 = \mathrm{id}$. Besides, by Lemma 1.5, $c_{-1} = p - q = 0$ and $a_{-1}^2 = 1$. Therefore,

$$\psi_w(\xi) = \begin{cases} (\pm 1)^w a_w & \text{for } w \geqslant 0, \\ \dfrac{(\pm 1)^w}{a_{-w}} & \text{for } w < 0, \end{cases}$$

where $a_w = \pm 1$ for $w \geqslant 0$. $\square$

Observe that in the skew-symmetric case the transformations of type (ii) described in Theorem 1.6 coincide up to a sign with those found in Lemma 1.3. Indeed, in this case $z'_{ji} = -z_{ij}$, and hence Lemma 1.3 gives $\varphi(\xi) = 1/\xi$ and

$$\psi_{ji}(\xi) = \psi_{z_{ij}}(\xi) = \begin{cases} (\xi + 1)^{z_{ij}} & \text{for } z_{ij} \geqslant 0, \\ \left(\dfrac{\xi+1}{\xi}\right)^{z_{ij}} & \text{for } z_{ij} < 0. \end{cases}$$

It is easy to check that after the scaling $\tau \mapsto b\tau$ one gets $\varphi(\xi) = b^2/\xi$ and

$$\psi_{ji}(\xi) = \psi_{z_{ij}}(\xi) = \begin{cases} \left(\dfrac{\xi+b}{b}\right)^{z_{ij}} & \text{for } z_{ij} \geqslant 0, \\ \left(\dfrac{\xi+b}{\xi}\right)^{z_{ij}} & \text{for } z_{ij} < 0. \end{cases}$$

2. The cluster manifold

In this section we construct an algebraic variety $\mathcal{X}$ (which we call the *cluster manifold*) related to the cluster algebra $\mathcal{A}(Z)$. Our approach is suggested by considering coordinate rings of double Bruhat cells, which provide main examples of cluster algebras.

2.1. Poisson brackets on the cluster manifold. It would be very natural to define the cluster manifold as $\mathrm{Spec}(\mathcal{A}(Z))$, since $\mathrm{Spec}(\mathcal{A}(Z))$ is the maximal manifold M satisfying the following two universal conditions:

a) all the cluster functions are regular functions on M;

b) for any pair x_1, x_2 of two distinct points on M there exists a cluster function $f \in \mathcal{A}(Z)$ such that $f(x_1) \neq f(x_2)$.

However, it was shown by S. Fomin that the Markov cluster algebra defined in [FZ2] is not finitely generated. This observation shows that $\mathrm{Spec}(\mathcal{A}(Z))$ might be a rather complicated object. Therefore we define $\mathcal{X}$ as a "handy" nonsingular part of $\mathrm{Spec}(\mathcal{A}(Z))$.

We will describe $\mathcal{X}$ by means of charts and transition functions. Assume that $\mathcal{A}(Z)$ is given by an m-regular tree $\mathbb{T}_m$ (see Section 1.1). For each vertex v of $\mathbb{T}_m$ we define the chart, that is, an open subset $X = X_v \subset \mathcal{X}$ by

$$X_v = \mathrm{Spec}(\mathbb{F}[f_{v,1}, f_{v,1}^{-1}, \dots, f_{v,m}, f_{v,m}^{-1}, f_{m+1}, \dots, f_n])$$

(as before, $\mathbb{F}$ is a field of characteristic 0). An edge $(v, \bar{v})$ of $\mathbb{T}_m$ labeled by a number $i \in [1, m]$ defines a transition function $X_v \to X_{\bar{v}} = \bar{X}$ by the equations $\bar{f}_j = f_j$ if $j \neq i$, and the *three-term relation* $\bar{f}_i f_i = M_v^i + M_{\bar{v}}^i$, where M_u^i, $u \in \{v, \bar{v}\}$, are some monomials in $f_1, \dots, f_n$.

Note that the tree $\mathbb{T}_m$ is connected, hence any pair of its vertices is connected by a unique path. Therefore, the transition map between the charts corresponding to two arbitrary vertices can be computed as the composition of the transitions along this path. Finally, put

$$\mathcal{X} = \cup_{v \in \mathbb{T}_m} X_v.$$

It follows immediately from the definition that $\mathcal{X} \subset \operatorname{Spec}(\mathcal{A}(Z))$. However, $\mathcal{X}$ contains only such points $x \in \operatorname{Spec}(\mathcal{A}(Z))$ for which there exists a vertex v of $\mathbb{T}_m$ whose cluster variables form a coordinate system in some neighbourhood $\mathcal{X}(x) \subset \mathcal{X}$ of this point.

Example. Consider a cluster algebra $\mathcal{A}_1$ over $\mathbb{C}$ given by two clusters $\{f_1, f_2, f_3\}$ and $\{\bar{f}_1, f_2, f_3\}$ subject to one relation: $f_1 \bar{f}_1 = f_2^2 + f_3^2$.

In this case $\operatorname{Spec}(\mathcal{A}_1) = \operatorname{Spec}(\mathbb{C}[x, y, u, v]/\{xy - u^2 - v^2 = 0\})$ is a singular affine hypersurface $H \subset \mathbb{C}^4$ given by the equation $xy = u^2 + v^2$ and containing a singular point $x = y = u = v = 0$. On the other hand, $\mathcal{X} = H \setminus \{x = y = u^2 + v^2 = 0\}$ is nonsingular.

In the general case, the following proposition stems immediately from the above definitions and Theorem 1.4.

Lemma 2.1. *The cluster manifold $\mathcal{X}$ is nonsingular and possesses a Poisson bracket such that for each vertex v of $\mathbb{T}_m$ the corresponding extended cluster is log-canonical w.r.t. this bracket.*

Let ω be one of these Poisson brackets. Recall that a *Casimir element* corresponding to ω is a function that is in involution with all the other functions on $\mathcal{X}$. All rational Casimir functions form a subfield $\mathbb{F}_C$ in the field of rational functions $\mathbb{F}(\mathcal{X})$. The following proposition provides a complete description of $\mathbb{F}_C$.

Lemma 2.2. $\mathbb{F}_C = \mathbb{F}(\mu_1, \dots, \mu_s)$, *where μ_j has a monomial form*

$$\mu_j = \prod_{i=m+1}^{n} f_i^{\alpha_{ji}}$$

for some integer α_{ji}, and $s = \operatorname{corank} \omega$.

Proof. Fix a vertex v of $\mathbb{T}_m$ and consider an open subset

$$X_v^0 = X^0 = \{x \in \mathcal{X} : f_i(x) \neq 0, i \in [m+1, n]\}$$

(we omit in the notation the dependence of X, f and other objects on v). Define the τ-coordinates as in Section 1.2 and note that each τ_i is distinct from 0 and from ∞ in X^0. Therefore, $\widetilde{\tau}_i = \log \tau_i$ form a coordinate system in X^0, and $\omega(\widetilde{\tau}_p, \widetilde{\tau}_q) = \omega_{pq}^{\tau}$, where $\Omega^{\tau} = (\omega_{pq}^{\tau})$ is the coefficient matrix of ω in the basis τ. The Casimir functions of ω that are linear in $\{\widetilde{\tau}_i\}$ are given by the left nullspace $N_l(\Omega^{\tau})$ in the following way. Since Ω^{τ} is an integer matrix, its left nullspace contains an integral lattice L. For any vector $u = (u_1, \dots, u_n) \in L$, the sum $\sum_i u_i \widetilde{\tau}_i$ is in involution with all the coordinates $\tilde{\tau}_j$. Hence the product $\prod_{i=1}^n \tau_i^{u_i}$ belongs to $\mathbb{F}_C$; moreover $\mathbb{F}_C$ is generated by the monomials $\mu^u = \prod_{i=1}^n \tau_i^{u_i}$ for s distinct vectors $u \in L$.

Let us calculate $\log \mu^u = u\tilde{\tau}^T$, where $\tilde{\tau} = (\tilde{\tau}_1, \dots, \tilde{\tau}_n)$. Recall that by (1.3), $\tilde{\tau}^T = (Z' + K)\tilde{f}^T$, where K is a diagonal matrix whose first m diagonal entries are equal to zero, and $\tilde{f} = (\log f_1, \dots, \log f_n)$. So $u\tilde{\tau}^T = \alpha \tilde{f}^T$, where $\alpha = u(Z' + K)$. Further, consider the decompositions

$$Z' = \begin{pmatrix} Z_1 & Z_2 \\ -Z_2^T D & Z_4 \end{pmatrix}, \qquad \Omega^\tau = \begin{pmatrix} \Omega_1 & \Omega_2 \\ -\Omega_2^T & \Omega_4 \end{pmatrix},$$

where $Z_1 = Z'[m;m]$, $\Omega_1 = \Omega^\tau[m;m]$. By Theorem 1.4, $Z_1 = \Lambda^{-1}\Omega_1 D$ and $Z_2 = \Lambda^{-1}\Omega_2$; moreover it is easy to check that Λ and Ω_1 commute. Let $u = (\, u^1 \quad u^2 \,)$ and $\alpha = (\, \alpha^1 \quad \alpha^2 \,)$ be the corresponding decompositions of u and α. Since $u\Omega^\tau = 0$, one has $u^1\Omega_1 - u^2\Omega_2^T = 0$. Therefore $\alpha^1 = u^1 Z_1 - u^2 Z_2^T D = (u^1\Omega_1 - u^2\Omega_2^T)\Lambda^{-1} D = 0$, and hence $\mu^u = \prod_{i=m+1}^n f_i^{\alpha_i}$. Finally, X^0 is open and dense in X, so the above defined rational Casimir functions can be extended to the whole X, and hence to $\mathcal{X}$. □

2.2. Toric action on the cluster algebra. Assume that an integer weight $w_v = (w_{v,1}, \dots, w_{v,n})$ is given at any vertex v of the tree $\mathbb{T}_m$. Then we define a *local toric action* on the cluster at v as the $\mathbb{F}^\star$-action given by the formula $\{f_{v,1}, \dots, f_{v,n}\} \mapsto \{f_{v,1} \cdot t^{w_{v,1}}, \dots, f_{v,n} \cdot t^{w_{v,n}}\}$. We say that local toric actions are compatible if for any two extended clusters C_v and C_u the following diagram is commutative:

$$\begin{array}{ccc} C_v & \longrightarrow & C_u \\ {\scriptstyle t^{w_v}}\big\downarrow & & {\scriptstyle t^{w_u}}\big\downarrow \\ C_v & \longrightarrow & C_u \end{array}$$

In this case, local toric actions together define a global toric action on $\mathcal{A}(Z)$. This toric action is said to be an *extension* of the above local actions.

A toric action on the cluster algebra gives rise to a well-defined $\mathbb{F}^\star$-action on $\mathcal{X}$. The corresponding flow is called a *toric flow.*

Lemma 2.3. *Let Z denote the transformation matrix at a vertex v of $\mathbb{T}_m$, and let w be an arbitrary integer weight. The local toric action at v defined by $\{f_1, \dots, f_n\} \mapsto \{f_1 t^{w_1}, \dots, f_n t^{w_n}\}$ can be extended to a toric action on $\mathcal{A}(Z)$ if and only if w^T belongs to the right nullspace $N_r(Z)$. Moreover, if such an extension exists, then it is unique.*

Proof. Given a monomial $g = \prod_j f_j^{p_j}$, we define its (weighted) degree by $\deg g = \sum_j p_j w_j$. It is easy to see that local toric actions are compatible if and only if all the monomials in each relation defining the transition $X \to \bar{X}$ have the same degree.

Consider an edge $(v, \bar{v})$ of $\mathbb{T}_m$ labeled by i. Among the relations defining the transition $X \to \bar{X}$ there is a three-term relation $\bar{f}_i f_i = M^i + \bar{M}^i$. Hence, the compatibility condition implies $\deg M^i = \deg \bar{M}^i$, which by (1.1) is equivalent to

$$\sum_{z_{ik}>0} z_{ik} w_k = \sum_{z_{ik}<0} (-z_{ik}) w_k.$$

The latter condition written for all the m edges incident to v gives $Zw^T = 0$. Therefore, condition $w^T \in N_{\mathrm{r}}(Z)$ is necessary for the existence of global extension.

Let us find the weight $\bar w$ that defines the local toric action at $\bar v$ compatible with the initial local toric action at v. First, identities $\bar f_j = f_j$ for $j \neq i$ immediately give $\bar w_j = w_j$ for $j \neq i$. Next, the three-term relation gives

$$\bar w_i = \sum_{z_{ik}>0} z_{ik} w_k - w_i,$$

so the weight at $\bar v$ is defined uniquely. It remains to prove that $w^T \in N_{\mathrm{r}}(Z)$ implies $\bar w^T \in N_{\mathrm{r}}(\bar Z)$.

Let $k \neq i$, then the kth entry of $\bar Z \bar w^T$ is equal to

$$\begin{aligned}\sum_{j=1}^n \bar z_{kj}\bar w_j &= \sum_{j\neq i} z_{kj} w_j + \frac12 \sum_{j\neq i} \left(|z_{ki}| z_{ij} + z_{ki}|z_{ij}|\right) w_j - z_{ki}\left(\sum_{z_{il}>0} z_{il} w_l - w_i\right)\\ &= \sum_{j=1}^n z_{kj} w_j = 0.\end{aligned}$$

The ith entry of $\bar Z \bar w^T$ is equal to

$$\sum_{j=1}^n \bar z_{ij} \bar w_j = -\sum_{j\neq i} z_{ij} w_j = 0,$$

since $\bar z_{ii} = 0$. Hence, $\bar w^T \in N_{\mathrm{r}}(\bar Z)$. □

2.3. Symplectic leaves. Evidently, $\mathcal{X}$ is foliated into a disjoint union of symplectic leaves of the Poisson brackets ω. We are interested only in generic leaves, which means the following.

Fix some generators $q_1, \dots, q_s$ of the field of rational Casimir functions $\mathbb{F}_C$. They define a map $Q\colon \mathcal{X} \to \mathbb{F}^s$, $Q(x) = (q_1(x), \dots, q_s(x))$. Let $\mathcal{L}$ be a symplectic leaf, and let $z = (z_1, \dots, z_s) = Q(\mathcal{L}) \in \mathbb{F}^s$. We say that $\mathcal{L}$ is *generic* if there exist s vector fields u_i in a neighborhood of $\mathcal{L}$ such that

a) at every point $x \in \mathcal{L}$, the vector $u_i(x)$ is transversal to the surface $q_i(x) = z_i$, which means that $\nabla_{u_i} q_i(x) \neq 0$;

b) the translation along u_i for a sufficiently small time t gives a diffeomorphism between $\mathcal{L}$ and a close symplectic leaf $\mathcal{L}_t$.

Let us denote by $\mathcal{X}^0$ the open part of $\mathcal{X}$ given by the conditions $f_i \neq 0$ for $i \in [m+1, n]$. It is easy to see that $\mathcal{X}^0 = \cup_{v\in\mathbb{T}_m} X_v^0$.

Lemma 2.4. *$\mathcal{X}^0$ is foliated into a disjoint union of generic symplectic leaves of the Poisson bracket ω.*

Proof. Consider first the special case when the Poisson structure on $\mathcal{X}$ is nondegenerate at the generic point, i.e., its rank equals to the dimension of the manifold. Then we show that every point of $\mathcal{X}^0$ is generic, i.e., the rank of the Poisson bracket is maximal at each point. Note that for every point $x \in \mathcal{X}^0$ there exists a cluster chart X_v such that on X_v one has $f_{v,j} \neq 0$ for $j \in [1, n]$. Therefore the coordinates $\log f_{v,j}$ form a local coordinate system on X_v, and the Poisson structure written in these coordinates becomes a constant Poisson structure. If a constant

Poisson structure is nondegenerate, it is nondegenerate at each point, which proves the statement. Moreover, note that the complement $\mathcal{X} \setminus \mathcal{X}^0$ consists of degenerate symplectic leaves of smaller dimension. Hence, if the Poisson structure is symplectic, i.e., nondegenerate at a generic point, then $\mathcal{X}^0$ is a union of generic symplectic leaves.

Assume now that the rank of the Poisson structure is $r < n$. There exist $s = n-r$ Casimir functions that generate the field $\mathbb{F}_C$. By Lemma 2.2, one can build these Casimir functions by choosing s independent integer vectors u in the left nullspace $N_{\mathrm{l}}(\Omega^\tau)$ and by constructing the corresponding monomials μ^u. Observe that if $u = (\,u^1 \quad u^2\,) \in N_{\mathrm{l}}(\Omega^\tau)$ and $u' = (\,D^{-1}u^1 \quad u^2\,)$, then $(u')^T \in N_{\mathrm{r}}(Z)$. Therefore, by Lemma 2.3, such a u' defines a toric flow on $\mathcal{X}$. To accomplish the proof it is enough to show that the toric flow corresponding to the vector u' is transversal to the level surface $\{y \in \mathcal{X}^0 : \mu^u(y) = \mu^u(x)\}$, and that a small translation along the trajectory of this toric flow transforms one symplectic leaf into another one.

We will first show that if $x(t)$ is a trajectory of the toric flow corresponding to u' with the initial value $x(1) = x$ and the initial velocity vector $\nu = dx(t)/dt|_{t=1} = (u'_1 f_1, \dots, u'_n f_n)$ then $d\mu^u(\nu)/dt \neq 0$. Indeed, by Lemma 2.2,

$$d\mu^u(\nu)/dt = \sum_{i=m+1}^{n} \alpha_i \frac{\prod_{j=m+1}^{n} f_j^{\alpha_j}}{f_i} \cdot u_i f_i = \mu^u \alpha^2 (u^2)^T.$$

Since $x \in \mathcal{X}^0$, one has $\mu^u(x) \neq 0$. To find $\alpha^2(u^2)^T$ recall that by the proof of Lemma 2.2, $\alpha^2 = u^1 Z_2 + u^2 Z_4 + u^2 K'$, where K' is the submatrix of K whose entries lie in the last $n-m$ rows and columns. Clearly, $u^2 Z_4 (u^2)^T = 0$, since Z_4 is skew-symmetric. Next, $u^1 Z_2 (u^2)^T = u^1 \Lambda^{-1} \Omega_2 (u^2)^T = u^1 \Lambda^{-1} \Omega_1^T (u^1)^T = 0$, since $u^2 \Omega_2^T = u^1 \Omega_1$ and $\Lambda^{-1}\Omega_1^T$ is skew-symmetric. Thus, $\alpha^2 (u^2)^T = u^2 K' (u^2)^T \neq 0$, since K' can be chosen to be a diagonal matrix with positive elements on the diagonal, see the proof of Lemma 1.1.

Consider now another basis vector $\bar{u} \in N_{\mathrm{l}}(\Omega^\tau)$ and the corresponding Casimir function $\mu^{\bar{u}}$. It is easy to see that $d\mu^{\bar{u}}(\nu)/dt = \mu^{\bar{u}} \bar{\alpha}^2 (u^2)^T$. Note that the latter expression does not depend on the point x, but only on the value $\mu^{\bar{u}}(x)$ and on the vectors $\bar{\alpha}$ and u. Therefore the value of the derivative $d\mu^{\bar{u}}/dt$ is the same for all points x lying on the same symplectic leaf, and the toric flow transforms one symplectic leaf into another one. $\square$

In general, it not true that $\mathcal{X}^0$ coincides with the union of all "generic" symplectic leaves. A simple counterexample is provided by the cluster algebra given by two clusters $\{f_1, f_2, f_3\}$ and $\{\bar{f}_1, f_2, f_3\}$ subject to one relation: $\bar{f}_1 f_1 = f_2^2 f_3^2 + 1$. One can choose the Poisson bracket on $\mathcal{X}$ as follows: $\{f_1, f_2\} = f_1 f_2$, $\{f_1, f_3\} = f_1 f_3$, $\{f_2, f_3\} = 0$. Equivalently, $\{\bar{f}_1, f_2\} = -\bar{f}_1 f_2$, $\{\bar{f}_1, f_3\} = -\bar{f}_1 f_3$, $\{f_2, f_3\} = 0$. Generic symplectic leaves of this Poisson structure are described by the equation $Af_2 + Bf_3 = 0$ where $(A : B)$ is a homogeneous coordinate on P^1. All generic symplectic leaves form P^1. In particular, two leaves $(1:0)$ and $(0:1)$ (correspondingly, subsets $f_2 = 0, f_3 \neq 0, f_1 \bar{f}_1 = 1$ and $f_3 = 0, f_2 \neq 0, f_1 \bar{f}_1 = 1$) are generic symplectic leaves in $\mathcal{X}$. According to the definition of $\mathcal{X}^0$ these leaves are not contained in $\mathcal{X}^0$.

We can describe $\mathcal{X}^0$ as the nonsingular locus of the toric action. The main source of examples of cluster algebras are coordinate rings of homogeneous manifolds. Toric actions on such cluster algebras are induced by the natural toric actions on these manifolds.

All this suggests that $\mathcal{X}^0$ is a natural geometrical object in the cluster algebra theory, intrinsically related to the corresponding Poisson structure.

2.4. Connected components of $\mathcal{X}^0$. In what follows we assume that $\mathbb{F} = \mathbb{R}$. In this case, the first natural question concerning the topology of $\mathcal{X}^0$ is to find the number $\#(\mathcal{X}^0)$ of connected components of $\mathcal{X}^0$. To answer this question we follow the approach developed in [SSV1, SSV2, Z].

Given a vertex v of $\mathbb{T}_m$, we define an open subset $S(X^0) = S(X^0_v) \subset \mathcal{X}^0$ by

$$S(X^0_v) = X^0_v \cup \bigcup_{(v,\bar v)\in\mathbb{T}_m} X^0_{\bar v},$$

where $(v,\bar v) \in \mathbb{T}_m$ means that $(v,\bar v)$ is an edge of $\mathbb{T}_m$.

Recall that $X^0 \simeq (\mathbb{R}\setminus 0)^n$. We can decompose X^0 as follows. Let Σ be the set of all possible sequences $(\sigma(1),\dots,\sigma(n))$ of n signs $\sigma(i) = \pm 1$. For $\sigma \in \Sigma$ we define $X^0(\sigma)$ as the octant $\sigma(j)f_j > 0$ for all $j \in [1,n]$. Two octants $X^0(\sigma_1)$ and $X^0(\sigma_2)$ are called *essentially connected* if the following two conditions are fulfilled:

1) there exists $i \in [1,n]$ such that $\sigma_1(i) \ne \sigma_2(i)$ and $\sigma_1(j) = \sigma_2(j)$ for $j \ne i$;

2) there exists $x^* \in S(X^0)$ that belongs to the intersection of the closures of $X^0(\sigma_1)$ and $X^0(\sigma_2)$.

The second condition can be restated as follows:

2') there exists $x^* \in S(X^0)$ such that $f_i(x^*) = 0$, $\hat f_i(x^*) \ne 0$, and $f_j(x^*) \ne 0$ for $j \ne i$, where $\hat f = f_{\hat v}$ and $(v,\hat v)$ is the edge of $\mathbb{T}_m$ labeled by i.

Lemma 2.5. *Let $(v,\bar v)$ be an edge of $\mathbb{T}_m$, $\sigma_1,\sigma_2 \in \Sigma$, and let the octants $X^0(\sigma_1)$ and $X^0(\sigma_2)$ be essentially connected. Then $\bar X^0(\sigma_1)$ and $\bar X^0(\sigma_2)$ are essentially connected as well.*

Proof. Assume that the edge $(v,\bar v)$ is labeled by j, and consider first the case $\sigma_1(j) \ne \sigma_2(j)$. Then $\hat v$ in condition 2' coincides with $\bar v$, and hence $f_j(x^*)\bar f_j(x^*) = M^j(x^*) + \bar M^j(x^*) = 0$. Since M^j and $\bar M^j$ both do not contain f_j, any point x such that $f_l(x) = f_l(x^*)$ for $l \ne j$, $f_j(x) \ne 0$, $\bar f_j(x) = 0$ belongs to $S(\bar X^0)$, and hence $\bar X^0(\sigma_1)$ and $\bar X^0(\sigma_2)$ are essentially connected.

Assume now that $\sigma_1(i) \ne \sigma_2(i)$ for some $i \ne j$. As before, we get $M^i(x) + \widehat{M}^i(x) = 0$ for any x such that $f_l(x) = f_l(x^*)$ for $l \ne i$. Consider the edge $(\bar v,\bar{\bar v})$ of $\mathbb{T}_m$ labeled by i; by the above assumption, $\bar{\bar v} \ne v$. Without loss of generality assume that $\bar f_j$ does not enter $\bar{\bar M}^i$. Then by [FZ2] one has

$$M^i + \widehat{M}^i = \widehat{M}^i\left(\frac{M^i}{\widehat{M}^i} + 1\right) = \widehat{M}^i\left(\frac{\bar M^i}{\bar{\bar M}^i} + 1\right)\Bigg|_{\bar f_j \leftarrow \frac{M_0}{f_j}},$$

where $M_0 = M^j + \bar M^j|_{f_i=0}$. Therefore, for x^{**} such that $\bar f_l(x^{**}) = f_l(x^*)$ if $l \ne i,j$, $\bar f_j(x^{**}) = M_0(x^*)/f_j(x^*)$, $\bar f_i(x^{**}) = 0$, one has $\bar M^i(x^{**}) + \bar{\bar M}^i(x^{**}) = M^i(x^*) + \widehat{M}^i(x^*) = 0$, and hence one can choose $\bar{\bar f}_i(x^{**}) \ne 0$. $\square$

Corollary 2.6. *If $X^0_v(\sigma_1)$ and $X^0_v(\sigma_2)$ are essentially connected, then $X^0_{v'}(\sigma_1)$ and $X^0_{v'}(\sigma_2)$ are essentially connected for any vertex v' of $\mathbb{T}_m$.*

Proof. Since the tree $\mathbb{T}_m$ is connected, one can pick up the path connecting v and v'. Then the corollary follows immediately from Lemma 2.5. $\square$

Let $\#_v$ denote the number of connected components in $S(X^0_v)$.

Theorem 2.7. *The number $\#_v$ does not depend on v and is equal to $\#(\mathcal{X}^0)$.*

Proof. Indeed, since $S(X_v^0)$ is dense in $\mathcal{X}^0$, one has $\#(\mathcal{X}^0) \leqslant \#_v$. Conversely, assume that there are points $x_1, x_2 \in \mathcal{X}^0$ that are connected by a path in $\mathcal{X}^0$. Therefore their small neighborhoods are also connected in $\mathcal{X}^0$, since $\mathcal{X}^0$ is a topological manifold. Since X_v^0 is dense in $\mathcal{X}^0$, one can pick $\sigma, \sigma' \in \Sigma$ such that the intersection of the first of the above neighborhoods with $X_v^0(\sigma)$ and of the second one with $X_v^0(\sigma')$ are nonempty. Thus, $X_v^0(\sigma)$ and $X_v^0(\sigma')$ are connected in $\mathcal{X}^0$. Hence, there exist a loop γ in $\mathbb{T}_m$ with the initial point v, a subset $v_1, \dots, v_p$ of vertices of this loop, and a sequence $\sigma_1 = \sigma, \sigma_2, \dots, \sigma_{p+1} = \sigma' \in \Sigma$ such that $X_{v_l}^0(\sigma_l)$ is essentially connected with $X_{v_l}^0(\sigma_{l+1})$ for all $l \in [1,p]$. Then by Corollary 2.6 $X_v^0(\sigma_l)$ and $X_v^0(\sigma_{l+1})$ are essentially connected. Hence all $X_v^0(\sigma_l)$ are connected with each other in $S(X_v^0)$. In particular, $X_v^0(\sigma)$ and $X_v^0(\sigma')$ are connected in $S(X_v^0)$. This proves the assertion. $\square$

By virtue of Theorem 2.7, we write $\#$ instead of $\#_v$.

Let $\mathbb{F}_2^n$ be an n-dimensional vector space over $\mathbb{F}_2$ with a fixed basis $\{e_i\}$. Let $Z' \in \mathbb{Z}_{nn}^{\widehat{D}}(Z)$ be as in Section 1.2, and let $\omega = \omega_v$ be a (skew-)symmetric bilinear form on $\mathbb{F}_2^n$, such that $\omega(e_i, e_j) = z'_{ij}$. Define a linear operator $\mathfrak{t}_i \colon \mathbb{F}_2^n \to \mathbb{F}_2^n$ by the formula $\mathfrak{t}_i(\xi) = w - \omega(\xi, e_i)e_i$, and let $\Gamma = \Gamma_v$ be the group generated by $\mathfrak{t}_i$, $i \in [1,m]$.

The following lemma is a minor modification of the result presented in [Z].

Lemma 2.8. *The number of connected components $\#_v$ equals to the number of Γ_v-orbits in $\mathbb{F}_2^n$.*

For reader's convenience we will repeat the proof of this statement here.

Let us identify $\mathbb{F}_2^n$ with Σ by the following rule: a vector $\xi \in \mathbb{F}_2^n$ corresponds to $\sigma \in \Sigma$ such that $\sigma(i) = (-1)^{\xi_i}$. Abusing notation we will also write $X^0(\xi)$ instead of $X^0(\sigma)$.

Lemma 2.9. *Let ξ and ξ' be two distinct vectors in $\mathbb{F}_2^n$. Then the closures of $X^0(\xi)$ and $X^0(\xi')$ intersect in $S(X^0)$ if and only if $\xi' = \mathfrak{t}_i(\xi)$ for some $i \in [1,m]$.*

Proof. Suppose $x \in S(X^0)$ belongs to the intersection of the closures of $X^0(\xi)$ and $X^0(\xi')$. Then $f_l(x) = 0$ whenever $\xi_l \neq \xi'_l$. From the definition of $S(X^0)$ we see that there is a unique i such that $\xi_i \neq \xi'_i$; evidently, $i \in [1,m]$. Furthermore, if $(v, \bar v)$ is an edge of $\mathbb{T}_m$ labeled by i, then $\bar f_i(x) \neq 0$. Since any neighborhood of x intersects both $X^0(\xi)$ and $X^0(\xi')$, it follows that monomials M^i and $\bar M^i$ on the right hand side of the 3-term relation $f_i \bar f_i = M^i + \bar M^i$ must have opposite signs at x. Therefore

$$\xi_i - \xi'_i = 1 = \sum_{j=1}^{n} z_{ij}\xi_j.$$

Comparing this with the definition of the transvection $\mathfrak{t}_i$, we conclude that $\xi' = \mathfrak{t}_i(\xi)$, as claimed.

Conversely, suppose $\xi' = \mathfrak{t}_i(\xi) \neq \xi$, then $\sum_{j=1}^n z_{ij}\xi_j = 1$. Therefore, there exists a point $x \in S(X^0)$ such that $(-1)^{\xi_l} f_l(x) > 0$ for all $l \neq i$, and the right hand side of three 3-term relation vanishes at x. Then any neighborhood of x contains points where the signs of all f_l for $l \neq i$ remain the same, while the right hand side of the 3-term relation is positive (or negative). Thus, x belongs to the intersection of the closures of $X^0(\xi)$ and $X^0(\xi')$, and we are done. $\square$

Now we are ready to complete the proof of Lemma 2.9. Let Ξ be a Γ-orbit in $\mathbb{F}_2^n$, and let $X^0(\Xi) \subset S(X^0)$ be the union of the closures of $X^0(\xi)$ over all $\xi \in \Xi$. Each $X^0(\xi)$ is a copy of $\mathbb{R}^n_{>0}$, and is thus connected. Using the "if" part of Lemma 2.9, we conclude that $X^0(\Xi)$ is connected (since the closure of a connected set and the union of two non-disjoint connected sets are connected as well). On the other hand, by the "only if" part of the same lemma, all the sets $X^0(\Xi)$ are pairwise disjoint. Thus, they are the connected components of $S(X^0)$, and we are done. $\square$

Theorem 2.7 and Lemma 2.8 imply the following theorem.

Theorem 2.10. *The number of connected components $\#(\mathcal{X}^0)$ equals the number of Γ-orbits in $\mathbb{F}_2^n$.*

3. Poisson and cluster algebra structures on Grassmannians

3.1. Sklyanin bracket and factorization parameters. As we have seen from the discussion above, one can associate a cluster algebra with any algebraic Poisson manifold equipped with a system of rational log-canonical coordinates. A rich collection of non-trivial examples of this sort is provided by the theory of Poisson-Lie groups and Poisson homogeneous spaces. This collection includes real Grassmannians, which will serve as our main example.

Recall (see e.g. [ReST]) that a Lie group G equipped with a Poisson structure is called a Poisson-Lie group if the multiplication map $G \times G \to G$ is Poisson. First, we review the definition of the Sklyanin Poisson bracket and the standard Poisson-Lie structure on a semi-simple Lie group G.

Let B_+ and B_- be two $\mathbb{R}$-split opposite Borel subgroups, N and N_- be their unipotent radicals, $H = B_+ \cap B_-$ be an $\mathbb{R}$-split maximal torus of G, and $W = \mathrm{Norm}_G(H)/H$ be the Weyl group of G. For every $x \in N_- H N_+$, we write its unique Gauss factorization as $x = x_- x_0 x_+$ and define $x_{\leqslant 0} = x_- x_0$, $x_{\geqslant 0} = x_0 x_+$.

As usual, let $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{b}_\pm$, $\mathfrak{n}_\pm$ be Lie algebras that correspond to G, H, $B_\pm$, $N_\pm$. We denote by Φ the root system of $\mathfrak{g}$, by Φ^+ (resp. Φ^-) the set of all positive (resp. negative) roots, and by $\alpha_1, \dots, \alpha_r$ simple positive roots. To each α_i there corresponds the elementary reflection $s_i \in W$. A *reduced decomposition* for $w \in W$ is a factorization of w into a product $w = s_{i_1} \cdots s_{i_l}$ of simple reflections, where m is the smallest length of such a factorization. Then an integral vector $\mathbf{i} = (i_1, \dots, i_l)$ is called a *reduced word* that corresponds to w, while l is called the *length* of w and is denoted by $l(w)$.

We fix a Chevalley basis $\{e_\alpha, \alpha \in \Phi;\ h_i, i \in [1, r]\}$ in $\mathfrak{g}$. The Killing form on $\mathfrak{g}$ will be denoted by $\langle\ ,\ \rangle$.

Let $R \in \mathrm{End}(\mathfrak{g})$ be a skew-symmetric map. The Sklyanin bracket is defined by

$$\omega(f_1, f_2)(x) = \langle R(\nabla_{\mathrm{r}} f_1(x)), \nabla_{\mathrm{r}} f_2(x)\rangle - \langle R(\nabla_{\mathrm{l}} f_1(x)), \nabla_{\mathrm{l}} f_2(x)\rangle. \tag{3.1}$$

Here the right and left gradients ∇_{r}, ∇_{l} are defined by

$$\langle \nabla_{\mathrm{r}} f(x), \xi\rangle = \frac{d}{dt} f(x \exp(t\xi))\Big|_{t=0}, \qquad \langle \nabla_{\mathrm{l}} f(x), \xi\rangle = \frac{d}{dt} f(\exp(t\xi) x)\Big|_{t=0}$$

for $x \in G$, $\xi \in \mathfrak{g}$.

If R satisfies the modified classical Yang-Baxter equation (MCYBE)

$$[R\xi, R\eta] - R\left([R\xi, \eta] + [\xi, R\eta]\right) = -\alpha[\xi, \eta], \quad \xi, \eta \in \mathfrak{g},$$

then the bracket (3.1) satisfies the Jacobi identity and equips G with a structure of the Poisson-Lie group.

For any $\xi \in \mathfrak{g}$ we write its unique decomposition into a sum of elements in $\mathfrak{n}_-, \mathfrak{h}$ and $\mathfrak{n}_+$ as

$$\xi = \xi_- + \xi_0 + \xi_+.$$

The standard Poisson-Lie structure on G denoted by ω_G corresponds to a particular solution of MCYBE:

$$R(\xi) = \xi_+ - \xi_- \tag{3.2}$$

and can be characterized as follows [HKKR, R].

For $t \in \mathbb{C}$, denote $x_i^+(t) = \exp(te_{\alpha_i})$, $x_i^-(t) = \exp(te_{-\alpha_i})$. For every $i \in [1, r]$ one defines a group homomorphism (canonical inclusion) ρ_i from SL_2 to G by

$$\rho_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = x_i^+(t), \qquad \rho_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = x_i^-(t).$$

Let D_i be the length of the ith simple root α_i. Then ω_G is the unique Poisson-Lie structure on G such that each map

$$\rho_i \colon (SL_2, D_i \omega_{SL_2}) \to (G, \omega_G) \tag{3.3}$$

is Poisson.

Symplectic leaves of ω_G have been studied in [HL]. Their explicit description was obtained in more recent works [HKKR, R, Y, KoZ], where the key role is played by *double Bruhat cells* that were comprehensively studied in [FZ1]. We shall now recall the definition of a double Bruhat cell and review the results of [HKKR, R, KoZ] that show that each such cell is equipped with a family of log-canonical coordinates of the kind described in the first part of this paper.

Recall that the group G has two *Bruhat decompositions*, with respect to B_+ and B_-:

$$G = \bigcup_{u \in W} B_+ u B_+ = \bigcup_{v \in W} B_- v B_-.$$

The double Bruhat cells $G^{u,v}$ are defined by $G^{u,v} = B_+ u B_+ \cap B_- v B_-$. According to [FZ1], the variety $G^{u,v}$ is biregularly isomorphic to a Zariski open subset of $\mathbb{C}^{r+l(u)+l(v)}$. Furthermore, if for every pair $\mathbf{j}$, $\mathbf{k}$ of reduced words for u and v one defines a word $\mathbf{i} = (i_\nu)_{\nu=1}^m$ with $m = l(u) + l(v)$ as an arbitrary shuffle of the words $-\mathbf{j}$ and $\mathbf{k}$, then the map $x_{\mathbf{i}} \colon H \times \mathbb{C}^m \to G^{u,v}$ defined by

$$x_{\mathbf{i}}(h, t) = h \prod_{\nu=1}^{m} x_{i_\nu}^{\operatorname{sign}(i_\nu)}(t_\nu), \tag{3.4}$$

where $h \in H$ and $t = (t_1, \dots, t_m) \in \mathbb{C}^m$, restricts to a biregular isomorphism between $H \times \mathbb{C}^m_{\neq 0}$ and a Zariski open subset of $G^{u,v}$. Let us further factor h in (3.4) into $h = \rho_1(\operatorname{diag}(a_1, a_1^{-1})) \cdots \rho_r(\operatorname{diag}(a_r, a_r^{-1}))$. Parameters $a_1, \dots, a_r; t_1, \dots, t_m$ are called *factorization parameters*. Explicit formulae for the inverse of the map (3.4) were found in [FZ1].

The relevance of double Bruhat cells and factorization parameters in the context of the standard Poisson-Lie structure was observed in [HKKR, R]. It turns out that (i) for every u, v, $G^{u,v}$ is a Poisson submanifold of (G, ω_G), and (ii) for every $\mathbf{i}$, factorization parameters form a family of log-canonical coordinates on a Zariski open subset of $G^{u,v}$. Both assertions can be verified via the following construction [HKKR, R]. For $i \in [1,r]$ define subgroups $B_i^{\pm}$ of $B_{\pm}$ as images of resp. upper and lower triangular subgroups of SL_2 under the homomorphism ρ_i:

$$B_i^{+} = \rho_i\left(\left\{\begin{pmatrix} d & c \\ 0 & d^{-1}\end{pmatrix}\right\}\right), \qquad B_i^{-} = \rho_i\left(\left\{\begin{pmatrix} d & 0 \\ c & d^{-1}\end{pmatrix}\right\}\right). \tag{3.5}$$

We can view parameters c, $d \neq 0$ as coordinates on B_i^+ (resp. B_i^-) and denote the corresponding elements of $B_i^{\pm}$ by $b_i^{\pm}(c, d)$. In view of the Poisson property (3.3) of the map ρ_i, one obtains for both B_i^+ and B_i^-

$$\omega(d, c) = D_i dc. \tag{3.6}$$

For any $u, v \in W$ and the corresponding fixed $\mathbf{i}$, denote

$$B_{\mathbf{i}} = \times_{\nu=1}^{m} B_{i_\nu}^{\operatorname{sign}(i_\nu)} \tag{3.7}$$

and define the multiplication map $y_{\mathbf{i}}\colon B_{\mathbf{i}} \to G$ by

$$y_{\mathbf{i}}(b_{\mathbf{i}}) = \prod_{\nu=1}^{m} b_{i_\nu}^{\operatorname{sign}(i_\nu)}(c_\nu, d_\nu), \tag{3.8}$$

where $b_{\mathbf{i}} = \left(b_{i_1}^{\operatorname{sign}(i_1)}(c_1,\ d_1), \dots, b_{i_m}^{\operatorname{sign}(i_m)}(c_m,\ d_m)\right) \in B_{\mathbf{i}}$. This map is clearly Poisson. Moreover, comparing (3.8) and (3.4), one sees that the image of the restriction of $y_{\mathbf{i}}$ to the set where all $b_\alpha \neq 0$ is a Zariski open subset of $G^{u,v}$ that coincides with the image of restriction of $x_{\mathbf{i}}$ to $H \times \mathbb{C}^m_{\neq 0}$. It is not hard to see that factorization parameters $a_1, \dots, a_r; t_1, \dots, t_m$ are monomial functions in parameters c_ν, d_ν and since by (3.6) the latter are log-canonical, the former are log-canonical as well.

The only drawback of the log-canonical coordinate system $a_1, \dots, a_r; t_1, \dots, t_m$ is that factorization parameters are rational functions on $G^{u,v}$. However, one of the main theorems of [FZ1] gives an explicit expression for these parameters as an invertible monomial transformation of a family of functions *regular* on $G^{u,v}$, the so-called *twisted generalized minors*. For every $\mathbf{i}$, log-canonical Poisson brackets of functions in this family w.r.t. the standard Poisson-Lie structure were computed explicitly in a recent paper [KoZ] and used to refine a description of symplectic leaves of the Poisson-Lie group G. The definition and properties of generalized minors of an element $x \in G$ can be found in [FZ1] and will not be reproduced here (in the SL_n case generalized minors are the minors of a matrix x). Twisted generalized minors are generalized minors of the *twist map* $G^{u,v} \ni x \mapsto x' \in G^{u^{-1},v^{-1}}$ defined in [FZ1] as follows. For any $w = s_{i_1} \cdots s_{i_l} \in W$, pick a representative $\widehat{w}$ of w^{-1} in G given by

$$\widehat{w} = \prod_{\nu=1}^{l} \rho_{i_{l+1-\nu}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and define

$$x' = \left((\widehat{u}x)_-^{-1}\widehat{u}x\widehat{v}(x\widehat{v})_+^{-1}\right)^\theta , \tag{3.9}$$

where θ is the involutive automorphism of G uniquely determined by

$$x_i^{\pm}(t)^\theta = x_i^{\mp}(t), \qquad h^\theta = h^{-1}, \quad h \in H$$

and a_-^{-1} stands for $(a_-)^{-1}$.

We conclude this section with the result that shows that a construction of [KoZ] can be used to build another system of log-canonical regular coordinates on $G^{u,v}$ from generalized minors of x rather than its twist x'.

Theorem 3.1. *The twist map* (3.9) *is an anti-Poisson map with respect to the Sklyanin bracket* (3.1), (3.2).

Proof. First, observe that an automorphism θ itself is an anti-Poisson map from G to G. This can be easily seen from the θ's action on subgroups $B_i^{\pm}$ defined in (3.5). Thus, we only need to check that the map $x \mapsto \chi_{u,v}(x)$, where $\chi_{u,v}(x)$ is defined by the expression inside the outer brackets in (3.9), is Poisson. It suffices to check this property for elements of a Zariski open subset of $G^{u,v}$. Let $G_0^{u,v} = G^{u,v} \cap N_-HN_+$. For $x \in G_0^{u,v}$,

$$\begin{aligned}\chi_{u,v}(x) &= (\widehat{u}x)_-^{-1}(\widehat{u}x)x^{-1}(x\widehat{v})(x\widehat{v})_+^{-1} = (\widehat{u}x)_{\geqslant 0}\ x^{-1}(x\widehat{v})_{\leqslant 0}\\ &= (\widehat{u}x_{\leqslant 0})_{\geqslant 0}x_+x^{-1}x_-(x_{\geqslant 0}\widehat{v})_{\leqslant 0} = (\widehat{u}x_{\leqslant 0})_{\geqslant 0}\ x_0^{-1}(x_{\geqslant 0}\widehat{v})_{\leqslant 0}.\end{aligned}$$

Furthermore, fix reduced words $\mathbf{j} = (j_\nu)_{\nu=1}^{l(u)}$, $\mathbf{k} = (k_\gamma)_{\gamma=1}^{l(v)}$ for u and v and assume that x belongs to the image of the multiplication map $y_\mathbf{i}\colon B_\mathbf{i} = B_{-\mathbf{j}}\ \times B_\mathbf{k} \to G^{u,v}$ defined as in (3.8), (3.7), where $\mathbf{i} = (-\mathbf{j}, \mathbf{k})$. Any element b in the preimage of x under the map $y_\mathbf{i}$ can be written as $b = (b_1, b_2)$, where $b_1 \in B_{-\mathbf{j}}$, $b_2 \in B_\mathbf{k}$, and hence $x = y_\mathbf{i}(b) = y_{-\mathbf{j}}(b_1)y_\mathbf{k}(b_2) = x_-d_1d_2x_+$, where $d_1 = (y_{-\mathbf{j}}(b_1))_0$, $d_2 = (y_\mathbf{k}(b_2))_0$. Therefore,

$$\begin{aligned}\chi_{u,v}(y_\mathbf{i}(b)) &= (\widehat{u}x_{\leqslant 0}d_2^{-1})_{\geqslant 0}(d_1^{-1}x_{\geqslant 0}\widehat{v})_{\leqslant 0} = (\widehat{u}y_{-\mathbf{j}}(b_1))_{\geqslant 0}(y_\mathbf{k}(b_2)\widehat{v})_{\leqslant 0}\\ &= \chi_{u,\mathrm{id}}(y_{-\mathbf{j}}(b_1))\chi_{\mathrm{id},v}(y_\mathbf{k}(b_2)).\end{aligned}$$

Denote $\tilde{\mathbf{j}} = (j_{l(u)-\nu+1})_{\nu=1}^{l(u)}$, $\tilde{\mathbf{k}} = (k_{l(v)-\gamma+1})_{\gamma=1}^{l(v)}$. It follows from the above equality that in order to prove that the map $x \mapsto \chi_{u,v}(x)$ is Poisson it is enough to show that maps $\chi_{u,\mathrm{id}}\colon G^{u,\mathrm{id}} \to G^{\mathrm{id},u^{-1}}$ and $\chi_{\mathrm{id},v}\colon G^{\mathrm{id},v} \to G^{v^{-1},\mathrm{id}}$ are Poisson. Both maps can be treated in the same fashion, so we shall concentrate on the second one.

Let $x \in G^{\mathrm{id},v} = B_+ \cap B_-vB_-$, put $q = i_{l(v)}$ and write v as $v = ws_q$. Assume further that x is in the image of the multiplication map $G^{\mathrm{id},w} \times B_q^+ \ni (x_1, b_q^+(c,d)) \mapsto x_1b_q^+(c,d) \in G^{\mathrm{id},v}$. (The map is Poisson and its image is Zariski open in $G^{\mathrm{id},v}$.) We shall show that $\chi_{\mathrm{id},v}(x) = (\tilde{x}\widehat{w})_{\leqslant 0}$, where $\tilde{x}$ belongs to $G^{s_q,w}$ and the map $x \mapsto \tilde{x}$ is Poisson. Then the statement will follow by induction on the length of v.

Observe that

$$\begin{aligned}x\widehat{v} &= x_1\left(b_q^+(c,d)\widehat{s_q}\right)\widehat{w} = x_1\left(b_q^-(d^{-1},\ c)b_q^+(-d,1)\right)\widehat{w}\\ &= \left(x_1b_q^-(d^{-1},\ c)\right)\widehat{w}\left((\widehat{w})^{-1}b_q^+(-d,1)\widehat{w}\right).\end{aligned}$$

So we can define $\tilde{x} = x_1 b_q^-(d^{-1}, c)$ and note that $l(s_q w^{-1}) = l(w^{-1}) + 1$ implies

$$\widehat{w}^{-1} b_q^+(-d, 1)\widehat{w} \in N_+.$$

Thus, $\chi_{\mathrm{id},v}(x) = (\tilde{x}\widehat{w})_{\leqslant 0}$, and since, by (3.6), the map $b_q^+(c,d) \mapsto b_q^-(d^{-1}, c)$ is Poisson, so is the map $x \mapsto \tilde{x}$. This completes the proof. $\square$

As we have seen from the discussion above, the real part of any double Bruhat cell $G^{u,v}$ can be equipped (in more than one way) with regular log-canonical coordinates that serve as a starting point for a construction of a cluster algebra. Results of [FZ1, FZ2, HKKR, R, Y] show that both Poisson and cluster algebra structure on $G^{u,v}$ are determined by the Lie group structure of G and are compatible in the way described in the first part of this paper. We now turn to our main example, real Grassmannians, to show how a structure of a Poisson homogeneous space can also lead to a construction of a cluster algebra.

3.2. Poisson structure and log-canonical coordinates. Let $\mathcal{P}$ be a Lie subgroup of a Poisson-Lie group (G, ω_G). A Poisson structure on the homogeneous space $\mathcal{P}\backslash G$ is called *Poisson homogeneous* if the action map $\mathcal{P}\backslash G \times G \to \mathcal{P}\backslash G$ is Poisson. Conditions on $\mathcal{P}$ for the Sklyanin bracket (3.1) to descend to a Poisson homogeneous structure on $\mathcal{P}\backslash G$ are conveniently formulated it terms of the Manin triple that corresponds to (G, ω_G) and can be found, e.g., in [ReST]. In particular, these conditions are satisfied for $G = SL_n$ and

$$\mathcal{P} = \mathcal{P}_k = \left\{ \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} : A \in SL_k, C \in SL_{n-k} \right\}.$$

The resulting homogeneous space is the Grassmannian $G(k, n)$.

In what follows, we will need an explicit expression of the Poisson homogeneous brackets on $G(k, n)$. First, consider the Sklyanin bracket (3.1), (3.2) on SL_n. The form $\langle\ ,\ \rangle$ now coincides with the trace form:

$$\langle A, B\rangle = \operatorname{Tr} AB.$$

The Sklyanin bracket can be extended from SL_n to the associative algebra Mat_n of $n \times n$ matrices; it is given there by

$$\omega(f_1, f_2)(X) = \langle R(\nabla f_1(X)X), \nabla f_2(X)X\rangle - \langle R(X\nabla f_1(X)), X\nabla f_2(X)\rangle, \tag{3.10}$$

where the gradient ∇ is defined w. r. t. the trace form. In terms of matrix elements x_{ij}, $i, j \in [1, n]$, of a matrix $X \in \mathrm{Mat}_n$, (3.10) looks as follows:

$$\omega(x_{ij}, x_{\alpha\beta}) = (\operatorname{sign}(\alpha - i) + \operatorname{sign}(\beta - j))x_{i\beta}x_{\alpha j}.$$

If $X \in SL_n$ admits a factorization into block-triangular matrices

$$X = \begin{pmatrix} X_1 & 0 \\ Y' & X_2 \end{pmatrix} \begin{pmatrix} \mathbf{1}_k & Y \\ 0 & \mathbf{1}_{n-k} \end{pmatrix} = VU,$$

then Y represents an element of the cell $G_0(k, n)$ in $G(k, n)$ characterized by non-vanishing of the Plücker coordinate $\pi_{[1,k]}$.

Relations between the Plücker coordinates π_I, $I = \{i_1, \dots, i_k \colon 1 \leqslant i_1 < \dots < i_k \leqslant n\}$, and minors $Y^{\beta_1\dots\beta_l}_{\alpha_1\dots\alpha_l} = \det(y_{\alpha_i,\beta_j})^l_{i,j=1}$ of Y are given by

$$Y^{\beta_1\dots\beta_l}_{\alpha_1\dots\alpha_l} = (-1)^{kl-l(l-1)/2-(\alpha_1+\dots+\alpha_l)} \frac{\pi_{([1,k]\setminus\{\alpha_1\dots\alpha_l\})\cup\{\beta_1+k\dots\beta_l+k\}}}{\pi_{[1,k]}}.$$

Note that, if the row index set $\{\alpha_1 \dots \alpha_l\}$ in the above formula is contiguous then the sign in the right hand side can be expressed as $(-1)^{(k-\alpha_l)l}$.

It is not hard to see that the variation of $Y = Y(X)$ is given by

$$\delta Y = \begin{pmatrix} \mathbf{1}_k & 0 \end{pmatrix} V^{-1} \delta X U^{-1} \begin{pmatrix} 0 \\ \mathbf{1}_{n-k} \end{pmatrix},$$

and, therefore,

$$\nabla(f \circ Y) = U^{-1} \begin{pmatrix} 0 \\ \mathbf{1}_{n-k} \end{pmatrix} \nabla f \begin{pmatrix} \mathbf{1}_k & 0 \end{pmatrix} V^{-1}$$

and

$$\nabla(f \circ Y)X = \mathrm{Ad}_{U^{-1}} \left(\begin{pmatrix} 0 \\ \mathbf{1}_{n-k} \end{pmatrix} \nabla f \begin{pmatrix} \mathbf{1}_k & 0 \end{pmatrix} \right) = \begin{pmatrix} -Y \\ \mathbf{1}_{n-k} \end{pmatrix} \nabla f \begin{pmatrix} \mathbf{1}_k & Y \end{pmatrix},$$
$$X \nabla(f \circ Y) = \mathrm{Ad}_V \left(\begin{pmatrix} 0 \\ \mathbf{1}_{n-k} \end{pmatrix} \nabla f \begin{pmatrix} \mathbf{1}_k & 0 \end{pmatrix} \right) \in \mathfrak{n}_-.$$

Thus we obtain from (3.10)

$$\begin{aligned} 2\omega(f_1 \circ Y, f_2 \circ Y) &= \left\langle \begin{pmatrix} -R(Y\nabla f_1) & -Y\nabla f_1 Y \\ -\nabla f_1 & R(\nabla f_1 Y) \end{pmatrix}, \begin{pmatrix} -Y\nabla f_2 & -Y\nabla f_2 Y \\ \nabla f_2 & \nabla f_2 Y \end{pmatrix} \right\rangle \\ &= \langle R(\nabla f_1 Y), \nabla f_2 Y\rangle + \langle R(Y\nabla f_1), Y\nabla f_2)]\rangle = \omega_{G(n,k)}(f_1, f_2) \circ Y. \end{aligned}$$

In terms of matrix elements y_{ij}, this formula looks as follows:

$$\omega(y_{ij}, y_{\alpha\beta}) = (\mathrm{sign}(\alpha - i) - \mathrm{sign}(\beta - j)) y_{i\beta} y_{\alpha j}. \tag{3.11}$$

Next, we will introduce new coordinates on $G(k,n)$, log-canonical w. r. t. ω. This will require some preparation.

Let $I = \{i_1, \dots, i_r\}, J = \{j_1, \dots, j_r\}$ be ordered multi-indices. We denote by $I(i_p \to \alpha)$ the result of replacing i_p with α in I, by $I \setminus i_p$ the multi-index obtained by deleting i_p from I and by (αI) the multi-index $I = \{\alpha, i_1, \dots, i_r\}$. For a matrix X, we denote $X^J_I = \det X(I; J) = \det(x_{i_p j_q})^r_{p,q=1}$. Then the Laplace expansion formula implies

$$\sum_{p=1}^{r} x_{i_p\beta} X^J_{I(i_p\to\alpha)} = x_{\alpha\beta} X^J_I - X^{(\beta J)}_{(\alpha I)}. \tag{3.12}$$

We will say that $\alpha < I$ (resp. $\alpha > I$), if α is less than the minimal index in I (resp., the maximal index in I is less than α). We define $\mathrm{sign}(\alpha - I) = -\mathrm{sign}(I - \alpha)$ to be $-1, 0$ or 1, if $\alpha < I, \alpha \in I$ or $\alpha > I$, resp. Otherwise, $\mathrm{sign}(\alpha - I)$ is not defined.

Lemma 3.2. *If* $\operatorname{sign}(\alpha - I)$ *and* $\operatorname{sign}(\beta - J)$ *are defined and*

$$|\operatorname{sign}(\alpha - I) - \operatorname{sign}(\beta - J)| \leqslant 1, \tag{3.13}$$

then

$$\omega(y_{\alpha\beta}, Y_I^J) = -\left(\operatorname{sign}(\alpha - I) - \operatorname{sign}(\beta - J)\right) y_{\alpha\beta} Y_I^J. \tag{3.14}$$

Proof. It is evident from (3.11) that $\omega(y_{\alpha\beta}, Y_I^J) = 0$ if $\alpha < I$, $\beta < J$, or $\alpha > I$, $\beta > J$, so in these cases (3.14) holds true.

In general, one obtains from (3.11)

$$\begin{aligned}
\omega(y_{\alpha\beta}, Y_I^J) &= \sum_{p,q=1}^{r} (-1)^{p+q} \{y_{\alpha\beta}, y_{i_p}^{j_q}\} Y_{I\setminus i_p}^{J\setminus j_q} \\
&= \sum_{p=1}^{r} \operatorname{sign}(i_p - \alpha) y_{i_p\beta} \sum_{q=1}^{r} (-1)^{p+q} y_{\alpha j_q} Y_{I\setminus i_p}^{J\setminus j_q} \\
&\quad - \sum_{q=1}^{r} \operatorname{sign}(j_q - \beta) y_{\alpha j_q} \sum_{p=1}^{r} (-1)^{p+q} y_{i_p\beta} Y_{I\setminus i_p}^{J\setminus j_q} \\
&= \sum_{p=1}^{r} \operatorname{sign}(i_p - \alpha) y_{i_p\beta} Y_{I(i_p\to\alpha)}^{J} - \sum_{q=1}^{r} \operatorname{sign}(j_q - \beta) y_{\alpha j_q} Y_I^{J(j_q\to\beta)} .
\end{aligned}$$

Assume that $\beta \in J$. Then, in the second sum above, $Y_I^{J(j_q\to\beta)}$ can be nonzero only if $\operatorname{sign}(j_q - \beta) = 0$, which implies that the sum is zero. If in addition $\alpha \in I$, the first sum is equal to zero as well, and thus $\omega(y_{\alpha\beta}, Y_I^J) = 0$ if $\alpha \in I, \beta \in J$, which is consistent with (3.14). Otherwise, by our assumptions, $\operatorname{sign}(i_p - \alpha)$ does not depend on p and is equal to $\operatorname{sign}(I - \alpha)$. In this case, (3.12) implies

$$\omega(y_{\alpha\beta}, Y_I^J) = \operatorname{sign}(I - \alpha)(y_{\alpha\beta} Y_I^J - Y_{(\alpha I)}^{(\beta J)}) = \operatorname{sign}(I - \alpha) y_{\alpha\beta} Y_I^J.$$

This agrees with (3.14). The remaining case $\alpha \in I, \operatorname{sign}(J - \beta) = \pm 1$ can be treated in the same way. □

The following Corollary drops out immediately from Lemma 3.2 together with the Leibnitz rule for Poisson brackets.

Corollary 3.3. *Let* $A = \{\alpha_1, \dots, \alpha_l\}, B = \{\beta_1, \dots, \beta_l\}$ *be such that for every pair* (α_p, β_q), $p, q \in [1, l]$, *condition* (3.13) *is satisfied. Then*

$$\omega(Y_A^B, Y_I^J) = -\left(\sum_{p=1}^{l} (\operatorname{sign}(\alpha_p - I) - \operatorname{sign}(\beta_p - J))\right) Y_A^B Y_I^J. \tag{3.15}$$

For every (i,j)-entry of Y define $l(i,j) = \min(i-1, n-k-j)$ and

$$F_{ij} = Y_{i-l(i,j),\dots,i}^{j,\dots,j+l(i,j)}. \tag{3.16}$$

It is easy to see that the change of coordinates $(y_{ij}) \mapsto (F_{ij})$ is a birational transformation.

Proposition 3.4. *Put*

$$t_{ij} = \frac{F_{ij}}{F_{i-1,j+1}}. \tag{3.17}$$

Then

$$\omega_{G(n,k)}(\ln t_{ij}, \ln t_{\alpha\beta}) = \operatorname{sign}(j-\beta)\delta_{i\alpha} - \operatorname{sign}(i-\alpha)\delta_{j\beta}. \tag{3.18}$$

Proof. First, we will show that coordinates F_{ij} are log-canonical. For this one needs to check that conditions of Corollary 3.2 are satisfied for every pair $F_{\alpha\beta}, F_{ij}$. One has the following seven cases to consider.

1) $\alpha \leqslant i$, $\beta \leqslant j$, $i+j \leqslant n-k+1$;
2) $\alpha \leqslant i$, $\beta > j$, $\max(\alpha+\beta, i+j) \leqslant n-k+1$;
3) $\alpha \leqslant i$, $\beta \leqslant j$, $\alpha+\beta > n-k+1$;
4) $\alpha \leqslant i$, $\beta > j$, $\min(\alpha+\beta, i+j) > n-k+1$;
5) $\alpha \leqslant i$, $\beta \leqslant j$, $\alpha+\beta \leqslant n-k+1 < i+j$;
6) $\alpha \leqslant i$, $\beta > j$, $\alpha+\beta \leqslant n-k+1 < i+j$;
7) $\alpha \leqslant i$, $\beta > j$, $i+j \leqslant n-k+1 < \alpha+\beta$.

Direct inspection shows that choosing $Y_A^B = F_{ij}$, $Y_I^J = F_{\alpha\beta}$ in case 3 and $Y_A^B = F_{\alpha\beta}$, $Y_I^J = F_{ij}$ in all the remaining cases ensures that conditions of Corollary 3.3 hold true. Moreover, one can use (3.15) to compute $\omega_{G(n,k)}(\ln F_{\alpha\beta}, \ln F_{ij})$ in each of these cases; the answers are

1) $-\min(\alpha, j-\beta)$;
2) $\max(\alpha+\beta, i+j) - \max(\beta, i+j)$;
3) $i - \max(\alpha, i+j+k-n-1))$;
4) $\min(\alpha, i+j+k-n-1) - \min(\alpha+\beta+k-n-1, i+j+k-n-1)$;
5) $n+1-j-k+\max(\alpha, i+j+k-n-1) - \max(\alpha+\beta, i)$;
6) $-(n+1-k-\max(\beta, i+j))$;
7) $-\min(\alpha, i+j+k-n-1)$.

Note that formulae above remain valid if we replace all strict inequalities used to describe cases 1 through 7 by non-strict ones. Now (3.18) can be derived from the formulae above via the case by case verification, simplified by noticing that due to the previous remark, for any (i,j) and (α,β), all the four quadruples $(i,j),(\alpha,\beta)$; $(i-1,j+1),(\alpha,\beta)$; $(i,j),(\alpha-1,\beta+1)$ and $(i-1,j+1),(\alpha-1,\beta+1)$ satisfy the same set of conditions out of 1–7 (with inequalities relaxed). □

Denote $n-k$ by m. If we arrange variables $\ln t_{ij}$ into a vector

$$\tilde{t} = (\ln t_{11}, \dots, \ln t_{1m}, \dots, \ln t_{k1}, \dots, \ln t_{km}), \tag{3.19}$$

the coefficient matrix Ω_{km} of Poisson brackets (3.18) will look as follows:

$$\Omega_{km} = \begin{pmatrix} A_m & \mathbf{1}_m & \mathbf{1}_m & \cdots & \mathbf{1}_m \\ -\mathbf{1}_m & A_m & \mathbf{1}_m & \cdots & \mathbf{1}_m \\ -\mathbf{1}_m & -\mathbf{1}_m & A_m & \cdots & \mathbf{1}_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\mathbf{1}_m & -\mathbf{1}_m & -\mathbf{1}_m & \cdots & A_m \end{pmatrix} = A_m \otimes \mathbf{1}_k - \mathbf{1}_m \otimes A_k, \tag{3.20}$$

where $A_1 = 0$ and

$$A_m = \Omega_{1m}^T = \begin{pmatrix} 0 & -1 & -1 & \cdots & -1 \\ 1 & 0 & -1 & \cdots & -1 \\ 1 & 1 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}.$$

We now proceed to compute a maximal dimension of a symplectic leaf of the bracket (3.11).

First note that left multiplying $(\lambda \mathbf{1}_m + A_m^T)$ by $C_m = \mathbf{1}_m + e_{1m} - \sum_{i=2}^m e_{i,i-1}$, where e_{ij} is a $(0,1)$-matrix with a unique 1 at position (i,j), results in the following matrix:

$$B_m(\lambda) = \begin{pmatrix} \lambda-1 & 0 & 0 & \cdots & 0 & \lambda+1 \\ -\lambda-1 & \lambda-1 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda-1 & \lambda-1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda-1 & \lambda-1 \end{pmatrix}.$$

Since the determinant of $B_m(\lambda)$ is easily computed to be equal to $(\lambda-1)^m + (\lambda+1)^m$, it follows that the spectrum of A_m is given by

$$\left(\frac{\lambda+1}{\lambda-1}\right)^m = -1. \tag{3.21}$$

Next, we observe that using block row transformations similar to row transformations applied to $(\lambda \mathbf{1}_m + A_m^T)$ above, one can reduce Ω_{km} to a matrix $B_k(A_m)$ obtained from $B_k(\lambda)$ by replacing λ with A_m and 1 with $\mathbf{1}_m$. Since $(A_m - \mathbf{1}_m)$ is invertible by (3.21), we can left multiply $B_k(A_m)$ by $\operatorname{diag}((A_m-\mathbf{1}_m)^{-1}, \ldots, (A_m-\mathbf{1}_m)^{-1})$ and conclude that the kernel of Ω_{km} coincides with that of the matrix

$$\begin{pmatrix} \mathbf{1}_m & 0 & 0 & \cdots & W \\ -W & \mathbf{1}_m & 0 & \cdots & 0 \\ 0 & -W & \mathbf{1}_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbf{1}_m \end{pmatrix},$$

where $W = (A_m + \mathbf{1}_m)(A_m - \mathbf{1}_m)^{-1}$. The kernel consists of vectors of the form $\begin{pmatrix} v & Wv & \ldots & W^{k-1}v \end{pmatrix}^T$, where v satisfies condition $(W^k + \mathbf{1}_m)v = 0$. In other words, the kernel of Ω_{km} is parametrized by the (-1)-eigenspace of W^k. Due to (3.21) the dimension of this eigenspace is equal to

$$d_{km} = \#\left\{\nu \in \mathbb{C} \colon \nu^k = \nu^m = -1\right\}.$$

Moreover, it is not hard to check that

$$W = (A_m + \mathbf{1}_m)(A_m - \mathbf{1}_m)^{-1} = -e_{m1} + \sum_{i=2}^m e_{i-1,i}, \tag{3.22}$$

and therefore $W^k = \sum_{i=k+1}^m e_{i-k,i} - \sum_{i=1}^k e_{i+m-k,i}$. A (-1)-eigenspace of W^k is non-trivial if and only if there exist natural numbers p, q such that $(2p-1)k = (2q-1)m$ (we can assume $(2p-1)$ and $(2q-1)$ are co-prime). Let $l = \gcd(k,m)$, then every non-trivial (-1)-eigenvector of W^k is a linear combination of vectors $v(i) = (v(i)_j)_{j=1}^m$, $i \in [1,l]$, that can be described as follows: $v(i)_{i+\alpha l} = (-1)^\alpha$ for $\alpha = 0, \dots, \frac{m}{l} - 1$, and all the other entries of $v(i)$ vanish. To analyze the corresponding element $\begin{pmatrix} v(i) & Wv(i) & \dots & W^{k-1}v(i) \end{pmatrix}^T$ of the kernel of P_{km}, we represent it as a $k \times m$ matrix

$$V(i) = \begin{pmatrix} v(i) \\ Wv(i) \\ \vdots \\ W^{k-1}v(i) \end{pmatrix}.$$

From the form of W one concludes that $V(i)$ is a matrix of 0s and ± 1s that has a Hankel structure, i.e. its entries do not change along anti-diagonals. More precisely,

$$V(i)_{pq} = \begin{cases} (-1)^\alpha & \text{if } p+q = i + \alpha l, \\ 0 & \text{otherwise} \end{cases}$$

Here α changes from 0 to $\frac{n}{l} - 1$. Since to each element V of the kernel of Ω_{km} (represented as a $k \times m$ matrix) there corresponds a Casimir function I_V of $\omega_{G(k,n)}$ given by $I_V = \prod_{i=1,j=1}^{k,m} t_{ij}^{V_{ij}}$, the observation above together with (3.17) implies that *on an open dense subset of $G(k,n)$ the algebra of Casimir functions is generated by monomials in*

$$J_1 = F_{11}, \dots, J_k = F_{k1}, J_{k+1} = F_{k2}, \dots, J_n = F_{km}. \tag{3.23}$$

In particular,

$$I_{V(i)} = \prod_{\alpha=0}^{\frac{n}{l}-1} J_{i+\alpha l}^{(-1)^\alpha}. \tag{3.24}$$

Thus we have proved

Theorem 3.5. *Let $l = \gcd(k,n)$. The co-dimension of a maximal symplectic leaf of $G(k,n)$ is equal to 0 if $\frac{k}{l}$ is even or $\frac{n-k}{l}$ is even, and is equal to l otherwise. In the latter case, a symplectic leaf via a point in general position is parametrized by values of Casimir functions $I_{V(i)}$, $i \in [1,l]$, defined in (3.24).*

3.3. Cluster algebra structure on Grassmannians compatible with the Poisson bracket. Our next goal is to build a cluster algebra $\mathcal{A}_{G(k,n)}$ associated with the Poisson bracket (3.11). The initial cluster consists of functions

$$f_{ij} = (-1)^{(k-i)(l(i,j)-1)} F_{ij} = \frac{\pi_{([1,k]\setminus[i-l(i,j),\ i])\cup[j+k,j+l(i,j)+k]}}{\pi_{[1,k]}}, \qquad i \in [1,k], \quad j \in [1,n-k]). \tag{3.25}$$

We designate functions $f_{11}, f_{21}, \dots, f_{k1}, f_{k2}, \dots, f_{km}$ (cf. (3.23)) to serve as tropical coordinates. This choice is motivated by the last statement of Theorem 3.5 and by the following observation: let I, J be the row and column sets of the minor that represents one of the functions (3.23), then for any pair (α, β), $\alpha \in [1,k]$, $\beta \in [1,m]$, condition (3.13) is satisfied and thus, functions (3.23) have log-canonical brackets with all coordinate functions $y_{\alpha\beta}$.

Now we need to define the matrix Z that gives rise to cluster transformations compatible with the Poisson structure. We want to choose Z in such a way that the submatrix of Z corresponding to cluster coordinates will be skew-symmetric and irreducible. According to (1.7) and to our choice of tropic coordinates, this means that Z must satisfy

$$Z\Omega^F = const \cdot (\,\mathrm{diag}(P, \dots, P) \quad 0\,)$$

where $\mathcal{P} = \sum_{i=1}^{m-1} e_{i,i+1}$ is a $(m-1) \times m$ matrix and Ω^F is the coefficient matrix of Poisson brackets ω in the basis F_{ij}.

Let $\tilde{t}$ be defined as in (3.19), and let

$$\tilde{F} = (\ln F_{11}, \dots, \ln F_{1m}, \dots, \ln F_{k1}, \dots, \ln F_{km}).$$

Then $\tilde{t} = J\tilde{F}^T$, where

$$J = \begin{pmatrix} \mathbf{1}_m & 0 & \cdots & 0 & 0 \\ -S & \mathbf{1}_m & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -S & \mathbf{1}_m \end{pmatrix}$$

and $S = \sum_{i=2}^{m} e_{i-1,i}$. Then $\Omega^F = J\Omega_{km}J^T$.

Define a $(k-1)m \times km$ block bidiagonal matrix

$$V = \begin{pmatrix} \mathbf{1}_m - W^{-1} & W^{-1} - \mathbf{1}_m & \cdots & 0 & 0 \\ 0 & \mathbf{1}_m - W^{-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{1}_m - W^{-1} & W^{-1} - \mathbf{1}_m \end{pmatrix}.$$

Observe that P is the upper $(m-1) \times m$ submatrix of S, and $PW^{-1} = PS^T$.

Since by (3.20), (3.22),

$$V\Omega_{km} = 2 \begin{pmatrix} \mathbf{1}_m & -W^{-1} & 0 & \dots & 0 & 0 \\ 0 & \mathbf{1}_m & -W^{-1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mathbf{1}_m & -W^{-1} \end{pmatrix},$$

we obtain

$$\frac{1}{2}\,\mathrm{diag}(P, \dots, P)V\Omega_{km} = (\,\mathrm{diag}(P, \dots, P) \quad 0\,)\,J^T.$$

Define

$$Z = \mathrm{diag}(P, \dots, P)VJ = \begin{pmatrix} Z_0 & Z_1 & 0 & \dots & 0 \\ Z_{-1} & Z_0 & Z_1 & \dots & 0 \\ 0 & Z_{-1} & Z_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & Z_0 \end{pmatrix},$$

where $Z_0 = P(\mathbf{1}_m - W^{-1})(\mathbf{1}_m + S)$, $Z_1 = -P(\mathbf{1}_m - W^{-1})$, and $Z_{-1} == -P(\mathbf{1}_m - W^{-1})S$. Then

$$Z\Omega^F = \operatorname{diag}(P, \dots, P) V \Omega_{km} (J^T)^{-1} = 2\,(\,\operatorname{diag}(P, \dots, P) \quad 0\,)\,,$$

as needed. Note that for $x = (x_{11}, \dots, x_{1,m}, \dots, x_{k1}, \dots, x_{k,m})$ one has

$$(Zx)_{ij} = x_{i+1,j} + x_{i,j-1} + x_{i-1,j+1} - x_{i+1,j-1} - x_{i,j+1} - x_{i-1,j}.$$

It is easy to see that the submatrix of Z corresponding to the non-tropic coordinates is indeed skew-symmetric and irreducible.

The matrix Z thus obtained can be conveniently represented by a directed graph with vertices forming a rectangular $k \times (n-k)$ array and labeled by pairs of integers (i, j), and edges $(i, j) \to (i, j+1)$, $(i+1, j) \to (i, j)$ and $(i, j) \to (i+1, j-1)$ (cf. Fig. 1).

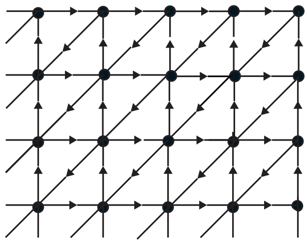

FIG. 1. GRAPH THAT CORRESPONDS TO $G(k, n)$

Proposition 3.6. *For every $i \in [1, k-1]$ and every $j \in [2, n-k]$, a cluster variable f'_{ij} obtained via the cluster transformation from the initial cluster* (3.25) *is a regular function on $G_0(k, n)$.*

Proof. The proof is based on Jacobi's determinantal identity

$$A^{(\gamma\delta J)}_{(\alpha\beta I)} A^J_I = A^{(\gamma J)}_{(\alpha I)} A^{(\delta J)}_{(\beta I)} - A^{(\delta J)}_{(\alpha I)} A^{(\gamma J)}_{(\beta I)}\ . \tag{3.26}$$

We will consider the following cases:

(i) $1 < i < n-k+1-j$. In this case $f_{ij} = (-1)^{(k-i)i} Y^{[j,j+i-1]}_{[1,i]}$, and (3.26), (3.25) imply

$$\begin{aligned}(-1)^{i(k+i)} f_{i+1,j-1} f_{i,j+1} f_{i-1,j} &= Y^{[j-1,j+i-1]}_{[1,i+1]} Y^{[j,j+i-2]}_{[1,i-1]} Y^{[j+1,j+i]}_{[1,i]} \\ &= \left(Y^{[j,j+i-1]}_{[1,i-1]\cup i+1} Y^{[j-1,j+i-2]}_{[1,i]} - Y^{[j-1,j+i-2]}_{[1,i-1]\cup i+1} Y^{[j,j+i-1]}_{[1,i]} \right) Y^{[j+1,j+i]}_{[1,i]}\end{aligned}$$

and

$$\begin{aligned}(-1)^{i(k+i)} f_{i+1,j} f_{i,j-1} f_{i-1,j+1} &= Y^{[j,j+i]}_{[1,i+1]} Y^{[j+1,j+i-1]}_{[1,i-1]} Y^{[j-1,j+i-2]}_{[1,i]} \\ &= \left(Y^{[j+1,j+i]}_{[1,i-1]\cup i+1} Y^{[j,j+i-1]}_{[1,i]} - Y^{[j,j+i-1]}_{[1,i-1]\cup i+1} Y^{[j+1,j+i]}_{[1,i]} \right) Y^{[j+1,j+i]}_{[1,i]}.\end{aligned}$$

Therefore,

$$f'_{ij} = \frac{f_{i+1,j-1}f_{i,j+1}f_{i-1,j} + f_{i+1,j}f_{i,j-1}f_{i-1,j+1}}{f_{ij}}$$
$$= Y^{[j+1,j+i]}_{[1,i-1]\cup i+1} Y^{[j-1,j+i-2]}_{[1,i]} - Y^{[j-1,j+i-2]}_{[1,i-1]\cup i+1} Y^{[j+1,j+i]}_{[1,i]}.$$

Other cases can be treated similarly. Below, we present corresponding expressions for f'_{ij}.

(ii) $n-k+1-i < j < n-k$. Then

$$f'_{ij} = Y^{j-1\cup[j+1,m]}_{[\alpha+1,i+1]} Y^{[j,m]}_{[\alpha-1,i-1]} - Y^{j-1\cup[j+1,m]}_{[\alpha-1,i-1]} Y^{[j,m]}_{[\alpha+1,i+1]},$$

where $m = n-k$ and $\alpha = i = j = k-n$.

(iii) $n-k+1-i = j < n-k$. Then

$$f'_{ij} = (-1)^{k-i}\left(Y^{[j-1,m]}_{[1,i+1]} Y^{[j+1,m-1]}_{[2,i-1]} - Y^{[j-1,m-1]}_{[2,i+1]} Y^{[j+1,m]}_{[1,i-1]}\right).$$

(iv) $i = 1$, $j < n-k$. Then $f'_{1j} = Y^{j-1,j+1}_{12}$.

(v) $i > 1$, $j = n-k$. Then $f'_{1j} = -Y^{n-k-1,n-k}_{i-1,i+1}$.

(vi) $i = 1$, $j = n-k$. Then $f'_{1,n-k} = (-1)^k y_{2,n-k-1}$.

In all the six cases, f'_{ij} is a polynomial in variables y_{pq}, which proves the assertion. $\square$

Proposition 3.7. *For every $i \in [1, k-1]$ and every $j \in [2, n-k]$, the coordinate function y_{ij} belongs to some cluster obtained from the initial one.*

Proof. Let $m = n-k$ as before, and let $\bar{Y}$ be the matrix obtained from Y by deleting the first row and the last column. Denote by $\bar{F}_{ij}$, $\bar{f}_{ij}$, $i \in [2, k]$, $j \in [1, n-k-1]$, the functions defined by (3.16), (3.25) with Y replaced by $\bar{Y}$. Define also $f = (f_{ij})$ and $\bar{f} = (\bar{f}_{ij})$.

Let us consider the following composition of cluster transformations:

$$T = T_{k-1} \circ \cdots \circ T_1, \tag{3.27}$$

where

$$T_\gamma = T_{k-1,m-\gamma+1} \circ \cdots \circ T_{\gamma+1,m-\gamma+1} \circ T_{\gamma\ 2} \circ \cdots \circ T_{\gamma,m-\gamma+1}$$

for $\gamma = 1, \dots, k-1$. Note that every cluster transformation T_{ij}, $i = 2, \dots, k$, $j = 1, \dots, n-k-1$, features in (3.27) exactly once.

We claim that

$$\begin{aligned} (Tf)_{ij} &= \bar{f}_{ij}, && i \in [1, k-1]; j = 2, m-1, \\ (TZ)_{(ij),(\alpha\beta)} &= Z_{(ij),(\alpha\beta)}, && i, \alpha \in [1, k-1]; j, \beta = 2, m-1; j+\beta > 2; i+\alpha < 2k. \end{aligned} \tag{3.28}$$

In particular, $(Tf)_{1j} = Y_{2,k-1}$, $j \in [2, m]$, and $(Tf)_{im} = Y_{i+1,m-1}$, $j \in [2, m]$. Applying the same strategy to $\bar{Y}$ etc., we will eventually recover all matrix entries of Y.

To prove (3.28), it is convenient to work with graphs associated with the matrix Z and its transformations, rather that with matrices themselves. Using the initial graph given on Fig. 1 and using the remark at the end of S 1.1, it is not hard to convince oneself that at the moment when T_{ij} (considered as a part of the composition (3.27)) is applied, the corresponding graph changes according to Fig. 2. In the latter figure, the white circle denotes the vertex (i, j) and only the vertices connected with (i, j) are shown. If $i = 1$ (resp., $j = m$) then vertices above (resp. to the right of) (i, j) and edges that connect them to (i, j) should be ignored.

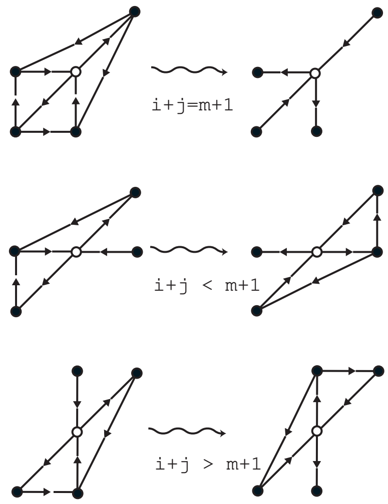

FIG. 2. TRANSFORMATION T_{ij}

It follows from Fig. 2 that

(i) the direction of an edge between (α, β) and $(\alpha + 1, \beta - 1)$ changes when $T_{\alpha\beta}$ is applied and is restored when $T_{\alpha+1,\beta-1}$ is applied;

(ii) a horizontal edge between (α, β) and $(\alpha, \beta + 1)$ is erased when $T_{\alpha-1,\beta-1}$ is applied and is restored with the original direction when $T_{\alpha+1,\beta}$ is applied;

(iii) a vertical edge between (α, β) and $(\alpha+1, \beta)$ is erased when $T_{\alpha,\beta+1}$ is applied and is restored with the original direction when $T_{\alpha-1,\beta+1}$ is applied;

(iv) an edge between $(\alpha, \beta + 1)$ and $(\alpha + 1, \beta - 1)$ is introduced when $T_{\alpha\beta}$ is applied and is erased when $T_{\alpha+1,\beta}$ is applied;

(v) an edge between $(\alpha - 1, \beta)$ and $(\alpha + 1, \beta - 1)$ is introduced when $T_{\alpha\beta}$ is applied and is erased when $T_{\alpha,\beta-1}$ is applied.

Thus, after applying all transformations that constitute T in (3.27), one obtains a directed graph whose upper right $(k - 1) \times (m - 1)$ part (not taking into account edges between the vertices in the first column and the last row) coincides with that of the initial graph on Fig. 1. This proves the second equality in (3.28).

To prove the first equality in (3.28), note that, by the definition of (3.27), a cluster coordinate f_{ij} changes to $(Tf)_{ij}$ at the moment when T_{ij} is applied and stays unchanged afterwards. In particular, on Fig. 2, coordinates that correspond to vertices above and to the right of (i, j) are coordinates of Tf, while coordinates that correspond to vertices above and to the right of (i, j) are coordinates of f.

Thus, (3.28) will follow if we show that

$$\begin{aligned}
f_{ij}\bar f_{ij} &= f_{i+1,j-1}\bar f_{i-1,j+1} + f_{i,j-1}\bar f_{i+1,j}, \quad i+j=m+1,\\
f_{ij}\bar f_{ij} &= f_{i+1,j-1}\bar f_{i-1,j+1} + f_{i,j-1}\bar f_{i,j+1}, \quad i+j<m+1,\\
f_{ij}\bar f_{ij} &= f_{i+1,j-1}\bar f_{i-1,j+1} + f_{i+1,j}\bar f_{i-1,j}, \quad i+j>m+1.
\end{aligned}$$

But referring to definitions of f and $\bar f$, one finds that the three equations above are just another instances of Jacobi's identity (3.26). The proof is complete. □

3.4. The number of connected components of refined Bruhat cells in real Grassmanians. Consider the union of regular $\mathbb{R}^*$-orbits in $\mathcal{X}^0_{G(k,n)}$ corresponding to the described above cluster algebra $\mathcal{A}_{G(k,n)}$ compatible with the Sklyanin Poisson bracket in $G(k,n)$. Recall that by construction functions $f_{11},\dots,f_{k1},f_{k2},\dots,f_{km}$ serve as tropical coordinates. Moreover, any matrix element in the standard representation of the maximal Bruhat cell in the Grassmanian enters as a cluster coordinate for some cluster in $\mathcal{A}_{G(k,n)}$. Therefore, $\mathcal{X}^0_{G(k,n)}$ is naturally embedded into $G(k,n)$ and we can consider it as the subset in $G(k,n)$ determined by the conditions that all tropical coordinates f_{ik} and f_{kl} do not vanish. Tropical coordinates are all "cyclically dense" minors among all the Plücker coordinates, i.e., minors containing all columns with indices $i,i+1,\dots,i+k$ or $i,i+1,\dots,i+l=n,1,2,\dots,k+i+1-n$. We call $\mathcal{X}^0_{G(k,n)}$ a *refined open Bruhat cell* in $G(k,n)$, since it is an intersection of n usual open Bruhat cells in $G(k,n)$ in general position.

A method to compute the number of connected components of $\mathcal{X}^0_{G(k,n)}$ was discussed in Section 2 of this paper. Let us recall certain notions and results from [SSVZ].

We denote by $\mathcal{E}$ the graph corresponding to the matrix Z (see the remark at the end of Section 1.1) and by $\mathbb{F}_2^{\mathcal{E}}$ the vector space generated by the characteristic vectors of the vertices of $\mathcal{E}$. Similarly, $\mathbb{F}_2^{\mathcal{C}}$ denotes a subspace of $\mathbb{F}_2^{\mathcal{E}}$ generated by the vertices corresponding to cluster variables. A finite (undirected) graph is said to be E_6-*compatible* if it is connected and contains an induced subgraph with 6 vertices isomorphic to the Dynkin graph E_6. A directed graph is said to be E_6-compatible if the corresponding undirected graph obtained by replacing each directed edge by an undirected one is E_6-compatible.

Theorem 3.8 ([SSVZ, Th. 3.11]). *Suppose that the induced subgraph of $\mathcal{E}$ spanned by the vertices corresponding to cluster variables is E_6-compatible. Then the number of Γ-orbits in $\mathbb{F}_2^{\mathcal{E}}$ is equal to*

$$2^t\cdot(2+2^{\dim(\mathbb{F}_2^{\mathcal{C}}\cap\ker\ \overline{\Omega})}),$$

where $\overline{\Omega}$ denotes the $\mathbb{F}_2$-valued bilinear form on $\mathbb{F}_2^{\mathcal{E}}$ obtained by reduction modulo 2 from the form Ω and t is the number of tropic variables.

Combining this theorem with Theorem 2.10 we get the following corollary.

Corollary 3.9. *The number of connected components of a refined open Bruhat cell $\mathcal{X}^0_{G(k,n)}$ is equal to $3\cdot 2^{n-1}$ if $k>3$, $n>7$.*

Proof. Indeed, by Theorem 2.10 we know that the number of connected components of $\mathcal{X}^0_{G(k,n)}$ equals the number of orbits of Γ-orbits in $\mathbb{F}_2^{\mathcal{E}}$, where the graph $\mathcal{E}$ is shown on Fig. 1, and the subset C is formed by all the vertices except for the first column and the last row. Since in the case $k \geqslant 4$, $n \geqslant 8$ the subgraph spanned by C is evidently E_6-compatible, Theorem 3.8 implies that to prove the statement it is enough to show that $\mathbb{F}_2^C \cap \ker \overline{\Omega} = 0$; in other words, that there is no nontrivial vector in $\ker \overline{\Omega}$ with vanishing tropical components.

Indeed, let us denote such a vector by $h \in \mathbb{F}_2^C$, and let δ_{ij} be the i,j-th basis vector in $\mathbb{F}_2^{\mathcal{E}}$. Note that the condition $\overline{\Omega}(h, \delta_{k,n-k}) = 0$ implies that $h_{k-1,n-k} = 0$. Further, assuming $h_{k-1,n-k} = 0$ we see that the condition $\overline{\Omega}(h, \delta_{k,n-k-1}) = 0$ implies $h_{k-1,n-k-1} = 0$ and so on. Since $\overline{\Omega}(h, \delta_{kj}) = 0$ for any $j \in [1,m]$, we conclude that $h_{k-1,j} = 0$ for any $j \in [1, n-k]$. Proceeding by induction we prove that $h_{ij} = 0$ for any $i \in [1,k]$ and any $j \in [1, n-k]$. Hence $h = 0$. Note that $t = n$, which accomplishes the proof of the statement. □

It is easy to notice that the number of connected components for $\mathcal{X}^0_{G(k,n)}$ equals the number of connected components for $\mathcal{X}^0_{G(n-k,n)}$. Therefore, taking in account Corollary 3.9, in order to find the number of connected components for refined open Bruhat cells for all Grassmanians we need to consider only two remaining cases: $G(2,n)$ and $G(3,n)$.

Proposition 3.10. *The number of connected components of a refined open Bruhat cell equals to* $(n-1)\cdot 2^{n-2}$ *for* $G(2,n)$, $n \geqslant 3$, *and to* $3 \cdot 2^{n-1}$ *for* $G(3,n)$, $n \geqslant 6$.

Proof. The proof follows immediately from Lemmas 1,2 and corollary of Lemma 3 of [GSV]. Following notations of [GSV], let us denote by U the subgraph of the graph $\mathcal{E}$ (for the Grassmanian $G(k,n)$) consisting of the first $k-1$ rows, and by L the subgraph containing only the last row. The corresponding vector subspaces of $\mathbb{F}_2^{\mathcal{E}}$ are denoted by $\mathbb{F}_2^U$ and $\mathbb{F}_2^L$; the corresponding subgroups of Γ are denoted by Γ^U and Γ^L. Lemma 1 of [GSV] states that for any $G(k,n)$ we have $2^{n-k}\#(\Gamma^U)$ orbits of Γ-action, where 2^{n-k} is the number of vectors in $\mathbb{F}_2^L$, and $\#(\Gamma^U)$ is the number of Γ^U-orbits in $\mathbb{F}_2^U$. Lemma 2 counts $\#(\Gamma^U) = n-1$ for $G(2,n)$, $n \geqslant 3$. Finally, the corollary of Lemma 3 calculates $\#(\Gamma^U) = 12$ for $G(3,n)$, $n \geqslant 6$. Substituting these values of $\#(\Gamma^U)$ into the above mentioned formula from Lemma 1 proves the proposition. □

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